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Poisson hyperplane tessellation

Asymptotic probabilities of the zero and typical cells

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Contents

Chapter 1

Introduction

Contents

Let η be a stationary and isotropic Poisson line process in the Euclidean plane $\mathbb{R}^2$. For each line $H \in \eta$, we consider the half-space H^- supported by H and containing the origin. The intersection $Z_{\boldsymbol{o}} = \cap_{H\in\eta} H^-$ is called the zero cell of the line tessellation associated to η. D.G. Kendall recalled in 1987 in the foreword of [SKM87, first edition] a conjecture that he made a few decades ago:

> *'[...] the conditional law of the shape of $Z_{\boldsymbol{o}}$, given the area $V_2(Z_{\boldsymbol{o}})$ of $Z_{\boldsymbol{o}}$, converges weakly, as $V_2(Z_{\boldsymbol{o}}) \to \infty$, to the degenerated law concentrated at the circular shape.'*

This conjecture was later proved by Kovalenko [Kov97, Kov99] and many contributions to this problem and very broad generalisations of it have been done by Miles, Goldman, Mecke, Osburg, Hug, Reitzner and Schneider. See Note 9 of Section 10.4 in [SW08] for precise references. One of the broadest generalisations was presented and solved by Hug and Schneider in [HS07]. They considered the higher dimensional case $d \geq 2$, diminished the isotropy and stationarity conditions to a condition of homogeneity of degree $r \geq 1$, and replaced the volume V_d by any size functional Σ satisfying mild properties. It is quite remarkable that they succeeded to solve the problem in such a high level of generalities. A cornerstone in this thesis is a new proof of the theorems proved by Hug and Schneider, with some slight improvements on the results.

In this introduction, we will briefly present the setting and explain this result. Then we will describe the most important steps of the proof, which are themselves highlights of the manuscripts. Next we will present additional related results we obtained. Finally we will give an outline of the thesis.

1.1 Generalized Kendall's problem

We consider the tessellation associated to a Poisson hyperplane process η with an intensity measure of the form

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(\boldsymbol{u},t) \in \cdot\,\right) t^{r-1} \mathrm{d}t \mathrm{d}\varphi(\boldsymbol{u}),$$

where $\gamma > 0$, $r \geq 1$, φ is a probability measure on $\mathbb{S}^{d-1}$, and $H(\boldsymbol{u},t) = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{u}\rangle = t\}$ denotes the hyperplane orthogonal to $\boldsymbol{u}$ at distance t from the origin. The cell containing the origin $\boldsymbol{o}$ is called the *zero cell* and denoted $Z_{\boldsymbol{o}}$. One motivation for such a general setting is that it includes two classical random polytopes: the zero cell of a stationary hyperplane tessellation (not necessarily isotropic) when $r = 1$, and the typical cell of a Poisson Voronoï tessellation when $r = d$ and φ rotation invariant.

A *size measurement* is any real function Σ on the set of convex bodies which is continuous, not identically zero, homogeneous of degree $k > 0$, and increasing under set inclusion. One specific size measurement, which will be of great importance in this manuscript, is the so called Φ-*content* :

$$\Phi(K) := \mu\left(\{H \in \mathcal{H} : H \cap K \neq \emptyset\}\right) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(\boldsymbol{u},t) \cap K \neq \emptyset\right) t^{r-1} \mathrm{d}t\, \mathrm{d}\varphi(\boldsymbol{u}).$$

For a given size measurement Σ, we characterise the shape of a convex body by the isoperimetric ratio $\Phi(K)\Sigma(K)^{-r/k}$. It is easy to see that

$$\tau := \min_{K \in \mathcal{K}} \frac{\Phi(K)}{\Sigma(K)^{\frac{r}{k}}} > 0.$$

Convex bodies minimizing this ratio are called *extremal.* For example, in the original setting of D.G. Kendall's problem, Φ is proportional to the perimeter, Σ is the area, $r = 1$ and $k = 2$. Thus the classical isoperimetric inequality tells us that the extremal bodies are precisely the balls. Hug and Schneider showed that there exist constants C and C' such that, for any $\epsilon > 0$ and $a > 0$,

$$\mathbb{P}\left(\frac{\Phi(Z_{\boldsymbol{o}})}{\Sigma(Z_{\boldsymbol{o}})^{\frac{r}{k}}} > \tau + \epsilon \mid \Sigma(Z_{\boldsymbol{o}}) > a\right) \leq C \exp\left(C' \epsilon \gamma a^{\frac{r}{k}}\right),$$

where C' depends only on τ and C on φ, r and ϵ. To rephrase it in a similar way as Kendall's original conjecture, we can say that the conditional law of the shape of $Z_{\boldsymbol{o}}$, given the size $\Sigma(Z_{\boldsymbol{o}})$, gets concentrated weakly, as $\Sigma(Z_{\boldsymbol{o}}) \to \infty$, on the set of extremal bodies. In this manuscript we will prove the slightly stronger following result. If there exists a convex body K such that $\Phi(K)\Sigma(K)^{-r/k} > \tau + \epsilon$, then

$$\lim_{a \to \infty} a^{-\frac{r}{k}} \ln \left(\mathbb{P} \left(\frac{\Phi(Z_{\boldsymbol{o}})}{\Sigma(Z_{\boldsymbol{o}})^{\frac{r}{k}}} > \tau + \epsilon \mid \Sigma(Z_{\boldsymbol{o}}) > a \right) \right) = -\epsilon\gamma. \tag{1.1}$$

1.2 Major steps in the proof

Our proof is based on essentially three steps, which correspond to parts of chapter 3, 5 and 6, respectively. We believe that the decomposition into these three steps makes the proof clearer.

The first step is the complementary theorem, which states that under the condition $\{f(Z_{\boldsymbol{o}}) = n\}$,

- The size $\Phi(Z_{\boldsymbol{o}})$ is $\Gamma_{\gamma,n}$ distributed,
- The size $\Phi(Z_{\boldsymbol{o}})$ and the shape $\Phi(Z_{\boldsymbol{o}})^{-1/r} Z_{\boldsymbol{o}}$ are independent.

Similar results have been proved before, see e.g. Miles [Mil71b], Møller and Zuyev [MZ96], Cowan [Cow06] and Baumstark and Last [BL09]. Most of the results in these references correspond to more general settings than the one we consider. This makes their statements more complicated to work with. Also, for our purposes we need a very detailed description, adapted to our situation, which we could not find in the literature. Therefore and for the sake of completeness we state them explicitly and give proofs in Chapter 3.

In the second step we give an asymptotic estimation for the probability $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n)$, when $n \to \infty$, and where $f(P)$ denotes the number of facets of P. Really little is known about the distribution of the number of facets of $Z_{\boldsymbol{o}}$. In dimension 2 and 3 one can deduce the mean number of facets from the mean number of vertices, see the comments after Corollary 3.3 in [HHRC15] or the last remarks of [Sch09]. As a special case of a formula due to Schneider [Sch09, Sec. 5] we also know the relation $\mathbb{E}f(Z_{\boldsymbol{o}}) = \gamma\mathbb{E}\Phi(Z_{\boldsymbol{o}})$, which we will recover as a simple corollary of the complementary theorem. In the 2-dimensional and isotropic case, Calka and Hilhorst [HC08] obtained a really precise asymptotic estimate of $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n)$, as $n \to \infty$. To the knowledge of the author, no significant additional results are known about the distribution of $f(Z_{\boldsymbol{o}})$. We fill this gap by showing that, under a mild condition, the quantity

$$n^{\frac{2}{d-1}} \sqrt[n]{\mathbb{P}(f(Z_{\boldsymbol{o}}) = n)}$$

is bounded from above and from below. In the two dimensional isotropic case, we recover the first order terms of the estimate of Calka and Hilhorst. Our proof relies on the complementary theorem and results about approximation of a polytope by a polytope with fewer facets. These latter results are described in Chapter 4.

In the final step, by combining the results of the two first steps, we obtain estimations of probabilities of the form

$$\mathbb{P}\left(\Sigma(Z_{\boldsymbol{o}}) > a\,,\ \Phi(Z_{\boldsymbol{o}})^{-\frac{1}{r}} Z_{\boldsymbol{o}} \in S\right),$$

when $a \to \infty$ and where Σ is an arbitrary size measurements and S a set of shapes. These estimations lead directly to the solution of the generalised D.G. Kendall's problem.

