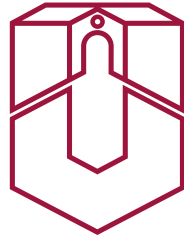


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ALGORITHMS & COMPUTATIONS

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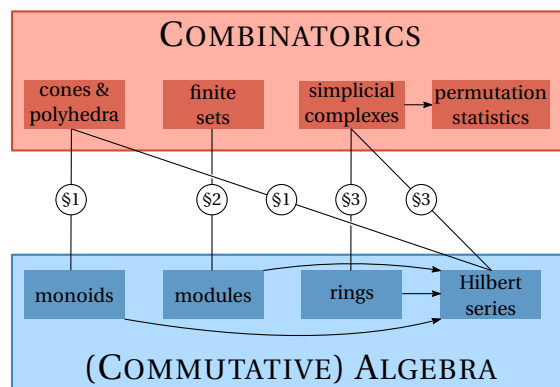
Introduction

We must know – we will know!

David Hilbert

It is a common and central theme in mathematics to build bridges between different fields of research in order to solve problems that could not be tackled solely within the boundaries of one area. This thesis is devoted to the bridge that connects the areas of *combinatorics* and (*commutative*) *algebra*. In particular, we are interested in how combinatorial structures can be used to compute properties and invariants in algebra – be it by results in combinatorics or the possibility of computer algorithms.

Following this idea, we associate objects of these two areas with each other throughout the thesis: cones and polyhedra with monoids and thereby monoid algebras in Chapter 1, finite sets with restrictions of polynomial rings and *Stanley spaces* in Chapter 2 and finally simplicial complexes with their *Stanley-Reisner rings* in Chapter 3. We examine properties and structures of these combinatorial objects and thereby draw conclusions on their linked algebraic counterparts.



For example, an object which appears frequently in this thesis is the *Hilbert series* of rings, modules, cones and other structures. Originally, it derives from dimensions of the homogeneous components of a graded ring or module, yet it also arises from the enumeration of lattice points in dilates of a polytope or from the face enumeration of a simplicial complex. This reinterpretation of the same object in different fields is a common pattern which often proves useful in our investigations of algebraic and combinatorial structures.

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Although the three research projects in this thesis seem to be rather unrelated on first sight, they all follow the theme of connecting combinatorial and algebraic objects.

In Chapter 1 “Algorithms in rational cones”, we consider the main computation goals and algorithms of the software Normaliz [BIR⁺], which originates in the computation of *normalizations* of affine monoids or monomial ideals (hence its name), which are algebraic objects. However, here we mainly describe these objects and relevant algorithms in the language of discrete geometry, specifically cones and polyhedra. After introducing all relevant concepts behind the software, we discuss a new algorithm that speeds up the computation of the *Hilbert basis* – the minimal generating system of an affine monoid and a major computation goal of Normaliz. The second computation goal is the Hilbert series of an affine monoid and we consider its different representations. One representation we focus on in more detail originates from certain constructions in commutative algebra.

Chapter 2 “The Stanley Depth in the Koszul Complex” is devoted to *Stanley* and *Hilbert decompositions* of modules and their respective depth, specifically of the syzygy modules in the *Koszul complex*. Our considerations lead us to the combinatorics of finite sets. In particular, we study properties of certain injective maps in the Boolean algebra and thereby draw conclusions on the linear independence of particular homogeneous elements in the syzygy modules.

The connection of algebra and combinatorics is also a key aspect in Chapter 3 “Local *h*-vectors”. We investigate *simplicial complexes*, their possible subdivisions and face numbers. The latter are encoded in the *h-vector*. Its behavior under subdivision can be controlled by the *local h-vector*, whose possible properties have an impact on the outcome of the usual *h-vector* after a particular subdivision. The computation of the local *h-vector* for *barycentric* subdivisions leads us back to pure combinatorics, here in the form of permutation statistics.

In conclusion, this thesis contains various research problems which show diverse ways of connecting combinatorial and algebraic objects. These connections, for their part, indicate the benefit of combinatorial structures and algorithms for computations in algebra.

The material presented in this thesis is based on a series of published papers and preprints. In Chapter 1, Section 1.3 mainly contains the content of the article “The Subdivision of Large Simplicial Cones in Normaliz” [BSS16b] and Section 1.4 appeared as a section in the overview article “Normaliz 2013–2016” [BSS16a]. Chapter 2 bases on the article “The Stanley Depth in the Upper Half of the Koszul Complex” published in [KS16]. The last chapter mainly follows the paper “Local *h*-vectors of Quasi-Geometric and Barycentric Subdivisions” which, at the time of the submission of this thesis, is a preprint available at [JMS17].

Summary of the thesis

The first chapter is devoted to Normaliz' main algorithms and some of its new features and improvements. In Section 1.1 we give an overview of cones and polyhedra, affine monoids and their Hilbert bases and series. Then, in Section 1.2, we briefly sketch the main algorithm to compute the Hilbert basis, Hilbert series and other data of a cone. Section 1.3 discusses one key step in this algorithm, the evaluation of *simplicial* cones. We investigate the complexity of this evaluation and present a way of reducing this complexity by subdividing a large simplicial cone into smaller cones using interior lattice points. The main focus of the section lies on two algorithms for the computation of such subdivision points. The first is based on solving *integer programs* and we use the mathematical software SCIP [GFG⁺16] for its implementation. The second consists of approximating the simplicial cone from the outside with a cone whose Hilbert basis is significantly easier to compute. We also discuss the implementation and benchmark results of these two algorithms. In the last section of the chapter we move on from the problem of *generation* to the one of *enumeration* and discuss several different representations of the Hilbert series of an affine monoid. We present a representation with non-negative numerator polynomial and low exponents in the denominator which results from *homogeneous systems of parameters* of the algebra associated to the monoid. After reviewing the definition of these systems and their connection to the Hilbert series, we discuss their construction. In particular, we present an algorithm for the computation of the degrees of the elements of such a system, which relies on the face structure of the cone.

Chapter 2 is concerned with the Hilbert and Stanley depth in the Koszul complex. As a main result, we show that the Stanley depth in the upper half of the Koszul complex over the polynomial ring $K[x_1, \dots, x_n]$, where K is a field, is given by $n - 1$ (Theorem 2.0.1), and thus coincides with the Hilbert depth, which was computed in [BKU10]. Section 2.1 introduces the definition of these decompositions, their relation and some examples. The subsequent section surveys certain injective maps in the Boolean algebra and their properties. These maps are used in Section 2.3 to construct a Hilbert decomposition of the syzygy modules as in [BKU10] and, in particular, to transform this decomposition into a Stanley decomposition. This transformation is based on the linear independence of homogeneous elements of certain multidegrees. We present a choice of such homogeneous elements and prove their independence by specifying an order of these elements such that the chain map in the Koszul complex forms an upper triangular matrix.

In the last chapter we address the problem of face enumeration of simplicial complexes and their subdivisions. Instead of studying the number of faces directly, it is often more convenient to work with the *h-vector* of a simplicial complex, which is given by a unique transformation of the face numbers. Given a subdivision of a simplicial complex, its *h-vector* can be specified by the *h-vector* of the *links* of the simplicial complex and the so-called *local h-vector* associated to the subdivision. The computation and classification of the local *h-vector* for certain types of subdivisions is the focus of this chapter. In Section 3.1, we begin by discussing permutation statistics, in particular *Eulerian* and *de-*

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derangement polynomials. We continue by proving a new recurrence formula for the latter. This formula is not only interesting since it resembles the known recurrence formula for Eulerian polynomials, but is also used in later proofs. Section 3.2 provides background material on simplicial complexes, their subdivisions and (local) h -vectors. Provided that the h -vector is symmetric, we can define its associated γ -vector. We discuss some important conjectures concerning the γ -vector, including the Charney-Davis conjecture and Gal's conjecture. We illustrate the connection of those conjectures to the non-negativity of the *local γ -vector*, which is the counterpart for the local h -vector. Then, in Section 3.3, we show that the local h -vector of the barycentric subdivision of any subdivision of the simplex is γ -non-negative (Theorem 3.0.2). The proof is based on a new formula for the local h -vector involving differences of usual h -vectors and derangement polynomials. Finally, Section 3.4 deals with the characterization of local h -vectors for certain classes of subdivisions. Here, we complement work by Chan [Cha94] and characterize local h -vectors of *quasi-geometric* subdivisions (Theorem 3.0.3).

Acknowledgements

First and foremost, I want to thank my advisor Winfried Bruns for his guidance during my time in Osnabrück. I learned a lot from our discussions about combinatorial and algebraic structures and algorithms. Often enough, I would have become stuck without his patient and clear answers to all of my questions during the years. Especially, I would like to thank him for teaching me about programming with C++, its pitfalls and common interfaces.

I would also like to thank Martina Juhnke-Kubitzke for teaching me about face enumeration problems and the geometry of simplicial complexes. I enjoyed our fruitful discussions and joint puzzling over local h -vectors. I am also grateful for her useful comments and tips about mathematical writing. Personally, I wish her all the best for her (academic) future.

I had a wonderful time in Osnabrück and our department, which is also due to my great colleagues. Our occasional mathematical conversations, seminars and especially our fun (board game) nights will stay with me in good memory. In particular, I would like to thank Lukas Katthän and Christof Söger for all their patient explanations and support. I also want to thank my two office mates, Davide Alberelli and Jonathan Steinbuch, for our pleasant conversations and funny little moments. My colleagues Gilles Bonnet, Jan-Marten Brunink, Alex Grosdos, Sean Tilson and Lorenzo Venturello quickly became very good friends of mine and we shared plenty of great moments, also outside the department. I am thankful to my long-standing former fellow student and flatmate Gregor Hendel, who did not only help us implementing SCIP into Normaliz but also thoroughly read through the first chapter of this thesis. Furthermore, I owe thanks to the secretaries of our department, especially Marianne Gausmann, for their help and kindness across all organizational matters.

During my Ph.D., I was fortunate to visit many places and meet interesting people. I gained plenty of knowledge from various discussions with other mathematicians during conferences and workshops. Above all, I want to express my gratitude to the people that welcomed me at their departments for some time. Federico Ardila, Matthias Beck, Joseph Gubeladze and Serkan Hoşten were cordial hosts and advisors while I was staying at the San Francisco State University for half a year. The time in the United States taught me a lot about communicating mathematics and looking at problems from different views. I am also thankful for the hospitality I experienced during my short term stays at the University of Washington with Isabella Novik and at the Dipartimento di Matematica in Genova. I want to thank Michael Joswig and the polymake team at TU Berlin for the regular and fruitful exchange of ideas about mathematical software.

Besides my colleagues, I met plenty of great people outside of mathematics in Osnabrück. Yet, the list of their names would be just too long to mention here. At least, I want to thank the university's pop choir and my turbo touch team for the memorable time we shared.

Finally, I want to express my deep gratitude to my family, who, although they never totally understood what I am doing, supported me in every possible way. In the same way, I want to thank my former fellow students and friends in Berlin and Osnabrück who accompanied me on my way. After all, they themselves became a kind of family.

Notation

$[n]$	the set $\{1, \dots, n\}$
2^M	power set of the set M / simplex on the set M
ℓ -set	a set with ℓ elements
S_n	set of permutations on $[n]$
lcm	least common multiple
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	natural, integer, rational, real and complex numbers
$\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$	their non-negative parts
$\mathbb{Z}_{>0}, \mathbb{Q}_{>0}, \mathbb{R}_{>0}$	their positive parts
K	a field, which we always assume to be infinite
$K[x_1, \dots, x_n]$	polynomial ring over the field K in n variables
$\text{Rad}(I)$	radical of the ideal I

1 Algorithms in rational cones

The work guides the way.

Prof. Bruns' handymen

The following material is based on two papers written by Winfried Bruns, Christof Söger and the author: Section 1.3 was published in [BSS16b] and Section 1.4 appears in [BSS16a].

In this chapter we investigate the main objects and algorithms of the mathematical software Normaliz ([BIR⁺]). It is developed at the University of Osnabrück since 1998 and is mainly used for computations in affine monoids and rational cones.

In Section 1.1 we review some basic facts on rational cones and polyhedra and their connection to affine monoids. In particular, we discuss Normaliz' two major computation goals of *generation*, i.e., computing the *Hilbert basis*, and *enumeration*, i.e., computing the *Hilbert series*. Section 1.2 gives a short overview of Normaliz' main algorithm for computing these goals. In the subsequent section we take a closer look at the evaluation of *simplicial* cones. We then present a subdivision algorithm that tremendously reduces the cost of a Hilbert basis computation in a simplicial cone with large volume. This algorithm subdivides a cone with lattice points inside the cone which either originate from solving integer programs or an approximation procedure. The last section is devoted to the Hilbert series of affine monoids and its different representations as rational function. In particular, we develop a representation with a non-negative numerator polynomial and low exponents in the denominator. This representation arises from homogeneous systems of parameters.

1.1 Preliminaries

We introduce the main objects of this chapter – rational cones and polyhedra – and summarize some of their crucial properties. We omit the relevant proofs and refer the reader for them and further background material to [BG09], in particular Chapter 1 and 2.

1.1.1 Rational cones and polyhedra

Cones and polyhedra can be described as the solution set to a finite linear system of inequalities, where systems belonging to cones only contain homogeneous inequalities. An *affine form* $a : \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$a(x) = \lambda(x) + a_0, \quad a_0 = a(0),$$

where $\lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ is a (unique) *linear form* on \mathbb{R}^d . These forms are called *rational* if they can be represented using only integral and coprime coefficients.

An affine form a defines a *hyperplane*

$$H_a := \{x \in \mathbb{R}^d \mid a(x) = 0\},$$

closed halfspaces

$$H_a^+ := \{x \in \mathbb{R}^d \mid a(x) \geq 0\}, \quad H_a^- := \{x \in \mathbb{R}^d \mid a(x) \leq 0\},$$

and *open halfspaces*

$$H_a^> := \{x \in \mathbb{R}^d \mid a(x) > 0\}, \quad H_a^< := \{x \in \mathbb{R}^d \mid a(x) < 0\}.$$

In the case that a is in fact linear, these sets will be called linear as well. Furthermore, we call these sets rational if their defining form is rational.

One way of describing cones and polyhedra is by means of halfspaces, the *H-description*.

Definition 1.1.1. A *polyhedron* is a subset $P \subset \mathbb{R}^d$ that can be described as the intersection of finitely many closed halfspaces H_1^+, \dots, H_s^+ , that is,

$$P = H_1^+ \cap \dots \cap H_s^+.$$

If these halfspaces are rational, P is said to be rational as well. Moreover, a *cone* is a polyhedron which can be described using only linear halfspaces.

The *dimension* of P is the dimension of the smallest affine subspace containing P .

Finally, a *polytope* is a bounded polyhedron.

Additionally, we call a cone C *pointed* if it does not contain any non-trivial subspace (or, equivalently, $x, -x \in C$ implies $x = 0$). Figure 1.1 shows some examples of these objects.

We see from the definition that cones form a special class of polyhedra. However, every polyhedron has a unique associated cone, which is given by the *homogenization* of its defining set of (possibly) inhomogeneous inequalities. Given an affine form $a : \mathbb{R}^d \rightarrow \mathbb{R}$, $a(x) = \lambda(x) + a_0$, its homogenization is the linear form

$$\text{hom}(a) : \mathbb{R}^{d+1} \rightarrow \mathbb{R}, \quad \text{hom}(a)(x, h) = \lambda(x) + a_0 h.$$

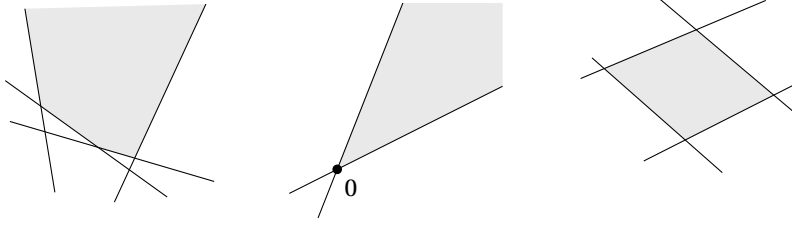


Figure 1.1: A polyhedron, cone and polytope in \mathbb{R}^2 .

Moreover, $\text{hom}(H_a) \subset \mathbb{R}^{d+1}$ is the associated homogenized linear hyperplane and we use the same notation for homogenized halfspaces. Let $P = H_1^+ \cap \dots \cap H_s^+$ be a polyhedron. The *cone over P* is defined as

$$C(P) := \text{hom}(H_1^+) \cap \dots \cap \text{hom}(H_s^+) \cap H^+, \tag{1.1}$$

where $H^+ = \{(x, h) \in \mathbb{R}^{d+1} \mid h \in \mathbb{R}_+\}$. The original polyhedron can be reconstructed from its cone by taking the *cross section* at height 1:

$$C(P) \cap \{(x, h) \in \mathbb{R}^{d+1} \mid h = 1\} = (P, 1).$$

See Figure 1.2 for an example of a cone over a 2-dimensional polytope.

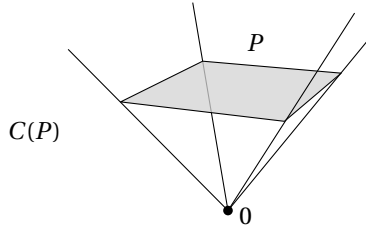


Figure 1.2: Cone over a polytope.

Next, we discuss the face structure of cones and polyhedra. Let P be a polyhedron. A hyperplane H satisfying $P \subset H^+$ is called a *support hyperplane* of P . The intersection $H \cap P$ is again a polyhedron (resp. cone) and is said to be a *face* of P . Faces of dimension one smaller than the dimension of P are called *facets* and faces of dimension 0 are the *vertices*. In the case that P is a cone, the faces of dimension 1 are the *extreme rays* of P .

We introduced polyhedra as a finite intersection of halfspaces. However, they can also be described in terms of finitely generated sets of non-negative linear combinations, the *V-description*. First, we look at this description in the case that the polyhedron is a cone. By the Minkowski-Weyl Theorem ([BG09, Theorem 1.15]), a set $C \subset \mathbb{R}^d$ is a cone if and only if it is a finitely generated *conical set*, i.e., there are $x_1, \dots, x_n \in \mathbb{R}^d$ such that

$$C = \text{cone}(x_1, \dots, x_n) := \{a_1 x_1 + \dots + a_n x_n \mid a_1, \dots, a_n \in \mathbb{R}_+\}. \tag{1.2}$$

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We call $x_1, \dots, x_n \in \mathbb{R}^d$ the *generators* of C . Note that in the rational case, x_1, \dots, x_n can be chosen to be integer vectors. If the set of generators can be chosen to be linearly independent, we call C *simplicial*.

If P is a polytope, we see from (1.1) and (1.2) that P can be described as the *convex hull* of finitely many points:

$$P = \text{conv}(x_1, \dots, x_n) := \left\{ a_1 x_1 + \dots + a_n x_n \mid a_1, \dots, a_n \in \mathbb{R}_+, \sum_{i=1}^n a_i = 1 \right\}.$$

Finally, every polyhedron P can be written as the (Minkowski) sum of a polytope and a unique cone, the so-called *recession cone* of P , by Motzkin's Theorem (see [BG09, Theorem 1.27]).

1.1.2 Affine monoids and their Hilbert bases & series

As a next step towards the scope and algorithms of Normaliz we consider the discrete analogue of cones: *affine monoids*. Recall that a *monoid* is defined to be a set M together with an associative binary operation $+$: $M \times M \rightarrow M$ and identity element 0 .

Definition 1.1.2. An *affine monoid* M is a finitely generated submonoid of the free abelian group \mathbb{Z}^d . It is called *positive* if $x \in M$ and $-x \in M$ implies $x = 0$, i.e., 0 is the only invertible element in M .

An affine monoid M can be embedded into a free abelian group $\text{gp}(M) = \mathbb{Z}M$ of \mathbb{Z}^d , which is isomorphic to \mathbb{Z}^r for some $r \in \mathbb{Z}_+$. We call r the *rank* of M .

Let M be a positive monoid. We call an element $x \in M, x \neq 0$ *irreducible* if $x = y + z$ implies $y = 0$ or $z = 0$ and otherwise *reducible*. By a theorem of van der Corput, there are only finitely many irreducible elements and they form the unique minimal system of generators of M , which we call the *Hilbert basis* $\text{Hilb}(M)$ of M (see [BG09, Proposition 2.14]).

An affine monoid M generates a cone via $C = \mathbb{R}_+ M$. By [BG09, Proposition 2.16], M is positive if and only if C is a pointed cone. On the other hand, if we intersect a rational cone C with a *lattice* L – a subgroup of \mathbb{Z}^d – we have:

Lemma 1.1.3 (Gordan's Lemma). *Let $C \subset \mathbb{R}^d$ be a rational cone and $L \subset \mathbb{Z}^d$ a lattice. Then $C \cap L$ is an affine monoid.*

The computation of the Hilbert basis for this class of affine monoids is the first main task of Normaliz. By abuse of notation we write $\text{Hilb}(C)$ for the Hilbert basis of $C \cap L$.

Example 1.1.4. Consider the cone $C = \mathbb{R}_+(2, 1) + \mathbb{R}_+(2, 5)$, which is illustrated in Figure 1.3 and is the running example in this chapter. The Hilbert basis of $C \cap \mathbb{Z}^2$ consists of the points

$$\text{Hilb}(C) = \{(2, 1), (2, 5), (1, 1), (1, 2)\}.$$

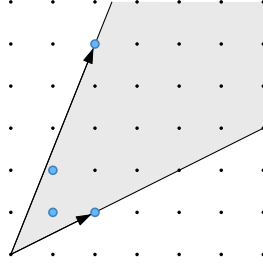


Figure 1.3: A cone with Hilbert basis.

With the Hilbert basis it is possible to compute the *normalization* of an affine monoid M . The *integral closure* of M with respect to the lattice L is given by

$$\widehat{M}_L := \mathbb{R}_+ M \cap L = \{x \in L \mid mx \in M \text{ for some } m \in \mathbb{Z}_{>0}\}$$

and M is called *integrally closed* in L if $M = \widehat{M}_L$. If $L = \text{gp}(M)$, \widehat{M}_L is the *normalization* of M and M is said to be *normal* if $M = \widehat{M}_{\text{gp}(M)}$.

After discussing the problem of *generation* for affine monoids, we now address the *enumeration* of elements by degree. A *grading* of a monoid M is a homomorphism $\text{deg} : M \rightarrow \mathbb{Z}^g$ for some $g \geq 1$. Here, we only consider \mathbb{Z} -gradings, i.e., the case $g = 1$. The *Hilbert function* of M counts the elements in M by degree:

$$H(M, k) = \#\{x \in M \mid \text{deg}(x) = k\}.$$

As we will see below, this counting function can be written as a *quasi-polynomial* for sufficiently large degrees k . A function $q : \mathbb{Z} \rightarrow \mathbb{Z}$ is called a quasi-polynomial of period $\pi > 0$ and degree g if it can be represented in the form

$$q(k) = q_0(k) + q_1(k)k + \dots + q_g(k)k^g,$$

with functions $q_i : \mathbb{Z} \rightarrow \mathbb{C}$ such that $q_i(k) = q_i(j)$ for all i whenever $j \equiv k \pmod{\pi}$, it satisfies $q_g(k) \neq 0$ for at least one k and π is chosen as small as possible.

