Einige Beiträge zur

Homotopie Theorie

eingereicht als Habilitationsschrift

zur Erlangung der Venia legendi

für das Fach Mathematik

am Fachbereich Mathematik/Informatik

der Universität Osnabrück

vorgelegt von

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2016 Osnabrück

In der Habilitation enthaltene Publikationen

1 Fiber wise infinite products and M-category . Bull. Korean Math. Soc. 36, (1999), 671-682. (with Hans Scheerer)
2 An theory, L.S category and strong category. Mathematische Zeitschrift 247 (2007), 81-106.
3 Lusternik-Schnirelmann category and products of local spaces. Homology Homotopy and Applications 11(2) (2009), 275-307.
4 A model categorical approach to group completion. Homotopy Theory and related Structures (2012), 207-221.
5 The Arone-Goodwillie spectral sequence and topological realization at an odd prime. AGT 13 (2013), 127-169. (with Sebastian Bscher, Fabian Hebestreit and Oliver Rndigs)
6 Algebraic structures on categories and applications to iterated loop spaces Advances in Mathematics 248 (2013), 1089-1155. (with Zig Fiedorowicz and Rainer Vogt)
7 Rectification of weak product algebras over an operad in Cat and Top and applications. AGT 16 (2016), 711755. (with Zig Fiedorowicz and Rainer Vogt)
8 The realization space of an unstable coalgebra. Asterisque 393 (2017) viii-148 pages, Societe Mathematique de France. (with Georg Biedermann and George Raptis)

Contents

1. Lusternik-Schnirelmann category

The Lusternik-Schnirelmann category cat(X) of a topological space X is defined to be the least number n such that there are $n+1$ open subsets which cover X and which are contractible in X. A classical theorem of Lusternik and Schnirelmann [53] states that cat$(M)+1$ is a lower bound for the number of critical points of a smooth function f on the smooth manifold M. Due to this result, the invariant cat plays an important role in geometry and analysis [18]. Under mild point-set topology assumptions, there are equivalent descriptions of cat according to Ganea and Whitehead which are manifestly homotopy invariant in nature. A first one, given by Whitehead, defines cat$(X) \leq n$ if the diagonal

$$\Delta^{n+1} : X \longrightarrow X^{n+1}$$

factors up to homotopy through the fat wedge $X^{<n+1>}$. A second one, corresponing to Ganea, puts $cat(X) \leq n$ if the homotopy fibration

$$\Omega X^{*n+1} \to B_n\Omega X \to B_\infty \Omega X \simeq X$$

admits a homotopy section. Here, ΩX^{*n+1} is the $(n+1)$-fold join of the Moore loop space ΩX, and $B_n\Omega X$ stands for the n^{th} stage of the classifying space of the topological monoid ΩX. There are also relative versions $cat(p)$ which take as input a fibration $p : Y \to X$.

It is a classical fact that LS-category satisfies an inequality

$$cat(X \times Y) \leq cat(X) + cat(Y).$$

There are well-known examples, involving torsion at different primes in the homology of X, Y, for which the inequality is strict. Relatively recently, Iwase gave also torsion-free examples in which one of the factors is a sphere [46]. They serve as counter examples for the famous Ganea conjecture which predicted that

$$cat(X \times S^n) = cat(X) + 1.$$

In contrast, for simply connected rational spaces of finite type X_0, Y_0, Felix, Halperin, and Lemaire showed in [26] that the equality

$$cat(X_0 \times Y_0) = cat(X_0) + cat(Y_0)$$

holds. They built on important earlier work of Hess and Jessup which, taken together, settles the case in which Y_0 is the rational sphere S^n_0. Jessup proved a product theorem for the so-called module category

$Mcat$ [47]. This invariant is at first sight a weak approximation of cat. It was at first defined in a purely algebraic way. To be more precise, one considers the minimal Sullivan model ΛV [73] of X_0 and the quotient map

$$\Lambda V \stackrel{p}{\to} \Lambda V/\Lambda^{>n}V$$

where $\Lambda^{>n}V$ stands for the $(n+1)$-th power of the augmentation ideal. Consider a factorization $p = qi$ into cofibration and trivial fibration, in the model category introduced in [11]. Then, by a famous theorem of Felix and Halperin [25], $cat(X_0)$ is the smallest n such that i has a retraction of dg algebras. That is, the map i is a model of the n-th Ganea fibration in a weak sense. If one asks only for a retraction of dg ΛV-modules, one arrives at the definition of the invariant $Mcat$.

A little later, Hess proved the surprising theorem [45] that in fact there is an equality

$$cat(X_0) = Mcat(X_0).$$

Before we go further, we provide some background information on the algebraic models of the homotopy category which show up in our work on the subject.