1.3 Related problems

Although, (1.1) is a really strong and general result, there are still some cases which are not covered. When $\Sigma = \Phi$, all convex bodies are extremal and thus (1.1) does not give any information. If φ has finite support, then the shape of cells with big Φ-content converges weakly to a random polytope (explicitly: $\Phi(Z_{\boldsymbol{o}})^{-1/r} Z_{\boldsymbol{o}}$ conditioned on $\{f(Z_{\boldsymbol{o}}) = |\text{supp}\varphi|\}$), because of the complementary theorem. One example of this form has been considered in [HS07, Sec. 7]. In the isotropic case, or more generally when φ has infinite support, the limit shape distribution of cells with big Φ-content is unknown. Actually, even the existence of a limit shape distribution is unknown. This problem is directly related to the problem of the existence of a limit shape of $Z_{\boldsymbol{o}}$ conditioned on $\{f(Z_{\boldsymbol{o}}) = n\}$, as $n \to \infty$. At this point the author cannot resist to present his own conjecture, which he believes has not yet been officially formulated. We formulate it in the stationary and isotropic case, i.e. φ is the Haar measure on $\mathbb{S}^{d-1}$ and $r = 1$, but we could also consider generalisations of this conjecture to the non isotropic or non stationary cases. In this conjecture $\mathfrak{c}$ denotes the center of mass.

Conjecture. In the stationary and isotropic case, the conditional law of the centered shape $\Phi(Z_{\boldsymbol{o}})^{-1}(Z_{\boldsymbol{o}} - \mathfrak{c}(Z_{\boldsymbol{o}}))$, given the number of facets $f(Z_{\boldsymbol{o}})$, converges weakly, as $f(Z_{\boldsymbol{o}}) \to \infty$, to the degenerated law concentrated at B^d.

Proving this conjecture would be a great step in understanding even better the shapes of big cells, and a really interesting result on its own. We are not able to prove this conjecture, but we were able to describe partially the shape distribution of cells with many facets. Instead of showing that cells with many facets tend to have a specific shape, we show, under a mild condition, that they are not too elongated, meaning that they are not close to a lower dimensional convex body: There exists $\epsilon > 0$ such that for any $1 \leq 1 < \lceil \frac{d-1}{2} \rceil$,

$$\mathbb{P}\left(d_H\left(\Phi(Z_{\boldsymbol{o}})^{-\frac{1}{r}} Z_{\boldsymbol{o}}\,,\, \mathcal{K}_i\right) < \epsilon \mid f(Z_{\boldsymbol{o}}) = n\right) \to 0, \text{ as } n \to \infty,$$

where $\mathcal{K}_i$ denotes the set of convex bodies of dimension i in $\mathbb{R}^d$, and $d_H\left(\Phi(Z_{\boldsymbol{o}})^{-1/r} Z_{\boldsymbol{o}}\,,\, \mathcal{K}_i\right) = \inf_{K\in\mathcal{K}_i} d_H\left(\Phi(Z_{\boldsymbol{o}})^{-1/r} Z_{\boldsymbol{o}}\,,\, K\right)$. In order to get this result, we improved bounds from polytopal approximation theory. The specificity of our bounds is to take into account the elongation of convex bodies. These purely geometric results are presented and proved in Chapter 4.

Another important part of this work is that in addition the case of $Z_{\boldsymbol{o}}$ we also cover the case of the typical cell Z_{typ} of a stationary hyperplane tessellation. All the results above concerning $Z_{\boldsymbol{o}}$ apply in a similar form to Z_{typ}.

In contrast with D.G. Kendall's problem, we also investigate the shape distribution of small typical cells, with respect to a given size measurement Σ. The behavior of small cells depends on properties of φ and Σ. When φ is absolutely continuous, e.g. in the isotropic case, we show that the limit shape is a random simplex with a distribution depending on Σ. We also show that the speed of convergence of

$$\mathbb{P}(f(Z_{\text{typ}}) > d + 1 \mid \Sigma(Z_{\text{typ}}) < a) \to 0$$

is at most of order $a^{\frac{1}{k}}$, which is the case when $\Sigma = \Phi$, and at least of order $-\ln\left(a^{\frac{1}{k}}\right) a^{\frac{1}{k}}$, which is the lower bound obtained when considering the worst case, i.e. Σ is the inradius. In the general case, we give a technical criterion which tells us if the limit shape of small cells, with a fixed number of facets, is degenerated (i.e. concentrated on lower dimensional shapes) or not. Finally, we apply this criterion to a specific case, namely the 2-dimensional case where the lines are either horizontal or vertical. We improve a result due to Beermann, Redenbach and Thäle [BRT14].

1.4 Outline

In order to avoid repetitions in this section, we will write Z whenever a result is obtained for both $Z_{\boldsymbol{o}}$ and Z_{typ}.

In the second chapter we introduce some basic notations and give some background material needed for this thesis.

In the third chapter we give the complete setting for this thesis. We introduce some specific sets of convex bodies and polytopes and equip them with measures. We study some basic properties of these objects. This leads us naturally to the complementary theorems, which are essential tools for this thesis.

In the fourth chapter we develop geometric tools which are both needed in the further chapters and are of interest themselves.

The first four sections deal with polytopal approximation theory. We give bounds for the Hausdorff distance between a convex body K and its best approximation by a polytope $P \supset K$ with a given number of facets. In the case where K is elongated, i.e. some isoperimeter is close to zero, our bound is better then the previously known ones. We also consider the case where K is a polytope and each facet of P contains a facet of K.

The fifth section generalises tools of integral geometry to the general setting considered in this thesis.

In the fifth chapter we show that $n^{\frac{2}{d-1}} \sqrt[n]{\mathbb{P}(f(Z) = n)}$ is bounded from above, and under a mild condition on φ also bounded from below. The upper bound is the result of a recurrence relation between the probabilities $\mathbb{P}(f(Z) = n)$ and $\mathbb{P}(f(Z) = n+1)$, which also implies that $n \mapsto \mathbb{P}(f(Z_{\boldsymbol{o}}) = n)$ is decreasing for n big enough. Under an elongation condition we refine the upper bound.

The sixth chapter focuses on D.G. Kendall's problem, which asks for the asymptotic shape distribution of Z conditioned on $\Sigma(Z) \to \infty$. More generally, in this chapter we are interested in the asymptotic behaviour of any probabilities of the form $\mathbb{P}(B)$ and $\mathbb{P}(A \mid B)$ where B is either $\{f(Z) = n\}$, or $\{\Phi(Z) > a\}$, or $\{\Sigma(Z) > a\}$, and A is usually an event about the shape of Z. Thus, in this chapter we also recall some results obtained in the previous ones.

In the first section we consider cells with many facets. We recall the bounds on the tail distribution of $f(Z)$, provide upper and lower bounds for probabilities of the form $\mathbb{P}(\mathfrak{s}(Z) \in S \mid f(Z) = n)$, recall the Gamma distribution of $\Phi(Z)$ when Z is conditioned on the event $\{f(Z) = n\}$, and finally we study the tail distribution of $\Sigma(Z)$ when Z is conditioned on the event $\{f(Z) = n\}$.

In the second section we are interested in cells with a big Φ-content. We give bounds for the tail distribution of $\Phi(Z)$ and a partial result describing the shape distribution of Z conditioned on the event $\{\Phi(Z) > a\}$.

In the last section we study cells with big Σ-content. We give precise estimation $\mathbb{P}\left(\Sigma(Z) > a, \mathfrak{s}(Z) \in S\right)$, for specific sets of shapes S and when $a \to \infty$. From these bounds we derive easily our result answering D.G. Kendall's problem.

The seventh, and last, chapter deals with the opposite problem, namely to characterise the shape of small typical cells.

In the first section we characterise the set $\mathcal{N} = \{n \in \mathbb{N} : \mathbb{P}(f(Z_{\text{typ}}) = n) > 0\}$. Indeed we will need later the fact that if $|\mathcal{N}| > 1$, then $n_{\min}{+}1 \in \mathcal{N}$, where $n_{\min} = \min \mathcal{N}$.

In the second section we cover the case of cells with small Φ-content. The complementary theorem makes this case easy to deal with.

In the third and fourth sections we consider the case where φ is absolutely continuous, and where Σ can be any size functional. First we show that the shape of small cells are random simplices and we describe the asymptotic shape distribution. Second we give a rate of convergence for $\mathbb{P}(f(Z_{\text{typ}}) > d+1 \mid \Sigma < a) \to 0$, as $a \to 0$.

Finally, in the last section we consider the most general case.

1.5 Underlying papers

The two first sections of Chapter 4 are taken from [Bon16]. Sections 3.3, 4.3, 4.4 and 5.2 as well as Theorems 6.1.1, 6.2.1 and 6.2.5 are taken from [BCR16], with a slight generalisation in order to consider the non stationary case and $Z_{\boldsymbol{o}}$ in Sections 4.3 and 4.4 and Theorems 6.1.1, 6.2.1 and 6.2.5.

Chapter 2

Preliminaries and basic notations

Contents

For a general reference about convex geometry we refer the reader to [Gru07] and [Sch14]. For tools from stochastic and integral geometry the most important reference is [SW08].