The generating function of $H(M, k)$ is called the *Hilbert series* of M

$$H_M(t) = \sum_{k=0}^{\infty} H(M, k)t^k = \sum_{x \in M} t^{\text{deg}(x)}.$$

From now on we restrict ourselves to the case that the monoid M has the form $C \cap L$. An *extreme L -generator* (or *extreme integral generator* if $L = \mathbb{Z}^d$) of C is given by the generator of the rank 1 monoid $R \cap L$, where R is an extreme ray of C . The following classical theorem shows that the Hilbert series can be expressed as a rational function and the Hilbert function is a quasi-polynomial for sufficiently large degrees as parameters, see [BG09, Theorem 6.37 & 6.38] and [BH98, Theorem 4.4.3]

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Theorem 1.1.5 (Ehrhart, Hilbert-Serre). *Let $M = C \cap L$ and $r = \text{rank}(M)$.*

1. *The Hilbert series is a rational function that can be written in the form*

$$H_M(t) = \frac{Q(t)}{(1-t^\ell)^r}, \quad (1.3)$$

where $Q(t) = 1 + h_1 t + \dots + h_s t^s$ is a polynomial of degree $s < r\ell$ with non-negative integer coefficients h_i , and ℓ is the least common multiple of the degrees of the extreme L -generators of C .

2. *There exists a (unique) quasi-polynomial $q_M(k)$ of degree $r - 1$ and period dividing ℓ such that $H(M, k) = q_M(k)$ for all $k > s - r\ell$.*

To derive the first claim, we consider a *Stanley decomposition* of the monoid M , which appears again in the next chapter in the setting of graded polynomial rings. It is shown in [Sta82] that there exists a decomposition of M into sets of the form

$$D = x + \sum_{i=1}^r \mathbb{Z}_+ v_i,$$

where v_1, \dots, v_r are linearly independent extreme L -generators of C . Geometrically, the cone C is decomposed into half-open simplicial cones $S \setminus \mathcal{F}$, where S is a simplicial cone and \mathcal{F} a union of facets of S . The computation of such a decomposition in Normaliz uses *order vectors* and is discussed in [BIS16]. Provided $\deg(v_i) > 0$ for $i = 1, \dots, r$, the Hilbert series of D is given by

$$H_D(t) = \frac{t^{\deg(x)}}{(1-t^{\deg(v_1)}) \dots (1-t^{\deg(v_r)})}. \quad (1.4)$$

Therefore, we obtain the Hilbert series of M as the sum of rational functions of the form (1.4), that is,

$$H_M(t) = \frac{R(t)}{(1-t^{g_1}) \dots (1-t^{g_k})}, \quad R(t) \in \mathbb{Z}[t], \quad (1.5)$$

where each g_i is a divisor of ℓ (the lcm of the degrees of the extreme L -generators) and $k \geq r = \text{rank}(M)$. To obtain the denominator in (1.3) with exactly r factors, we replace every exponent in the rational functions in (1.4) by ℓ . In particular, the coefficients in the numerator polynomial are non-negative. The non-negativity can also be justified from the perspective of commutative algebra, see Section 1.4. The section also contains a discussion about other representations using r factors in the denominator, with the advantage that those factors have smaller exponents.

In this thesis we use the terms ‘‘Hilbert series’’ and ‘‘Hilbert quasi-polynomial’’. One could equally well name these objects after Ehrhart. In fact, the Hilbert series of M is nothing but the Ehrhart series of the polytope that one obtains by intersecting C with the hyperplane of degree 1 elements in \mathbb{R}^d .

Example 1.1.6. We equip the cone $C = \mathbb{R}_+(1, 2) + \mathbb{R}_+(2, 1)$ with the *total grading*, i.e., $\deg(x_1, x_2) = x_1 + x_2$, see Figure 1.4. For example, $M = C \cap \mathbb{Z}^2$ has no points of degree

one, one point of degree two and two points of degree three and so on. Its Hilbert quasi-polynomial has period 3 and is given by

$$\begin{aligned} p(k) &= 1 + \frac{1}{3}k, & k \equiv 0 \quad (3), \\ p(k) &= -\frac{1}{3} + \frac{1}{3}k, & k \equiv 1 \quad (3), \\ p(k) &= \frac{1}{3} + \frac{1}{3}k, & k \equiv 2 \quad (3). \end{aligned}$$

Its Hilbert series can be written as

$$H_M(t) = \frac{1 + t^2 + t^4}{(1 - t^3)^2}.$$

We discuss different representations of this series as a rational function in Section 1.4.

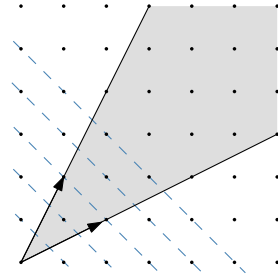


Figure 1.4: A cone with a grading.

1.2 The primal algorithm

After introducing the main objects and computation goals of Normaliz, we present the key ideas behind Normaliz' primal algorithm. The algorithm is named *primal* in order to distinguish it from the *dual* algorithm, which is based on ideas by Pottier [Pot96]. Here, we do not study the dual algorithm and refer the reader to [BI10] for a detailed description.

The primal algorithm starts from a pointed rational cone $C \subset \mathbb{R}^d$ given by a system of generators x_1, \dots, x_n and a sublattice $L \subset \mathbb{Z}^d$ that contains x_1, \dots, x_n . Other types of input data are first transformed into this format. The algorithm is composed as follows:

1. Initial coordinate transformation to $E = L \cap (\mathbb{R}x_1 + \dots + \mathbb{R}x_n)$;
2. Fourier-Motzkin elimination computing the support hyperplanes of C ;
3. computation of a triangulation, i.e., a face-to-face decomposition into simplicial cones;

1 Algorithms in rational cones

4. evaluation of the simplicial cones in the triangulation;
5. collection of the local data;
6. reverse coordinate transformation to \mathbb{Z}^d .

The algorithm does not strictly follow this chronological order, but interleaves steps 2–5 in an elaborate way to ensure low memory usage and efficient parallelization.

In view of the initial and final coordinate transformations 1. and 6. it is no essential restriction to assume that $\dim C = d$ and $L = \mathbb{Z}^d$, as we do in the following.

The major complexity of the algorithm and therefore the runtime is concentrated in building a triangulation and evaluating the simplicial cones. Their complexity originates in quite different reasons:

Combinatorial The size of the triangulation depends heavily on the combinatorial structure of the vector configuration given by the generators x_1, \dots, x_n .

Arithmetic The expense to evaluate a simplicial cone correlates with its number of interior lattice points, which is equal to the determinant of its defining generators.

1.3 Subdivision of large simplicial cones

We discuss how the arithmetic complexity of the primal algorithm can be reduced by subdividing simplicial cones with a large amount of lattice points into smaller cones. For this purpose, we find subdivision points inside such a cone by applying techniques from integer programming or an approximation method.

1.3.1 Simplicial cones

First, we take a closer look at the evaluation of a simplicial cone. Let $x_1, \dots, x_d \in \mathbb{Z}^d$ be linearly independent and $S = \text{cone}(x_1, \dots, x_d)$ be a simplicial cone. The *fundamental domain* of S is the half-open set

$$\pi(S) = \{q_1 x_1 + \dots + q_d x_d \mid 0 \leq q_i < 1\}.$$

The integer points in the fundamental domain $\pi_{\mathbb{Z}}(S) = \pi(S) \cap \mathbb{Z}^d$ together with x_1, \dots, x_d generate the whole monoid $M = S \cap \mathbb{Z}^d$.

Let $U = \mathbb{Z}x_1 + \dots + \mathbb{Z}x_d$ be the lattice generated by x_1, \dots, x_d . Every residue class in the quotient \mathbb{Z}^d/U has a unique representative in $\pi_{\mathbb{Z}}(S)$. Geometrically, the fundamental domain gives a tiling of the cone S . These representatives can be quickly computed via the elementary divisor algorithm and from an arbitrary representative we obtain the one in $\pi_{\mathbb{Z}}(S)$ by division with remainder.

This shows that the elements in $\pi_{\mathbb{Z}}(S)$ together with the generators x_1, \dots, x_d form the candidates for the Hilbert basis of M (see also [BG09, Proposition 2.43]). Normaliz gen-

1.3 Subdivision of large simplicial cones

erates the points in $\pi_{\mathbb{Z}}(S)$ and afterwards shrinks them together with the generators to the Hilbert basis of S by successively discarding reducible elements.

Example 1.3.1. The fundamental domain of the simplicial cone in Example 1.1.4 is illustrated in Figure 1.5. It contains 8 lattice points:

$$\pi_{\mathbb{Z}}(C) = \{(0,0), (1,1), (1,2), (2,1), (2,3), (2,4), (3,4), (3,5)\}.$$

We see, for instance, that $(2,2) = (1,1) + (1,1)$ and $(2,3) = (1,1) + (1,2)$ are reducible, whereas $(1,1)$ and $(1,2)$ are the irreducible elements. Therefore

$$\text{Hilb}(C) = \{(2,1), (2,5), (1,1), (1,2)\}.$$

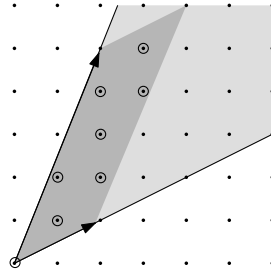


Figure 1.5: A cone with fundamental domain.

The number of elements in $\pi_{\mathbb{Z}}(S)$ is given by the (lattice normalized) volume of S :

$$\#\pi_{\mathbb{Z}}(S) = \text{vol}(S) = \det(x_1, \dots, x_d).$$

Therefore the arithmetic complexity of the Normaliz algorithm is determined by the determinant of the generators of the simplicial cones in a triangulation.

The key idea behind the following algorithms consists of trying to decompose a simplicial cone with large volume into simplices such that the sum of their volumes is considerably smaller. For the decomposition we compute lattice points in the cone to successively define *stellar subdivisions* of the cone into simplices.

The optimal choice for subdivision points are the vertices of the *bottom* $B(S)$ of the simplicial cone. This set is defined as the union of the bounded faces of the polyhedron $\text{conv}\{x \in M \mid x \neq 0\}$, the truncated cone. Figure 1.6 depicts the bottom of the cone in Example 1.1.4.

It is evident that a triangulation with respect to the bottom has the minimal sum of determinants over all possible triangulations of S (see [BSS16a, Proposition 11]). This means that the cones in the triangulation have the form \mathbb{R}_+F , where F is a facet of the bottom.

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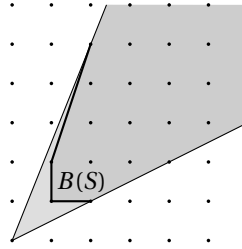


Figure 1.6: The bottom of a cone.

We denote the lattice points of the bottom by \mathcal{B} . They form a subset of the Hilbert basis of M and are even equal to it in the 2-dimensional case (see [BG09, Proposition 2.62]).

However, computing all vertices of the bottom is nearly as expensive as computing the whole Hilbert basis itself and would equalize the benefit from the small volumes (the runtime graphs in Figure 1.9 and 1.10 already indicate this).

Therefore, we determine only some points from the bottom until the respective volumes of the resulting simplices is smaller than a certain bound. We employ two methods for this purpose:

- (1) computation of subdivision points by integer programming methods;
- (2) computation of a set of possible subdivision points by approximation of the given simplicial cone by an overcone that is generated by vectors of “low denominator”.

The computation of possible subdivision points is currently started for simplices with a volume $\geq 10^8$ or $\geq 10^7$ when also the Hilbert basis should be computed. These bounds are chosen to guarantee a high chance of finding such a point: There are $\dim(S)!$ simplices in the canonical triangulation of the fundamental domain $\pi(S)$. Assuming that its lattice points are evenly distributed, the simplicial cone containing the bottom has $\approx \frac{\text{vol}(S)}{\dim(S)!}$ lattice points. This shows that these bounds work very well up to dimension ≈ 10 . However, adjusting these bounds to higher dimension accordingly would have no satisfactory result: Experience shows that computing the Hilbert basis for simplices with volumes $\approx 10^{10}$ is not fast enough to compute for practical purposes.

1.3.2 Methods from integer programming

For each simplicial cone $S = \text{cone}(x_1, \dots, x_d)$ in the triangulation with sufficiently large volume we try to compute a point x that minimizes the *sum of determinants*:

$$\sum_{i=1}^d \det(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d).$$

This sum can also be expressed as a scalar product:

Lemma 1.3.2. *We have*

$$\sum_{i=1}^d \det(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d) = N^T x,$$

for a normal vector N on the affine hyperplane defined by x_1, \dots, x_d .

Proof. The affine hyperplane through the generators x_1, \dots, x_d is spanned by the vectors $x_1 - x_d, \dots, x_{d-1} - x_d$. Let H be the hyperplane through the origin and parallel to this hyperplane. Then H is given as the kernel of the map

$$x \mapsto \det(x_1 - x_d, x_2 - x_d, \dots, x_{d-1} - x_d, x).$$

Thus, there exists a normal vector $N \in H^\perp$ such that

$$\det(x_1 - x_d, x_2 - x_d, \dots, x_{d-1} - x_d, x) = N^T x.$$

Finally, an easy computation shows that

$$N^T x = \det(x_1 - x_d, x_2 - x_d, \dots, x_{d-1} - x_d, x) = \sum_{i=1}^d \det(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d).$$

□

We point out that such a normal vector can be found by solving the following linear system of equalities:

$$\tilde{N}^T x_1 = \tilde{N}^T x_2 = \dots = \tilde{N}^T x_n = 1.$$

over the rationals. Then $N \in \mathbb{Z}^d$ is given by multiplying the solution \tilde{N} with the lcm of the denominators in its coefficients.

In conclusion, in order to find a point that minimizes the sum of determinants, we try to solve the following integer program:

$$\min\{N^T x \mid x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\}. \quad (1.6)$$

Note that the condition $x \neq 0$ can be fulfilled by requiring $N^T x \geq 1$.

If the problem has a solution \hat{x} , we form a *stellar subdivision* of the simplicial cone with respect to \hat{x} : For every support hyperplane H_i of S (the one opposite to x_i) which does not contain \hat{x} we form the simplicial cone

$$T_i = \text{cone}(x_1, \dots, x_{i-1}, \hat{x}, x_{i+1}, \dots, x_d).$$

If the volume of T_i is larger than a particular bound, we repeat this process and continue until all simplices have a smaller volume than this bound or the corresponding integer problems have no solutions. Figure 1.7 illustrates the algorithm and Algorithm 1 gives a detailed description.

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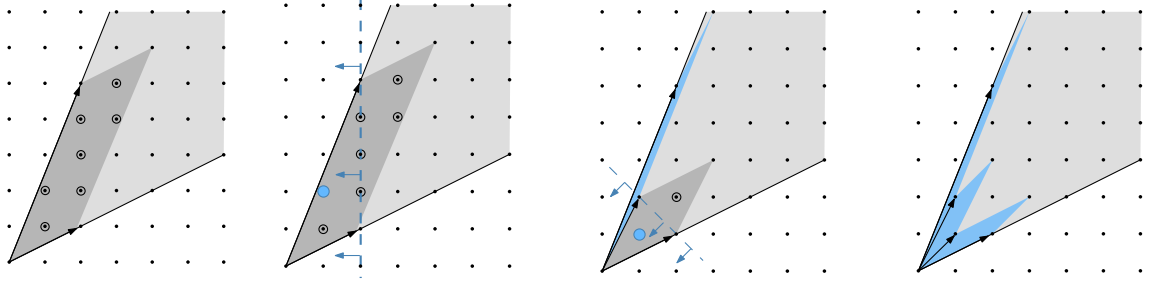


Figure 1.7: The integer programming algorithm for a cone.

Algorithm 1 Bottom Points

Input: simplicial cone $S = \text{cone}(x_1, \dots, x_d)$ with $\text{vol}(S) \geq \text{BOUND}$

Return: Points from $B(S)$

- 1: $\mathcal{B}, \mathcal{S} \leftarrow \emptyset$
 - 2: store S into \mathcal{S}
 - 3: **while** $\mathcal{S} \neq \emptyset$ **do**
 - 4: let $T = \text{cone}(y_1, \dots, y_d)$ be the first element of \mathcal{S} and delete it
 - 5: compute a normal vector N on the hyperplane spanned by y_1, \dots, y_d
 - 6: compute hyperplanes $\{H_1, \dots, H_d\}$ and volume of T
 - 7: **if** $\text{vol}(T) < \text{BOUND}$ **then continue**
 - 8: **if** IP (1.6) is solvable for T **then**
 - 9: $y \leftarrow$ optimal solution of (1.6)
 - 10: store y into \mathcal{B}
 - 11: **for all** hyperplanes H_i of T **do**
 - 12: **if** $y \notin H_i$ **then**
 - 13: $T_i \leftarrow \text{cone}(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_d)$ ▷ stellar subdivision
 - 14: store T_i into \mathcal{S}
 - 15: **return** \mathcal{B}
-

After computing a set of integer points \mathcal{B} , we triangulate the bottom of the polytope $\text{conv}(\mathcal{B} \cup \{x_1, \dots, x_d\})$ and continue by evaluating this triangulation with the usual Normaliz algorithm.

The algorithm encourages a parallelization by nature, since the integer programs of the individual simplices can be solved independently. To realize parallel computation, we use a *generation model* in which a family of new simplices in each recursion step of the algorithm is solved simultaneously. Figure 1.8 illustrates this model.

Implementation and results

We use the mixed integer programming solver SCIP [GFG⁺16] (version 3.2.1) via its C++ interface. The algorithm runs in parallel with one SCIP environment for every thread

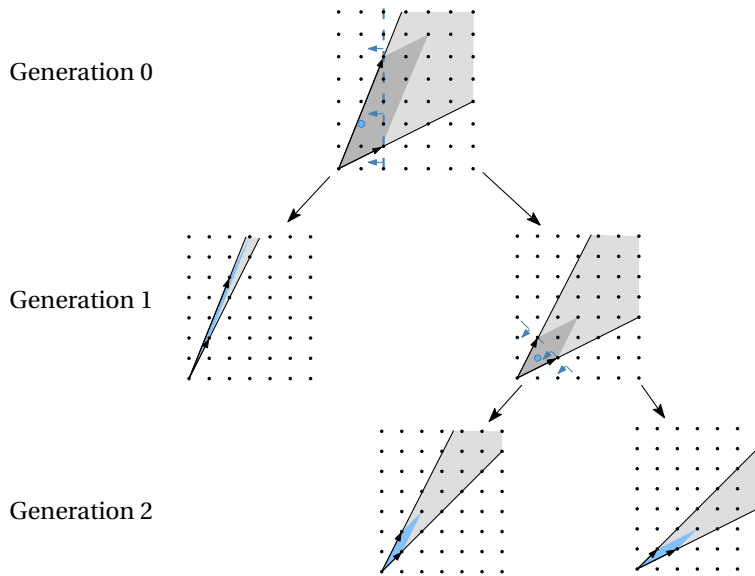


Figure 1.8: Generation model for bottom point algorithm.

using OpenMP. Moreover each SCIP instance has its own time limit $(\log(\text{vol}(S)))^2$ sec and feasibility bounds.

As mentioned above, the condition that $x \neq 0$ could be implemented by the inequality $N^T x \geq 1$. However, this approach is prone to large numbers in N . Therefore we first check, whether all generators are positive in one entry i and thus require $x_i \geq 1$. If this is not the case we enforce a non-zero solution vector through a bound disjunction constraint of the form $(x_i \leq -1 \vee x_i \geq 1)$.

Table 1.1 presents example data computed on a SUN xFire 4450 with four Intel Xeon X7460 processors, using 20 threads and solving integer programs only for simplices with volume larger than 10^6 . All the listed examples are simplicial and run in the default mode.

The bound on the volume to stop the calculation of a single simplicial cone has a significant effect on the runtime of the algorithm. A smaller bound means that more integer programs have to be solved by SCIP, whereas a large bound prevents a major improvement of the respective volume. Running several experiments, it turns out that 10^6 is a good value in between these two extreme cases. Figures 1.9 and 1.10 show runtime graphs illustrating the effect of different choices for this bound. The measured time is a single thread computation of `hickerson-18` and `knapsack_11_60`.

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	hickerson-16	hickerson-18	knapsack_11_60
volume	9.83×10^7	4.17×10^{14}	2.8×10^{14}
volume under bottom	8.10×10^5	3.86×10^7	2.02×10^7
volume used	3.93×10^6	1.04×10^9	1.18×10^8
# integer programs solved	3	2157	250
# bottom points	2	130	59
improvement factor	25	4.02×10^5	2.36×10^6
runtime without subdivision	2s	> 12d*	> 8d*
runtime with subdivision	0.5s	36s	4.3s

* Estimated time.

Table 1.1: Runtime improvements using integer programming methods.

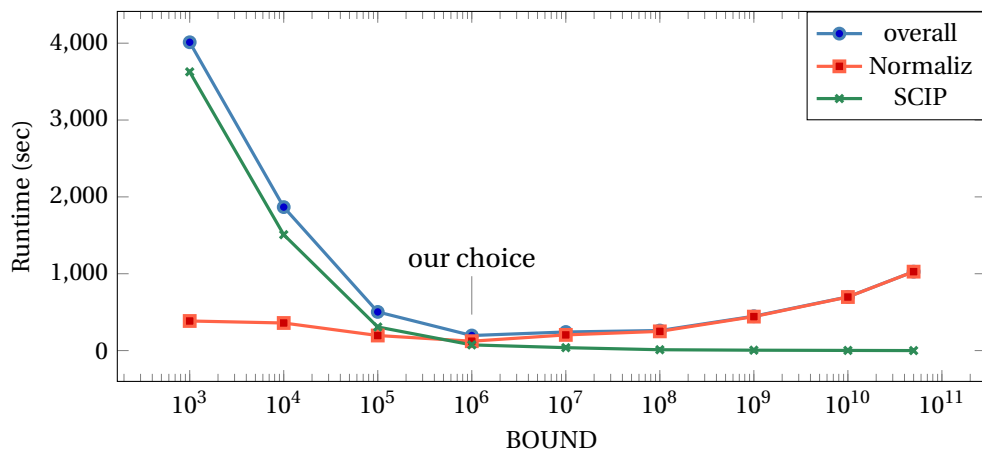


Figure 1.9: Runtime graph for hickerson-18 for different bounds.

1.3.3 Approximation

The second approach to compute subdivision points is completely implemented within Normaliz. It first approximates the simplicial cone S by a (not necessarily simplicial) overcone C for which the sets $\sigma_{\mathbb{Z}}(T)$ in a triangulation of C are significantly faster to compute. Then these points are used to decompose the original simplicial cone as before. It is clear that the efficiency depends crucially on the intersection of the sets $\sigma_{\mathbb{Z}}(T)$ with S .