There are algebraic models involving dg commutative algebras, dg Lie algebras, or dg Hopf algebras for the homotopy theory of 1-connected p-local complexes of fixed dimension localized at a large enough prime. Hence, no notion of E_∞-structures is needed in this restricted context. More generally, the homotopy category of tame spaces can be modeled in this way. Without going into details, a space is tame if the divisibility of the homotopy groups increases fast enough to prevent stable k-invariants from appearing.

It was Dwyer, following Quillen's treatment of rational homotopy theory, who constructed a dg Lie-version of tame homotopy theory based on so-called Lazard algebras [22]. These are decreasingly filtered Lie algebras with increasing divisibility on the filtrations.

As in Sullivan's approach to rational homotopy theory [73], a de Rham theorem for a dg algebra of polynomial forms is central to the version of tame theory based on commutative dg algebras. This generalization of the Sullivan-de Rham theorem is due to Cenkl-Porter [14]. The Cenkl-Porter algebra $T^{*,*}(X)$ on a simplicial set X is a filtered commutative dg algebra of polynomial forms on the cubical subdivided simplexes of a simplicial set. It is filtered by polynomial degree in a way so that the k-th filtration $T^{*,k}(X)$ is a module over the ring $\mathbb{Z}[\frac{1}{p}|p \leq k]$.

Integration defines a homology equivalence from $T^{*,k}(X)$ to the singular cochaines on X with coefficients in $\mathbb{Z}[\frac{1}{p}|p \leq k]$. Of course, the filtrations are not closed under the wedge product of forms. The homotopy theory of filtered dg commutative algebras was developed by Cenkl and Porter in a series of preprints and then further by the F.U. topology group, who gave a model categorical presentation, in some preprints and several theses [9].

A little later Anick constructed a version of tame homotopy theory based on so-called Hopf algebras up to homotopy [2] which is somehow intermediate between the dg Lie and dg commutative algebra versions. The main algebraic models used by Anick are universal enveloping algebras of dg Lie-algebras.

In a series of papers, Scheerer and Tanré explored the theory further and gave applications [61],[62],[63],[64].

A large part of rational homotopy theory can be extended to this large prime p-local setting. For example, there are minimal models which are unique up to isomorphism and an interpretation of the quadratic part of the differential of a minimal model in terms of mod-p Whitehead products. For us most important is a description of LS category [61] parallel to the one given in the Felix-Halperin theorem quoted above. Recently, classical rational theorems of Halperin on free torus actions [42] have been generalized by Hanke to free $(\mathbf{Z}/p\mathbf{Z})^r$-actions by use of tame homotopy theory [43]. Moreover, there are decomposition theorems for tame loop spaces and suspensions which state that every loop space is homotopy equivalent to a product of Eilenberg-MacLane spaces and every suspension is equivalent to a wedge product of Moore spaces. Hence, stable tame homotopy theory is rather trivial, a property which is shared by rational stable homotopy theory. However, the Bockstein operators add a new layer of structure to the tame theory.

After the appearance of the rational results on the additivity of *cat* quoted above, Scheerer and Tanré and also Hess conjectured that there should be a large prime version of the main results in [26]. This was provided by the following theorem [70]:

Theorem 1.1. *Let X, Y be n-connected p-local CW-complexes of finite $\mathbb{Z}_{(p)}$-type, and R a quotient ring of $\mathbb{Z}_{(p)}$. Suppose that*

$$dim(X) + dim(Y) \leq min(n + 2p - 3, np - 1),$$

and that $\tilde{H}_*(\Omega X, \mathbb{Z}_{(p)}), \tilde{H}_*(\Omega Y, \mathbb{Z}_{(p)})$ are free R-modules for $* \leq dim(X) + dim(Y) - 1$. Then $cat(X \times Y) = cat(X) + cat(Y)$.

As a corollary one obtains a mod p version of the Ganea conjecture under restrictions:

Corollary 1.2. *Let $P^l(R)$ be the Moore space for R with top cell in dimension l. Suppose that Y is as in the theorem above and that $dim(Y)+l \leq min(n+2p-3, np-1)$. Then $cat(Y \times P^l(R)) = cat(Y)+1$.*

In a first main step, the equality $cat(X) = Mcat(X)$ is deduced from a deformation argument. As one may expect, the action of the Bockstein operators causes some problems. It is only here where the assumptions on the loop space homology are needed. In a further step, the additivity for $Mcat$ is reduced to show additivity of the Toomer invariant e [77]. Finally, a duality argument shows additivity for e. The rational results serve as a blue-print. We also rely heavily on work of Scheerer and Tanré which, in the p-local theory, connects holonomy and LS category [64].

What is missing so far is to assemle the p-local and rational results by some arithmetic square argument. But the behaviour of cat with respect to localization is not fully understood. There are infinite complexes X such that $cat(X_{(p)}) < m$ for all primes p but $cat(X) = m$. No such examples are known for finite X (see [58]).

