Appendix to "A Thorough Formalization of Conceptual Spaces"

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Abstract

This appendix provides mathematical proofs for the propositions made in the paper "A Thorough Formalization of Conceptual Spaces" by Lucas Bechberger and Kai-Uwe Kühnberger [1].

1 Fuzzy Simple Star-Shaped Sets

Proposition 1. *Any fuzzy simple star-shaped set* $\widetilde{S} = \langle S, \mu_0, c, W \rangle$ *as defined in [1] is star-shaped with respect to* $P = \bigcap_{i=1}^{m} C_i$ *under* $d_C^{\Delta_S}$.

Proof. As $\mu_{\widetilde{S}}(x)$ is inversely related to the distance of x to S, any $\widetilde{S}^\alpha$ with $\alpha \leq \mu_0$ is equivalent to an ϵ-neighborhood of S under $d_C^{\Delta_S}$ (cf. Figure 1). We can define the ϵ-neighborhood of a cuboid C_i under $d_C^{\Delta_S}$ as

$$C_i^\epsilon = \{z \in CS \mid \forall d \in D_S : p_{id}^- - u_d \leq z_d \leq p_{id}^+ + u_d\}$$

where u represents the difference between $x \in C_i$ and $z \in C_i^\epsilon$. Thus, u must fulfill the following constraints:

$$\sum_{\delta \in \Delta_S} w_\delta \cdot \sqrt{\sum_{d \in \delta} w_d \cdot (u_d)^2} \leq \epsilon \quad \wedge \quad \forall d \in D_S : u_d \geq 0$$

Let now $x \in C_i, z \in C_i^\epsilon$.

$$\begin{aligned} \forall d \in D_S :\ & x_d = p_{id}^- + a_d \text{ with } a_d \in [0, p_{id}^+ - p_{id}^-] \\ & z_d = p_{id}^- + b_d \text{ with } b_d \in [-u_d, p_{id}^+ - p_{id}^- + u_d] \end{aligned}$$

We know that a point $y \in CS$ is between x and z with respect to $d_C^{\Delta_S}$ if the following condition is true:

$$\begin{aligned} & d_C^{\Delta_S}(x, y, W) + d_C^{\Delta_S}(y, z, W) = d_C^{\Delta_S}(x, z, W) \\ \iff & \forall \delta \in \Delta_S : d_E^\delta(x, y, W_\delta) + d_E^\delta(y, z, W_\delta) = d_E^\delta(x, z, W_\delta) \\ \iff & \forall \delta \in \Delta_S : \exists t \in [0,1] : \forall d \in \delta : y_d = t \cdot x_d + (1-t) \cdot z_d \end{aligned}$$

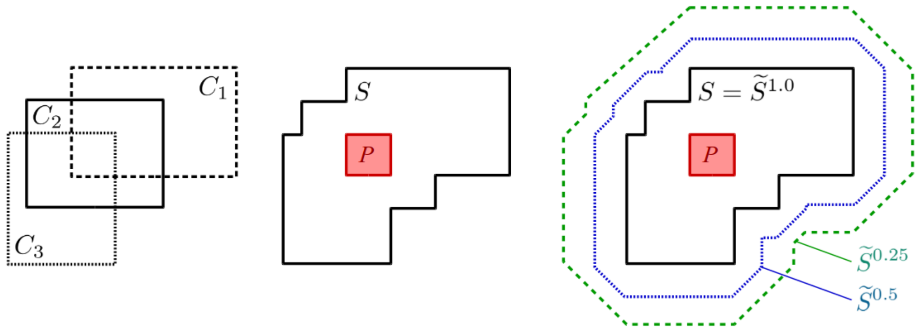

Figure 1: Left: Three cuboids C_1, C_2, C_3 with nonempty intersection. Middle: Resulting simple star-shaped set S based on these cuboids. Right: Fuzzy simple star-shaped set $\tilde{S}$ based on S with three α-cuts for $\alpha \in \{1.0, 0.5, 0.25\}$.

The first equivalence holds because $d_C^{\Delta_S}$ is a weighted sum of Euclidean metrics d_E^{δ}. As the weights are fixed and as the Euclidean metric is a metric obeying the triangle inequality, the equation with respect to $d_C^{\Delta_S}$ can only hold if the equation with respect to d_E^{δ} holds for all $\delta \in \Delta$.

We can thus write the components of y like this:

$$\forall d \in D_S : \exists t \in [0,1] : y_d = t \cdot x_d + (1-t) \cdot z_d = t \cdot (p_{id}^- + a_d) + (1-t)(p_{id}^- + b_d)$$
$$= p_{id}^- + t \cdot a_d + (1-t) \cdot b_d = p_{id}^- + c_d$$

As $c_d = t \cdot a_d + (1-t) \cdot b_d \in [-u_d, p_{id}^+ - p_{id}^- + u_d]$, it follows that $y \in C_i^\epsilon$. So C_i^ϵ is star-shaped with respect to C_i under $d_C^{\Delta_S}$. More specifically, C_i^ϵ is also star-shaped with respect to P under $d_C^{\Delta_S}$. Therefore, $S^\epsilon = \bigcup_{i=1}^m C_i^\epsilon$ is star-shaped under $d_C^{\Delta_S}$ with respect to P.