Note that the general idea of approximation is not restricted to the simplicial case and can, for example, also be used to compute the lattice points of a polytope. This case will be of particular interest in the next section on the enclosing Fourier-Motzkin algorithm. Furthermore, the following methods are currently applied after the coordinate

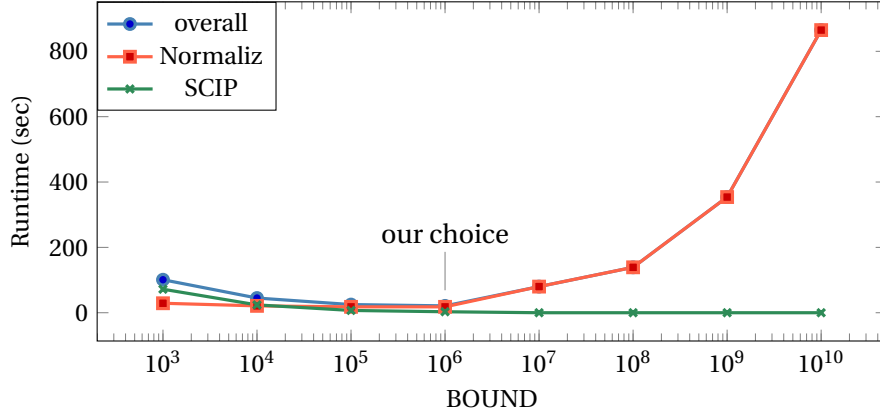


Figure 1.10: Runtime graph for knapsack_11_60 for different bounds.

transformation to the full-dimensional lattice (see step 1. in 1.2). In examples that are already full-dimensional, this usually does not impose a problem. In a lower-dimensional example, however, large coordinates might be produced by this transformation and it consequently would be preferable to approximate beforehand. Normaliz currently tries to approximate prior to the initial coordinate transformation in the case that the lattice points of a polytope should be computed.

To build the approximating cone, we look at the polytope given by the cross section of the original cone at a specific height, initially 1. That is, we restrict the following computations to the codimension one subspace given by the hyperplane associated to the height. The height function either comes from the normal vector N on the affine hyperplane spanned by the generators or is a predefined grading.

For every vertex of this cross section polytope we triangulate the lattice cube around it using the braid hyperplane arrangement $\{x_i = x_j\}$. We continue by detecting the minimal face of this triangulation containing the current vertex and collect its vertices, which are at most d . Essentially, this step consists of division with remainder in every coordinate. More precisely, let $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^d$ be a vertex of the polytope, $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^d$ with $a_i = \lfloor v_i \rfloor$ and $t_i = \{v_i\} = v_i - a_i$. First, assume that $t_i \neq t_j$ for $i \neq j$. Then there exists a unique permutation $\pi \in S_d$ such that

$$1 > t_{\pi(1)} > t_{\pi(2)} > \dots > t_{\pi(d)} > 0.$$

The simplex associated to π is given by

$$\begin{aligned} \Delta_\pi &= \text{conv}(\mathbf{0}, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, \dots, e_{\pi(1)} + e_{\pi(2)} + \dots + e_{\pi(d)}) \\ &= \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d \mid 1 \geq x_{\pi(1)} \geq x_{\pi(2)} \geq \dots \geq x_{\pi(d)} \geq 0 \right\}. \end{aligned}$$

It is easily seen that the vertex \mathbf{v} is contained in the shifted simplex $\mathbf{a} + \Delta_\pi$, also referred to as *Weyl chamber* (see [BG09, Proposition 3.1]). The vertices of $\mathbf{a} + \Delta_\pi$ are then used to

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approximate the vertex ν . If the fractional parts t_i are equal in some coordinates, ν lies in the intersection of several Weyl chambers. These chambers arise from permuting those sets of coordinates for which the fractional parts are equal. In this case, the vertices of the common intersection are collected for the approximation.

Finally, the approximating cone C is generated by all vertices created in this way. Figure 1.11 illustrates the approximation for a 3-dimensional cone (with a 2-dimensional cross section). The vertices of the cross section polytope and their approximation points are

$$\left(\frac{3}{10}, \frac{23}{10}\right), \{(0, 2), (1, 3)\}; \quad (3, 2), \{(3, 2)\}; \quad \left(\frac{27}{10}, \frac{1}{5}\right), \{(2, 0), (3, 0), (3, 1)\}.$$

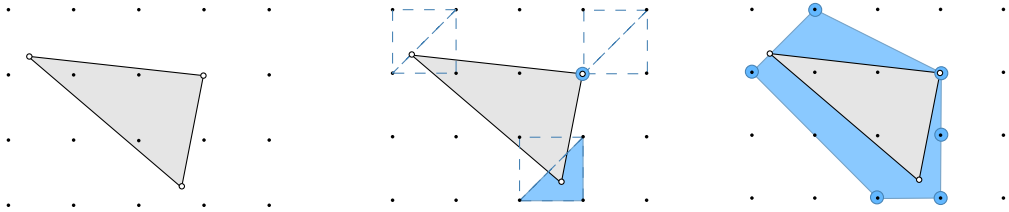


Figure 1.11: Approximating cone

After building the exterior cone C we continue as in the usual Normaliz algorithm and create a candidate list for the Hilbert basis of C . We keep only those candidates which lie inside the original simplicial cone S and reduce them as usual. This procedure results in a list of points \mathcal{B} which is then used for a recursive subdivision of the simplicial cone as in the integer programming approach. In each subdivision step we choose a point that minimizes the height and create a respective stellar subdivision. More precisely, the algorithm is the same as Algorithm 1, but step 8. and 9. are replaced by checking whether the list of points is non-empty and choosing a height minimizing point. Figure 1.12 illustrates this process for the previous example.

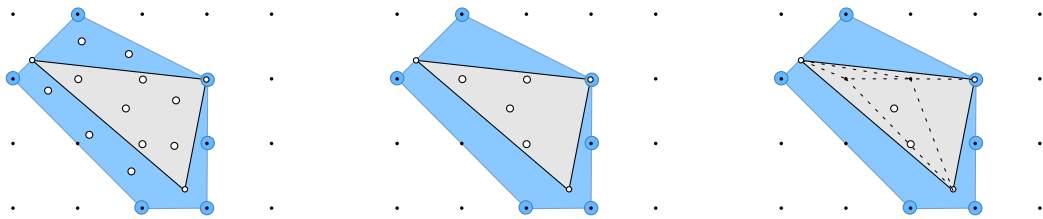


Figure 1.12: Decomposition of a simplicial cone after approximation

In both approaches, it might happen that no point for a subdivision can be found – since the integer program cannot be solved or the cross section at height one of the cone

1.3 Subdivision of large simplicial cones

does not contain any lattice points. If the volume of such a simplicial cone is quite large ($\approx 10^9$) the approximation method is applied again but with a greater height for the approximation. This new height ℓ is determined such that the approximating cone has roughly a volume of 10^5 . More explicitly:

$$\ell \approx \left(\frac{10^5 \cdot \prod N^T x_i}{\text{vol}(S)} \right)^{1/d}.$$

If the new approximation also does not yield new subdivision points, the simplicial cone is processed as in the usual Normaliz algorithm.

Table 1.2 contains performance data for the examples in Section 1.3.2.

	hickerson-16	hickerson-18	knapsack_11_60
volume used	3.93×10^6	1.53×10^9	1.01×10^{10}
# approximation points	25	59	17
# candidates	276	19775	6612
# bottom points	2	193	1937
improvement factor	25	2.72×10^5	2.77×10^4
runtime with subdivision	0.6s	36s	2m23s

Table 1.2: Runtime improvements using the approximation method.

Enclosing Fourier-Motzkin elimination

Instead of taking all points found by approximating the vertices of the polytope in the approximation algorithm, one could try to reduce the number of generators for the exterior cone by building the new cone iteratively until the original cone is contained and then stop. This seems especially useful for non-simplicial examples. Moreover, the support hyperplanes of the exterior cone have to be computed in any case. Since Normaliz is using successive Fourier-Motzkin elimination to compute the hyperplanes (see [BIS16]), stopping this process at a certain point does not impose a problem for the runtime.

For this purpose, we create the approximation points as before and store the set of respective points for each original vertex. These sets are then arranged in descending order according to the number of negative halfspaces containing them. More precisely: If H_1, \dots, H_s are the support hyperplanes of the original cone and x an approximation point, this number is given by $\#\{H_i \mid x \in H_i^<\}$. The overall set of new generators is then arranged periodically with respect to the original vertices, i.e., we take the first point of every original vertex and continue with the respective second point until all approximation points are sorted. We continue by building the new cone with the usual Fourier-Motzkin elimination (see e.g. [BIS16]), but stop the process as soon as all original vertices are contained in the current cone, i.e., they are non-negative on the current hyperplanes. This

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non-negativity check is omitted if one of the current hyperplanes is known to be negative on some original vertex. Figure 1.13 illustrates the algorithm for our running example, where the vertex labels indicate the number of negative halfspaces.

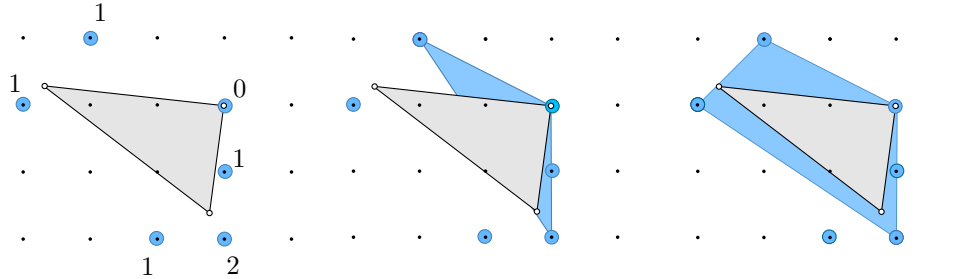


Figure 1.13: Enclosing Fourier-Motzkin elimination.

As suggested above this idea appears to be particularly fruitful for non-simplicial examples, e.g., computing the lattice points of a polytope. Nevertheless, example data shows that only a fraction of approximation points can be neglected.

Table 1.3 shows computational results of calculating the lattice points of a polytope (option `-r1`) via global approximation of `max_cand` from the Normaliz example database and randomly generated rational polytopes. These polytopes were created by choosing the integer coefficients of their defining inequalities at random. In doing so, we fix bounds $A, B \in \mathbb{Z}_{>0}$ such that the coefficients in the inequality

$$a_1 x_1 + \dots + a_d x_d \geq b$$

satisfy $-A \leq a_i \leq A$ and $1 \leq b \leq B$.

The first two columns contain the runtime in seconds with and without enclosing Fourier-Motzkin elimination. The last two columns give the amount of overall approximation points and the ones that were actually used.

	new rt	old rt	dim	# ineq	A	B	# vertices	# approx pts	# used pts
<code>max_cand</code>	2.8	2.7	5	11	-	-	24	99	82
<code>rand_15_5</code>	0.2	0.2	5	15	50	500	27	129	123
<code>rand_30_6</code>	0.4	0.4	6	30	100	1000	124	429	336
<code>rand_20_7</code>	31	35	7	20	100	1000	204	1264	1257
<code>rand_25_7</code>	4.9	6	7	25	100	500	256	1054	860
<code>rand_30_8</code>	407	455	8	30	100	1000	430	2502	2494
<code>rand_40_8</code>	564	725	8	40	100	1000	1016	4269	4261

Table 1.3: Performance data of enclosing Fourier-Motzkin elimination.

1.4 Representations of Hilbert series and homogeneous system of parameters

This section deals with the enumeration of lattice points of a cone. Recall from Section 1.1 that this enumeration is encoded in the Hilbert function and that its generating function is called the Hilbert series, which can be expressed as a rational function. Here, our focus lies on different representations of the Hilbert series as a rational function.

As in the introductory section, we consider monoids $M = C \cap L$ where $C \subset \mathbb{R}^d$ is a rational pointed cone and $L \subset \mathbb{Z}^d$ is a lattice. As explained above, we may assume that $d = \dim C$ and $L = \mathbb{Z}^d$. We fix a grading $\deg: \mathbb{Z}^d \rightarrow \mathbb{Z}$ such that $\deg(x) > 0$ for $x \in M, x \neq 0$.

We have seen in Theorem 1.1.5 that the Hilbert series $H_M(t)$ of M can be expressed as a rational function with a non-negative and integer valued polynomial in the numerator. While this theorem gives a representation of $H_M(t)$ in which all parameters have a natural combinatorial description, it is not completely satisfactory since the denominator often has a very large degree and one can do better. In this section we discuss a representation of $H_M(t)$ as a rational function whose

1. denominator is of the form $(1 - t^{g_1}) \cdots (1 - t^{g_d})$ and of small degree $g_1 + \cdots + g_d$;
2. numerator polynomial has non-negative integer coefficients and they have a combinatorial interpretation.

The following example shows that in general there is no canonical choice of the denominator.

Example 1.4.1. The Hilbert series of the cone in Example 1.1.6 can be written as

$$H_M(t) = \frac{1 - t + t^2}{(t-1)^2(1+t+t^2)} = \frac{1 - t + t^2}{(1-t)(1-t^3)} = \frac{1 + t^2 + t^4}{(1-t^3)^2} = \frac{1 + t^3}{(1-t^2)(1-t^3)}$$

As the following discussion shows, all representations are systematic. Note that there are negative coefficients appearing in the numerator of the first two representations.

In the case that all extreme integral generators have degree 1, e.g., a cone over a lattice polytope, the denominator of (1.3) in Theorem 1.1.5 is $(1-t)^d$ and there is nothing to discuss.

The default representation of the Hilbert series in Normaliz is computed as follows: We recall that the “raw” form (1.5) is attained as the sum of the Hilbert series of the half-open simplicial cones given by a triangulation of the cone. We factor the denominator of this form into a product of *cyclotomic polynomials* and obtain a representation

$$H_M(t) = \frac{\tilde{Q}(t)}{\zeta_{q_1}^{e_1} \cdots \zeta_{q_u}^{e_u}}$$

with $\tilde{Q}(t) \in \mathbb{Z}[t]$ and cyclotomic polynomials $\zeta_k, 1 = q_1 < q_2 < \cdots < q_u$ such that $\zeta_k \nmid \tilde{Q}(t)$. This is the first representation in Example 1.4.1. Note that $\zeta_1 = t - 1$ and $\zeta_3 = 1 + t + t^2$.

1 Algorithms in rational cones

We point out that it follows from Theorem 1.1.5 that the lcm of q_1, \dots, q_u is the period of the Hilbert quasi-polynomial.

In order to achieve a representation with a denominator of the form $(1 - t^{g_1}) \cdots (1 - t^{g_d})$, we take g_d as the lcm of all q_i and replace the product $\zeta_{q_1} \cdots \zeta_{q_u}$ by $(1 - t^{g_d})$. We proceed with the possibly remaining cyclotomic factors in the same way. Thereby, the exponents g_k express the periods of the coefficients in the Hilbert quasi-polynomial: g_i is the lcm of the periods of the coefficients q_{d-1}, \dots, q_{d-i} . We will refer to the denominator of this representation as *standard denominator*. This choice is easy to compute and natural in its way, but not satisfactory if one wants a combinatorial interpretation of the coefficients in the numerator. For example the second representation in Example 1.4.1 has the standard denominator but the numerator polynomial has negative coefficients.

1.4.1 Homogeneous system of parameters

To fulfill the two requirements above, we can choose g_1, \dots, g_d as the degrees of the elements in a *homogeneous system of parameters* (hsop for short) of the monoid algebra. Their computation requires an analysis of the face lattice of the cone and thus is only possible if the cone has a manageable number of facets.

We review some basic concepts of commutative algebra following [BH98]. Let R be a commutative Noetherian local ring with maximal ideal \mathfrak{m} . Let $\mathfrak{p} \subset R$ be a prime ideal. Its *height* is the maximal length h of a strictly descending chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supset \cdots \supset \mathfrak{p}_h.$$

The height of an arbitrary ideal $I \subset R$ is then given by

$$\text{ht}(I) = \inf \{ \text{ht}(\mathfrak{p}) \mid \mathfrak{p} \supset I, \mathfrak{p} \text{ prime} \}.$$

By *Krull's principal ideal theorem* ([BH98, Theorem A.1]) it is finite if I is finitely generated.

The *Krull dimension* $\dim(R)$ of R is the supremum of the heights of its prime ideals and is equal to $\text{ht}(\mathfrak{m})$ ([BH98, Theorem A.3]). Given an R -module N , we define its dimension via $\dim N = \dim(R/\text{Ann}N)$, where $\text{Ann}N = \{r \in R \mid rn = 0 \forall n \in N\}$ is the *annihilator* of N .

Let N be an R -module. An element $x \in R$ is called *N -regular*, if $xn = 0$ for $n \in N$ implies $n = 0$. A sequence $\mathbf{x} = x_1, \dots, x_n$ of elements in R is called an *N -regular sequence* if

- (i) x_i is regular on $N/(x_1, \dots, x_{i-1})N$ and
- (ii) $N/\mathbf{x}N \neq 0$.

If N is a finite R -module, we define the *depth* of N to be the length of a maximal N -regular sequence in \mathfrak{m} . Note that by a theorem of Rees ([BH98, Theorem 1.2.5]) all those sequences have the same length. N is called *Cohen-Macaulay* if $\text{depth}(N) = \dim(N)$.

Now, let $R = \bigoplus_{i=0}^{\infty} R_i$ be a finitely generated \mathbb{Z} -graded algebra over some infinite field $K = R_0$ of Krull dimension $\dim(R) = d$. Its graded maximal ideal is given by $\mathfrak{m} = \bigoplus_{i>0} R_i$.

In our case, R is the *monoid algebra* $K[M]$ corresponding to the monoid $M = C \cap L$. The monoid algebra is defined to be the free R -module with basis x^m , $m \in M$, that is,

$$K[M] = \left\{ \sum_{i=0}^k \lambda_i x^{m_i} \mid \lambda_i \in K, m_i \in M \right\},$$

with multiplication $x^m \cdot x^{m'} = x^{m+m'}$ for $m, m' \in M$. Since $M = C \cap L$ is normal, $K[M]$ is Cohen-Macaulay by a theorem of Hochster's (see [BG09, Theorem 6.10]).

Definition 1.4.2. Homogeneous elements $\theta_1, \dots, \theta_d \in \mathfrak{m}$ form a *homogeneous system of parameters* (of R) if $\mathfrak{m} = \text{Rad}(\theta_1, \dots, \theta_d)$.

These systems can be characterized by the following statement ([BH98, Theorem 1.5.17, Proposition 2.2.11]).

Lemma 1.4.3. Let $\theta_1, \dots, \theta_d \in \mathfrak{m}$ be homogeneous elements and $\theta = (\theta_1, \dots, \theta_d)$ the ideal generated by them. Then the following are equivalent:

- (i) $\theta_1, \dots, \theta_d$ is an hsop.
- (ii) R is a finite $K[\theta_1, \dots, \theta_d]$ -module.
- (iii) $\dim(R/\theta) = 0$.
- (iv) R/θ is a finite-dimensional K -vector space.

If, additionally, R is Cohen-Macaulay, being an hsop is equivalent to R being a finite free $K[\theta_1, \dots, \theta_d]$ -module.

In the \mathbb{Z} -graded case the existence of such a system is guaranteed by the *prime avoidance lemma*, see [BH98, Lemma 1.5.10]:

Lemma 1.4.4. Let R be a \mathbb{Z} -graded ring and $I \subset R$ an ideal generated in positive degree. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be prime ideals such that $I \not\subset \mathfrak{p}_i$ for $i = 1, \dots, r$. Then there exists a homogeneous element $x \in I$ with $x \notin \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$.

For any ideal I in R generated in positive degree of height $\text{ht}(I) = h$, the lemma provides the existence of homogeneous elements $\theta_1, \dots, \theta_h$ of I such that $\text{ht}(\theta_1, \dots, \theta_i) = i$ for all $i = 1, \dots, h$. In particular, it is possible to construct an hsop for R , since $\text{ht}(\mathfrak{m}) = d$.

Now, if $\theta_1, \dots, \theta_d$ is an hsop for $K[M]$, the Hilbert series of M can be written in the form

$$H_M(t) = \frac{h_0 + h_1 t + \dots + h_m t^m}{(1 - t^{g_1}) \dots (1 - t^{g_d})},$$

where $g_j = \deg(\theta_j)$. Furthermore, h_i counts the number of elements of degree i in a homogeneous basis of $K[M]$ over $K[\theta_1, \dots, \theta_d]$. In particular, h_i is non-negative (see [BG09, Theorem 6.40]).

1 Algorithms in rational cones

Returning to our task of finding a nice representation of the Hilbert series, we can therefore compute (the degrees of) an hsop for the monoid algebra $K[M]$.

The main idea for the construction of such an hsop is to create its elements θ_i with $\text{ht}(\theta_1, \dots, \theta_i) = i$ out of the extreme integral generators of the cone C . We denote them by $x_1, \dots, x_n \in \mathbb{Z}^n$ and point out that $\text{ht}(x_1, \dots, x_n) = d$, where x_1, \dots, x_n are seen as monomials in $K[M]$. This claim will become apparent when we discuss the computation of the height of a monomial ideal in $K[M]$.

We successively insert the monomials x_j into a monomial ideal and compute its height. By Krull's principal ideal theorem ([BH98, Theorem A.1]), the height of this monomial ideal can only increase by at most one in each insertion step. If the height increases, i.e.,

$$\text{ht}(x_1, \dots, x_j) = i > i - 1 = \text{ht}(x_1, \dots, x_{j-1}),$$

we let

$$\theta_i := \lambda_1 x_1^{a_1} + \dots + \lambda_j x_j^{a_j},$$

where $\lambda_k \in K$ are generic coefficients and the exponents a_k are chosen in such a way that θ_i is homogeneous of degree $\text{lcm}(\deg(x_1), \dots, \deg(x_j))$. We point out that the height does not change if we replace the x_i by powers of them. Furthermore, in general all current monomials x_1, \dots, x_j are needed to ensure that $\text{ht}(\theta_1, \dots, \theta_i) = i$.

We are left with the task to compute $\text{ht}(x_1, \dots, x_j)$. The monomial prime ideals in the monoid algebra $K[M]$ are exactly those of the form $\mathfrak{p}_F = K\{M \setminus F\}$, where F is a face of the cone C . Given a monomial ideal I in $K[M]$, its minimal prime ideals are given by those prime ideals \mathfrak{p}_F for which F is a maximal face with respect to disjointness to I , i.e., $I \cap F = \emptyset$ (see [BG09, Corollary 4.35]).

Furthermore, the height of a prime ideal is given by the codimension of its respective face, i.e., $\text{ht}(\mathfrak{p}_F) = \text{codim}(F) = d - \dim(F)$ (see [BG09, Proposition 4.36]). In particular, the ideal generated by the extreme generators x_1, \dots, x_n has height d : the only face disjoint to them is the vertex $\{0\}$.

In conclusion:

$$\text{ht}(x_1, \dots, x_j) = \min_{F \text{ face}} \{\text{codim}(F) \mid F \cap \{x_1, \dots, x_j\} = \emptyset\}.$$

These considerations lead to a successive algorithm to compute the *heights vector* $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{Z}_+^n$ with $h_j = \text{ht}(x_1, \dots, x_j)$. We start with the facets of the cone. In each step the current maximal disjoint faces with respect to $\{x_1, \dots, x_j\}$ are computed from the previous set of faces, where we partition this set depending on which faces contain the current generator x_j . The respective height is updated accordingly: If all current faces contain x_j or the maximal dimension of the faces not containing x_j decreases, the height increases by one. Otherwise it stays the same. Algorithm 2 gives a detailed overview.