In [66] we gave, together with Hans Scheerer, a topological interpretation of the invariant $Mcat$ (see also chapter 5.6. of [18]).

Consider for a fibration $p : E \to B$ of spaces the fibration $Sym^\infty_f(E) \to B$ obtained by application of a fiberwise version of the infinite symmetric power functor to the fibration p.

Definition 1.3. Define $Sym^\infty cat(X)$ to be the smallest n such that

$$Sym^\infty_f(G_n(p)) \to B$$

has a section.

Let A be an commutative dg algebra over the rationals. The forget functor from dg commutative algebras under A to pointed dg A-modules has a left adjoint $\hat{U}(M)$. The functor $\hat{U}$ shows up in the next theorem which gives an interpretation of the invariant $Sym^\infty cat(X)$ in the rational context:

Theorem 1.4. *Let $p : Y \to X$ be a fibration of 1-connected rational spaces of finite type. Moreover, let ΛV be a cofibrant model of X and $\Lambda V \to \Lambda V \otimes_\tau \Lambda W$ be a cofibration which models p. Then*

$$\Lambda V \to \hat{U}(\Lambda V \otimes_\tau \Lambda W)$$

is a model for the fibration

$$Sym_f^\infty(Y) \to X.$$

The subscript τ above indicates that the differential is twisted.

We obtain the following interpretation of Hess's result:

Corollary 1.5. *Let X be a 1-connected rational space of finite type. Then the Ganea fibration*

$$G_n(X) \to X$$

admits a section if, and only if, the symmetrized Ganea fibration

$$Sym_f^\infty(G_n(X)) \to X$$

admits one.

Remark 1.6. There is also a version of 1.4 in the tame p-local setting.

The functor Sym^∞ can be replaced by others, for example by the functor $Q(-)$ representing stabilization. The idea to define invariants by fiberwise linearization of Ganea fibrations has been taken up by several authors [78],[65] who proved additivity theorems for the linearized invariants. Recently, the authors of [29] gave an interpretation of the homotopy category of unbounded dg ΛV-modules in terms of rational spectra parametrized by the rational space X_0 whose model is ΛV.

The strong category $\mathrm{Cat}(X)$ was introduced by Fox [34] as an approximation to $\mathrm{cat}(X)$. It is the smallest number n such that there is a space of the homotopy type of X which can be covered by n+1 open contractible subsets. The strong category can be described as cone length. That is, $\mathrm{Cat}(X) \leq n$ if, and only if, there are cofibration sequences

$$L_i \to X_i \to X_{i+1}, \qquad 0 \leq i < n$$

with $X_0 \simeq *$ and $X_n \simeq X$ [37]. It was proved by Cornea [16],[17] that one can find a n-cone presentation in which $L_i \simeq \Sigma^i Z_i$.

In [72], we study relations between $\mathrm{cat}(X)$ and $\mathrm{Cat}(X)$ of a topological space X. In general, there is an inequality [37]:

$$\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X) + 1.$$

Until recently, there were no spaces X known with $2 \leq \mathrm{cat}(X) \neq \mathrm{Cat}(X)$. This has now changed. The first example, found by Dupont

[21], is a rational space with $\mathrm{cat}(X) = 3$ and $\mathrm{Cat}(X) = 4$. There can be no rational example of lower category [28]. A little later, Stanley constructed, for each n spaces X_n with $\mathrm{cat}(X_n) = n$ and $\mathrm{Cat}(X_n) = n + 1$ [69]. Stanley's work was inspired by the ideas used by Iwase in the construction of his counter-example to the Ganea conjecture [46]. In a certain range, both invariants agree. For example, Clapp and Puppe, improving on an earlier result of Ganea [37], showed [14] that, for X a finite $(l-1)$-connected $(2 \leq l)$ CW-complex with $dim(X) \leq (2\mathrm{cat}(X) + 1)l - 3$, the equality $\mathrm{cat}(X) = \mathrm{Cat}(X)$ holds. The case of category one was already studied by Ganea [35]. The condition $\mathrm{cat}(X) = 1$ is equivalent to X being a co-H space, and $\mathrm{Cat}(X) = 1$ translates into X being of the homotopy type of a suspension. For a co-H space X which admits the structure of a coalgebra up to homotopy over the cotriple defined by the functor $\Sigma\Omega$, a theorem in [35] tells us that already the inequality $\dim(X) \leq 4l - 5$ implies that $\mathrm{Cat}(X) = 1$. Later, Saito showed that this bound can be further improved for co-A_4 spaces [59].