2.1 Geometry

2.1.1 General setting

We work in the euclidean space $\mathbb{R}^d$ of dimension $d \geq 2$, with origin $\boldsymbol{o}$, scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. We denote the vectors in $\mathbb{R}^d$ by bold letters, e.g. $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y} \dots$. Vectors of vectors are denoted with a bar above them, e.g. $\bar{\boldsymbol{u}} = (\boldsymbol{u}_0, \cdots, \boldsymbol{u}_d) \in \left(\mathbb{R}^d\right)^{d+1}$. The vectors of the canonical base of $\mathbb{R}^d$ are denoted by $\boldsymbol{e}_i$, $i = 1, \ldots, d$. We denote by $B(\boldsymbol{x}, r)$ and $\mathbb{S}^{d-1}(\boldsymbol{x}, r)$, respectively, the ball and the sphere of center $\boldsymbol{x}$ and radius r. The unit ball $B^d = B(\boldsymbol{o}, 1)$ has volume κ_d and the unit sphere $\mathbb{S}^{d-1} = S(\boldsymbol{o}, 1)$ has surface area $\omega_d = d\kappa_d$. The *Lebesgue measure* is denoted by λ_d and the *spherical Lebesgue measure* on $\mathbb{S}^{d-1}$ by σ. The normalised spherical Lebesgue measure $\omega_d^{-1}\sigma$ is called *Haar measure*. The *interior* and the *boundary* of a set A are denoted by A^o and ∂A, respectively. We denote by $f(P)$ the number of facets of a polytope P. We denote by $\mathbb{R}_+ = [0, \infty)$ the set of non negative numbers.

2.1.2 Set of convex bodies

A *convex body* in $\mathbb{R}^d$ is a compact and convex set with non empty interior. We denote by $\mathcal{K}$ the set of convex bodies. Because sometime we need to consider convex bodies of lower dimension we denote by $\mathcal{K}'$ the set of convex and compact sets with at least two points. The set $\mathcal{K}$ have an algebraic structure, with *scale and translation action*,

$$tA := \{t\boldsymbol{a} : \boldsymbol{a} \in A\}, \quad A + \boldsymbol{x} = \boldsymbol{x} + A := \{\boldsymbol{a} + \boldsymbol{x} : \boldsymbol{a} \in A\},$$

for any $A \in \mathcal{K}$, $t \in \mathbb{R} \setminus \{0\}$ and $\boldsymbol{x} \in \mathbb{R}^d$, and also a sum called *Minkowski sum*,

$$A + B := \{\boldsymbol{a} + \boldsymbol{b} \mid \boldsymbol{a} \in A, \boldsymbol{b} \in B\},$$

for any A and B in $\mathcal{K}$.

The set $\mathcal{K}$ is equipped with the Hausdorff distance

$$d_H(K, L) = \min_{r \geq 0} \left(K \subset L + rB^d, L \subset K + rB^d\right)$$

and its associated topology and Borel structure.

The following useful theorem is due to Blaschke, see Theorem 6.3 in [Gru07] for two different proofs of it.

Theorem 2.1.1 (Blaschke Selection Theorem)**.** *Any bounded sequence of convex bodies in $\mathbb{R}^d$ contains a convergent subsequence.*

2.1.3 Real functions on the set of convex bodies

Invariance

Let $f : \mathcal{K} \to \mathbb{R}$ be a map. If there exists $k \in \mathbb{R}$ such that $f(tK) = t^k K$ for any $K \in \mathcal{K}$ and $t > 0$, we say that f is *homogeneous* (of degree k). We say that f is *scale invariant* if f is homogeneous of degree 0. If $f(K+\boldsymbol{x}) = f(K)$ for any $K \in \mathcal{K}$ and $\boldsymbol{x} \in \mathbb{R}^d$, we say that f is *translation invariant.* We say that f is a *shape factor* if f is scale and translation invariant.

The Steiner Formula and Intrinsic Volumes.

We denote by $V_d(\cdot)$ the volume, i.e., the d-dimensional Lebesgue measure. The *Steiner formula* tells us that there exist functionals $V_i\colon \mathcal{K} \to [0,\infty)$, for $0 \le i \le d$, such that for any $K \in \mathcal{K}$ and $\epsilon \ge 0$

$$V_d(K + \epsilon B^d) = \sum_{i=0}^{d} \epsilon^{d-j} \kappa_{d-j} V_j(K).$$

$V_i(K)$ is called the *i-th intrinsic volume* of K. Some of the intrinsic volumes have a clear geometric meaning. V_d is the volume. If K has non-empty interior, then

$$V_{d-1}(K) = \frac{1}{2}\mathcal{H}^{d-1}(\partial K),$$

where $\mathcal{H}^{d-1}(\partial K)$ is the $(d-1)$-dimensional Hausdorff measure of the boundary of K. Thus, $2V_{d-1}$ is the surface area. V_1 is proportional to the *mean width* b. More precisely,

$$\frac{d\kappa_d}{2} b(K) = \kappa_{d-1} V_1(K) = \int_{\mathbb{S}^{d-1}} h(K,\boldsymbol{u})\sigma(\mathrm{d}\boldsymbol{u}),$$

where σ is the spherical Lebesgue area measure on $\mathbb{S}^{d-1}$ and $h(K,\boldsymbol{u}) := \max\{\langle \boldsymbol{x},\boldsymbol{u}\rangle \mid \boldsymbol{x} \in K\}$ is the value of the *support function* of K at $\boldsymbol{u}$. $V_0(K) = 1$ is the Euler characteristic. For $1 \le i < j \le d$ and $K \in \mathcal{K}$, we call the shape factor $\frac{V_j(K)^{1/j}}{V_i(K)^{1/i}}$ the (i,j)-*isoperimetric ratio* of K.

The Isoperimetric Inequality.

Let $B \subset \mathbb{R}^d$ be a d-dimensional ball. For any $K \in \mathcal{K}$ and for any $1 \le i < j \le d$,

$$V_j(K)^{1/j} \le \frac{V_j(B)^{1/j}}{V_i(B)^{1/i}} V_i(K)^{1/i}, \tag{2.1}$$

with equality if and only if K is a ball.

A Steiner-type Formula.

For any $K \in \mathcal{K}$

$$V_{d-1}(K + B^d) = \sum_{k=0}^{d-1} \frac{(d-k)\kappa_{d-k}}{2d} V_k(K). \tag{2.2}$$

The isoperimetric inequality and the Steiner-type formula imply the next fact.

Fact :

Let $d \geq 3$, I be an interval (convex hull of two distinct points), B be a ball, and $K \in \mathcal{K}$ neither an interval nor a ball. Assume that $V_1(I) = V_1(K) = V_1(B)$. Note that $V_1(I)$ is just the length of the segment I. Then, we have

$$V_{d-1}(I + B^d) < V_{d-1}(K + B^d) < V_{d-1}(B + B^d). \tag{2.3}$$

2.2 Poisson Hyperplane Mosaic

2.2.1 Space of hyperplanes

A *hyperplane* is a set of the form

$$H(\boldsymbol{u}, t) = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{u} \rangle = t\} \subset \mathbb{R}^{d-1},$$

with $\boldsymbol{u} \in \mathbb{S}^{d-1}$ and $t \in \mathbb{R}$. We denote by $\mathcal{H}$ the space of hyperplanes. The surjection $\mathbb{S}^{d-1} \times \mathbb{R} \to \mathcal{H}$, defined by $(\boldsymbol{u}, t) \mapsto H(\boldsymbol{u}, t)$ induces on $\mathcal{H}$ a topology. We equip $\mathcal{H}$ with this topology and the corresponding σ-algebra. Let Θ be a homogeneous (of degree $r > 0$) measure on $\mathcal{H}$. The homogeneity property means that $\Theta(tB) = t^r \Theta(B)$ for any Borel set $B \subset \mathcal{H}$ and $t > 0$, and is equivalent to the fact that it is of the form

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(\boldsymbol{u}, t) \in \cdot\right) t^{r-1} \mathrm{d}t \mathrm{d}\varphi(\boldsymbol{u}), \tag{2.4}$$

where $\gamma > 0$ and φ is a Borel probability measure on $\mathbb{S}^{d-1}$. We call γ, φ, and r, respectively, the *intensity*, the *directional distribution*, and the *distance exponent*, associated to Θ.

We denote by $\aleph$ the set of Borel probability measure on $\mathbb{S}^{d-1}$ with support not contained in some closed hemisphere, by $\aleph_e$ the set of measures $\varphi \in \aleph$ which are even, i.e. $\varphi(-\cdot) = \varphi(\cdot)$, and by $\aleph_{e,c}$ the set of measures $\varphi \in \aleph_e$ which are absolutely continuous with respect to the spherical Lebesgue measure σ. We say that φ is *well spread* (resp. *strongly well spread*) if it is lower bounded by a multiple of spherical Lebesgue measure on some spherical cap (resp. on the whole unit sphere).

From now on and for the rest of the manuscript, we assume the following two assumptions.

- $\Theta = \gamma\mu$ is homogeneous of degree $r > 0$, and thus is of the form (2.4). After Chapter 3 we will be even more restrictive and consider that $r \geq 1$, since otherwise some geometric arguments would fail.

- $\varphi \in \aleph$, and when we are in the stationary setting $\varphi \in \aleph_e$.