In step 13. in the algorithm, some of the facets can be neglected in the process of taking intersections with the current faces in iteration j due to the following criteria:

Algorithm 2 Heights

```

1:  $h_0 \leftarrow 1$ 
2:  $\mathcal{G} \leftarrow$  facets of  $C$ 
3:  $m \leftarrow d - 1$ 
4: for  $j = 1, \dots, n$  do
5:    $\mathcal{G}_1 \leftarrow \{G_k \in \mathcal{G} \mid x_j \notin G_k\}$ 
6:    $\mathcal{G}_2 \leftarrow \{G_k \in \mathcal{G} \mid x_j \in G_k\}$ 
7:   if  $\mathcal{G}_1 \neq \emptyset$  then
8:     if  $\max_{G_k \in \mathcal{G}_1} \{\dim(G_k)\} < m$  then
9:        $m \leftarrow m - 1$ 
10:       $h_j \leftarrow h_{j-1} + 1$ 
11:     else  $h_j \leftarrow h_{j-1}$ 
12:   else  $h_j \leftarrow h_{j-1} + 1$ 
13:   for all facets  $F_\ell$  with  $x_j \notin F_\ell$  do
14:     for all  $G_k \in \mathcal{G}_2$  do
15:        $G_{k,\ell} \leftarrow G_k \cap F_\ell$ 
16:    $\mathcal{G} \leftarrow \mathcal{G}_1 \cup \{\text{maximal faces from } G_{k,\ell}\}$ 

```

1. The facet contains the current generator x_j ;
2. The facet is given by generators appearing in faces in \mathcal{G}_1 or $\{x_1, \dots, x_{j-1}\}$;
3. Facets only involving the generators x_1, \dots, x_j can be ignored for all following iterations.

Algorithm 2 is illustrated in Figure 1.14 for a cone over the cube $[0,1]^3$. The maximal disjoint faces are colored in blue and the current height of the ideal (x_1, \dots, x_j) is written below. Note that the intersection with facets is only necessary after inserting the third and fifth generator thanks to the criteria above.

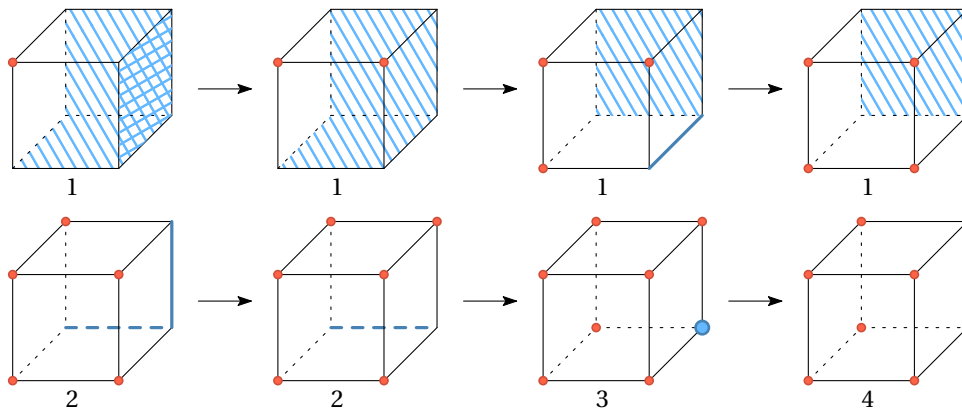


Figure 1.14: The heights algorithm for a cone over a cube.

Implementation

We discuss the actual implementation of the algorithm. The faces are encoded as bitsets which represent incidence vectors of the generators defining them. These bitsets are sorted lexicographically, which has two advantages: First, it makes it easy to partition the set of faces, since we can go through the faces and stop as soon as the current generator is set. All subsequent faces have to contain the generator. Second, we have to use fewer comparisons when filtering the maximal faces in the last step. A face can only contain another face if the incidence vector of the latter is lexicographically smaller. For the containment problem, we create a key vector collecting the indices of generators not contained in the current face and check whether the lexicographically smaller faces are not set on any of the entries of this key and can thus be deleted. Moreover, the bit for the current generator is removed from the face bitsets in each step, which makes the set operations faster. Nevertheless, the critical aspect for the runtime is the face structure of the cone.

Once the heights vector \mathbf{h} is computed, the degrees of the corresponding hsop can be determined as mentioned before, although not all initial generators need to appear in the lcm to compute the homogeneous degree. More precisely, let ℓ denote the smallest index such that $h_\ell = h_{\ell+1}$. Since $\text{ht}(x_1, \dots, x_j, x_{j+1}) = h_{j+1} = h_j + 1 = \text{ht}(x_1, \dots, x_j) + 1$ for $j = 1, \dots, \ell - 1$, we can choose the elements θ_i such that

$$\deg(\theta_i) = \begin{cases} \deg(x_i), & \text{if } i \leq \ell, \\ \text{lcm}(\deg(x_{\ell+1}), \dots, \deg(x_i)), & \text{if } i > \ell. \end{cases}$$

We finally calculate the numerator of the new representation of the Hilbert series, by multiplying the representation with cyclotomic polynomials in the denominator with the product $(1 - t^{g_1}) \cdots (1 - t^{g_d})$, where $g_j = \deg(\theta_j)$.

We note that for the simplicial case the extreme integral generators x_1, \dots, x_d already form an hsop. Therefore the choice of their degrees in the denominator of the Hilbert series can be considered as canonical.

In Example 1.4.1 the series can be expressed as:

$$H_M(t) = \frac{1 + t^2 + t^4}{(1 - t^3)^2},$$

where the degrees appearing in the denominator come from the extreme integral generators of C . The numerator has non-negative coefficients and counts the number of homogeneous basis elements of $K[M]$ as a $K[x_1, x_2]$ -module per degree, in this case $(0, 0)$, $(1, 1)$ and $(2, 2)$ of degree 0, 2 and 4 respectively. This example also shows that using the Hilbert basis instead of the extreme integral generators as a generating system for M sometimes yields smaller exponents in the denominator, namely $(1 - t^2)(1 - t^3)$ (the last representation). However, using the Hilbert basis for our algorithm increases the complexity of taking intersections remarkably, which is the most expensive step.

Example 1.4.5. Let $C = Q \times \{1\}$ be the cone over a square Q , see Figure 1.15. We equip C with the grading given by $\deg(x_i) = i$ for $i = 1, \dots, 4$. (This choice is eligible since the only condition for this configuration is that the two sums of the degrees of antipodal points agree.) The algorithm computes the following sequence of heights:

$$h_1 = \text{ht}(x_1) = 1, h_2 = \text{ht}(x_1, x_2) = 1, h_3 = \text{ht}(x_1, x_2, x_3) = 2, h_4 = \text{ht}(x_1, x_2, x_3, x_4) = 3.$$

This is also illustrated in Figure 1.15 where blue lines indicate the maximal disjoint faces.

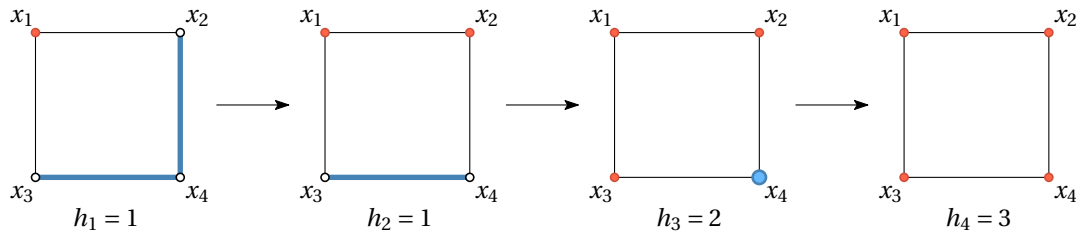


Figure 1.15: Sequence of heights for a cone over a square.

The degrees for the corresponding hsop are given by $\deg(\theta_1) = 1, \deg(\theta_2) = 6$ and $\deg(\theta_3) = 12$ and the Hilbert series has the form

$$H_M(t) = \frac{1 + t^2 + t^3 + 2t^4 + 2t^6 + t^7 + 2t^8 + 2t^{10} + t^{11} + t^{12} + t^{14}}{(1-t)(1-t^6)(1-t^{12})}.$$

The heights vector and the degrees of the corresponding hsop can also be seen on the terminal if Normaliz is run with the verbosity option:

Heights vector: 1 1 2 3

Degrees of HSOP: 1 6 12

The representation of the Hilbert series with standard denominator explained at the beginning of this section for this example is

$$H_M(t) = \frac{1 + t^3 + t^4 - t^5 + t^6 + t^7 + t^{10}}{(1-t)(1-t^2)(1-t^{12})}.$$

Note that again it has a negative coefficient in the numerator.

If the order of the generators would be x_2, x_3, x_1, x_4 , the degrees and hence the exponents in the denominator of the Hilbert series are smaller, namely $\deg(\theta_1) = 2, \deg(\theta_2) = 3, \deg(\theta_3) = 4$ and

$$H_M(t) = \frac{1 + t + t^2 + t^3 + t^4}{(1-t^2)(1-t^3)(1-t^4)}.$$

1 Algorithms in rational cones

However, considerations about the best possible order of generators would involve knowledge about the algebraic structure and defining equations (in this case $x_1x_4 = x_2x_3$) of the input data. But those data are not accessible in Normaliz. Moreover, there is no clear answer to the question what an optimal choice for the exponents in the denominator should look like. Nevertheless, a possibility to improve the current representation would be a dynamic choice of the generators, where the next generator is chosen to lie in as many faces as possible, e.g., x_1, x_4, x_2, x_3 in the above example.

2 The Stanley depth in the upper half of the Koszul complex

There is no problem in all mathematics
that cannot be solved by direct counting.

Ernst Mach

This chapter mainly follows the article “The Stanley Depth in the Upper Half of the Koszul Complex” written by Lukas Katthän and the author and published in [KS16].

Stanley decompositions and Stanley depth form an important and much investigated topic in combinatorial commutative algebra. These decompositions split a module into a direct sum of graded vector spaces of the form Sm , where $S = K[x_{i_1}, \dots, x_{i_d}]$ is a subalgebra of the polynomial ring and m a homogeneous element. They were introduced by Stanley in [Sta82] and he conjectured that the maximal depth of all possible decompositions of a module – the *Stanley depth* – is greater than or equal to the usual depth of the module. Being open for 34 years, the *Stanley conjecture* ([Sta82, Conjecture 5.1]) was recently disproved in [DGKM16] by constructing a non-partitionable Cohen-Macaulay simplicial complex.

Bruns et. al. introduced a weaker notion of decompositions in [BKU10], namely *Hilbert decompositions*. In contrast to Stanley decompositions they only depend on the Hilbert series of the module and are usually easier to compute. The analogue of the Stanley depth – the *Hilbert depth* – gives a natural upper bound for the Stanley depth of every module and leads to a weakened formulation of the Stanley conjecture. Nevertheless, the counterexample given by Duval et. al. has equal Hilbert and Stanley depth, which follows immediately from [BKU10, Proposition 2.8]. Therefore this example also disproves the weaker conjecture.

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over some field K . Let us denote by $M(n, k)$ the k -th syzygy module of the residue field K of R , i. e., the k -th syzygy module in the Koszul complex. It was shown in [BKU10] that the Hilbert depth in the upper half of the Koszul complex is $n - 1$, where n is the number of variables, and conjectured that the same is true for the Stanley depth. In the following we prove this conjecture:

Theorem 2.0.1. *For all n and $n > k \geq \lfloor \frac{n}{2} \rfloor$ one has*

$$\text{sdepth } M(n, k) = n - 1.$$

2 The Stanley Depth in the Koszul Complex

If a module is *finely graded*, i. e., every graded part has K -dimension at most 1, it is rather easy to transform a Hilbert decomposition into a Stanley decomposition and thus also to compute the respective depth (see [BKU10, Proposition 2.8]). However, this is not the case for the modules of our interest. In particular

$$\dim M(n, k)_m = \binom{|m| - 1}{k - 1}$$

where m is a multidegree and $|m|$ its total degree. Hence our theorem provides the Stanley depth for a whole family of modules with higher dimensions in the graded components. Up to now only a few examples of this type are known. To obtain the result we transform the Hilbert decomposition in [BKU10] into a Stanley decomposition by applying new combinatorial techniques. Especially, we are interested in matchings in the Boolean algebra and their properties.

In Section 2.1 we review the definitions of Stanley and Hilbert decompositions and respective depth and their connections. The next section deals with a matching in the Boolean algebra and its properties, in particular a concrete formula for an injective map from larger to smaller sets in the upper half of this poset. This part mainly relies on a paper by Aigner (see [Aig73]). In the last section we first review the Hilbert decomposition used for the proof of Bruns et. al. for the Hilbert depth in the upper half of the Koszul complex. We then introduce the notion of the *index* of a subset G of a given set M . It is the highest power of the matching restricted to the power set of M for which G is in its image. We argue that in order to prove the theorem, we have to show that the subsets of size k with even index of every set M have an order fulfilling a certain condition. As a final step it is shown that the *squashed* order satisfies this condition.

The methods and notions developed are not only important for the sake of the proof. They also give new interesting insights from a purely combinatorial point of view.

Note that we not investigate the Stanley depth in the lower half of the Koszul complex. Already the Hilbert depth behaves quite irregularly in this case, as it was pointed out in the last sections of [BKU10]. Moreover, the techniques developed are rather special and cannot be applied to the lower half.

2.1 Stanley and Hilbert decompositions

We briefly review the basic concepts as given in [BKU10]. For a fuller treatment of Stanley decompositions see for example [Her13]. The basic algebraic concepts are discussed in Section 1.4.1.

We denote by R the polynomial ring $K[x_1, \dots, x_n]$ in n variables over a field K . Here, we equip it with a multigrading over \mathbb{Z}^n , i. e., $\deg(x_i) = e_i$ where $e_i \in \mathbb{Z}^n$ is the i -th unit vector.

2.1 Stanley and Hilbert decompositions

Let M be a finitely generated graded R -module and $m \in M$ a homogeneous element. Furthermore let $Z \subseteq \{x_1, \dots, x_n\}$. The module $K[Z]m$ is called a *Stanley space* of M if $K[Z]m$ is a free $K[Z]$ -submodule of M .

Definition 2.1.1. Let M be a finitely generated graded R -module. A *Stanley decomposition*

$$\mathcal{D} = (K[Z_i], m_i)_{i \in I}$$

with $Z_i \subseteq \{x_1, \dots, x_n\}$ is a finite decomposition of M as a graded K -vector space

$$M = \bigoplus_{i \in I} K[Z_i]m_i$$

where the $K[Z_i]m_i$'s are Stanley spaces of M .

This direct sum forms an R -module via component-wise addition and scalar multiplication with elements in R . In particular, it has a well-defined depth. Note that this depth is given by $\min \{\#Z_i \mid i \in I\}$. The *Stanley depth* of M is the maximal depth of all possible Stanley decompositions:

$$\text{sdepth}(M) := \max \{ \text{depth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M \}.$$

Example 2.1.2. We consider the introductory example from [Her13]. Let $R = K[x_1, x_2]$ and $I = (x_1^3 x_2, x_1 x_2^3) \subset R$. Possible Stanley decompositions of I and R/I are given by

$$\begin{aligned} \mathcal{D}_I : \quad & I = x_1 x_2^3 K[x_1, x_2] \oplus x_1^3 x_2^2 K[x_1] \oplus x_1^3 x_2 K[x_1] \\ \mathcal{D}_{R/I} : \quad & R/I = K[x_2] \oplus x_1 K[x_1] \oplus x_1 x_2 K \oplus x_1 x_2^2 K \oplus x_1^2 x_2 K \oplus x_1^2 x_2^2 K \end{aligned}$$

Therefore $\text{sdepth}(I) \geq \text{depth}(\mathcal{D}_I) = 1$ and $\text{sdepth}(R/I) \geq \text{depth}(\mathcal{D}_{R/I}) = 0$. In fact, $\text{sdepth}(I) = 1$ and $\text{sdepth}(R/I) = 0$, as it is shown below. The decompositions are illustrated in Figure 2.1, where the blue, red and green parts belong to one of the two decompositions accordingly.

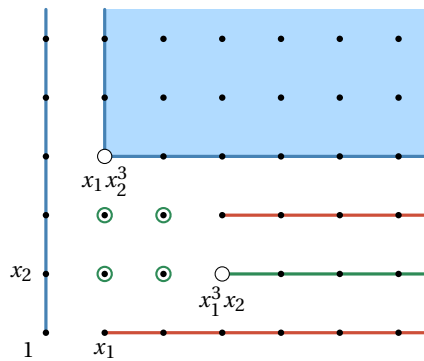


Figure 2.1: Stanley decompositions of two modules.

2 The Stanley Depth in the Koszul Complex

Hilbert decompositions and depths are defined in a similar manner, but they only depend on the Hilbert series of M , which makes them easier to compute.

Definition 2.1.3. Let M and R be as in Definition 2.1.1. A *Hilbert decomposition*

$$\mathcal{D} = (K[Z_i], s_i)_{i \in I}$$

with $Z_i \subseteq \{x_1, \dots, x_n\}$ is a finite family of modules $K[Z_i]$ and multidegrees $s_i \in \mathbb{Z}^n$, such that

$$M \cong \bigoplus_{i \in I} K[Z_i](-s_i)$$

as a graded K -vector space.

Furthermore, the *Hilbert depth* of M is defined as the maximal depth of all possible Hilbert decompositions of M :

$$\text{hdepth}(M) := \max \{ \text{depth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Hilbert decomposition of } M \}.$$

Since a Hilbert decomposition just requires an isomorphism to a direct sum of modules, the Hilbert depth only depends on the Hilbert function $H(M, \alpha) = \dim(M_\alpha)$, $\alpha \in \mathbb{Z}^n$, resp. the Hilbert series

$$H_M(t_1, \dots, t_n) = \sum_{\alpha \in \mathbb{Z}^n} H(M, \alpha) t_1^{\alpha_1} \cdots t_n^{\alpha_n}$$

of M . That is,

$$\text{hdepth}(M) = \max \{ \text{depth}(N) \mid N \text{ finitely generated graded } R\text{-module with } H_M = H_N \}.$$

Note that by definition we immediately have

$$\text{hdepth}(M) \geq \text{sdepth}(M)$$

for any R -module M . Moreover, the Stanley and Hilbert depth can be defined similarly in the \mathbb{Z} -graded case, see [BKU10].

Example 2.1.4. We compute the Hilbert depth of the modules in Example 2.1.2. Their Hilbert series are given by

$$H_I(t_1, t_2) = \frac{t_1^3 t_2 + t_1 t_2^3 - t_1^3 t_2^3}{(1 - t_1)(1 - t_2)}$$

and

$$H_{R/I}(t_1, t_2) = \frac{1 - t_1^3 t_2 - t_1 t_2^3 + t_1^3 t_2^3}{(1 - t_1)(1 - t_2)}.$$

Since both rational functions have negative coefficients in the numerator polynomial, we see that $\text{hdepth}(I) < 2$ and $\text{hdepth}(R/I) < 2$. A decomposition of $H_I(t_1, t_2)$ is given by

$$H_I(t_1, t_2) = \frac{t_1 t_2^3}{(1-t_1)(1-t_2)} + \frac{t_1^3 t_2^2}{1-t_1} + \frac{t_1^3 t_2}{1-t_1}.$$

Therefore $\text{hdepth}(I) = \text{sdepth}(I) = 1$. We claim that $\text{hdepth}(R/I) = 0$. Indeed it suffices to show that the Hilbert depth is 0 in the \mathbb{Z} -graded case, since $\text{hdepth}_1(M) \geq \text{hdepth}(M)$, where hdepth_1 denotes the Hilbert depth in the case that the polynomial ring is equipped with the \mathbb{Z} -grading $\deg(x_i) = 1$. The Hilbert series of R/I in this case is

$$H_{R/I}(t) = \frac{1 - 2t^4 + t^6}{(1-t)^2} = \frac{1 + t + t^2 + t^3 - t^4 - t^5}{1-t}.$$

Uliczka showed in [Uli10, Theorem 3.2] that the Hilbert depth of a module M in the \mathbb{Z} -graded case equals the *positivity* of the Hilbert series $H_M(t)$, i.e., the maximal $r \in \mathbb{Z}_+$ such that the series expansion of $(1-t)^r H_M(t)$ has only non-negative coefficients. Straight-forward computation shows that

$$H_{R/I}(t) = 1 + 2t + 3t^3 + 4t^3 + 3t^4 + \sum_{k \geq 5} 2t^k.$$

Since the coefficients of the series are not monotonically increasing, any multiplication with $(1-t)$ yields negative coefficients. We conclude that $\text{hdepth}_1(R/I) = \text{hdepth}(R/I) = \text{sdepth}(R/I) = 0$.

2.2 Matching in the Boolean algebra

The Boolean algebra is the lattice of subsets of $\{1, \dots, n\}$. Associated to this lattice, we consider a *matching*, i.e., an injection f with $f(G) \subset G$, in the upper half of the Boolean algebra. This matching will be used for the construction of the Hilbert and Stanley decomposition of the considered modules in the next section. The following construction has been known in combinatorics for some time. In particular, it was used to give a proof of Sperner's Theorem on the width of the Boolean algebra (see for instance [And87]).

The *lexicographic mapping* ψ is a (partially defined) injective map on the Boolean algebra which assigns to an $(\ell + 1)$ -set an ℓ -set in the following way. First, we write down the Boolean algebra and sort each level of ℓ -sets in the lexicographic order, see [And87, Ch. 7]. Then for each $(\ell + 1)$ -set $G \subseteq [n]$, $\psi(G)$ is the lexicographically smallest ℓ -subset of G that is not already in the image of ψ . If there exists no such subset, then ψ is undefined. As justified below this map is always defined for sets G with $\#G \geq \lceil \frac{n+1}{2} \rceil$. Thus it can be used to assign the respective multidegrees in the upper half of the Koszul complex uniquely to smaller multidegrees in the next section. For an illustration of ψ in the Boolean algebra on the set $\{1, \dots, 5\}$ see Figure 2.2.

2 The Stanley Depth in the Koszul Complex

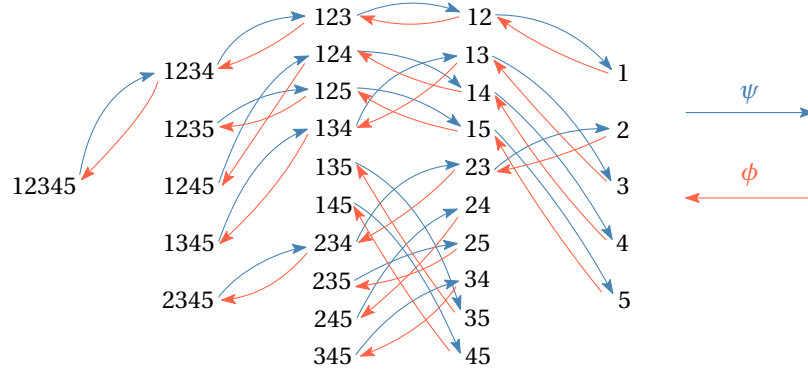


Figure 2.2: ψ and ϕ in the Boolean algebra of $\{1, \dots, 5\}$.