These results were generalized independently by Arkowitz and Golasinski [3] on the one hand, and by Klein, Schwänzl, and Vogt on the other [49]. Both groups apply a version of A_∞-theory for co-H spaces. In contrast to [49] where the comultiplication is the main object of study, Arkowitz and Golasinski work with a section for the first Ganea fibration instead. Following a suggestion of Ganea in [35], the notion of a homotopy coalgebra of order r over $\Sigma\Omega$ is introduced which plays the part of an A_r space structure. The main theorem in [3] is then that a finite 2-connected co-H space whose co-H structure satisfies all higher coherencies can be desuspended. In [49], the same result is achieved. Apart from the connectivity hypothesis, this establishes a perfect Eckmann-Hilton dual result to Stasheff's classical delooping theorem for A_∞ spaces.

In [72], we generalize some of these results for spaces X with $\mathrm{cat}(X) \geq 2$. In doing so, we follow the route laid out by Arkowitz and Golasinski as we study sections of the n-th Ganea fibration. The functor $\Sigma\Omega$ is generalized by the n-th Ganea space functor $G_n(-)$ defined by Ganea's fibre-cofibre construction [37] (or equivalently up to homotopy $B_n\Omega$). This functor defines a cotriple [19]. The key notion is that of a (weak) homotopy coalgebra of order r for this cotriple. This structure encodes higher homotopies to some sort of coassociativity. Homotopy coalgebras of order 1 over the Ganea cotriple were studied by Deligiannis in his thesis [20]. These make up the second layer of structure. On the other hand, we followed the authors of [49] in that we used higher doses of A_n-theory in a form which was pioneered by Boardman and Vogt [8]. The new ingredient is that A_n-theory for H-spaces enters the picture.

In the definition of a homotopy coalgebra over $G_n(-)$, some of the structure maps are only A_n-maps, not A_n-homomorphisms. That means they commute with the A_n-actions up to n-coherent homotopy. The combinatorial description of A_n-maps is a good deal more complicated than the one of the associahedra making up the Stasheff A_n-operad. In order to deal with this fact, it was appropriate to relate A_n-structures to partial monoids.

To make this more precise, we need the next.

Definition 1.7. A filtration of a space M with $M_1 \subset M_2 \subset \ldots \subset M_n = M$ with a partial monoid structure such that

$$M_{\langle 2 \rangle} = \bigcup_{i+j \leq n} M_i \times M_j$$

is called an A_n rectification of M_1 if the inclusion of M_{i-1} into M_i is a homotopy equivalence for $1 \leq i - 1 \leq n - 1$.

The name (and the definition) is justified by:

Theorem 1.8. *A well pointed space X admits an A_n structure if, and only if, there is an A_n rectification with $X = M_1$.*

The description of A_n-stuctures through partial monoid thickenings is used in the proof of the following theorem which is essential for the deduction of the main results below.

Theorem 1.9. *Let (X, F) be a well pointed A_n space and $g : X \to N$ an A_n map to a monoid N. Moreover, let $f : K \to N$ be a map of monoids. Then in the homotopy pullback square*

$$\begin{array}{ccc} P_{g,f} & \longrightarrow & K \\ \downarrow & & \downarrow f \\ X & \xrightarrow{\ g\ } & N. \end{array}$$

there exists an induced A_n-structure on the space $P_{g,f}$.

The theorem below generalizes Ganea's theorem quoted above for all n.

Theorem 1.10. *Let X be of the homotopy type of an $(l-1)$-connected CW-complex for some $l \geq 2$. Assume furthermore that cat$(X) = n$ and dim$(X) \leq 2(n+1)l - 5$. Then Cat$(X) = n$ if and only if X is a homotopy coalgebra of order 1 over $B_n\Omega$.*

The next theorem generalizes the main results in [3] and [14] from $n = 1$ to all n and from $r = 1$ to all r, respectively .

Theorem 1.11. *Let X be of the homotopy type of a CW-complex. Suppose that cat$(X) = n$ and that X admits a weak homotopy coalgebra over $B_n\Omega$ of order r. Furthermore, suppose that X is $(l-1)$-connected $l \geq 2$ and*

$$dim(X) \leq (2n+1)l + (r-1)(n(l-1)-1) - 3.$$

Then cat$(X) =$Cat(X) holds.

Remark 1.12. As was shown by Cornea in [16], there exists for every space X with $\mathrm{cat}(X) = n$ a space Z such that $\mathrm{Cat}(X \vee \Sigma^n Z) = n$. So, in a way, we are looking for conditions which make it possible to cancel the summand $\Sigma^n Z$.

2. N-fold loop spaces and N-fold monoidal categories

Algebraicly structured categories show up in homotopy theory, algebra, and physics. For example, the fact that the nerves of monoidal, braided, and symmetric monoidal categories give rise to loop spaces, 2-fold loop spaces, and infinite loop spaces upon group completion is central in K-theory. Braidings on categories play a role in the representation theory of quantum groups and in knot theory.
There are algebraic structures on categories which interpolate between monoidal and symmetric monoidal structures. These k-fold monoidal categories were introduced in [4] where it was shown that the classifying space of an k-fold monoidal category is acted on by an E_k-operad. An k-fold monoidal structure on a category is given by k monoidal structures which commute with each other up to some appropriate kind of homotopy. An operad $\mathcal{M}_k$ in the category of small categories *Cat* (do not mix up this category with strong category!) was introduced, such that k-fold monoidal categories are exactly the algebras over $\mathcal{M}_k$. Before, the class of braided monoidal categories was already well studied [30],[48]. A braiding on category is a special case of a 2-fold monoidal structure.