We will often restrict the setting in order to have stationarity. This is needed when considering the typical cell (which will be defined in the next chapter). Therefore an important observation is the following:

$$\Theta \text{ is stationary } \Leftrightarrow\ r = 1 \text{ and } \varphi \in \aleph_e\ . \tag{2.5}$$

Similarly, Θ is isotropic if and only if φ is the Haar measure.

2.2.2 Poisson hyperplane process

Recall that a discrete real random variable Y is Poisson distributed with parameter $\theta \in (0,1)$ if

$$\mathbb{P}(Y = k) = \frac{\theta^k e^{-k}}{k!},\ k \in \mathbb{N}.$$

The definition extends to the case $\theta = 0$ (resp. $\theta = \infty$), in which case $Y = 0$ (resp. $Y = \infty$) almost surely. A *Poisson hyperplane process* η of *intensity measure* Θ is a discrete random set in $\mathcal{H}$ such that

- for any Borel set $B \subset \mathcal{H}$, the random variable $|\eta \cap B|$ is Poisson distributed with parameter $\Theta(B)$.
- for any pairwise disjoint Borel sets $B_1, \dots, B_n \subset \mathcal{H}$, the random variables $|\eta \cap B_1|, \dots, |\eta \cap B_n|$ are independent.

We say that η is stationary (resp. isotropic) when Θ is stationary (resp. isotropic).

2.2.3 Poisson hyperplane tessellation

A *polyhedron* is a finite intersection of halfspaces. A *polytope* is a bounded polyhedron. A *tessellation* is a collection of polytopes which are covering the whole space and have pairwise disjoint interiors. The polytopes forming a tessellation X are called the *cells* of X.

Let η be a Poisson hyperplane process of intensity measure Θ. The closure of each of the connected components of the complement $\mathbb{R}^d \setminus \cup_{H\in\eta} H$ is almost surely a polytope. They are the cells of the so called *Poisson hyperplane tessellation* $X = X_\eta$ associated to η.

Almost surely the origin $\boldsymbol{o}$ is not contained in any hyperplane of η. The cell of X containing $\boldsymbol{o}$ is called the zero cell and denoted by $Z_{\boldsymbol{o}}$. Note that the assumption we made on the degree of homogeneity r of the intensity measure Θ, namely that $r > 0$, is a sufficient and necessary condition for that $Z_{\boldsymbol{o}}$ is almost surely not reduced to $\{\boldsymbol{o}\}$.

In the stationary case, i.e. $r = 1$ and φ even, we can define the typical cell Z_{typ}. It will be done rigorously in Section 3.3 of the next chapter.

2.2.4 Slivnyak-Mecke formula

A really useful tool in stochastic geometry when dealing with Poisson processes is the Slivnyak-Mecke formula. We present it in the context of a Poisson hyperplane processes η of intensity $\Theta = \gamma\mu$ as described above, but it holds as well for general Poisson processes.

Let N be the set of all locally finite sets of hyperplane equipped with the σ-algebra generated by the maps $\mathsf{N} \to \mathbb{N} \cup \{\infty\}$, $\eta \mapsto |\eta \cap B|$, where B ranges over all sets in $\mathcal{H}$.

Let $n \in \mathbb{N}$. The Slivnyak-Mecke formula states that for any non negative measurable map $f\colon \mathsf{N} \times \mathcal{H}^n \to \mathbb{R}$

$$\begin{aligned} &\mathbb{E} \sum_{(H_1,\dots,H_n)\in\eta^n_{\neq}} f(\eta, H_1, \dots, H_n) \\ &= \gamma^n \int_{\mathcal{H}^n} \mathbb{E} f\left(\eta \cup \bigcup_{i=1}^{n} H_i, H_1, \dots, H_n\right) \mathrm{d}\mu^n(\boldsymbol{H}), \end{aligned} \tag{2.6}$$

where $\eta^n_{\neq}$ denotes the sets of n-tuples $(H_1, \dots, H_n)$ with $H_i \neq H_j$ if $i \neq j$.

2.3 Analysis

2.3.1 Approximation

Here we collect the notations used to compare functions asymptotically. Most of them are very standard.

Symbol	Definition
$f \sim g$	$\frac{f}{g} \to 1$
$f \sim_< g$	$f \sim g$ and $f < g$
$f = o(g)$	$\frac{f}{g} \to 0$
$f = O(g)$	$\limsup \left\lvert\frac{f}{g}\right\rvert < \infty$
$f = \boldsymbol{\Theta}(g)$	$0 < \liminf \left\lvert\frac{f}{g}\right\rvert \leq \limsup \left\lvert\frac{f}{g}\right\rvert < \infty$

2.3.2 Stirling approximation

Occasionally we refer to the Stirling approximation:

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e\, n^{n+\frac{1}{2}} e^{-n},$$

for any $n \in \mathbb{N}$. We will actually only need the following weaker form:

$$n^n e^{-n} \leq n! \leq n^n$$

2.3.3 Gamma distribution

Let $n > 0$, and $\gamma > 0$. A positive real random variable is $\Gamma_{\gamma,n}$ *distributed*, or Γ distributed with *shape parameter* n and *rate parameter* γ, if its probability density function is

$$t \mapsto \frac{\gamma^n}{\Gamma(n)} e^{-\gamma t} t^{n-1},$$

where

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \mathrm{d}t,$$

and in particular

$$\Gamma(n) = (n-1)!,$$

when $n \in \mathbb{N}$.

2.3.4 Convention about the constants

In this manuscript, unless explicitly stated, all constants are strictly positive. Most of them are of the form $\boldsymbol{c}_i$, c_i or C_i, where i is an index which varies. The table below indicates on which parameter(s) each of these 3 families of constants depend.

Constants	Dependence
$\boldsymbol{c}_i$	universal
c_i	depends only on d
C_i	depends on d, φ, r, i, j, Σ, ...

Chapter 3

Setting and Complementary Theorems

Contents

We recall that we consider a Poisson hyperplane mosaic X associated to a Poisson hyperplane process η of intensity measure

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(\boldsymbol{u}, t) \in \cdot\right) t^{r-1}\, \mathrm{d}t\, \mathrm{d}\varphi(\boldsymbol{u}),$$

where $\gamma > 0$, $r > 0$ and $\varphi \in \aleph$. Apart from this chapter we will actually consider only the more restrictive case where $r \geq 1$, since otherwise some geometric arguments would fail. We recall as well that the cell of X which contains the origin is called the zero cell and denoted $Z_{\boldsymbol{o}}$. In the stationary case ($r = 1$ and $\varphi \in \aleph_e$) we will define, in Section 3.3, the so called typical cell Z_{typ}.

Essential tools in this thesis are the Complementary Theorems. Note the plural form. The first one is a result about $Z_{\boldsymbol{o}}$ while the second concerns Z_{typ}. Similar results have been proved before, see e.g. Miles [Mil71b], Møller and Zuyev [MZ96] and Cowan [Cow06]. Most of the results in the references cited above correspond to more general settings than the one we consider. This makes their statements more complicate to work with. Also, for our purposes we need a very detailed description, adapted to our situation, which we could not find in the literature. Therefore and for the sake of completeness we state them explicitly and give proofs in Sections 3.2 and 3.3 of the present chapter.

3.1 Setting and Notation

In this section we introduce various functions on the set of convex bodies, subsets of the set convex bodies, homeomorphisms decomposing such subsets, and measures on such subsets, see Subsections 3.1.1, 3.1.2, 3.1.3, and 3.1.4, respectively. Along the way, we will also study basic properties of these objects.

3.1.1 Functions

Although, we will exhaustively introduce notations for various sets of convex bodies in Subsection 3.1.2, we start by giving the few notations already needed in the present subsection. Recall that $\mathcal{K}$ and $\mathcal{K}'$ denote the set of convex bodies and the set of compact and convex sets with at least two points, respectively. Sometimes we restrict our considerations to sets containing the origin. For this purpose we introduce $\mathcal{K}_{\boldsymbol{o}} := \{K \in \mathcal{K} : \boldsymbol{o} \in K^o\}$ and $\mathcal{K}'_{\boldsymbol{o}} := \{K \in \mathcal{K}' : \boldsymbol{o} \in K^o\}$.