While the Definition makes clear that ψ is injective, it is not very convenient to work with. Aigner provided a concrete formula for the above matching, see [Aig73]. Stanton and White gave a helpful pictorial interpretation of this formula in [SW86] which we discuss below. The following reformulation is motivated by this interpretation.

We associate to a set $G \subseteq [n]$ an incidence vector χ_G by setting

$$\chi_G(g) = \begin{cases} 1, & \text{if } g \in G, \\ -1, & \text{if } g \notin G. \end{cases}$$

Furthermore, we set $\chi_G(0) := 0$. Now, we look at all elements in G for which the function

$$\rho_G(g) := \sum_{j=0}^g \chi_G(j), \quad g \in \{0, 1, \dots, n\}$$

is maximized:

$$\alpha(G) := \max_{g \in G \cup \{0\}} \{\rho_G(g)\}$$

$$N(G) := \{g \in G \cup \{0\} \mid \rho_G(g) = \alpha(G)\},$$

and pick the smallest element among them

$$v(G) := \min N(G).$$

Then the map ψ is defined by deleting the element $v(G)$ (if possible):

$$\psi(G) := G \setminus \{v(G)\}.$$

This map is defined if and only if $v(G) > 0$ (equivalently if and only if $\alpha(G) > 0$). As $\alpha(G) \geq \rho_G(n) = 2\#G - n$, this is indeed the case for all subsets $G \subseteq [n]$ with $\#G \geq \lceil \frac{n+1}{2} \rceil$.

As mentioned above Stanton and White provided a geometric interpretation of ψ [SW86]. The vector χ_G can be seen as a (diagonal) lattice path, where a 1 means going one up to the right and -1 means going one down to the right. Then $\rho_G(g)$ determines the height in place g and $N(G)$ consists of all global maxima. Therefore, ψ “flips down” the edge before the first global maximum (if it is above the x -axis), as illustrated in Figure 2.3.

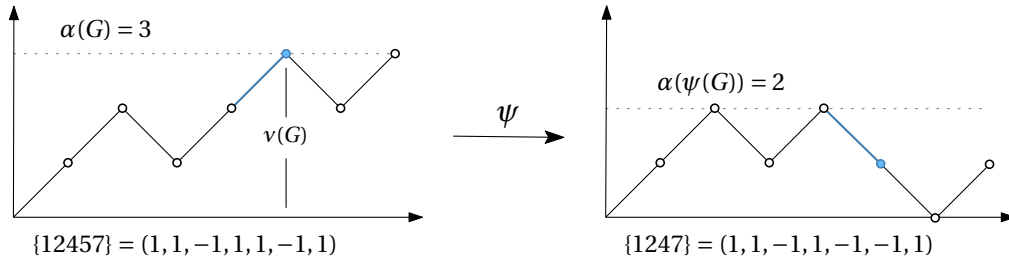


Figure 2.3: Geometric interpretation of ψ

2.2.1 The inverse ϕ

Sometimes it is helpful to also consider the mapping in the other direction in the Boolean algebra which is denoted by ϕ . Again the map is given in explicit terms by Aigner in [Aig73]. Once more, we consider the set $N(G)$ for which ρ_G is maximized, but this time we take its maximal element

$$\mu(G) = \max N(G).$$

Then

$$\phi(G) := G \cup \{\mu(G) + 1\}.$$

Thus ϕ is defined if and only if $\mu(G) < n$. This is indeed the case for all sets in the lower half of the Boolean algebra.

If the set is considered as a lattice path like above, ϕ “flips up” the edge after the last global maximum, i.e., changes the subsequent entry to a 1, see Figure 2.4.

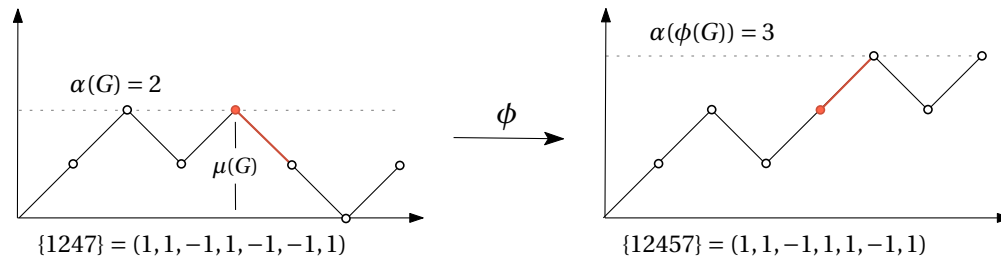


Figure 2.4: Geometric interpretation of ϕ

The following statement from [Aig73, Theorem 3] is used in later proofs:

Proposition 2.2.1. *A subset of $[n]$ is in the image of ψ if and only if ϕ is defined on this set and vice versa. Furthermore ψ and ϕ are inverse to each other on the respective domain.*

2.3 Hilbert and Stanley depth in the Koszul complex

In order to prove our main result we need to review the arguments from [BKU10] which show that the Hilbert depth in the upper half of the Koszul complex is $n - 1$.

Let K be a field and $R = K[x_1, \dots, x_n]$. Then $K = R/\mathfrak{m}$, where \mathfrak{m} is the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. The *Koszul complex* is the following minimal free resolution of K :

$$0 \rightarrow \bigwedge^n R^n \xrightarrow{\partial} \bigwedge^{n-1} R^n \xrightarrow{\partial} \dots \xrightarrow{\partial} R^n \xrightarrow{\partial} R \rightarrow 0,$$

where ∂ is the usual boundary operator given by

$$\partial(e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{j=1}^k (-1)^{j+1} x_j e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_k}. \quad (2.1)$$

By $M(n, k)$ we denote the k -th syzygy module of K , i.e., the image of the k -th boundary map. We recall the Hilbert decomposition of $M(n, k)$ constructed in [BKU10]. Let

$$\mathcal{S} := \{S \subseteq [n] \mid \#S = k + j, j \text{ even}\}.$$

For $S \in \mathcal{S}$ we further set

$$Z_S = \begin{cases} \{x_i \mid i \in [n]\}, & \text{if } S \text{ is not in the image of } \psi, \\ \{x_i \mid i \in [n] \setminus \{s\}\}, & \text{if } S = \psi(S \cup \{s\}). \end{cases} \quad (2.2)$$

Then, as shown in the proof of [BKU10, Theorem 3.5],

$$(K[Z_S], S)_{S \in \mathcal{S}} \quad (2.3)$$

is a Hilbert decomposition of $M(n, k)$ for $n > k \geq \lfloor \frac{n}{2} \rfloor$.

Throughout the rest we let $n > k \geq \lfloor \frac{n}{2} \rfloor$ be fixed.

We show that also the Stanley depth in the upper half of the Koszul complex is $n - 1$ by turning the Hilbert decomposition (2.3) into a Stanley decomposition. For this, we need to choose for every $S \in \mathcal{S}$ an element $m_S \in M(n, k)$ of multidegree S .

Note that $M(n, k)$ is generated by the elements $\partial(e_G)$, where $e_G := e_{i_1} \wedge \dots \wedge e_{i_k}$ with $G = \{i_1, \dots, i_k\} \subset [n]$. Moreover, if $G \subseteq S$, then $x^{S \setminus G} \partial(e_G)$ has multidegree S , where $x^{S \setminus G}$ is the monomial of degree $S \setminus G$. Hence, we essentially need to choose subsets $G(S) \subseteq S$ of cardinality k for every $S \in \mathcal{S}$. It turns out that the following choice works:

$$G(S) := \psi^{\#S-k}(S) \quad \text{for } S \in \mathcal{S}. \quad (2.4)$$

Thus we are going to prove the following theorem, which implies Theorem 2.0.1.

Theorem 2.3.1. *Let $n, k \in \mathbb{N}$ such that $n > k \geq \lfloor \frac{n}{2} \rfloor$. Then*

$$(K[Z_S], m_S)_{S \in \mathcal{S}} \quad (2.5)$$

is a Stanley decomposition of $M(n, k)$, where Z_S is as in (2.2) and $m_S := x^{S \setminus G(S)} \partial(e_{G(S)})$.

2.3 Hilbert and Stanley depth in the Koszul complex

To show that it is indeed a Stanley decomposition, we use the following criterion.

Proposition 2.3.2 (Proposition 2.9, [BKU10]). *Let $(K[Z_i], s_i)_{i \in I}$ be a Hilbert decomposition of a module M . For every $i \in I$ choose a homogeneous non-zero element $m_i \in M$ of degree s_i . Then $(K[Z_i], m_i)_{i \in I}$ is a Stanley decomposition of M , if for every multidegree m the family*

$$\mathcal{C}(m) = \{m_i \mid (K[Z_i]m_i)_m \neq 0\}$$

is linearly independent over R .

It turns out to be more convenient to consider instead the sets

$$\mathcal{G}(m) := \{G(S) \mid (K[Z_S]m_S)_m \neq 0\}.$$

Clearly $\mathcal{C}(m)$ and $\mathcal{G}(m)$ determine each other. Moreover, it is easy to see that $\mathcal{C}(m)$ and $\mathcal{G}(m)$ only depend on the support of m :

$$\text{supp}(m) = \{i \in [n] \mid \text{the } i\text{-th component of } m \text{ is non-zero}\}.$$

So by abuse of notation we write $\mathcal{C}(M) := \mathcal{C}(m)$ and $\mathcal{G}(M) := \mathcal{G}(m)$ if $M = \text{supp}(m)$.

We check the linear independence with the following condition:

Definition 2.3.3. A family \mathcal{G} of k -sets fulfills the *triangle condition* (Δ -condition), if there is a total order $<$ on \mathcal{G} such that every $G \in \mathcal{G}$ contains a $(k-1)$ -subset T (the *distinguished set*) which is not contained in any preceding set, i.e., $T \not\subseteq H$ for every $H < G$.

Lemma 2.3.4. *If a family \mathcal{G} fulfills the Δ -condition, then the set $\{\partial(e_G) \mid G \in \mathcal{G}\}$ is linearly independent.*

Proof. By the definition of the differential map ∂ (2.1), we have that

$$\partial(e_G) = \pm e_T + \dots,$$

where $T \subset G$ is the distinguished subset. Because \mathcal{G} satisfies the Δ -condition, the term e_T does not appear in $\partial(e_H)$ for any $H < G$. Hence, the restriction of the chain map ∂ to $\{e_G \mid G \in \mathcal{G}\}$ forms an (upper) triangular matrix. \square

In particular, if $\mathcal{G}(m)$ satisfies the Δ -condition then $\mathcal{C}(m)$ is linearly independent. So to prove Theorem 2.3.1, we have to show that $\mathcal{G}(m)$ fulfills the Δ -condition for every multidegree m .

For this we need a more explicit description of the sets $\mathcal{G}(m)$. Fix a multidegree m . Looking at the Hilbert decomposition we see that Z_S is the whole polynomial ring if S is not in the image of ψ , or one variable is missing, which is the one dropped by ψ . This means that $(K[Z_S]m_S)_m \neq 0$ if and only if S is not in the image of the restriction of ψ to

2 The Stanley Depth in the Koszul Complex

all subsets of $\text{supp}(m)$. In this case, we call m_S a *contributing generator* for m . Overall, the set of all contributing generators for a multidegree m is given by

$$\mathcal{G}(M) = \{G(S) \mid S \in \mathcal{S}, S \subseteq M, S \notin \text{Im}\psi|_{2^M}\},$$

where $M = \text{supp}(m)$. Recall that $G(S) = \psi^{\#S-k}(S)$ and that $\#S - k$ is even by the Definition of \mathcal{S} . Hence $\mathcal{G}(M)$ consists of the k -subsets of M , which lie in an even power of ψ restricted to 2^M , but not in the subsequent odd one, that is,

$$\mathcal{G}(M) = \left\{ G \in \binom{M}{k} \mid \exists i \exists M' \subseteq M : \psi^{2i}(M') = G \text{ and } \nexists M'' \subseteq M : \psi^{2i+1}(M'') = G \right\}.$$

Consequently, the following definition is quite useful:

Definition 2.3.5. For a subset $G \subseteq M$ the *index* of G in M is defined as

$$\text{ind}_M(G) = \max \left\{ i \mid \exists M' \subseteq M : \psi^i(M') = G \right\}.$$

This allows us to write

$$\mathcal{G}(M) = \left\{ G \in \binom{M}{k} \mid \text{ind}_M(G) \text{ is even} \right\}.$$

Note that by Proposition 2.2.1 the index can also be expressed in terms of ϕ :

$$\text{ind}_M(G) = \max \left\{ i \mid \phi^i(G) \text{ is defined and } \phi^i(G) \subseteq M \right\}.$$

This formula can be quite useful for later computations and proofs.

Example 2.3.6. For $n = 7, k = 3$ and the set $M = \{1, 2, 4, 5, 7\}$ resp. the multidegree 12457 we have one element of index 2 (147) and five elements of index 0 (125, 127, 245, 257), as it can be seen in Figure 2.5.

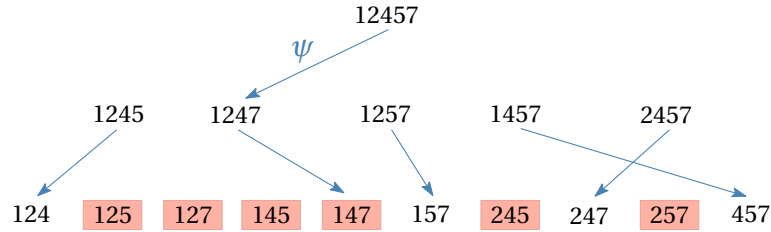


Figure 2.5: 3-Subsets of $\{1, 2, 4, 5, 7\}$ with even index.

As a final step, we show that the *squashed order* fulfills the Δ -condition for the family $\mathcal{G}(M)$. It is defined for two sets of the same size as

$$G > H :\Leftrightarrow \max G \Delta H \in G,$$

2.3 Hilbert and Stanley depth in the Koszul complex

where Δ denotes the symmetric difference. For details on the squashed and other orders of finite sets, see [And87, Ch. 7].

Moreover, the distinguished subset of a set G , i.e., the subset which is not contained in any preceding set, is given by $\tilde{\psi}(G)$, where $\tilde{\psi}$ is defined as ψ , but the restriction of Section 2.2 is dropped:

$$\begin{aligned}\tilde{\alpha}(G) &:= \max_{g \in G} \{\rho_G(g)\}, \\ \tilde{N}(G) &:= \{g \in G \mid \rho_G(g) = \tilde{\alpha}(G)\}, \quad \tilde{\nu}(G) := \min \tilde{N}(G), \\ \tilde{\psi}(G) &:= G \setminus \{\tilde{\nu}(G)\}.\end{aligned}$$

So $\tilde{\psi}$ is always defined, but we lose the injectivity. As we will see later, this does not impose a problem for our result.

The following technical lemma is the key ingredient for our main result. Recall that

$$\rho_G(g) = \sum_{j=0}^g \chi_G(j), \quad g \in \{0, 1, \dots, n\}.$$

Lemma 2.3.7. *Let G and H be subsets of a set $M \subseteq \{1, \dots, n\}$ with $\#G = \#H \geq \lfloor \frac{n}{2} \rfloor$ and $G \neq H$. Furthermore, let $G \succ H$ and $\tilde{\psi}(G) \subset H$. Then the following hold:*

- (1) *If $\tilde{\alpha}(G) \geq 0$, then $\text{ind}_M(H) = \text{ind}_M(G) + 1$;*
- (2) *If $\tilde{\alpha}(G) < 0$, then $\text{ind}_M(H) = 1$.*

Figure 2.6 illustrates the (quite technical) proof of this result.

Proof. By the assumption that $\tilde{\psi}(G) \subset H$ we have $G \Delta H = \{h, \tilde{\nu}(G)\}$ with $h \in H \setminus G$ and $\tilde{\nu}(G) \in G \setminus H$. Since $G \succ H$ it follows that

$$\max G \Delta H = \max\{h, \tilde{\nu}(G)\} = \tilde{\nu}(G).$$

The definition of ρ implies the following equation

$$\rho_H(g) = \begin{cases} \rho_G(g), & \text{if } g < h, \\ \rho_G(g) + 2, & \text{if } h \leq g < \tilde{\nu}(G), \\ \rho_G(g), & \text{if } g \geq \tilde{\nu}(G). \end{cases} \quad (2.6)$$

Since $\rho_G(g) < \tilde{\alpha}(G)$ for $g < \tilde{\nu}(G)$ it is $\rho_H(g) \leq \tilde{\alpha}(G) + 1$ for $g < \tilde{\nu}(G)$. Furthermore $\rho_H(g) = \rho_G(g) \leq \tilde{\alpha}(G)$ for all $g \geq \tilde{\nu}(G)$. So overall $\tilde{\alpha}(H) \leq \tilde{\alpha}(G) + 1$.

On the other hand

$$\tilde{\alpha}(H) \geq \rho_H(\tilde{\nu}(G) - 1) = \rho_G(\tilde{\nu}(G)) + 1 > \rho_G(\tilde{\nu}(G)) = \tilde{\alpha}(G), \quad (2.7)$$

by (2.6). This shows that $\tilde{\alpha}(H) = \tilde{\alpha}(G) + 1$. Furthermore (2.6) and (2.7) yield that

$$\rho_H(\tilde{\nu}(G) - 1) = \tilde{\alpha}(G) + 1 > \rho_G(g) = \rho_H(g), \text{ for all } g \geq \tilde{\nu}(G). \quad (2.8)$$

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Next, we show that $\tilde{\alpha}(H) \geq 0$. This is obvious if $\tilde{\alpha}(G) \geq 0$ by (2.7). If $\tilde{\alpha}(G) < 0$ then $\#G \geq \lfloor \frac{n}{2} \rfloor$ implies $\#G = \#H = \lfloor \frac{n}{2} \rfloor$. Note that this can only happen if n is odd. Furthermore this implies that

$$\rho_G(n) = \rho_H(n) = 2\#G - n = -1$$

and hence $\tilde{\alpha}(G) = -1$. So in both cases $\tilde{\alpha}(H) \geq 0$ and thus (2.8) shows that

$$\mu(H) = \max\{h \in H \cup \{0\} \mid \rho_H(h) = \alpha(H)\} = \tilde{\nu}(G) - 1, \quad (2.9)$$

and thus

$$\phi(H) = H \cup \{\tilde{\nu}(G)\}. \quad (2.10)$$

(1) Assume that $\tilde{\alpha}(G) \geq 0$ and thus $\tilde{\alpha}(H) > 0$. We know by the Definition of ϕ and (2.6) that

$$\rho_{\phi(H)}(g) = \rho_G(g) + 2, \text{ for all } g \geq h. \quad (2.11)$$

Since $\tilde{\alpha}(G) \geq 0$ we have that $\mu(G) > 0$ and thus $\mu(G) \geq \tilde{\nu}(G) > h$ and $\mu(\phi(H)) > \mu(H) = \tilde{\nu}(G) - 1 \geq h$. Hence (2.11) implies that $\mu(\phi(H)) = \mu(G)$, as illustrated in Figure 2.6.

Note that (2.11) stays valid if ϕ is applied to both G and $\phi(H)$. Moreover $\mu(\phi^2(H)) > \mu(\phi(H))$ and $\mu(\phi(G)) > \mu(G)$. Hence $\mu(\phi^2(H)) = \mu(\phi(G))$. Continuing this argument we see that $\phi^i(G)$ is defined and $\phi^i(G) \subseteq M$ if and only if $\phi^i(\phi(H))$ is defined and is a subset of M . This shows that

$$\text{ind}_M(G) = \text{ind}_M(\phi(H)) = \text{ind}_M(H) - 1.$$

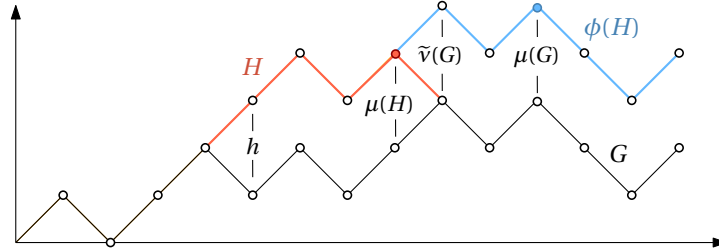


Figure 2.6: Illustration of Lemma 2.3.7

(2) Since (2.9) holds in the case $\tilde{\alpha}(G) < 0$ as well, we know that $\text{ind}_M(H) \geq 1$. Moreover

$$\rho_{\phi(H)}(n) = 1 = \alpha(\phi(H)),$$

so $\mu(\phi(H)) = n$ and thus $\text{ind}_M(H) = 1$.

□

2.3 Hilbert and Stanley depth in the Koszul complex

Example 2.3.8. We consider the sets depicted in Figure 2.6. Let $n = 14, k = 8, M = [n]$ and

$$G = \{1, 3, 4, 6, 8, 9, 11, 14\}, \quad H = \{1, 3, 4, 5, 6, 8, 11, 14\}.$$

We have $\max G \Delta H = \max\{5, 9\} = 9 \in G$, which means $G > H$. The lattice path of G shows that $\tilde{\nu}(G) = 9$, so $\tilde{\psi}(G) = \{1, 3, 4, 6, 8, 11, 14\} \subset H$. Furthermore, $\mu(H) = 8 = \tilde{\nu}(G) - 1$ and $\phi(H) = \{1, 3, 4, 5, 6, 8, 9, 11, 14\}$. We see that $\mu(G) = \mu(\phi(H)) = 11$ and thus $\phi(G) = \{1, 3, 4, 6, 8, 9, 11, 12, 14\}$. But then $\mu(\phi(G)) = \mu(\phi^2(H)) = 14 = n$, which implies $\text{ind}_M(G) = \text{ind}_M(\phi(H)) = 1$ and in particular $\text{ind}_M(H) = 2 = \text{ind}_M(G) + 1$.

Remark 2.3.9. Note that in the case $\tilde{\alpha}(G) < 0$ the index of G in M is 1 or 0 depending whether $1 \in M$ or not. Hence the case distinction is necessary.

Now we can show that the squashed order works:

Proposition 2.3.10. *The family $\mathcal{G}(m)$ fulfills the Δ -condition with respect to the squashed order for every multidegree m .*

Proof. Let $M = \text{supp}(m)$. The set $\mathcal{G}(M)$ contains only k -subsets with even index. Hence by Lemma 2.3.7, $G > H$ implies that $\tilde{\psi}(G) \not\subseteq H$. □

Example 2.3.11. We continue Example 2.3.6. Figure 2.7 shows the 3-element subsets of $\{1, 2, 4, 5, 7\}$ in squashed order as well as their images under $\tilde{\psi}$ (where we omit the set brackets and commas). The sets with even indices are indicated by red boxes. Note that the image of a set with even index is not contained in any preceding set with even index.