In the late 90's, Thomason proved that every infinite loop space comes up to homotopy from a symmetric monoidal category [76]. For the argument he relied on his former work on homotopy colimits in symmetric monoidal categories [75]. Motivated by this, Zig Fiedorowicz and Rainer Vogt conjectured that a generalization of Thomason's theorem should hold true, i.e. that every k-fold loop space comes from an k-fold monoidal category. Generalizations of this conjecture were proved in the papers [31],[32], jointly with Zig Fiedrowicz and Rainer Vogt.
Let $\mathcal{C}_k$ be the little k-cubes operad [8]. We write $Cat^{\mathcal{M}_k}$ and $Top^{\mathcal{C}_k}$ for k-fold monoidal categories and E_k spaces, respectively. For any operad $\mathcal{M}$ in Cat, a map in $Cat^{\mathcal{M}}$ is defined to be a weak equivalence if its image under the classifying space functor B is a weak equivalence of spaces.

Theorem 2.1. *For $1 \leq k \leq \infty$, the classifying space functor B and change of operads induce equivalences of homotopy categories.*

$$Cat^{\mathcal{M}_k}[we^{-1}] \simeq Top^{B\mathcal{M}_k}[we^{-1}] \simeq Top^{\mathcal{C}_k}[we^{-1}]$$

Remark 2.2. A much more general theorem was proved in [31]. One can replace the operad $\mathcal{M}_k$ by any Σ-free operad whose underlying categories satisfy one additional factorization condition. Very roughly, this condition gives a weak substitute for the terminal objects in the categories which make up the categorical Barratt-Eccles operad modeling symmetric monoidal categories. It was important to find a condition weak enough so that non-contractible operads such as $\mathcal{M}_k$ could be treated.

Let $\mathcal{B}r$ be the operad which models braided categories. It satisfies the factorization condition. Hence we get:

Theorem 2.3. *The classifying space functor and the change of operads functor induce equivalences of categories*

$$Cat^{\mathcal{B}r}[we^{-1}] \simeq Top^{B(\mathcal{B}r)}[we^{-1}] \simeq Top^{\mathcal{C}_2}[we^{-1}].$$

Let $\widetilde{\Sigma}$ be the Cat-version of the Barratt-Eccles operad modeling permutative categories $Perm$. It satisfies the factorization condition. We obtain a version of Thomason's theorem before group completion:

Theorem 2.4. *The classifying space functor and the change of operads functors induce equivalences of categories*

$$Perm[we^{-1}] = Cat^{\Sigma}[\widetilde{we^{-1}}] \simeq Top^{B(\Sigma)}[we^{-1}] \simeq Top^{\mathcal{C}_\infty}[we^{-1}].$$

In the approach taken in [31], we define homotopy colimits in algebras $Cat^{\mathcal{M}}$ for a suitable Cat operad $\mathcal{M}$ and compare them with the homotopy colimits in $B\mathcal{M}$-algebras in Top. The construction of these homotopy colimits by generators and relations is a bit technical and too complex to be given here. To formulate the correct universal property of hocolim, one needs to introduce categories of lax algebras over an operad in Cat. It is not possible to rely on the general literature on model categories since there is no suitable model structure known on algebras over a Cat operad. The point is that Thomason's model structure on Cat is not well behaved with respect to the monoidal structure since a subdivision functor is involved. In order to accomplish this comparison, objects are replaced by simplicial resolutions. One is thus reduced to the study of diagonals and to the case of free diagrams. The following result is an essential step in the proof of 2.1:

Theorem 2.5. *Let $\mathcal{M}$ be a reduced Σ-free operad satisfying the factorization condition. Then for each diagram $X : \mathcal{L} \to Cat^{\mathcal{M}}$ there is a natural weak equivalence*

$$\alpha(X) : hocolim^{B\mathcal{M}} BstrX \longrightarrow B(hocolim^{\mathcal{M}} X).$$

The strictification functor *str* can and should be thought as some kind of cofibrant replacement. It is defined as the homotopy colimt of a diagram from the trivial category.

There are modifications of the theorems above which apply to iterated loop spaces. The weak equivalences are modified to the class of morphisms we_g which become weak equivalences after a group completion functor is applied. For any $k \leq \infty$ let $\Omega^k Top$ stand for the category of k-fold loop spaces.