Φ-content

The Φ-*content* of a convex set $K \subset \mathbb{R}^d$ is defined by

$$\begin{aligned}\Phi(K) &:= \mu\left(\{H \in \mathcal{H} : H \cap K \neq \emptyset\}\right)\\ &= \int\limits_{\mathbb{S}^{d-1}} \int\limits_{0}^{\infty} \mathbb{1}\left(H(\boldsymbol{u}, t) \cap K \neq \emptyset\right) t^{r-1} \mathrm{d}t \, \mathrm{d}\varphi(\boldsymbol{u}).\end{aligned}$$

In the stationary case ($r = 1$ and $\varphi \in \aleph_e$) or when $\boldsymbol{o} \in K$, it can be written in a simpler form

$$\Phi(K) = \frac{1}{r} \int\limits_{\mathbb{S}^{d-1}} h(K, \boldsymbol{u})^r \mathrm{d}\varphi(\boldsymbol{u}),$$

where $h(K, \boldsymbol{u}) := \max\{\langle \boldsymbol{x}, \boldsymbol{u}\rangle : \ \boldsymbol{x} \in K\}$ is the value of the support function of K at $\boldsymbol{u}$. In the literature, e.g. [HS07], Φ is called the *parameter functional* of the process η. It is due to the fact, that for any $K \in \mathcal{K}'$ the number of hyperplane of the process η hitting K is a Poisson random variable with parameter $\gamma\Phi(K)$,

$$\mathbb{P}\left(|\{H \in \eta \colon H \cap K \neq \emptyset\}| = n\right) = \exp\left(-\gamma\Phi(K)\right) \frac{(\gamma\Phi(K))^n}{n!}.$$

In special cases Φ has a simple geometric description. In the stationary case, it is the directional mean of the widths of K, where the mean is weighted by the directional distribution φ. More precisely, we can write

$$\Phi(K) := \frac{1}{2} \int\limits_{\mathbb{S}^{d-1}} b(K, \boldsymbol{u}) \mathrm{d}\varphi(\boldsymbol{u}),$$

where, $b(K, \boldsymbol{u}) := |h(K, \boldsymbol{u}) - h(K, \boldsymbol{u})|$ is the width of K is direction $\boldsymbol{u}$. In the stationary and isotropic case, i.e. when φ is the Haar measure on $\mathbb{S}^{d-1}$ and $r = 1$, the Φ-content of a set $K \in \mathcal{K}'$ is half of the mean width, and also, up to a constant the first intrinsic volume $V_1(K)$.

Note that Φ is homogeneous of degree r, $\Phi(tK) = t^r \Phi(K)$ for any $K \in \mathcal{K}'$ and $t \geq 0$. In the stationary case, Φ is also stationary, $\Phi(K + \boldsymbol{x}) = \Phi(K)$ for any $K \in \mathcal{K}'$ and $\boldsymbol{x} \in \mathbb{R}^d$.

Constants related to the Φ-content

Because of the Blashke Selection Theorem 2.1.1, we have that the sets

$$\mathcal{K}'_{\boldsymbol{o},\Phi} = \{K \in \mathcal{K}'_{\boldsymbol{o}} : \Phi(K) = 1\}$$

and

$$\mathcal{K}'_{\boldsymbol{o},V_1} = \{K \in \mathcal{K}'_{\boldsymbol{o}} : V_1(K) = 1\}$$

are compact. By continuity on a compact, and by homogeneity of degree 1 of V_1, and $\Phi^{1/r}$, and $h(\cdot, \boldsymbol{u})$ for any $\boldsymbol{u} \in \mathbb{S}^{d-1}$, we see that

$$c_\Phi := \max_{K \in \mathcal{K}'_{\boldsymbol{o}}} \frac{V_1(K)}{\Phi(K)^{\frac{1}{r}}} = \max_{K \in \mathcal{K}'_{\boldsymbol{o},\Phi}} V_1(K) < \infty,$$

and

$$c_h := \max_{K \in \mathcal{K}'_{\boldsymbol{o}}, \boldsymbol{u} \in \mathbb{S}^{d-1}} \frac{h(K, \boldsymbol{u})}{V_1(K)} = \max_{K \in \mathcal{K}'_{\boldsymbol{o},V_1}, \boldsymbol{u} \in \mathbb{S}^{d-1}} h(K, \boldsymbol{u}) < \infty.$$

Thus, for any $K \in \mathcal{K}_{\boldsymbol{o}}$ and any $\boldsymbol{u} \in \mathbb{S}^{d-1}$,

$$V_1(K) \leq c_\Phi \Phi(K)^{\frac{1}{r}} \quad \text{and} \quad h(K, \boldsymbol{u}) \leq c_h c_\Phi \Phi(K)^{\frac{1}{r}}. \tag{3.1}$$

Note that, in the stationary case case, (3.1) holds for any $K \in \mathcal{K}$, i.e. when we remove the condition $\boldsymbol{o} \in K$.

Continuity of the Φ-content

We will also use some kind of uniform continuity of Φ. In the case $r = 1$, it is easy to see from the definition that $|\Phi(K) - \Phi(L)| \leq d_H(K, L)$, for any $K, L \in \mathcal{K}$. The following lemma generalizes this property in two directions. First, it holds for any $r \geq 1$. Second, it takes into account the range of directions for which the support functions $h(K, \cdot)$ and $h(L, \cdot)$ are distinct.

Lemma 3.1.1. *Let $U \subset \mathbb{S}^{d-1}$ and $K \subset L \in \mathcal{K}_{\boldsymbol{o}}$ such that, for any $u \in \mathbb{S}^{d-1}$,*

$$0 \leq h(L, \boldsymbol{u}) - h(K, \boldsymbol{u}) \leq \mathbb{1}(\boldsymbol{u} \in U)\delta,$$

where $\delta = \delta' c_\Phi \Phi(K)^{1/r} > 0$. Then

$$\Phi(L) - \Phi(K) \leq \delta'(c_h + \delta')^{r-1} c_\Phi^r \Phi(K)\varphi(U),$$

where c_h is the constant, dependent on d, defined in (3.1).

In particular, for any $K \subset L \in \mathcal{K}_{\boldsymbol{o}}$,

$$\Phi(L) - \Phi(K) \leq d_H(K, L)\left(d_H(K, L) + c_h c_\Phi \Phi(K)^{\frac{1}{r}}\right)^{r-1}.$$

Note that in case $r = 1$, one can replace the condition $K \subset L \in \mathcal{K}_{\boldsymbol{o}}$ by $K \subset L \in \mathcal{K}$ in the Lemma above, because of stationarity.

Proof. First observe that since $\boldsymbol{o} \in K \subset L$ and $r \geq 1$, we have

$$\begin{aligned} h(L, \boldsymbol{u})^r - h(K, \boldsymbol{u})^r = \int_{h(K,\boldsymbol{u})}^{h(L,\boldsymbol{u})} r t^{r-1} \mathrm{d}t &\leq [h(L, \boldsymbol{u}) - h(K, \boldsymbol{u})] r h(L, \boldsymbol{u})^{r-1} \\ &\leq \mathbb{1}(\boldsymbol{u} \in U)\delta r h(L, \boldsymbol{u})^{r-1}. \end{aligned}$$

But, by assumption and using (3.1), we have

$$h(L, \boldsymbol{u}) \leq h(K, \boldsymbol{u}) + \delta \leq (c_h + \delta') c_\Phi \Phi(K)^{\frac{1}{r}}.$$

Thus

$$\begin{aligned} h(L, \boldsymbol{u})^r - h(K, \boldsymbol{u})^r &\leq \mathbb{1}(\boldsymbol{u} \in U)\delta r \left((c_h + \delta') c_\Phi \Phi(K)^{\frac{1}{r}}\right)^{r-1} \\ &= \mathbb{1}(\boldsymbol{u} \in U) r \delta' (c_h + \delta')^{r-1} c_\Phi^r \Phi(K). \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(L) - \Phi(K) &= \frac{1}{r} \int_{\mathbb{S}^{d-1}} h(L, \boldsymbol{u})^r - h(K, \boldsymbol{u})^r \mathrm{d}\varphi(\boldsymbol{u}) \\ &\leq \int_U \delta'(c_h + \delta')^{r-1} c_\Phi^r \Phi(K) \mathrm{d}\varphi(\boldsymbol{u}) \\ &= \delta'(c_h + \delta')^{r-1} c_\Phi^r \Phi(K)\varphi(U), \end{aligned}$$

which is the first part of the lemma.

For the second part, we only have to set $\delta' = d_H(K, L) c_\Phi^{-1} \Phi(K)^{-1/r}$ and $U = \mathbb{S}^{d-1}$, and apply the first part. □

Centring function

A *centering function* is a function $\mathfrak{c}\colon \mathcal{K} \to \mathbb{R}^d$ which is translation and scale homogeneous, i.e.

$$\mathfrak{c}(tK + \boldsymbol{x}) = t\mathfrak{c}(K) + \boldsymbol{x}.$$

We naturally call $\mathfrak{c}(K)$ the *center* of K.

There exist many common choices for such function. Here we list a few examples: the center of mass, the center of the circumball (the smallest ball containing the body), and the point inside the body with the lowest coordinates with respect to the lexicographic order.

In Section 3.3, we are going to make use of a centering function to define the typical cell Z_{typ}. But most of the time, the particular choice for $\mathfrak{c}$ has no influence. So when we talk about $\mathfrak{c}$, the reader can usually think of his favourite centering function. The only restriction to this remark concerns a few proofs where it is simpler to assume that $\mathfrak{c}(K) \in K$ for any $K \in \mathcal{K}$, and that the 1-Lipschitz property holds, i.e. $\|\mathfrak{c}(K) - \mathfrak{c}(L)\| \leq d_H(K, L)$. Note that all the examples above have both properties.