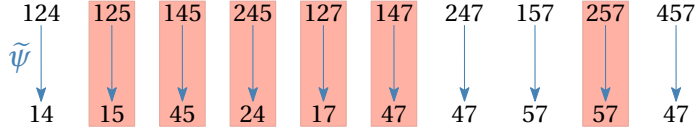


Figure 2.7: Squashed order of 3-subsets of $\{1, 2, 4, 5, 7\}$.

Proof of Theorem 2.3.1. In conclusion, we have shown that the sets $\mathcal{G}(m)$ fulfill the Δ -condition and hence the sets $\{\partial(e_G) \mid G \in \mathcal{G}(m)\}$ are linearly independent by Lemma 2.3.4. So by Proposition 2.3.2 and the subsequent discussion, Theorem 2.3.1 follows. □

3 Local h -vectors

Don't bring negative to my door.

Maya Angelou

The following chapter is based on the paper “Local h -vectors of Quasi-Geometric and Barycentric Subdivisions” [JMS17] written by Martina Juhnke-Kubitzke, Satoshi Murai and the author.

The classification of face numbers of (triangulated) spaces is an important and central topic not only in algebraic and geometric combinatorics but also in other fields, as e.g., commutative and homological algebra and discrete, algebraic and toric geometry. The studied classes of spaces comprise abstract simplicial complexes, triangulated spheres and (pseudo)manifolds but also not necessarily simplicial objects such as (boundaries of) polytopes and Boolean cell complexes. In 1992, Stanley [Sta92] introduced the so-called *local h -vector* of a topological subdivision of a $(d - 1)$ -dimensional simplex as a tool to study face numbers of subdivisions of simplicial complexes. His original motivation was the question, posed by Kalai and himself, if the (classical) h -vector increases under subdivision of a Cohen-Macaulay complex. Using local h -vectors, Stanley could provide an affirmative answer to this question for so-called *quasi-geometric subdivisions*. These subdivisions have the crucial property that their local h -vectors are non-negative; a property, which is no longer true if one considers arbitrary topological subdivisions.

Local h -vectors have been studied for a vast variety of classical subdivisions. For the barycentric subdivision of a simplex it is already considered in Stanley’s original paper [Sta92] and can be expressed using permutation statistics. Furthermore, in a series of papers, Athanasiadis and Savvidou gave a combinatorial interpretation for the local h -vectors for some typical subdivisions: cluster subdivisions in [AS12], simplicial barycentric subdivisions of cubical subdivisions in [AS13] and r -fold edgewise subdivisions in [Ath16].

As the local h -vector is symmetric it makes sense to define a *local γ -vector*, which was introduced by Athanasiadis in [Ath12] and is defined in the same way as is the usual γ -vector for homology spheres. The central conjecture for local γ -vectors is the following, due to Athanasiadis [Ath12, Conjecture 5.4].

Conjecture 3.0.1. *The local γ -vector of a flag vertex-induced homology subdivision of a simplex is non-negative.*

3 Local h -vectors

This conjecture is indeed a strengthening of Gal’s conjecture for flag homology spheres [Gal05, Conjecture 2.1.7] and, in particular, implies the Charney-Davis conjecture [CD95] (see the discussion in Section 3.2.4). Conjecture 3.0.1 is known to be true in small dimensions [Ath12] and for various special classes of subdivisions, including barycentric, edgewise and cluster subdivisions of the simplex [Ath12, AS12, Ath16] but besides it is still widely open. We add more evidence to it and answer [Ath12, Question 6.2] affirmatively by showing the following:

Theorem 3.0.2. *Let Γ be a CW-regular subdivision of a simplex. The local γ -vector of the barycentric subdivision $\text{sd}(\Gamma)$ of Γ is non-negative.*

We point out that the local h -vector, and therefore the local γ -vector, not only depends on the combinatorial type of $\text{sd}(\Gamma)$ but also on the subdivision map. In Theorem 3.0.2, we are considering the natural subdivision map of the barycentric subdivision, which we explain in Section 3.3.

The proof of Theorem 3.0.2 is based on an expression of the local h -vector which involves differences of h -vectors of restrictions of the subdivision and their boundaries as well as derangement polynomials (Theorem 3.3.2). The non-negativity of the local γ -vector is then concluded from a result by Ehrenborg and Karu [EK07], stating that the **cd**-index of those differences is non-negative.

Along the way, we prove a new recurrence formula for the derangement polynomials (Theorem 3.1.1); first in a purely combinatorial manner and, subsequently, as a direct byproduct of the proof of Theorem 3.0.2.

In another result we complement work by Chan [Cha94] and show that – except for the conditions that are true for any local h -vector – non-negativity already characterizes local h -vectors of quasi-geometric subdivisions entirely. More precisely, we show the following statement, which partly solves [Ath14, Problem 2.11].

Theorem 3.0.3. *Let $\ell = (\ell_0, \dots, \ell_d) \in \mathbb{Z}^{d+1}$. The following conditions are equivalent:*

- (1) *There exists a quasi-geometric subdivision Γ of the $(d-1)$ -simplex such that the local h -vector of Γ is equal to ℓ .*
- (2) *ℓ is symmetric (i.e., $\ell_i = \ell_{d-i}$ for $0 \leq i \leq d$), $\ell_0 = 0$ and $\ell_i \geq 0$ for $1 \leq i \leq d-1$.*

We want to remark, that it already follows from [Sta92] that the local h -vector of any quasi-geometric subdivision satisfies (2). To prove Theorem 3.0.3, it therefore suffices to construct a quasi-geometric subdivision Γ having a prescribed vector $\ell = (\ell_0, \dots, \ell_d) \in \mathbb{Z}^{d+1}$, satisfying (2), as its local h -vector. For this, we extend constructions developed by Chan [Cha94] to characterize local h -vectors of regular and topological subdivisions.

The outline of this chapter is the following. In Section 3.1, we review some basic facts on permutation statistics. In particular, we are concerned with Eulerian and derangement polynomials and derive a recurrence relation for the latter. Section 3.2 contains the necessary background on simplicial complexes, topological subdivisions and

local h -vectors. The subsequent section deals with barycentric subdivisions and the proof of Theorem 3.0.2. Finally, Section 3.4 is devoted to the characterization of local h -polynomials in the quasi-geometric case (Theorem 3.0.3).

3.1 Permutation statistics

Most problems in this chapter are concerned with certain properties of polynomials with real coefficients. Therefore, we start this section with a brief review.

For two polynomials $p(x) = \sum_{k=0}^n a_k x^k \in \mathbb{R}[x]$ and $q(x) = \sum_{k=0}^m b_k x^k \in \mathbb{R}[x]$ we write $p(x) \geq q(x)$ if $p(x)$ is greater or equal than $q(x)$ coefficient-wise, that is, $a_k \geq b_k$ for $k = 0, \dots, \max\{n, m\}$, where we set the remaining coefficients of the possibly lower-degree polynomial to 0. In particular, $p(x) \geq 0$ means that the coefficients of $p(x)$ are non-negative and the polynomial is called *non-negative* itself. We say that $p(x)$ is *unimodal* if there exists an index $0 \leq j \leq n$ such that

$$a_0 \leq \dots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \dots \geq a_n.$$

Furthermore, $p(x)$ is called *symmetric* with *center of symmetry* $n/2$ if its coefficients form a symmetric sequence, i.e., $a_k = a_{n-k}$ for $0 \leq k \leq n$. A symmetric polynomial with center of symmetry $n/2$ can be expressed in the polynomial basis $\{x^k(1+x)^{n-2k} \mid 0 \leq k \leq \lfloor n/2 \rfloor\}$, i.e., there are unique $\gamma_0, \dots, \gamma_{\lfloor n/2 \rfloor} \in \mathbb{R}$ such that

$$p(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1+x)^{n-2k}.$$

If the coefficients γ_k in this expression are non-negative, $p(x)$ is called *γ -non-negative*. Note that this property implies that $p(x)$ is also unimodal. We often refer to the vector $(\gamma_1, \dots, \gamma_{\lfloor n/2 \rfloor})$ as *γ -vector* associated to $p(x)$. See [Brä15] for a survey on these and further properties of polynomials and their importance for combinatorics.

We review the well-known *Eulerian polynomials* of order n , which encode the number of *descents* for permutations in S_n . Given a permutation $\pi \in S_n$, a descent of π is a position $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. Now, let

$$\text{des}(\pi) = \#\{i \in [n-1] \mid \pi(i) > \pi(i+1)\}$$

be the number of descents of π . For example, the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 2 & 6 & 4 & 8 & 3 \end{pmatrix}$ has descents 2, 5 and 7 and thus $\text{des}(\pi) = 3$. The *Eulerian number* $A(n, k)$ for $k = 0, \dots, n-1$ is defined as the number of permutations in S_n having exactly k descents

$$A(n, k) = \#\{\pi \in S_n \mid \text{des}(\pi) = k\}.$$

The generating polynomial of the Eulerian numbers $A(n, 0), \dots, A(n, n-1)$ is the *Eulerian polynomial* of order n , that is,

$$A_n(x) = \sum_{k=0}^{n-1} A(n, k) x^k = \sum_{\pi \in S_n} x^{\text{des}(\pi)}.$$

3 Local h -vectors

It is useful to set $A_0(x) = 0$. The first Eulerian polynomials are given by

$$A_1(x) = 1, \quad A_2(x) = 1 + x, \quad A_3(x) = 1 + 4x + x^2, \quad A_4(x) = 1 + 11x + 11x^2 + x^3.$$

These polynomials satisfy a nice recurrence relation, which is crucial for the proof of the main result in this section:

$$A_n(x) = \sum_{j=0}^{n-1} \binom{n}{j} A_j(x) (x-1)^{n-1-j}. \quad (3.1)$$

For a further and detailed treatment of Eulerian numbers and polynomials we refer the reader to [Sta86] and [Pet15].

Closely related to the above permutation statistic is the problem of counting the number of *derangements* in S_n , which are permutations without fixed points, i.e., $\pi \in S_n$ is a derangement if and only if $\pi(i) \neq i$ for all $i \in [n]$ (see for instance [Sta86, Section 2.2]). We denote the set of derangements with \mathfrak{D}_n . Its size is called the *derangement number* $\mathfrak{d}_n = \#\mathfrak{D}_n$, where we set $\mathfrak{d}_0 = 1$ for convenience.

Using the principle of inclusion-exclusion it can be easily seen that

$$\mathfrak{d}_n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k!,$$

which implies the useful identity

$$\mathfrak{d}_n = n \cdot \mathfrak{d}_{n-1} + (-1)^n. \quad (3.2)$$

The *derangement polynomial* of order n is given by

$$\mathfrak{d}_n(x) = \sum_{\pi \in \mathfrak{D}_n} x^{\text{exc}(\pi)}.$$

Here, $\text{exc}(\pi)$ denotes the number of *excedances* of π

$$\text{exc}(\pi) = \#\{i \in [n-1] \mid \pi(i) > i\}.$$

For example, the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 2 & 6 & 4 & 8 & 3 \end{pmatrix}$ has excedances 1, 2, 5 and 7 and thus $\text{exc}(\pi) = 4$. It is convenient to set $\mathfrak{d}_0(x) = 1$. The first instances of derangement polynomials are

$$\mathfrak{d}_1(x) = 0, \quad \mathfrak{d}_2(x) = x, \quad \mathfrak{d}_3(x) = x + x^2, \quad \mathfrak{d}_4(x) = x + 7x^2 + x^3.$$

These polynomials were first studied by Brenti in [Bre90], although slightly different versions were already considered by Garsia and Remmel in [GR80] and Wachs in [Wac89]. Brenti showed that the derangement polynomials are symmetric and unimodal. Furthermore, it was shown in [Zha95] that they only have real roots and are γ -non-negative – an essential property for the proof of Theorem 3.0.2.

The “transformation fondamentale” on S_n given by Foata and Schützenberger (see [FS70]) shows that the number of descents and the number of excedances is equidistributed, which implies

$$A_n(x) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)}.$$

This allows us to transform Eulerian and derangement polynomials into each another:

$$A_n(x) = \sum_{k=0}^n \binom{n}{k} \mathfrak{d}_k(x) \quad (3.3)$$

and

$$\mathfrak{d}_n(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} A_k(x), \quad (3.4)$$

where the second equality follows again from the principle of inclusion-exclusion.

Note that the number of excedances and descents are not equidistributed on the set of derangement polynomials \mathfrak{D}_n . For example, if $n = 3$, \mathfrak{D}_3 contains the two permutations $\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and their respective statistics are given by $\text{exc}(\pi_1) = 2$, $\text{des}(\pi_1) = 1$ and $\text{exc}(\pi_2) = 1$, $\text{des}(\pi_2) = 1$ respectively.

The following result provides a recurrence relation for the derangement polynomials quite similar to that for the Eulerian polynomials.

Theorem 3.1.1. *For every $n \in \mathbb{N}$, $n \geq 1$,*

$$\mathfrak{d}_n(x) = \sum_{k=0}^{n-2} \binom{n}{k} \mathfrak{d}_k(x) (x + \dots + x^{n-1-k}). \quad (3.5)$$

In order to prove this relation, we make use of the geometric sum formula. Thus we prove the case $x = 1$ separately. Since $\mathfrak{d}_n(1) = \mathfrak{d}_n$, we obtain a recursive formula for the derangement numbers.

Lemma 3.1.2. *For every $n \in \mathbb{N}$, $n \geq 1$,*

$$\mathfrak{d}_n = \sum_{k=0}^{n-2} \binom{n}{k} \mathfrak{d}_k \cdot (n-1-k).$$

Proof. We use induction on n and the identity (3.2). This identity also yields the induction basis $\mathfrak{d}_1 = 0 = 1 \cdot \mathfrak{d}_0 - 1$. We proceed with the induction step.

$$\begin{aligned} \mathfrak{d}_n &\stackrel{(3.2)}{=} n \cdot \mathfrak{d}_{n-1} + (-1)^n \\ &= n \sum_{k=0}^{n-3} \binom{n-1}{k} \mathfrak{d}_k \cdot (n-2-k) + (-1)^n = \sum_{k=1}^{n-2} \binom{n-1}{k-1} n \cdot \mathfrak{d}_{k-1} \cdot (n-1-k) + (-1)^n \end{aligned}$$

3 Local h -vectors

$$\begin{aligned}
&= \sum_{k=1}^{n-2} \binom{n}{k} k \cdot \mathfrak{d}_{k-1} \cdot (n-1-k) + (-1)^n \stackrel{(3.2)}{=} \sum_{k=1}^{n-2} \binom{n}{k} (\mathfrak{d}_k + (-1)^{k+1})(n-1-k) + (-1)^n \\
&= \sum_{k=1}^{n-2} \binom{n}{k} \mathfrak{d}_k \cdot (n-1-k) + \sum_{k=1}^{n-2} \binom{n}{k} (-1)^{k+1} (n-1-k) + (-1)^n \\
&= \sum_{k=1}^{n-2} \binom{n}{k} \mathfrak{d}_k \cdot (n-1-k) + \sum_{k=0}^n \binom{n}{k} (-1)^{k+1} (n-1-k) + (n-1).
\end{aligned}$$

Now, it can easily be seen that $\sum_{k=0}^n \binom{n}{k} (-1)^{k+1} (n-1-k) = 0$, so:

$$= \sum_{k=1}^{n-2} \binom{n}{k} \mathfrak{d}_k \cdot (n-1-k) + (n-1) = \sum_{k=0}^{n-2} \binom{n}{k} \mathfrak{d}_k \cdot (n-1-k).$$

□

We close this section with the proof of the general statement.

Proof of Theorem 3.1.1. It follows from (3.3) that equation (3.5) is equivalent to

$$A_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} \mathfrak{d}_k(x) (1 + x + \dots + x^{n-1-k}).$$

Now let $x \neq 1$. We have

$$\begin{aligned}
\sum_{k=0}^{n-1} \binom{n}{k} \mathfrak{d}_k(x) (1 + x + \dots + x^{n-1-k}) &= \sum_{k=0}^{n-1} \binom{n}{k} \mathfrak{d}_k(x) \left(\frac{x^{n-k} - 1}{x - 1} \right) \\
&\stackrel{(3.4)}{=} \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{x^{n-k} - 1}{x - 1} \right) \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} A_j(x) \\
&= \sum_{j=0}^{n-1} A_j(x) \sum_{k=j}^{n-1} \binom{n}{k} \binom{k}{j} \left(\frac{x^{n-k} - 1}{x - 1} \right) (-1)^{k-j} \\
&= \sum_{j=0}^{n-1} A_j(x) \sum_{k=0}^{n-1-j} \binom{n}{k+j} \binom{k+j}{j} \left(\frac{x^{n-k-j} - 1}{x - 1} \right) (-1)^k.
\end{aligned}$$

Using the identity $\binom{n}{k+j} \binom{k+j}{j} = \binom{n}{j} \binom{n-j}{k}$ we get

$$\begin{aligned}
&= \sum_{j=0}^{n-1} \binom{n}{j} A_j(x) \sum_{k=0}^{n-1-j} \binom{n-j}{k} \left(\frac{x^{n-k-j} - 1}{x - 1} \right) (-1)^k \\
&= \sum_{j=0}^{n-1} \binom{n}{j} A_j(x) \frac{1}{x-1} \underbrace{\left(\sum_{k=0}^{n-1-j} \binom{n-j}{k} (x^{n-k-j} - 1) (-1)^k \right)}_{(*)}.
\end{aligned}$$

Now (*) can be written as:

$$\begin{aligned}
 \sum_{k=0}^{n-1-j} \binom{n-j}{k} (x^{n-k-j} - 1)(-1)^k &= \sum_{k=0}^{n-1-j} \binom{n-j}{k} x^{n-k-j} (-1)^k - \sum_{k=0}^{n-1-j} \binom{n-j}{k} (-1)^k \\
 &= \sum_{k=0}^{n-1-j} \binom{n-j}{k} x^{n-k-j} (-1)^k - \underbrace{\sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k}_{=0} + (-1)^{n-j} \\
 &= \sum_{k=0}^{n-j} \binom{n-j}{k} x^{n-k-j} (-1)^k = (x-1)^{n-j}.
 \end{aligned}$$

Thus, the above expression equals

$$\sum_{j=0}^{n-1} \binom{n}{j} A_j(x) \frac{1}{x-1} (x-1)^{n-j} = \sum_{j=0}^{n-1} \binom{n}{j} A_j(x) (x-1)^{n-1-j} \stackrel{(3.1)}{=} A_n(x).$$

This concludes the proof. □

3.2 Simplicial complexes and subdivisions

We provide some background material on simplicial complexes, their subdivisions and (local) h -vectors.

3.2.1 Simplicial complexes and their face numbers

Given a finite set V , a *simplicial complex* Δ on V is a family of subsets of V which is closed under inclusion, i.e., $G \in \Delta$ and $F \subseteq G$ implies $F \in \Delta$. The elements of Δ are called *faces* and the inclusion-maximal faces are called *facets* of Δ . The *dimension* of a face $F \in \Delta$ is given by $\dim(F) = \#F - 1$ and the *dimension* of Δ is the maximal dimension of its facets. If all facets of Δ have the same dimension, Δ is called *pure*. For a pure simplicial complex of dimension $(d-1)$, a *ridge* is defined to be a face of dimension $(d-2)$. The *boundary* of a pure simplicial complex Δ of dimension $(d-1)$, denoted by $\partial\Delta$, is the simplicial complex that is generated by all ridges which are contained in exactly one facet, that is, the collection of all subfaces of those ridges. Faces that are not contained in the boundary are called *interior faces*. We define the *link* of a face $F \in \Delta$ to be

$$\text{link}_\Delta(F) = \{G \in \Delta \mid G \cap F = \emptyset, G \cup F \in \Delta\}.$$

The *f-vector* $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{d-1}(\Delta))$ of a $(d-1)$ -dimensional simplicial complex Δ encodes the number of i -dimensional faces $(-1 \leq i \leq d-1)$, i.e.,

$$f_i(\Delta) = \#\{F \in \Delta \mid \dim(F) = i\} \quad \text{for } -1 \leq i \leq d-1.$$

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Often it is more convenient to work with the h -vector $h(\Delta) = (h_0(\Delta), \dots, h_d(\Delta))$ of Δ , which is defined as

$$h_i(\Delta) = \sum_{j=0}^i \binom{d-j}{i-j} (-1)^{i-j} f_{j-1}(\Delta) \quad \text{for } 0 \leq i \leq d.$$

Regarding this vector as a sequence of coefficients yields the h -polynomial of Δ

$$h(\Delta, x) = \sum_{i=0}^d h_i(\Delta) x^i.$$

As discussed below, it is the numerator polynomial of the Hilbert series of a certain graded ring associated with Δ . This fact already indicates why various important statements on face numbers are expressed using h -vectors and polynomials instead of f -vectors. We refer the reader to [BH98] and [Sta04] for further background material.

3.2.2 Subdivisions and local h -vectors

Subdivisions of simplicial complexes and local h -vectors go back to Stanley in [Sta92]. Nearly all following definitions are contained in his paper. For an excellent survey article see [Ath14].

From now on, V always denotes a non-empty finite set of cardinality d . A *topological subdivision* of a simplicial complex Δ is a pair (Γ, σ) , where Γ is a simplicial complex and σ is a map $\sigma : \Gamma \rightarrow \Delta$ such that

- (1) $\Gamma_F := \sigma^{-1}(2^F)$ is a subcomplex of Γ which is homeomorphic to a ball of dimension $\dim(F)$. Γ_F is called the *restriction* of Γ to F .
- (2) $\sigma^{-1}(F)$ consists of the interior faces of Γ_F .

Following Stanley [Sta92], we call the face $\sigma(G) \in \Delta$ the *carrier* of $G \in \Gamma$. We also want to warn the reader not to confuse the notation Γ_F with the induced subcomplex of Γ on the vertex set F , which might even consist just of the vertices in F . Also, note that it directly follows from condition (1) that σ is inclusion-preserving, i.e., $\sigma(G) \subseteq \sigma(F)$ if $G \subseteq F$. In the following, we often just write subdivision instead of topological subdivision if we are referring to a subdivision without additional properties, and we say that Γ is a subdivision of Δ without referring to the map σ if this one is clear from the context.

We provide an overview of various classical types of subdivision. Let (Γ, σ) be a subdivision of a simplicial complex Δ . We say that (Γ, σ) is *quasi-geometric* if there do not exist faces $E \in \Gamma$ and $F \in \Delta$ with $\dim(F) < \dim(E)$ such that $\sigma(v) \subseteq F$ for all vertices v of E . The subdivision (Γ, σ) is *vertex-induced* if for all faces $E \in \Gamma$ and $F \in \Delta$ such that every vertex of E is a vertex of Γ_F , we have $E \in \Gamma_F$. Moreover, (Γ, σ) is called *geometric* if the subdivision Γ admits a geometric realization that geometrically subdivides a geometric realization of Δ . Finally, we say that (Γ, σ) is *regular* if the subdivision is induced by a weight function, i.e., it can be obtained via a projection of the lower hull of a polytope (see [Sta92, Definition 5.1]). As we restrict our attention to topological and quasi-geometric subdivisions,

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we do not examine the geometric and regular case in more detail and rather refer the reader to [Sta04] and [Zie95] for a thorough treatment.