Theorem 2.6. *The classifying space functor composed with the group completion induces equivalences of categories:*

$$Cat^{M_k}[we_g^{-1}] \simeq \Omega^k Top[we^{-1}]$$
$$Cat^{Br}[we_g^{-1}] \simeq \Omega^2 Top[we^{-1}]$$
$$Cat^{\widetilde{\Sigma}}[we_g^{-1}] \simeq \Omega^\infty Top[we^{-1}].$$

In order to make sense out of the homotopy categories of algebras above, we construct in [71] a model category on algebras in simplicial sets over a cofibrant E_n-operad $\mathcal{M}$ with some desireable properties. The weak equivalences are the maps which become weak equivalences of simplicial sets after group completion $X \to QX$, and the fibrant objects are the group complete algebras.

Theorem 2.7. *Let $\mathcal{M}$ be a cofibrant E_k-operad in simplicial sets. The category $SS_*^{\mathcal{M}}$ of pointed algebras over $\mathcal{M}$ with the Q-structure is a left proper simplicial model category. Moreover, a morphism $f : X \to Y$ in $SS_*^{\mathcal{M}}$ is a Q-fibration if it is a fibration and*

$$\begin{array}{ccc} X & \xrightarrow{q_X} & QX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Qf} \\ Y & \xrightarrow[q_Y]{} & QY \end{array}$$

is a homotopy fibre square. In case $\pi_0 Qf$ is onto this condition is also necessary.

Remark 2.8. The cofibrancy of the operad $\mathcal{M}$ is used to construct a group completion map $X \to QX$ which is a genuine $\mathcal{M}$-homomorphism. This is needed in order to apply general results on Bousfield-Friedlander model structures.

A different approach to the main results of [31] was developed in [32]. Recall that May and Thomason defined the notions of a category of operators $\widehat{\mathcal{M}}$ associated to a topological operad $\mathcal{M}$ and of algebras $Top^{\widehat{\mathcal{M}}}$ over it [54]. They used this structure in the comparison of the delooping machine based on spaces with an E_∞-operad action with Segal's machine which involves a weakening of the cartesian product and the notion of product up to equivalence. A rectification construction was created which associates to an $\widehat{\mathcal{M}}$-algebra a genuine $\mathcal{M}$-algebra. In [32], a rectification

$$M : Cat^{\widehat{\mathcal{M}}} \to Cat^{\mathcal{M}}$$

which works in Cat is given. This construction should be seen as a Cat-version of the classical M-construction, of Boardman and Vogt [8]. This functor associates with a topological algebra over a cofibrant model $W(\mathcal{M})$ of a given operad $\mathcal{M}$ an algebra over the original operad. The functor M is defined as the Grothendieck construction

$$M(\hat{G}) = \mathcal{T} \int F^{\hat{G}}$$

of a certain category $\mathcal{T}$ made of trees and a functor

$$F^{\hat{G}} : \mathcal{T} \to Cat.$$

So in the investigation of its properties in homotopy, one can rely on Thomason's classical homotopy colimit theorem for the Grothendieck construction [74]. The rectification can be used to give a substitute for the homotopy colimit construction in the case of simplicial algebras over a Σ-free Cat-operad $\mathcal{M}$. To be more precise, one defines a functor

$$H_{Cat} : Cat^{\mathcal{M}^{\Delta^{op}}} \to Cat^{\mathcal{M}}$$

by $H_{Cat}(D) = \mathcal{T} \int F^{\Delta^{op} \int \hat{G}}$ where $\hat{G}$ is the algebra over $\widehat{\mathcal{M}}$ associated with the $\mathcal{M}$-algebra G. It is then shown that H_{Cat} induces an equivalence on homotopy categories. This suffices since, by former work of Fiedorowicz and Vogt [33], the homotopy category $Cat^{\mathcal{M}^{\Delta^{op}}}[we^{-1}]$ of simplicial algebras is known to be equivalent to the homotopy category of topological $B(\mathcal{M})$-algebras.

3. Unstable modules and (co)algebras

The mod p cohomology $H^*(X)$ of a topological space X is an unstable module over the Steenrod algebra A_p. Moreover, the cup product interacts with the A_p-module action to define the structure of an unstable algebra on $H^*(X)$. Questions on the realizability of a given A_p-module as the cohomology of a space are classical and central in algebraic topology. Examples of small modules which can not be realized show up in the famous Hopf and Kervaire invariant one problems. In contrast, the modules which underly a polynomial unstable algebra, are rather large and not finitely generated. Steenrod asked for a classification which has been achieved recently.
In the mid-90's, Nick Kuhn made the following conjecture:

If $H^*(X)$ is finitely generated as an A_p-module, then it is finite dimensional as an $\mathbb{F}_p$-vector space.