Shape

In complementarity to the size measurement Φ and a centering function $\mathfrak{c}$, we define shape functions. Informally the shape of a convex body is the information that remains when we ignore some of its characteristics, like its size, position or orientation. When studying the zero cell, we sometime look at polytopes up to scale transformation. Hence we define, for any convex body $K \in \mathcal{K}$,

$$\mathfrak{s}_\Phi(K) := \frac{1}{\Phi(K)^{\frac{1}{r}}} K.$$

Similarly, with the typical cell, we occasionally consider polytopes up to translation and scale transformation. So we set

$$\mathfrak{s}_{\mathfrak{c},\Phi}(K) := \frac{1}{\Phi(K)^{\frac{1}{r}}} (K - \mathfrak{c}(K)).$$

We call both $\mathfrak{s}_\Phi(K)$ and $\mathfrak{s}_{\mathfrak{c},\Phi}(K)$ *shape* of K.

3.1.2 Sets of convex bodies

$\mathcal{K}' \supset \mathcal{K} \supset \mathcal{P} \supset \mathcal{P}_n$ denote, respectively, the set of compact and convex sets with at least 2 points, the set of convex bodies, the set of polytopes, and the set of *n-topes* (polytopes with n facets) in $\mathbb{R}^d$. Here we define in a systematic way a series of subsets of $\mathcal{K}'$. For a set $\mathcal{X} \subset \mathcal{K}'$ of convex bodies, e.g. $\mathcal{P}$, $\mathcal{P}_n$, $\mathcal{K}$ or $\mathcal{K}'$, we define the corresponding subset of convex bodies

- containing the origin in its interior

$$\mathcal{X}_{\boldsymbol{o}} := \{K \in \mathcal{X} : \ \boldsymbol{o} \in K^o\},$$

- with Φ-content equal to 1

$$\mathcal{X}_\Phi := \{K \in \mathcal{X} : \ \Phi(K) = 1\},$$

- centered at the origin

$$\mathcal{X}_\mathfrak{c} := \{K \in \mathcal{X} : \ \mathfrak{c}(K) = \boldsymbol{o}\}.$$

We can combine the indices, e.g. $\mathcal{P}_{n,\mathfrak{c},\Phi}$ is the set of n-topes centered at the origin and of Φ-content equal to 1. All these sets are equipped with the Hausdorff distance and the induced Borel σ-algebra.

3.1.3 Homeomorphisms

We define homeomorphisms decomposing any convex body into size and shape

$$\begin{aligned} \mathfrak{h}_\Phi : \mathcal{K} &\to (0,\infty) \times \mathcal{K}_\Phi \\ K &\mapsto (\Phi(K), \mathfrak{s}_\Phi(K)), \end{aligned}$$

or size, center and shape

$$\begin{aligned} \mathfrak{h}_{\mathfrak{c},\Phi} : \mathcal{K} &\to \mathbb{R}^d \times (0,\infty) \times \mathcal{K}_{\mathfrak{c},\Phi} \\ K &\mapsto (\mathfrak{c}(K), \Phi(K), \mathfrak{s}_\Phi(K)). \end{aligned}$$

Later we will restrict the domain and the codomain of $\mathfrak{h}_\Phi$ or $\mathfrak{h}_{\mathfrak{c},\Phi}$ according to the context. We are going to ignore it in the notation of the homeomorphisms but it will be clear from the situation. For example, in the context of n-topes containing the origin, the homeomorphism $\mathfrak{h}_\Phi$ is defined by

$$\begin{aligned} \mathfrak{h}_\Phi : \mathcal{P}_{n,\boldsymbol{o}} &\to (0,\infty) \times \mathcal{P}_{n,\boldsymbol{o},\Phi} \\ P &\mapsto (\Phi(P), \mathfrak{s}_\Phi(P)). \end{aligned}$$

3.1.4 Measures

Measure on the set of half spaces

We consider two different representations of half spaces in $\mathbb{R}^d$. The first takes into account whether or not the half space contains the origin. For any $H \in \mathcal{H}$ not containing $\boldsymbol{o}$,

H^- is the half space supported by H and containing $\boldsymbol{o}$,
H^+ is the half space supported by H and not containing $\boldsymbol{o}$.

This representation is undefined for half spaces with supporting hyperplane containing the origin, but for the purpose of this thesis, this is a negligible set of half spaces, i.e. a set of measure zero. In the second representation, by allowing t to be negative, we can omit the exponent $\pm$.

$$\widetilde{H}(\boldsymbol{u},t) := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{u} \rangle \le t\},$$

for any $\boldsymbol{u} \in \mathbb{S}^{d-1}$ and $t \in \mathbb{R}$.

The measure μ gives rise to the measure

$$\begin{aligned} \widetilde{\mu}(\cdot) &:= \int_{\mathcal{H}} \sum_{\epsilon \in \{\pm 1\}} \mathbb{1}(H^\epsilon \in \cdot)\, \mathrm{d}\mu(H) \\ &= \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{+\infty} \mathbb{1}\left(\widetilde{H}(\boldsymbol{u},t) \in \cdot\right) |t|^{r-1} \mathrm{d}t\, \mathrm{d}\varphi(\boldsymbol{u}) \end{aligned} \tag{3.2}$$

on the set of halfspaces $\widetilde{\mathcal{H}}$.

Measures on sets of polytopes

The measure $\widetilde{\mu}$ on $\widetilde{\mathcal{H}}$ naturally induces a measure μ_n on $\mathcal{P}_n$ via

$$\mu_n(\cdot) := \frac{1}{n!} \int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(\bigcap_{i=1}^n H_i^{\epsilon_i} \in \cdot \right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right), \tag{3.3}$$

where $\widetilde{\mu}^n := \widetilde{\mu} \otimes \cdots \otimes \widetilde{\mu}$ denotes the product measures and $\boldsymbol{H}^{\boldsymbol{\epsilon}} := (H_1^{\epsilon_1}, \dots, H_n^{\epsilon_n})$. Note that, when we restrict μ_n to $\mathcal{P}_{n,\boldsymbol{o}}$, we get a simpler representation due to the fact that $\cap_{i=1}^n H_i^{\epsilon_i} \notin \mathcal{P}_{n,\boldsymbol{o}}$ as soon as $\epsilon_i = +1$ for one i,

$$\mu_n(\cdot) = \frac{1}{n!} \int_{\mathcal{H}^n} \mathbb{1}\left(\bigcap_{i=1}^n H_i^- \in \cdot \right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right). \tag{3.4}$$

Splitting of the measures on sets of polytopes

Because the measure μ_n and the functional Φ are homogeneous of degree nr and r, respectively, we obtain

$$\mathfrak{h}_\Phi(\mu_n)((0,b) \times C) = b^n \mathfrak{h}_\Phi(\mu_n)((0,1) \times C),$$

for any Borel sets $C \subset \mathcal{P}_{n,\Phi}$ and $b > 0$. In the stationary case, the measure μ is stationary invariant and so is μ_n. This implies, similarly as above, that in the stationary case,

$$\mathfrak{h}_{\mathfrak{c},\Phi}(\mu_n)(A \times (0,b) \times C) = \lambda_d(A) b^n \mathfrak{h}_\Phi(\mu_n)\left([0,1]^d \times (0,1) \times C\right),$$

for any $b > 0$ and any Borel sets $A \subset \mathbb{R}^d$ and $C \subset \mathcal{P}_{n,\mathfrak{c},\Phi}$.

To simplify our notations we introduce on $\mathcal{P}_{n,\Phi}$ the normalized pushforward measure

$$\mu_{n,\Phi}(\cdot) = \mathfrak{h}_\Phi(\mu_n)\left((0,1)\times\cdot\right) = \mu_n\left(\mathfrak{h}_\Phi^{-1}\left((0,1)\times\cdot\right)\right).$$

Similarly, we define on $\mathcal{P}_{n,\mathfrak{c},\Phi}$ the normalized pushforward measure

$$\mu_{n,\mathfrak{c},\Phi}(\cdot) = \mathfrak{h}_{\mathfrak{c},\Phi}(\mu_n)\left([0,1]^d\times(0,1)\times\cdot\right) = \mu_n\left(\mathfrak{h}_{\mathfrak{c},\Phi}^{-1}\left([0,1]^d\times(0,1)\times\cdot\right)\right).$$

We define on $\mathbb{R}_+$ the measure

$$\lambda_1^{(n)}(B) = \int\limits_B nt^{n-1}\,\mathrm{d}t,\ \ \lambda_1^{(n)}((0,b)) = b^n$$

which is, up to a constant, the unique homogeneous measure of degree n on $\mathbb{R}_+$. With these notations the pushforward measure $\mathfrak{h}_\Phi(\mu_n)$ splits into the following product of measures:

$$\mathfrak{h}_\Phi\left(\mu_n\right) = \lambda_1^{(n)}\otimes\mu_{n,\Phi}. \tag{3.5}$$

In the stationary case we also have

$$\mathfrak{h}_{\mathfrak{c},\Phi}\left(\mu_n\right) = \lambda_d\otimes\lambda_1^{(n-d)}\otimes\mu_{n,\mathfrak{c},\Phi}. \tag{3.6}$$

Because $\mu_{n,\Phi}(\mathcal{P}_{n,\boldsymbol{o},\Phi})$ and $\mu_{n,\mathfrak{c},\Phi}(\mathcal{P}_{n,\mathfrak{c},\Phi})$ are finite, $\mu_{n,\Phi}(\cdot)/\mu_{n,\Phi}(\mathcal{P}_{n,\boldsymbol{o},\Phi})$ and $\mu_{n,\mathfrak{c},\Phi}(\cdot)/\mu_{n,\mathfrak{c},\Phi}(\mathcal{P}_{n,\mathfrak{c},\Phi})$ define probability measures on $\mathcal{P}_{n,\boldsymbol{o},\Phi}$ and $\mathcal{P}_{n,\mathfrak{c},\Phi}$, respectively. The Complementary Theorems 3.2.1 and 3.3.1 in the next sections say, among other things, that these are the distributions of $\mathfrak{s}_\Phi(Z_{\boldsymbol{o}})$, the shape of the zero cell, and $\mathfrak{s}(Z_{\mathrm{typ}})$, the shape of the typical cell, respectively.