We have the following relations between those properties:

$$\{\text{topological subdivisions}\} \supsetneq \{\text{quasi-geometric subdivisions}\} \supsetneq \{\text{vertex-induced subdivisions}\} \supsetneq \{\text{geometric subdivisions}\} \supsetneq \{\text{regular subdivisions}\},$$

where all containments are strict. Figure 3.1 shows examples of subdivisions of the 2-simplex that are (a) regular, (b) geometric but not regular, (c) quasi-geometric but not vertex-induced and (d) not even quasi-geometric. A subdivision that is vertex-induced but not geometric is harder to depict but can be found in [Cha94].

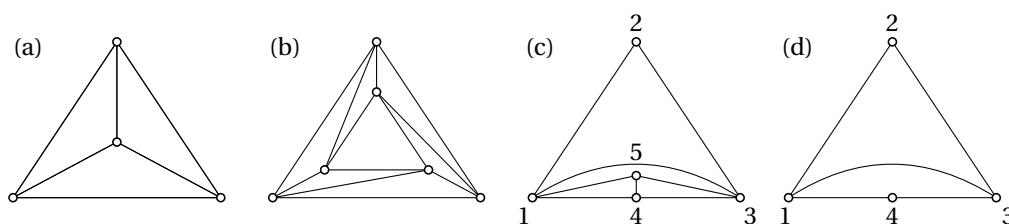


Figure 3.1: Examples of subdivisions of the 2-simplex.

Stanley introduced the *local h -polynomial*, in order to understand the behavior of the h -polynomial under a subdivision. In particular it is a tool to determine cases in which the coefficients increase.

Definition 3.2.1. Let Γ be a subdivision of the $(d-1)$ -simplex 2^V . Then

$$\ell_V(\Gamma, x) = \sum_{F \in \Gamma} (-1)^{d-\#F} h(\Gamma_F, x) = \sum_{i=0}^d \ell_i(\Gamma) x^i \quad (3.6)$$

is called the *local h -polynomial* of Γ and the vector $\ell_V(\Gamma) = (\ell_0(\Gamma), \ell_1(\Gamma), \dots, \ell_d(\Gamma))$ is referred to as the *local h -vector* of Γ (with respect to V).

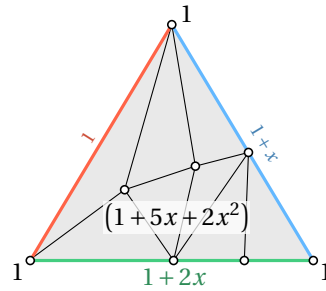
Example 3.2.2. Figure 3.2 shows a subdivision of the 2-simplex and the computation of its local h -polynomial. The labels indicate the h -polynomials arising from the restriction of the subdivision to a face of the simplex.

The local h -polynomial can also be expressed by comparing the size of each face with the size of its carrier. For a face $F \in \Gamma$, we define the *excess* of F to be $e(F) = \#\sigma(F) - \#F$. Then [Sta92, Proposition 2.2]:

$$\ell_V(\Gamma, x) = \sum_{F \in \Gamma} (-1)^{d-\#F} x^{d-e(F)} (x-1)^{e(F)}. \quad (3.7)$$

It follows easily from (3.7) that $\ell_1(V)$ counts the number of interior vertices of Γ . In particular, $\ell_1(\Gamma) \geq 0$.

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$$\begin{aligned}
 \ell_V(\Gamma, x) &= (1 + 5x + 2x^2) \\
 &\quad - ((1 + 2x) + 1 + (1 + x)) \\
 &\quad + (1 + 1 + 1) - 1 \\
 &= 2x + 2x^2 \\
 \ell_V(\Gamma) &= (0, 2, 2, 0)
 \end{aligned}$$

Figure 3.2: Computation of the local h -polynomial.

We summarize some of the most important properties of local h -vectors that will be used later on (see also [Ath14, Theorem 2.6]).

Theorem 3.2.3 (Stanley [Sta92]).

(1) Let Δ be a pure simplicial complex and let Γ be a subdivision of Δ . Then:

$$h(\Gamma, x) = \sum_{F \in \Delta} \ell_F(\Gamma_F, x) h(\text{link}_\Delta(F), x). \quad (3.8)$$

(2) Let $V \neq \emptyset$ and let Γ be a subdivision of 2^V . Then:

- (a) $\ell_V(\Gamma, x)$ is symmetric. Furthermore, $\ell_0(\Gamma) = 0$ and $\ell_1(\Gamma) \geq 0$.
- (b) If Γ is quasi-geometric, then $\ell_V(\Gamma, x)$ is non-negative.
- (c) If Γ is regular, then $\ell_V(\Gamma, x)$ is unimodal.

It was shown by Chan [Cha94] that the conditions in (2)(a) already characterize local h -vectors of topological subdivisions. Adding unimodality, one obtains the characterization of local h -vectors of regular subdivisions. We complete this picture by showing in the last section that indeed every vector satisfying the conditions in (2)(a) and (2)(b) occurs as local h -vector of a quasi-geometric subdivision.

Recall that a simplicial complex Δ is called *Cohen-Macaulay* if its Stanley-Reisner ring $K[\Delta]$ (as defined below) is Cohen-Macaulay (see Section 1.4.1 for the definition). Note that it follows from (1) and (2)(b) that if $h(\text{link}_\Delta(F), x) \geq 0$ for every face $F \in \Delta$, e.g., if Δ is Cohen-Macaulay, and Γ is a quasi-geometric subdivision of Δ , then $h(\Gamma, x) \geq h(\Delta, x)$ (see [Sta92, Theorem 4.10]).

Remark 3.2.4. The proof of the non-negativity of the local h -polynomial in the quasi-geometric case is based on an elegant argument that calls in commutative algebra and also appears in several other important concepts in algebraic combinatorics. More explicitly, Stanley proves the statement by showing that the coefficients of the local h -polynomial equal the dimensions of the graded parts of a certain module.

In order to move to the area of commutative algebra, we associate a graded polynomial ring $K[\Gamma]$ to a simplicial complex Γ . Assume that Γ is defined on the ground set $[n]$ and

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has dimension $\dim(\Gamma) = d - 1$. Furthermore, we fix the polynomial ring $R = K[x_1, \dots, x_n]$. First, we consider the *Stanley-Reisner ideal* $I_\Gamma = (x^F \mid F \notin \Gamma) \subset R$, where $x^F = x_{i_1} \dots x_{i_r}$ for $F = \{i_1, \dots, i_r\} \subset [n]$. The *Stanley-Reisner ring* associated to Γ is given by $K[\Gamma] = R/I_\Gamma$. Stanley-Reisner rings form a crucial connection between commutative algebra and discrete topology and are well-studied. To give a small hint of the usefulness of these rings we note that the h -polynomial of Γ appears as the numerator of the Hilbert series of $K[\Gamma]$ (equipped with the \mathbb{Z} -grading), that is,

$$H_{K[\Gamma]}(t) = \frac{h_0(\Gamma) + h_1(\Gamma)t + \dots + h_d(\Gamma)t^d}{(1-t)^d}.$$

For a detailed treatment of Stanley-Reisner rings we refer to [Sta04] and [BH98].

We return to the scenario where Γ is seen as a subdivision of the simplex 2^V with $V = [d]$. A homogeneous system of parameters (see Definition 1.4.2) $\theta_1, \dots, \theta_d$ of degree one of $K[\Gamma]$ is called *special* if each θ_i is a linear combination of those variables x_j , $j \in [n]$, that are associated to vertices which do not lie on the closure of the face $V \setminus \{x_i\}$. As an example, we consider the subdivisions (c) and (d) in Figure 3.1 with the indicated vertex labeling. A special hsop for the first subdivision is given by the linear forms $\theta_1 = x_1 - x_4 - x_5$, $\theta_2 = x_2 - x_5$ and $\theta_3 = x_3 - x_4 - x_5$. On the other hand, no special hsop can be found for the second example, since θ_2 could only involve x_2 and thus it could not be an element in an hsop for the Stanley-Reisner ring. This is no coincidence, since Stanley shows in [Sta92, Corollary 4.4] that Γ is quasi-geometric if and only if $K[\Gamma]$ admits a special hsop.

Finally, we consider the ideal generated by the interior faces of Γ , i.e.,

$$(\text{int } \Gamma) = (x^F \mid \sigma(F) = V) \subset K[\Gamma].$$

Its image in $K[\Gamma]/(\theta_1, \dots, \theta_d)K[\Gamma]$ is then defined to be the *local face module* $L_V(\Gamma)$ of Γ . It is a graded ideal and $\dim_K L_V(\Gamma)_i = 0$ for all $i > d$. Now, a chain of homological arguments shows that in fact $\dim_K L_V(\Gamma)_i = \ell_i(\Gamma)$ if $\theta_1, \dots, \theta_d$ forms a special hsop for $K[\Gamma]$ [Sta92, Theorem 4.6]. Thus, the local h -polynomial is indeed non-negative in the quasi-geometric case.

3.2.3 CW-regular subdivisions

The definition of topological subdivisions can be naturally extended to *regular CW-complexes* [Sta92, §7]. A regular CW-complex is a non-empty topological space X together with a finite set Γ of subsets of X (the so-called *cells* or *faces*) such that (see [BH98, §6.2])

- (i) $\emptyset \in \Gamma$;
- (ii) $X = \bigcup_{C \in \Gamma} C$;
- (iii) the cells $C \in \Gamma$ are pairwise disjoint;

3 Local h -vectors

- (iv) for each non-empty cell $C \in \Gamma$ there exists a homeomorphism from a closed i -dimensional ball onto the closure of C such that the restriction of this homeomorphism maps the interior of the closed ball onto C . In this case we write $\dim(C) = i$.

For a regular CW-complex Γ , we write $P(\Gamma)$ for its face poset. A *CW-regular subdivision* of a simplicial complex Δ is a pair (Γ, σ) , where Γ is a regular CW-complex and $\sigma : P(\Gamma) \rightarrow \Delta$ is a map satisfying the conditions (1) and (2) of topological subdivisions.

3.2.4 Local γ -vectors

In the following, let Γ be a subdivision of 2^V . As by Theorem 3.2.3 (2)(a) the local h -polynomial $\ell_V(\Gamma, x)$ is symmetric, we can express it as

$$\ell_V(\Gamma, x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \xi_k(\Gamma) x^k (1+x)^{d-2k},$$

where $\xi_k(\Gamma) \in \mathbb{Z}$ are uniquely determined. The sequence $\xi_V(\Gamma) = (\xi_0(\Gamma), \dots, \xi_{\lfloor d/2 \rfloor}(\Gamma))$ is called the *local γ -vector* of Γ (with respect to V).

Remark 3.2.5. We shed some light on the connection between the non-negativity of local γ -vectors and important conjectures in discrete topology. A simplicial complex Δ is called *flag* if every minimal non-face of Δ has two elements. Furthermore, Δ is a *homology sphere* if the link of every face $F \in \Delta$ has the homology of a sphere of dimension $(d - \#F - 1)$ (*homology balls* are defined accordingly). Now, let $d \in \mathbb{Z}_{>0}$, $e = \lfloor d/2 \rfloor$ and Δ be a $(d-1)$ -dimensional flag homology sphere with h -vector $h(\Delta) = (h_0(\Delta), \dots, h_d(\Delta))$. The Charney-Davis conjecture [CD95, Conjecture D] claims that

$$(-1)^e h(\Delta, -1) = (-1)^e (h_0(\Delta) - h_1(\Delta) + h_2(\Delta) - \dots + h_d(\Delta)) \geq 0. \quad (3.9)$$

Note that since the h -vector of Δ is symmetric [Sta04, Section II.6], the left hand side of (3.9) is equal to 0 if d is odd. Furthermore, the symmetry of $h(\Delta)$ implies that Δ has a well-defined γ -vector $\gamma(\Delta)$. Gal conjectured in [Gal05, Conjecture 2.1.7] that $\gamma(\Delta)$ is non-negative and thereby generalized the Charney-Davis conjecture: By definition of $\gamma(\Delta)$, we have that $(-1)^e h(\Delta, -1) = \gamma_e(\Delta)$ and thus $\gamma(\Delta) \geq 0$ implies the validity of (3.9).

Gal's conjecture is known to be true for, e.g., 3- and 4-dimensional homology spheres (see [DO01] and [Gal05] resp.), for barycentric subdivisions of boundary complexes of polytopes [MN12] and barycentric subdivisions of homology spheres [NP11]. In addition, the Charney-Davis conjecture has been validated for certain subdivisions of particular classes of spheres in [Sta94] and [Fro10].

Generalizing further, Postnikov, Reiner and Williams [PRW08, Conjecture 14.2] and later Athanasiadis [Ath12, Conjecture 1.4] asked for the monotonicity of the γ -vector: Given a flag homology sphere Δ and a flag and vertex-induced subdivision Γ of Δ (meaning the restriction of Γ to any face of Δ is flag), do we have $\gamma(\Gamma) \geq \gamma(\Delta)$? This conjecture is analogous to the monotonicity problem of the h -vector, which was the motivation to

3.3 Local γ -vectors of barycentric subdivisions

introduce local h -vectors. With this in mind, it is quite natural to extend the conjecture about the monotonicity and ask for the non-negativity of the local γ -vector (Conjecture 3.0.1). Let Γ be a subdivision of Δ . A similar reasoning as in the proof of (3.8) shows that [Ath12, Proposition 5.3]

$$\gamma(\Gamma, x) = \sum_{F \in \Delta} \xi_F(\Gamma_F, x) \gamma(\text{link}_\Delta(F), x). \quad (3.10)$$

In order to conclude with (3.10) the monotonicity of the γ -vector from the non-negativity of the local γ -vector, we would require that $\gamma(\text{link}_\Delta(F), x) \geq 0$, i.e., the validity of Gal's conjecture. Fortunately, there is a way out: Let Σ_{d-1} be the boundary complex of the d -dimensional cross-polytope (see Example 3.3.5). Then every flag homology sphere can be seen as a flag vertex-induced subdivision of Σ_{d-1} [Ath12, Theorem 1.5]. Moreover [Ath12, Corollary 5.5]

$$\gamma(\Delta, x) = \sum_{F \in \Sigma_{d-1}} \xi_F(\Delta_F, x).$$

So indeed, the non-negativity of local γ -vectors implies the monotonicity of the usual γ -vector. Summarizing, we have the following implications of conjectures, where Δ denotes a flag homology sphere:

$$\begin{array}{ccccccc} \text{local } \gamma\text{-vector} & \implies & \gamma\text{-vector monotonic} & \implies & \text{Gal's conjecture:} & \implies & \text{Charney-Davis} \\ \text{non-negative} & & \text{under subdivision} & & \gamma(\Delta) \geq 0 & & \text{conjecture:} \\ & & & & & & \gamma_{\lfloor d/2 \rfloor}(\Delta) \geq 0 \end{array}$$

The local γ -vector is known to be non-negative for flag vertex-induced subdivisions in dimension ≤ 3 [Ath12] and for special classes of subdivisions including barycentric, edgewise and cluster subdivisions [Ath12, AS12] of the simplex. In the next section we prove the non-negativity for barycentric subdivisions of any CW-regular subdivision. In particular, the local h -polynomial is unimodal in all of these cases.

3.3 Local γ -vectors of barycentric subdivisions

In this section, we provide the proof of Theorem 3.0.2, i.e., we show that the local γ -vector of the barycentric subdivision of any CW-regular subdivision of a simplex is non-negative. This answers Question 6.2 in [Ath12] in the affirmative.

Given a regular CW-complex Γ , its barycentric subdivision $\text{sd}(\Gamma)$ is the simplicial complex, whose i -dimensional faces are given by chains

$$\tau_0 \preceq \tau_1 \preceq \cdots \preceq \tau_i,$$

where $\tau_j \in P(\Gamma)$ is a non-empty face of Γ for $j = 0, \dots, i$. It is well-known that the geometric realizations of Γ and $\text{sd}(\Gamma)$ are homeomorphic. Then, for a CW-regular subdivision (Γ, σ) of a simplicial complex Δ , $\text{sd}(\Gamma)$ can be naturally considered as a subdivision of Δ by the map

$$\sigma'(\tau_0 \preceq \tau_1 \preceq \cdots \preceq \tau_i) := \sigma(\tau_i),$$

3 Local h -vectors

since

$$(\sigma')^{-1}(2^F) = \{(\tau_0 \preceq \tau_1 \preceq \cdots \preceq \tau_i) \in \text{sd}(\Gamma) \mid \sigma(\tau_i) \subseteq F\} = \text{sd}(\Gamma_F),$$

where $\Gamma_F = \{\tau \in P(\Gamma) \mid \sigma(\tau) \subseteq F\}$. When we consider the local h -polynomials of $\text{sd}(\Gamma)$, we always consider the local h -polynomial using the above map σ' .

Figure 3.3 depicts the barycentric subdivision of the stellar subdivision of the 2-simplex together with their respective local h -vectors.

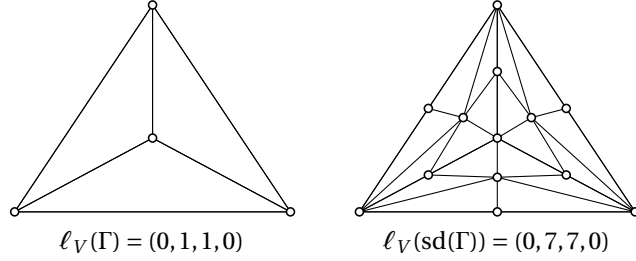


Figure 3.3: Barycentric subdivision of the stellar subdivision.

It is easy to see that the barycentric subdivision $\text{sd}(2^V)$ of a $(d-1)$ -simplex 2^V is a special instance of a regular subdivision. Its h -polynomial is given by the Eulerian polynomial of order d , that is,

$$h(\text{sd}(2^V), x) = A_d(x). \quad (3.11)$$

Moreover, in [BW08] Brenti and Welker give a linear transformation of the h -vector for barycentric subdivisions of a general simplicial complex Δ of dimension $(d-1)$:

$$h_j(\text{sd}(\Delta)) = \sum_{i=0}^d A(d+1, j, i+1) h_i(\Delta),$$

where $A(d+1, j, i+1)$ denotes the *restricted Eulerian number*, i.e., the number of permutations $\pi \in S_{d+1}$ with $\pi(1) = i+1$ and $\text{des}(\pi) = j$.

We consider the definition (3.6) of the local h -polynomial. Given (3.11) and the formula (3.4), we immediately see that the local h -polynomial of $\text{sd}(2^V)$ is given by the derangement polynomial of order d (see also [Sta92, Proposition 2.4]):

$$\ell_V(\text{sd}(2^V), x) = \mathfrak{d}_d(x). \quad (3.12)$$

In particular, the local h -polynomial of $\text{sd}(2^V)$ is γ -non-negative.

We come to the proof of Theorem 3.0.2 and show that, in fact, the local h -polynomial of the barycentric subdivision of any CW-regular subdivision is γ -non-negative. First, we derive a formula for the local h -polynomial of any CW-regular subdivision of the simplex involving derangement polynomials and differences between h -polynomials of restrictions of the subdivision and their boundaries. We then conclude the proof by applying a result of Ehrenborg and Karu which implies the γ -non-negativity of these differences in the case of the barycentric subdivision of any CW-regular subdivision of the simplex.

3.3 Local γ -vectors of barycentric subdivisions

Lemma 3.3.1. *Let Δ be a subdivision of the simplex 2^V . The local h -polynomial of Δ can be written as*

$$\ell_V(\Delta, x) = h(\Delta, x) - h(\partial\Delta, x) + \sum_{F \subsetneq V} \ell_F(\Delta_F, x)(x + \dots + x^{d-\#F-1}).$$

Proof. First, note that for any $F \subsetneq V$ the link of F in 2^V respectively in $\partial(2^V)$ is a $(\#V - \#F - 1)$ -simplex respectively its boundary. Hence,

$$h(\text{link}_{2^V}(F), x) = 1 \quad \text{and} \quad h(\text{link}_{\partial(2^V)}(F), x) = 1 + x + \dots + x^{d-\#F-1}.$$

Applying (3.8) to Δ , we obtain

$$\begin{aligned} h(\Delta, x) &= \sum_{F \subseteq V} \ell_F(\Delta_F, x) h(\text{link}_{2^V}(F), x) \\ &= \ell_V(\Delta, x) + \sum_{F \subsetneq V} \ell_F(\Delta_F, x). \end{aligned} \quad (3.13)$$

Similarly, viewing $\partial\Delta$ as a subdivision of $\partial(2^V)$ and using that $(\partial\Delta)_F = \Delta_F$ for $F \subsetneq V$, the h -polynomial of $\partial\Delta$ can be written in the following way:

$$\begin{aligned} h(\partial\Delta, x) &= \sum_{F \subsetneq V} \ell_F(\Delta_F, x) h(\text{link}_{\partial(2^V)}(F), x) \\ &= \sum_{F \subsetneq V} \ell_F(\Delta_F, x)(1 + x + \dots + x^{d-\#F-1}). \end{aligned} \quad (3.14)$$

Subtracting (3.14) from (3.13) yields

$$\ell_V(\Delta, x) = h(\Delta, x) - h(\partial\Delta, x) + \sum_{F \subsetneq V} \ell_F(\Delta_F, x)(x + \dots + x^{d-\#F-1}),$$

as desired. □

Using (3.12) and the fact that $\text{sd}(2^V)$ has the same h -polynomial as its boundary (since it is just the cone over it), we retrieve the recurrence formula in Theorem 3.1.1 as a special case of Lemma 3.3.1 when $\Delta = \text{sd}(2^V)$.

The next formula for the local h -polynomial is crucial in the proof of Theorem 3.0.2.

Theorem 3.3.2. *Let Δ be a subdivision of the simplex 2^V . The local h -polynomial of Δ can be written as*

$$\ell_V(\Delta, x) = \sum_{F \subseteq V} [h(\Delta_F, x) - h(\partial(\Delta_F), x)] \cdot \mathfrak{d}_{\#V \setminus F}(x). \quad (3.15)$$

Proof. To simplify notation, we set

$$h_F(x) = h(\Delta_F, x) - h(\partial(\Delta_F), x).$$

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We show the claim by induction on $\#V$. If $\#V = 1$, both sides in (3.15) are equal to 0 and the claim is trivially true.

Assume $\#V \geq 2$. By Lemma 3.3.1 and the induction hypothesis, we have

$$\begin{aligned} \ell_V(\text{sd}(\Delta), x) &= h_V(x) + \sum_{F \subsetneq V} \left[\sum_{G \subseteq F} h_G(x) \cdot \mathfrak{d}_{\#F \setminus G}(x) \right] (x + \cdots + x^{d-\#F-1}) \\ &= h_V(x) + \sum_{G \subsetneq V} h_G(x) \left[\sum_{G \subseteq F \subsetneq V} \mathfrak{d}_{\#F \setminus G}(x) (x + \cdots + x^{d-\#F-1}) \right] \\ &= h_V(x) + \sum_{G \subsetneq V} h_G(x) \left[\sum_{F \subsetneq V \setminus G} \mathfrak{d}_{\#F}(x) (x + \cdots + x^{d-\#G-1-\#F}) \right] \\ &= h_V(x) + \sum_{G \subsetneq V} h_G(x) \cdot \mathfrak{d}_{\#V \setminus G}(x), \end{aligned}$$

where the last equality follows from Theorem 3.1.1. This finishes the proof. \square

Example 3.3.3. We illustrate Theorem 3.3.2 using two examples.