□

2 Union of Fuzzy Simple Star-Shaped Sets

Proposition 2. *Let $\widetilde{S}_1 = \langle S_1, \mu_0^{(1)}, c^{(1)}, W^{(1)} \rangle$ and $\widetilde{S}_2 = \langle S_2, \mu_0^{(2)}, c^{(2)}, W^{(2)} \rangle$ be two fuzzy simple star-shaped sets as defined in [1]. If we assume that $\Delta_{S_1} = \Delta_{S_2}$ and $W^{(1)} = W^{(2)}$, then $\widetilde{S}_1 \cup \widetilde{S}_2 \subseteq U(\widetilde{S}_1, \widetilde{S}_2) = \widetilde{S}'$.*

Proof. As both $\Delta_1 = \Delta_2$ and $W^{(1)} = W^{(2)}$, we know that

$$d(x,y) := d_C^{\Delta_{S_1}}(x, y, W^{(1)}) = d_C^{\Delta_{S_2}}(x, y, W^{(2)}) = d_C^{\Delta_{S'}}(x, y, W')$$

Moreover, $S_1 \cup S_2 \subseteq S'$. Therefore:

$$\begin{aligned}
\mu_{\widetilde{S}_1 \cup \widetilde{S}_2}(x) &= \max(\mu_{\widetilde{S}_1}(x), \mu_{\widetilde{S}_2}(x)) \\
&= \max(\max_{y \in S_1} \mu_0^{(1)} \cdot e^{-c^{(1)} \cdot d(x,y)}, \max_{y \in S_2} \mu_0^{(2)} \cdot e^{-c^{(2)} \cdot d(x,y)}) \\
&\leq \mu_0' \cdot \max(e^{-c^{(1)} \cdot \min_{y \in S_1} d(x,y)}, e^{-c^{(2)} \cdot \min_{y \in S_2} d(x,y)}) \\
&\leq \mu_0' \cdot e^{-c' \cdot \min(\min_{y \in S_1} d(x,y), \min_{y \in S_2} d(x,y))} \\
&\leq \mu_0' \cdot e^{-c' \cdot \min_{y \in S'} d(x,y)} = \mu_{\widetilde{S}'}(x)
\end{aligned}$$

□

3 Intersection of Projections to Subspaces

Proposition 3. *Let $\widetilde{S} = \langle S, \mu_0, c, W \rangle$ be a fuzzy simple star-shaped set as defined in [1]. Let $\widetilde{S}_1 = P(\widetilde{S}, \Delta_1)$ and $\widetilde{S}_2 = P(\widetilde{S}, \Delta_2)$ with $\Delta_1 \cup \Delta_2 = \Delta_S$ and $\Delta_1 \cap \Delta_2 = \emptyset$. Let $\widetilde{S}' = I(\widetilde{S}_1, \widetilde{S}_2)$ as described in Section 4.1 of [1].*
If $\sum_{\delta \in \Delta_1} w_\delta = |\Delta_1|$ and $\sum_{\delta \in \Delta_2} w_\delta = |\Delta_2|$, then $\widetilde{S} \subseteq \widetilde{S}'$.

Proof. We already know that $S \subseteq I(P(S, \Delta_1), P(S, \Delta_2)) = S'$. Moreover, one can easily see that $\mu_0' = \mu_0$ and $c' = c$.

$$\mu_{\widetilde{S}}(x) = \max_{y \in S} \mu_0 \cdot e^{-c \cdot d_C^{\Delta_S}(x,y,W)} \overset{!}{\leq} \max_{\substack{y \in \\ I(P(S,\Delta_1), P(S,\Delta_2))}} \mu_0' \cdot e^{-c' \cdot d_C^{\Delta_S}(x,y,W')} = \mu_{\widetilde{S}'}(x)$$

This holds if and only if $W = W'$. $W^{(1)}$ only contains weights for Δ_1, whereas $W^{(2)}$ only contains weights for Δ_2. As $\sum_{\delta \in \Delta_i} w_\delta = |\Delta_i|$ (for $i \in \{1, 2\}$), the weights are not changed during the projection. As $\Delta_1 \cap \Delta_2 = \emptyset$, they are also not changed during the intersection, so $W' = W$.

□

References

[1] Lucas Bechberger and Kai-Uwe Kühnberger. A Thorough Formalization of Conceptual Spaces for Machine Learning and Reasoning. In *Proceedings of the 40th German Conference on Artificial Intelligence*, in press.