3.2 Complementary Theorem for the zero cell

For any $r>0$, we have

Theorem 3.2.1. *Let $n \geq d+1$ be an integer.*

1. *For any Borel set of shapes $S \in \mathcal{P}_{n,\boldsymbol{o},\Phi}$ we have*

$$\begin{aligned}&\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\,,\mathfrak{s}_\Phi(Z_{\boldsymbol{o}})\in S\right)\\ &= n!\int\limits_{\mathcal{P}_{n,\boldsymbol{o}}}\mathbb{1}\left(\Phi(P)<1\right)\mathbb{1}\left(\mathfrak{s}_\Phi(P)\in S\right)\mathrm{d}\mu_n(P).\end{aligned} \tag{3.7}$$

2. ***(Complementary Theorem for the zero cell)*** *If we condition the zero cell $Z_{\boldsymbol{o}}$ to have n facets, then*

   (a) *$\Phi(Z_{\boldsymbol{o}})$ and $\mathfrak{s}_\Phi(Z_{\boldsymbol{o}})$ are independent random variables,*

(b) $\Phi(Z_{\boldsymbol{o}})$ *is* $\Gamma_{\gamma,n}$ *distributed, and*

(c) $\mathfrak{s}_\Phi(Z_{\boldsymbol{o}})$ *has probability distribution* $\mu_{n,\Phi}(\cdot)/\mu_{n,\Phi}(\mathcal{P}_{n,\boldsymbol{o},\Phi})$.

Proof. The number of cells of the mosaic X in a subset $D \subset \mathcal{P}_n$ is

$$X(D) = \frac{1}{n!} \sum_{(H_1,\dots,H_n)\in\eta^n_{\neq}} \sum_{\boldsymbol{\epsilon}\in\{\pm 1\}^n} \mathbb{1}\left(\bigcap_{i=1}^n H_i^{\epsilon_i} \in D\right) \mathbb{1}\left(\eta \cap \left(\bigcap_{i=1}^n H_i^{\epsilon_i}\right)^o = \emptyset\right),$$

because there are $n!$ possibilities of ordering a list of n different halfspaces. The Slivnyak-Mecke formula (2.6), gives for $P_{[n]} = \bigcap_{i=1}^n H_i^{\epsilon_i}$ that

$$\begin{aligned}\mathbb{E}X(D) &= \frac{\gamma^n}{n!} \int_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in D\right) \mathbb{P}\left(\eta \cap P_{[n]}^o = \emptyset\right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right) \\ &= \gamma^n \int_{\mathcal{P}_n} \mathbb{1}\left(P \in D\right) \mathbb{P}\left(\eta \cap P^o = \emptyset\right) \mathrm{d}\mu_n(P)\end{aligned}$$

by the definition of μ_n. Because η is a Poisson process, the probability in the integrand is equal to $e^{-\gamma\Phi(P)}$. So, we can write

$$\mathbb{E}X(D) = \gamma^n \int_{\mathcal{P}_n} \mathbb{1}\left(P \in D\right) e^{-\gamma\Phi(P)} \mathrm{d}\mu_n(P) \tag{3.8}$$

In the following we are interested in the case where $D = (\mathfrak{h}_\Phi)^{-1}(B \times C)$ with Borel sets $B \subset [0,\infty)$ and $C \subset \mathcal{P}_{n,\boldsymbol{o},\Phi}$. By (3.5) we obtain in this case

$$\begin{aligned}\mathbb{E}X\left((\mathfrak{h}_\Phi)^{-1}(B \times C)\right) &= \gamma^n \int_{\mathcal{P}_{n,\boldsymbol{o}}} \mathbb{1}\left(P \in (\mathfrak{h}_\Phi)^{-1}(B \times C)\right) e^{-\gamma\Phi(P)} \mathrm{d}\mu_n(P) \\ &= \gamma^n \int_C \int_B e^{-\gamma t}\, \mathrm{d}\lambda_1^{(n)}(t)\, \mathrm{d}\mu_{n,\Phi}(P).\end{aligned}$$

For the first part of Theorem 3.2.1, observe that

$$\begin{aligned}\mathbb{P}(f(Z_{\boldsymbol{o}}) = n,\, \mathfrak{s}_\Phi(Z_{\boldsymbol{o}}) \in C) &= \mathbb{E}X(\{P \in \mathcal{P}_{n,\boldsymbol{o}},\, \mathfrak{s}_\Phi(P) \in C\}) \\ &= \gamma^n \int_C \int_0^\infty e^{-\gamma t}\, \mathrm{d}\lambda_1^{(n)}(t)\, \mathrm{d}\mu_{n,\Phi}(P).\end{aligned}$$

Because the integration with respect to t gives $\gamma^{-n}\, n!\, \lambda_1^{(nr)}([0,1])$ by elementary computations, the right-hand side equals

$$n!\, \mu_n(\{P \in \mathcal{P}_{n,\boldsymbol{o}},\ \Phi(P) < 1, \mathfrak{s}_\Phi(P) \in C\}).$$

by the definition of μ_n. This proves the first part of the theorem.

Analogously, for the second part we have

$$\begin{aligned}\mathbb{P}(f(Z_{\boldsymbol{o}}) = n,\, \Phi(Z_{\boldsymbol{o}}) \in B, \mathfrak{s}_\Phi(Z_{\boldsymbol{o}}) \in C) &= \gamma^n \int\limits_C \int\limits_B e^{-\gamma t}\, \mathrm{d}\lambda_1^{(n)}(t) \mathrm{d}\mu_{n,\Phi}(P) \\ &= \gamma^n \mu_{n,\Phi}(C)\; n \int\limits_B e^{-\gamma t} t^{n-1} \mathrm{d}t.\end{aligned}$$

Thus, if we condition $Z_{\boldsymbol{o}}$ to have n facets, we have that $\mathfrak{s}_\Phi(Z_{\boldsymbol{o}})$ and $\Phi(Z_{\boldsymbol{o}})$ are independent random variables with distributions

$$\mathbb{P}(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}) \in C \mid f(Z_{\boldsymbol{o}}) = n) = \frac{\mu_{n,\Phi}(C)}{\mu_{n,\Phi}(\mathcal{P}_{n,\boldsymbol{o},\Phi})}$$

and

$$\mathbb{P}(\Phi(Z_{\boldsymbol{o}}) \in B \mid f(Z_{\boldsymbol{o}}) = n) = \frac{\gamma^n}{(n-1)!} \int\limits_B e^{-\gamma t} t^{n-1} \mathrm{d}t.$$

□

As a direct corollary we recover the following relation, which is a special case of a formula due to Schneider [Sch09, Sec. 5].