Kuhn proved his conjecture under the assumption that the Bockstein operator acts trivially on $H^*(X)$ in high degrees using some heavy guns from classical homotopy theory [50]. A little later, Lionel Schwartz gave a proof of Kuhn's conjecture at the prime 2 [68]. A new idea invented by Schwartz is to study the implications of a finitely generated but non-finite module structure on $M = H^*(X)$, for $H^*(\Omega^n(X))$ via the Eilenberg-Moore spectral sequence and deduce a contradiction from them. In a later paper [51], Kuhn gave a streamlined proof in which he replaced the iterated use of the Eilenberg-Moore spectral sequence by a single application of a spectral sequence constructed from the Goodwillie tower of the functor $X \to \Sigma^\infty\Omega^n(X)$. The argument reduces the non realizability of M to the non-realizability of certain finite modules. These non-realizability results are interesting in themselves. An attempt of a generailzation for odd primes p faces some difficulties which are well known in the theory of unstable modules and algebras. Only in even dimensions are the top odd primary operations directly related to the algebra structure. In odd dimensions, there is only an interpretation of the top operation in terms of higher symmetric Massey products. As a consequence, the module of indecomposables in an unstable algebra may fail to desuspend, and the whole desuspension theory of unstable modules is more complicated at odd primes than at 2. In classical desuspension theory, a similar complication arises, which is reflected by the fact that there are two EHP sequences at odd primes and one at 2. In [12], together with Sebastian Büscher, Fabian Hebestreit, and Oliver Röndigs, we partially extend the results in [50] for odd primes.

Let $\Phi(k, k+2)$ be the unstable module given as the subquotient of $H^*(K(\mathbb{Z}/p\mathbb{Z}, 1)$ with $\mathbb{F}_p$-base $\{t^{p^k}, t^{p^{k+1}}, t^{p^{k+2}}\}$ where t is a generator in $H^2(K(\mathbb{Z}/p\mathbb{Z}, 1)$.

Theorem 3.1. *Let M be an unstable module of finite type, concentrated in degrees between l and m. Suppose that M contains a desuspension class of even origin. If X is a topological space such that*

$$\tilde{H}^*(X) \cong M \otimes \Phi(k, k+2)$$

as A_p-modules, then $2p^k \leq (p^2-1) + (p-1)(m-l)$.

The proof uses the interplay of operations dual to the Dyer-Lashof operations with the Steenrod action on the Goodwillie spectral sequence. As a corollary, we obtain a new proof of Kuhn's theorem without reference to the Hopf invariant 1 mod p theorem.

Theorem 3.2. *Let X be a topological space such that $H^*(X)$ is finitely generated as an A_p-module and the Bockstein operator acts trivial in high degrees. Then $H^*(X)$ is finite dimensional as an $\mathbb{F}_p$-vector space.*

Only recently, Gaudens and Schwartz were able to show that a fast generalization of the conjecture at all primes holds [38]. The new approach uses Lannes' theory and gives no information on finite modules. It is an open question whether there is a version of 3.1 for modules which contain only desuspesion classes of odd origin.

In case one knows that a given unstable (co)algebra has a toplogical realization X, one may ask further how many such spaces are there up to equivalence. In joint work with Georg Biedermann and George

Raptis [5], we construct moduli spaces attached to the topological realizations of an unstable coalgebra C. The set of realizations is in bijection with the path components of the moduli space. Moreover, a tower of fibrations which approximate the moduli space is constructed, and the extensions in the tower are linked to cohomological data.

Moduli spaces which parametrize homotopy types with a given cohomology algebra or homotopy Lie algebra were first constructed in rational homotopy theory. The case of cohomology was treated by Félix [24], Lemaire-Sigrist [52], and Schlessinger-Stasheff [67]. These moduli sets turn out to be the quotient of a rational variety by the action of an unipotent proalgebraic group. Of course, the moduli schemes which are studied in algebraic geometry by the method of geometric invariant theory [56] also admit such a representation.

An obstruction theory for unstable coalgebras was developed by Blanc in [6]. He defined obstruction classes for a coalgebra C and proved that the vanishing of these classes is nessecary and sufficient for the existence of a realization. He went on and defined difference classes which may differentiate two given realizations. Both classes live in certain André-Quillen cohomology groups associated to C. For very nice unstable algebras, an obstruction theory using the Massey-Peterson machinery was developed by Harper [44] and McCleary [55] already in the late seventies. In a landmark paper, Blanc, Dwyer, and Goerss [7] constructed moduli spaces for realizations of a given Π-algebra over the integers. The components of the moduli space correspond to different realizations. A decomposition into a tower of fibrations which allows an inductive approach is achieved by means of simplicial resolutions. The authors relied on earlier work of Dwyer, Kan, and Stover [23] on resolution model categories which, later on, was generalized by Bousfield [10]. Along the same lines, the moduli spaces of realizations of a commutative algebra in comodules over a Hopf algebroid by an E_∞-algebra in spectra are treated by Goerss and Hopkins in a series of papers [40], [41], and [39]. These results gave rise to some profound applications in stable homotopy [57].