- (1) Let $\Delta = 2^V$ be the trivial subdivision of the simplex 2^V with $\#V = d \geq 1$. Then $\ell_V(\Delta, x) = 0$ and

$$h(\Delta_F, x) - h(\partial(\Delta_F), x) = -x - x^2 - \cdots - x^{\#F-1} \quad \text{for every } \emptyset \neq F \subseteq V.$$

However, $h(\Delta_\emptyset, x) - h(\partial(\Delta_\emptyset), x) = 1$ and all negative terms on the right-hand side of (3.15) cancel out. We retrieve the recurrence formula for derangement polynomials:

$$\begin{aligned} 0 &= \ell_V(\Delta, x) = \mathfrak{d}_d(x) + \sum_{\emptyset \neq F \subseteq V} (-x - \cdots - x^{\#F-1}) \cdot \mathfrak{d}_{\#V \setminus F}(x) \\ &= \mathfrak{d}_d(x) + \sum_{k=2}^d \binom{d}{k} (-x - \cdots - x^{k-1}) \cdot \mathfrak{d}_{d-k}(x) \\ &= \mathfrak{d}_d(x) - \sum_{k=0}^{d-2} \binom{d}{k} (x + \cdots + x^{d-k-1}) \cdot \mathfrak{d}_k(x). \end{aligned}$$

- (2) Let Δ be the barycentric subdivision of the stellar subdivision of the 2-simplex, as depicted in Figure 3.3. Then $\mathfrak{d}_0(x) = 1, \mathfrak{d}_1(x) = 0$ and

$$h(\Delta, x) - h(\partial\Delta, x) = (1 + 10x + 7x^2) - (1 + 4x + x^2) = 6x + 6x^2.$$

The right-hand side of (3.15) is

$$\begin{aligned} &\sum_{F \subseteq V} [h(\Delta_F, x) - h(\partial(\Delta_F), x)] \cdot \mathfrak{d}_{\#V \setminus F}(x) \\ &= 6x + 6x^2 + \sum_{F \subseteq V, \#F \leq 1} [h(\Delta_F, x) - h(\partial(\Delta_F), x)] \cdot \mathfrak{d}_{\#V \setminus F}(x) \\ &= 6x + 6x^2 + \mathfrak{d}_3(x) = 7x + 7x^2, \end{aligned}$$

which is indeed equal to the local h -polynomial of Δ (see Figure 3.3).

3.3 Local γ -vectors of barycentric subdivisions

We now prove Theorem 3.0.2. Let Γ be a CW-regular subdivision of a simplex 2^V . Then, by applying the special case of Theorem 3.3.2 when $\Delta = \text{sd}(\Gamma)$, we obtain

$$\ell_V(\text{sd}(\Gamma), x) = \sum_{F \subseteq V} [h(\text{sd}(\Gamma_F), x) - h(\partial(\text{sd}(\Gamma_F)), x)] \cdot \mathfrak{d}_{\#V \setminus F}(x).$$

Since we already know that $\mathfrak{d}_k(x)$ is γ -non-negative and since the product of two γ -non-negative polynomials is γ -non-negative as well, the next result due to Ehrenborg and Karu [EK07] completes the proof of Theorem 3.0.2.

Theorem 3.3.4 (Ehrenborg–Karu). *Let Γ be a regular CW-complex which is homeomorphic to a ball. Then $h(\text{sd}(\Gamma), x) - h(\partial(\text{sd}(\Gamma)), x)$ is γ -non-negative.*

Theorem 3.3.4 is an immediate consequence of [EK07, Theorem 2.5], but since this result is written in the language of **cd**-indices, we explain how Theorem 3.3.4 can be deduced from it. Before the proof, we recall flag h -numbers and **ab**-indices. Let Γ be a regular CW-complex of dimension $d - 1$ and fix a set $S \subseteq [d]$. An S -chain of Γ is a chain

$$\tau_0 \lesssim \tau_1 \lesssim \cdots \lesssim \tau_i$$

of $P(\Gamma)$ with $S = \{\dim(\tau_0) + 1, \dots, \dim(\tau_i) + 1\}$. The *flag f -number*, denoted by $f_S(\Gamma)$, is defined to be the number of S -chains of Γ . Furthermore, we define the *flag h -number* $h_S(\Gamma)$ by

$$h_S(\Gamma) = \sum_{T \subseteq S} (-1)^{\#S - \#T} f_T(\Gamma).$$

Note that one has

$$f_i(\text{sd}(\Gamma)) = \sum_{\substack{S \subseteq [d] \\ \#S = i+1}} f_S(\Gamma) \quad \text{and} \quad h_i(\text{sd}(\Gamma)) = \sum_{\substack{S \subseteq [d] \\ \#S = i}} h_S(\Gamma).$$

Let $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ and $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ be non-commutative polynomial rings, where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are variables with $\deg \mathbf{a} = \deg \mathbf{b} = \deg \mathbf{c} = 1$ and $\deg \mathbf{d} = 2$. We say that a polynomial $f \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ is *non-negative* if all coefficients of monomials in f are non-negative. For $S \subseteq [d]$, we define the non-commutative monomial $u_S = u_1 u_2 \cdots u_d \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The homogeneous polynomial

$$\Psi_\Gamma(\mathbf{a}, \mathbf{b}) = \sum_{S \subseteq [d]} h_S(\Gamma) u_S$$

is called the **ab**-index of Γ . Note that by substituting $\mathbf{a} = 1$ to $\Psi_\Gamma(\mathbf{a}, \mathbf{b})$ we obtain the h -polynomial of $\text{sd}(\Gamma)$, that is, $\Psi_\Gamma(1, \mathbf{b}) = \sum_{i=0}^d h_i(\text{sd}(\Gamma)) \mathbf{b}^i$.

Now, let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. As it was shown for (boundary complexes of) polytopes by Bayer and Klapper [BK91] and, more generally, for Eulerian posets by Stanley [Sta94], the **ab**-index can be written uniquely as a polynomial $\Phi_\Gamma(\mathbf{c}, \mathbf{d})$ in \mathbf{c} and \mathbf{d} in these cases. We refer to this polynomial as the **cd**-index of Γ . Moreover, by a result of Karu [Kar06], the coefficients of the **cd**-index are non-negative integers if the face poset $P(\Gamma)$ is *Gorenstein**, that is, $\text{sd}(\Gamma)$ is a homology sphere over \mathbb{R} .

3 Local h -vectors

Example 3.3.5. We consider the boundary complex Σ_2 of the three-dimensional cross-polytope

$$P = \text{conv}(e_1, -e_1, e_2, -e_2, e_3, -e_3) \subset \mathbb{R}^3,$$

where e_1, e_2, e_3 are the unit vectors in \mathbb{R}^3 . The cross-polytope is illustrated in Figure 3.4. The flag f - and h -numbers of Σ_2 are collected in Table 3.1.

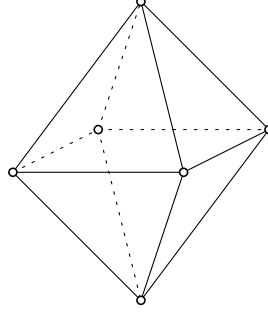


Figure 3.4: The three-dimensional cross-polytope

S	\emptyset	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$f_S(\Sigma_2)$	1	6	12	8	24	24	24	48
$h_S(\Sigma_2)$	1	5	11	7	7	11	5	1

Table 3.1: Flag f - and h -numbers of Σ_2 .

From the flag h -numbers, we obtain the \mathbf{ab} - and \mathbf{cd} -index of Σ_2 :

$$\Psi_{\Sigma_2}(\mathbf{a}, \mathbf{b}) = \mathbf{aaa} + 5\mathbf{baa} + 11\mathbf{aba} + 7\mathbf{aab} + 7\mathbf{bba} + 11\mathbf{bab} + 5\mathbf{abb} + \mathbf{bbb},$$

$$\Phi_{\Sigma_2}(\mathbf{c}, \mathbf{d}) = \mathbf{c}^3 + 6\mathbf{cd} + 4\mathbf{dc}.$$

Proof of Theorem 3.3.4. Let Γ be a regular CW-complex, which is homeomorphic to a $(d-1)$ -dimensional ball. Then the face poset of Γ is *near-Gorenstein** in the sense of [EK07, Definition 2.2], i.e., $\text{sd}(\Gamma)$ is a homology ball over \mathbb{R} , and [EK07, Theorem 2.5] says that there is a non-negative homogeneous polynomial $\Phi(\mathbf{c}, \mathbf{d}) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ of degree d such that

$$\Phi(\mathbf{a} + \mathbf{b}, \mathbf{ab} + \mathbf{ba}) = \Psi_{\Gamma}(\mathbf{a}, \mathbf{b}) - \Psi_{\partial\Gamma}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{a}.$$

By substituting $\mathbf{a} = 1$ in the above equation, we see that

$$\Phi(1 + \mathbf{b}, 2\mathbf{b}) = \Psi_{\Gamma}(1, \mathbf{b}) - \Psi_{\partial\Gamma}(1, \mathbf{b}) = \sum_{i=0}^d (h_i(\text{sd}(\Gamma)) - h_i(\partial(\text{sd}(\Gamma)))) \mathbf{b}^i \quad (3.16)$$

3.4 Characterization of local h -vectors

coincides with the polynomial $h(\text{sd}(\Gamma), \mathbf{b}) - h(\partial(\text{sd}(\Gamma)), \mathbf{b})$. On the other hand, since $\Phi(\mathbf{c}, \mathbf{d})$ is homogeneous and non-negative, there exist non-negative integers $\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_{\lfloor d/2 \rfloor}$ such that

$$\Phi(1 + \mathbf{b}, 2\mathbf{b}) = \sum_{k=0}^{\lfloor d/2 \rfloor} \gamma_k (1 + \mathbf{b})^{d-2k} (2\mathbf{b})^k. \quad (3.17)$$

We conclude from the two equations (3.16) and (3.17) that $h(\text{sd}(\Gamma), \mathbf{b}) - h(\partial(\text{sd}(\Gamma)), \mathbf{b})$ is γ -non-negative. \square

As γ -non-negativity implies unimodality, we obtain the following immediate corollary.

Corollary 3.3.6. *Let Γ be a CW-regular subdivision of a simplex 2^V . Then $\ell_V(\text{sd}(\Gamma))$ is unimodal.*

Remark 3.3.7. The polynomials $h(\Delta_F, x) - h(\partial(\Delta_F), x)$ in Theorem 3.3.2 may in general have negative coefficients but are always symmetric. Indeed, if Δ is a simplicial complex which is homeomorphic to a $(d-1)$ -ball with $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$, then $h_d(\Delta) = 0$ and $h_i(\partial\Delta) = \sum_{j=0}^i (h_j(\Delta) - h_{d-j}(\Delta))$ for all i (see [Sta04, p. 137]). Hence

$$h_i(\Delta) - h_i(\partial\Delta) = \sum_{j=0}^{i-1} (h_j(\Delta) - h_{d-j-1}(\Delta)), \quad (3.18)$$

and therefore $h_i(\Delta) - h_i(\partial\Delta) = h_{d-i}(\Delta) - h_{d-i}(\partial\Delta)$ for all $i = 0, 1, \dots, d$. Moreover, (3.18) says that if $h_j(\Delta) \geq h_{d-j-1}(\Delta)$ holds for $j < d/2$, then $h(\Delta, x) - h(\partial\Delta, x)$ is unimodal.

For example, it follows from [BS10, Theorem 1.2] that, if Δ is the r^{th} edgewise subdivision of any topological subdivision of a simplex and if r is sufficiently large, then we have $h_j(\Delta_F) \geq h_{\#F-j-1}(\Delta_F)$ for $j < \#F/2$ (see [BR05, §6] for a connection between edgewise subdivisions and Veronese subrings). As explained above, this implies the unimodality of $h(\Delta_F, x) - h(\partial(\Delta_F), x)$, and hence the unimodality of the local h -polynomial of Δ by Theorem 3.3.2.

3.4 Characterization of local h -vectors

In this section, we provide a characterization of local h -vectors of quasi-geometric subdivisions. This complements work by Chan [Cha94], who showed that local h -vectors of topological and regular subdivisions are completely characterized by their properties in Theorem 3.2.3 (2). In particular, we prove Theorem 3.0.3 by extending her main idea.

As local h -vectors of quasi-geometric subdivisions are known to be symmetric and non-negative with first entry equal to 0 (see Theorem 3.2.3 (2)), we only need to show that the conditions in (2)(b) are also sufficient. Let $\ell = (\ell_0, \dots, \ell_d) \in \mathbb{Z}^{d+1}$ be fixed and assume that ℓ is symmetric, $\ell_0 = 0$ and ℓ is non-negative. We will explicitly construct a quasi-geometric subdivision Γ of a $(d-1)$ -simplex 2^V with $\ell_V(\Gamma) = \ell$. The basic idea

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is to find operations on a subdivision Δ of a simplex that preserve quasi-geometricity, change the local h -vector of Δ in a prescribed way and such that ℓ can be realized as local h -vector of a subdivision obtained by successively applying these operations.

In [Cha94] Chan already provided three operations that suffice to construct all local h -vectors of arbitrary topological subdivisions. Though one of these does not necessarily preserve quasi-geometricity, we recall her constructions and their effects on the local h -vector, since we will use them in what follows.

Let (Γ, σ) be a subdivision of the $(d-1)$ -simplex 2^V .

- (O1) Let $(\text{stel}_\Gamma(F), \sigma')$ be obtained from Γ by stellar subdivision of a facet F of Γ , where $\sigma'(z) = \sigma(F)$ for the new vertex z . Then:

$$\ell_V(\text{stel}_\Gamma(F)) = \ell_V(\Gamma) + (0, 1, \dots, 1, 0).$$

- (O2) Let $d \geq 4$ and G be a $(d-2)$ -dimensional face of Γ with $(d-2)$ -dimensional carrier $\sigma(G)$. Let $(P_\Gamma(G), \sigma')$ be the subdivision of 2^V obtained from Γ by adding a new vertex w with carrier $\sigma'(w) = \sigma(G)$ and one new facet $G \cup \{w\}$ (with carrier V). (Note that $\sigma'(G) = V$.) Then:

$$\ell_V(P_\Gamma(G)) = \ell_V(\Gamma) + (0, 0, -1, \dots, -1, 0, 0).$$

We will say that $P_\Gamma(G)$ is obtained from Γ by *pushing* G into the interior.

- (O3) Let Ω be the subdivision of a 1-simplex $2^{\{d+1, d+2\}}$ into two edges. Then $\Gamma^{*1} := \Gamma * \Omega$ is a subdivision of the $(d+1)$ -simplex $2^{V \cup \{d+1, d+2\}}$ with

$$\ell_{V'}(\Gamma^{*1}) = (0, \ell_V(\Gamma), 0).$$

The operations are depicted in Figure 3.5.

Chan showed that – starting from a 2- or 3-simplex – these constructions suffice to generate any symmetric vector $\ell \in \mathbb{Z}^{d+1}$ with $\ell_0 = 0$ and $\ell_1 \geq 0$. Moreover, both, the stellar subdivision (O1) and the join operation (O3), maintain regularity of a subdivision and any symmetric and unimodal vector $\ell \in \mathbb{Z}^{d+1}$ with $\ell_0 = 0$ can be constructed by their successive application [Cha94].

It is straight forward to show that stellar subdivision (O1) and the join operation (O3) behave well with respect to quasi-geometricity.

Lemma 3.4.1. *Let Γ be a quasi-geometric subdivision of 2^V . Then:*

- (1) *If F is a facet of Γ , then $\text{stel}_\Gamma(F)$ is a quasi-geometric subdivision of 2^V .*
- (2) *Γ^{*1} is a quasi-geometric subdivision of $2^{V \cup \{d+1, d+2\}}$.*

Even though $P_\Gamma(G)$ (if defined) might not be quasi-geometric (even if Γ is), the next lemma shows that this obstruction can be “repaired” with just one additional stellar subdivision.

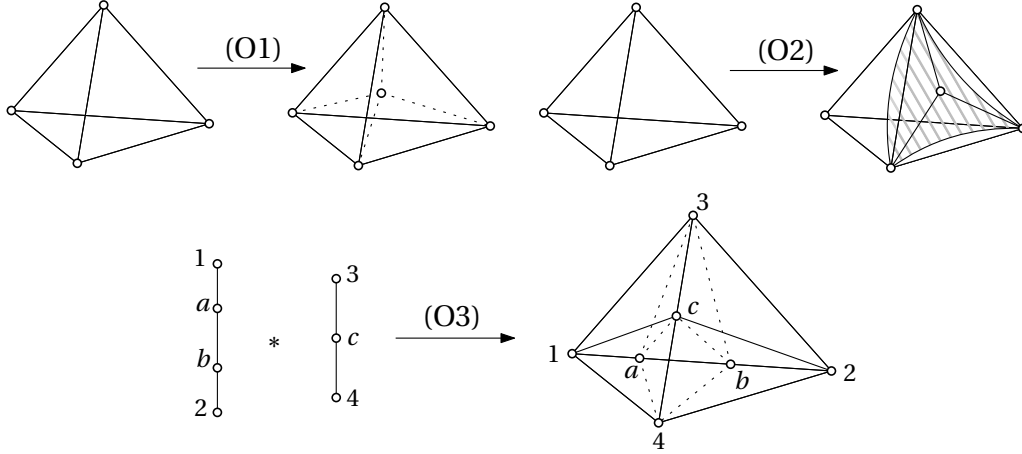


Figure 3.5: Constructions for subdivisions.

Lemma 3.4.2. *Let Γ be a quasi-geometric subdivision of 2^V and let $d = \#V \geq 4$. Let G be a $(d-2)$ -dimensional face of Γ with $(d-2)$ -dimensional carrier. Let w be the new vertex of $P_\Gamma(G)$. Then the subdivision $\text{stel}_{P_\Gamma(G)}(G \cup \{w\})$ obtained by first pushing G into the interior of Γ and then stellularly subdividing the new facet $G \cup \{w\}$ is a quasi-geometric subdivision of 2^V . Moreover,*

$$\ell_V(\text{stel}_{P_\Gamma(G)}(G \cup \{w\})) = \ell_V(\Gamma) + (0, 1, 0, \dots, 0, 1, 0).$$

Figure 3.6 shows $\text{stel}_{P_\Gamma(G)}(G \cup \{w\})$ in the case that Γ is just a 3-simplex.

Proof. To simplify notation, we set $\Gamma' := \text{stel}_{P_\Gamma(G)}(G \cup \{w\})$. The claim about the local h -vector follows immediately from the definition of the used operations (see also [Cha94]).

It remains to verify that Γ' is a quasi-geometric subdivision of 2^V . We denote by $\sigma_1 : \Gamma \rightarrow 2^V$ the map corresponding to the subdivision Γ of 2^V and by $\sigma_2 : \Gamma' \rightarrow \Gamma$ the subdivision map of Γ' (as a subdivision of Γ). The subdivision map of Γ' as a subdivision of 2^V is then given by $\sigma := \sigma_2 \circ \sigma_1$. Let z be the newly added vertex when applying stellar subdivision to $G \cup \{w\}$.

We first note that by definition of Γ' and σ we have

$$\sigma(E) = \sigma_1(E), \quad \forall E \in \Gamma \cap \Gamma' \setminus \{G\}, \quad \sigma(w) = \sigma_1(G) \quad \text{and} \quad \sigma(z) = V.$$

Given a face $E \in \Gamma'$, we need to show that the following condition, referred to as condition (QG) in the sequel, is satisfied:

(QG) For all $F \subseteq V$ such that $\sigma'(v) \subseteq F$ for all $v \in E$, it holds that $\dim(F) \geq \dim(E)$.

Let $E \in \Gamma \cap \Gamma' \setminus \{G\}$. In this case, we have $\sigma(E) = \sigma_1(E)$ and $\sigma(u) = \sigma_1(u)$ for all $u \in E$. As Γ is quasi-geometric, it follows, that E satisfies condition (QG).

3 Local h -vectors

Similarly, we have $\sigma(u) = \sigma_1(u)$ for all $u \in G$ and as Γ is quasi-geometric, condition (QG) holds for G .

It remains to consider faces $E \in \Gamma' \setminus \Gamma$. First assume $z \in E$. As $\sigma(z) = V$ by construction and $\#V = d$, those faces satisfy condition (QG). On the contrary, suppose that $z \notin E$. As $E \notin \Gamma$, we must have that $w \in E$. Since $z \notin E$, we can further conclude that $\dim(E) \leq d-2$. (Indeed, any facet containing w also contains z .) Combining this with the fact that $\sigma(w) = \sigma_1(G)$ is of dimension $d-2$ (by assumption) we get that E meets condition (QG). The claim follows. \square

We finally provide the proof of Theorem 3.0.3, i.e., the desired characterization of local h -vectors of quasi-geometric subdivisions.

Proof of Theorem 3.0.3. The “only if”-part follows directly from Theorem 3.2.3 (b). We now show the “if”-part. If d is even we start with a 2-simplex and take $\ell_{d/2}$ times the stellar subdivision (O1) of a facet. Similarly, if d is odd, we start with the 3-simplex and take the stellar subdivision (O1) of a facet $\ell_{(d-1)/2}$ times. In the next step, we apply once (O3) and then $\ell_{\lfloor d/2 \rfloor - 1}$ times the operation defined in Lemma 3.4.2. We continue in this way and, by Lemmas 3.4.1 and 3.4.2, this yields a quasi-geometric subdivision Γ of the $(d-1)$ -simplex whose local h -vector is equal to ℓ . \square

Chan and Stanley originally conjectured that all local h -vectors of quasi-geometric subdivisions are unimodal. However, Athanasiadis disproved this conjecture by providing a counterexample to it ([Ath12, Example 3.4] and [Ath14]). In fact, this example is obtained by applying the operation defined in Lemma 3.4.2 to the 3-simplex (see Figure 3.6) and has local h -vector $(0, 1, 0, 1, 0)$.

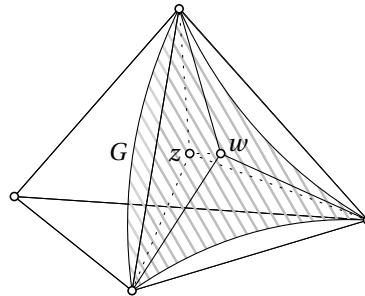


Figure 3.6: Quasi-geometric subdivision with non-unimodal h -polynomial.

Nevertheless, no geometric or even just vertex-induced subdivisions of the $(d-1)$ -simplex are known whose local h -vector is not unimodal. Already, Athanasiadis in [Ath12, Question 3.5] asked if such examples exist or if all vertex-induced subdivision of the $(d-1)$ -simplex have unimodal local h -vector. Based on a lot of experiments and a great vain effort to construct counterexamples we are inclined to believe that the latter is indeed the case.

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