Fix a prime field $\mathbb{F}$ and an unstable coalgebra C over it. The space of realizations $\mathcal{M}_{\mathrm{Top}}(C)$ is the nerve of the category whose objects are all spaces with $\mathbb{F}$-homology isomorphic to C as unstable coalgebra and whose morphisms are the $\mathbb{F}$-homology equivalences between these objects. In order to break this space into pieces which allow an inductive approach, the spaces realizing C are replaced by cosimplicial resolutions. These resolutions are products of Eilenberg-MacLane spaces of type $\mathbb{F}$ in each cosimplicial degree. There is a notion of Postnikov tower with respect to the cosimplicial direction for such resolutions. The stages are characterized by an approximation and a vanishing condition similar as in the classical situation. A potential n-stage for C is then defined to be a cosimplicial space which satisfies this condition up to level n. This definition makes also sense for $n = \infty$ giving rise to the notion of an ∞-stage. The category whose objects are potential n-stages of C defines a moduli space $\mathcal{M}_n(C)$, and the skeleton functor induces a map

$$\mathrm{sk}_n : \mathcal{M}_n(C) \to \mathcal{M}_{n-1}(C).$$

The following theorem addresses the relation between the moduli spaces of ∞-stages and actual realizations of C:

Theorem 3.3. *Let C be a simply-connected unstable coalgebra, i.e. $C_1 = 0$ and $C_0 = \mathbb{F}$. Then the totalization functor induces a weak equivalence*

$$\mathcal{M}_\infty(C) \simeq \mathcal{M}_{\mathrm{Top}}(C).$$

A further step towards an inductive approach is:

Theorem 3.4. *Let C be an unstable coalgebra. Then there is a weak equivalence*

$$\mathcal{M}_\infty(C) \simeq \operatorname*{holim}_n \mathcal{M}_n(C).$$

Associated with an abelian unstable coalgebra M which is also a C-comodule, there is an Eilenberg-MacLane object $K_C(M, n)$ in cosimplicial unstable coalgebras. We define André-Quillen spaces as the derived mapping spaces in the simplicial resolution model category, due Bousfield, of cosimplicial unstable coalgebras under C

$$TAQ^n_C(C; M) := \mathrm{map}^{\mathrm{der}}_{c(C/CA)}(K_C(M, n), cC),$$

where cC is the constant cosimplicial object defined by C. The homotopy groups of this mapping space define André-Quillen cohomology of C with coefficients in M. Let us write $C[n]$ for the C-comodule and unstable module obtained by shifting C n-steps up with respect to the internal degree.

The difference between $\mathcal{M}_n(C)$ and $\mathcal{M}_{n-1}(C)$ can be understood by the theorem below. The two spaces on the right in the diagram are moduli spaces of certain diagrams of Eilenberg-MacLane objects:

Theorem 3.5. *For every $n \geq 1$, there is a homotopy pullback square*

$$\begin{array}{ccc}
\mathcal{M}_n(C) & \longrightarrow & \mathcal{M}(K_C(C[n], n+2) \twoheadrightarrow K(C, 0)) \\
\Big\downarrow{\scriptstyle \mathrm{sk}_n} & & \Big\downarrow{\scriptstyle \Delta} \\
\mathcal{M}_{n-1}(C) & \xrightarrow{H_*} & \mathcal{M}(K(C, 0) \twoheadleftarrow K_C(C[n], n+2) \twoheadrightarrow K(C, 0)).
\end{array}$$

where the map on the right is induced by the functor $(V \twoheadleftarrow U) \mapsto (V \twoheadleftarrow U \twoheadrightarrow V)$.

From this we obtain the promised cohomological description of the sections in the tower of moduli spaces.

Theorem 3.6. *Let* $X^\bullet$ *be a potential* n*-stage for an unstable coalgebra* C*. Then there is a homotopy pullback square*

$$\begin{array}{ccc} TAQ_C^{n+1}(C; C[n]) & \longrightarrow & \mathcal{M}_n(C) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{sk}_n} \\ * & \xrightarrow{\mathrm{sk}_n X^\bullet} & \mathcal{M}_{n-1}(C). \end{array}$$

Notice that the Eilenberg-MacLane objects which define the layers vary in a much more transparent way than in the classical Postnikov decomposition of spaces.

A main tool in the proof of the results quoted above is an excision theorem in the resolution model category of cosimplicial spaces. There is also a version for cosimplicial unstable coalgebras from which the former is deduced.

Theorem 3.7. *Let*

$$\begin{array}{ccc} E^\bullet & \longrightarrow & X^\bullet \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y^\bullet & \xrightarrow{g} & Z^\bullet \end{array}$$

be a homotopy pullback square of cosimplicial spaces where f *is* m*-connected and* g *is* n*-connected. Then the square is homotopy* $(m+n)$*-cocartesian.*

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