Corollary 3.2.2.

$$\mathbb{E}\Phi(Z_{\boldsymbol{o}}) = \gamma^{-1} \mathbb{E} f(Z_{\boldsymbol{o}}).$$

Proof. Using the fact that, under the condition $f(Z_{\boldsymbol{o}}) = n$, we have that $\Phi(Z_{\boldsymbol{o}})$ is $\Gamma_{\gamma,n}$ distributed, we get

$$\begin{aligned}\mathbb{E}\Phi(Z_{\boldsymbol{o}}) &= \sum_n \mathbb{E}(\Phi(Z_{\boldsymbol{o}}) \mid f(Z_{\boldsymbol{o}}) = n)\, \mathbb{P}(f(Z_{\boldsymbol{o}}) = n) \\ &= \sum_n \frac{n}{\gamma}\, \mathbb{P}(f(Z_{\boldsymbol{o}}) = n) \\ &= \gamma^{-1} \mathbb{E} f(Z_{\boldsymbol{o}})\end{aligned}$$

□

3.3 Complementary Theorem for the typical cell

We consider in this section that we are in the stationary case ($r = 1$ and $\varphi \in \aleph_e$). Due to the natural homeomorphism

$$\begin{aligned}\mathcal{P} &\to \mathbb{R}^d \times \mathcal{P}_{\mathfrak{c}} \\ P &\mapsto (\mathfrak{c}(P), P - \mathfrak{c}(P)),\end{aligned}$$

we consider from now on X as a germ-grain process in $\mathbb{R}^d$ with grain space $\mathcal{P}_{\mathfrak{c}}$. Since η is stationary, this is also the case for X. That implies the existence of a probability measure $\mathbb{Q}$ on $\mathcal{P}_{\mathfrak{c}}$ such that the intensity measure of the germ-grain process X decomposes into $\mathbb{Q}$ and Lebesgue measure λ_d,

$$\mathbb{E}\, X(\{P - \mathfrak{c}(P) \in C,\, \mathfrak{c}(P) \in A\}) = \gamma^{(d)} \lambda_d(A)\, \mathbb{Q}(C) \tag{3.9}$$

for $C \subset \mathcal{P}_{\mathfrak{c}}$. We call $\mathbb{Q}$ the grain distribution, and the constant $\gamma^{(d)} = \mathbb{E}\, X(\{P \in \mathcal{P},\, \mathfrak{c}(P) \in [0,1]^d\})$ the intensity of X.

Note that it is easy to see that $\gamma^{(d)}$ is a multiple of γ^d, where γ is the intensity of the Poisson hyperplane process. The reader can find in [SW08, Thm. 10.3.3] an expression of the factor $\gamma^d/\gamma^{(d)}$ in term of the volume of the so called zonoid of the hyperplane process η, but this is not in the scope of our manuscript.

A random centered polytope $Z_{\text{typ}} \in \mathcal{P}_{\mathfrak{c}}$ with distribution $\mathbb{Q}$ is called *typical cell* of X (with respect to $\mathfrak{c}$).

Theorem 3.3.1. *Let $n \geq d+1$ be an integer.*

1. *For any Borel set of shapes $S \in \mathcal{P}_{n,\mathfrak{c},\Phi}$ we have*

$$\begin{aligned} &\mathbb{P}(f(Z_{\text{typ}}) = n\,, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in S) \\ &= \frac{\gamma^d}{\gamma^{(d)}}(n-d)! \int_{\mathcal{P}_n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(\mathfrak{s}_{\mathfrak{c},\Phi}(P) \in S\right) \mathrm{d}\mu_n(P). \end{aligned} \tag{3.10}$$

2. ***(Complementary Theorem)*** *If we condition the typical cell Z_{typ} to have n facets, then*

   (a) $\Phi(Z_{\text{typ}})$ and $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}})$ are independent random variables,

   (b) $\Phi(Z_{\text{typ}})$ is $\Gamma_{\gamma,n-d}$ distributed, and

   (c) $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}})$ has probability distribution $\mu_{n,\mathfrak{c},\Phi}(\cdot)/\mu_{n,\mathfrak{c},\Phi}(\mathcal{P}_{n,\mathfrak{c},\Phi})$.

Proof. The proof is similar as the one of the Complementary Theorem 3.2.1 for the zero cell. Applying (3.8) to sets of the form $D = (\mathfrak{h}_{\mathfrak{c},\Phi})^{-1}([0,1]^d \times B \times C)$ where $B \subset [0,\infty)$ and $C \subset \mathcal{P}_{n,\boldsymbol{o},\Phi}$ are Borel sets, gives

$$\begin{aligned} &\mathbb{E}X\left((\mathfrak{h}_{\Phi})^{-1}([0,1]^d \times B \times C)\right) \\ &= \gamma^n \int_{\mathcal{P}_n} \mathbb{1}\left(P \in (\mathfrak{h}_{\mathfrak{c},\Phi})^{-1}([0,1]^d \times B \times C)\right) e^{-\gamma\Phi(P)} \mathrm{d}\mu_n(P) \\ &= \gamma^n \int_C \int_B \int_{[0,1]^d} \mathrm{d}\lambda_d(\boldsymbol{c})\, e^{-\gamma t}\, \mathrm{d}\lambda_1^{(n-d)}(t)\, \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P), \end{aligned} \tag{3.11}$$

where the second equality is induced by the splitting (3.6) of the pushforward measure $\mathfrak{h}_{\mathfrak{c},\Phi}(\mu_n)$.

For the first part of Theorem 3.3.1, observe that by the definition (3.9) of the intensity measure $\mathbb{Q}$ and using (3.11) we have

$$\begin{aligned}\mathbb{P}(f(Z_{\mathrm{typ}})=n,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}})\in C) &= \frac{\gamma^{(d)}}{\gamma^{(d)}}\lambda_d\left([0,1]^d\right)\mathbb{Q}\left(\mathcal{P}_{n,\mathfrak{c}}\cap \mathfrak{s}_{\mathfrak{c},\Phi}^{-1}(C)\right)\\ &= \frac{1}{\gamma^{(d)}}\mathbb{E}X\left(\{P\in\mathcal{P}_n,\ \mathfrak{c}(P)\in[0,1]^d,\ \mathfrak{s}_{\mathfrak{c},\Phi}(P)\in C\}\right)\\ &= \frac{\gamma^n}{\gamma^{(d)}}\int_C\int_0^\infty\int_{[0,1]^d}\mathrm{d}\lambda_d(\boldsymbol{c})\ e^{-\gamma t}\,\mathrm{d}\lambda_1^{(n-d)}(t)\,\mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P).\end{aligned}$$

Because the integration with respect to t gives $\gamma^{-(n-d)}\,(n-d)!\,\lambda_1^{(n-d)}([0,1])$ by elementary computations, the right-hand side equals

$$\frac{\gamma^d}{\gamma^{(d)}}\,(n-d)!\,\mu_n\left(\{P\in\mathcal{P}_n,\ \mathfrak{c}(P)\in[0,1]^d,\ \Phi(P)<1,\mathfrak{s}_{\mathfrak{c},\Phi}(P)\in C\}\right).$$

by the definition of μ_n. This proves the first part of the theorem.

Analogously, for the second part we have

$$\begin{aligned}&\mathbb{P}(f(Z_{\mathrm{typ}})=n,\ \Phi(Z_{\mathrm{typ}})\in B,\mathfrak{s}_\Phi(Z_{\boldsymbol{o}})\in C)\\ &=\frac{\gamma^n}{\gamma^{(d)}}\int_C\int_B\int_{[0,1]^d}\mathrm{d}\lambda_d(\boldsymbol{c})\,e^{-\gamma t}\,\mathrm{d}\lambda_1^{(n-d)}(t)\,\mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P)\\ &=\frac{\gamma^n}{\gamma^{(d)}}\,\mu_{n,\mathfrak{c},\Phi}(C)\,(n-d)\int_B e^{-\gamma t}t^{n-d-1}\mathrm{d}t.\end{aligned}$$

Thus, if we condition Z_{typ} to have n facets, we have that $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}})$ and $\Phi(Z_{\mathrm{typ}})$ are independent random variables with distribution

$$\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}})\in C\mid f(Z_{\mathrm{typ}})=n)=\frac{\mu_{n,\mathfrak{c},\Phi}(C)}{\mu_{n,\mathfrak{c},\Phi}(\mathcal{P}_{n,\mathfrak{c},\Phi})}$$

and

$$\mathbb{P}(\Phi(Z_{\mathrm{typ}})\in B\mid f(Z_{\mathrm{typ}})=n)=\frac{\gamma^{n-d}}{(n-d-1)!}\int_B e^{-\gamma t}t^{n-d-1}\mathrm{d}t.$$

□

As a direct corollary we get that the mean of the Φ-content of Z_{typ} is independent from the directional distribution φ.

Corollary 3.3.2.

$$\mathbb{E}\Phi(Z_{\mathrm{typ}})=\gamma^{-1}d.$$

Proof. Using the fact that, under the condition $f(Z_{\text{typ}}) = n$, we have that $\Phi(Z_{\text{typ}})$ is $\Gamma_{\gamma,n-d}$ distributed, we get

$$\begin{aligned}\mathbb{E}\Phi(Z_{\text{typ}}) &= \sum_n \mathbb{E}(\Phi(Z_{\text{typ}}) \mid f(Z_{\text{typ}}) = n)\, \mathbb{P}(f(Z_{\text{typ}}) = n) \\ &= \sum_n \frac{n-d}{\gamma}\, \mathbb{P}(f(Z_{\text{typ}}) = n) \\ &= \gamma^{-1} \left(\mathbb{E} f(Z_{\text{typ}}) - d\right).\end{aligned}$$

Since $\mathbb{E}(Z_{\text{typ}}) = 2d$, see e.g. Theorem 10.3.1 of [SW08], this proves the corollary. □

Chapter 4

Geometric tools

Contents

Apart from the Complementary Theorems, an other essential tool to get our main results in Chapter 5 is the Polytopal Approximation Theory. The problem is the following: Given a convex body K, how good can we approximate K by a polytope P? There exists many way to consider this question, depending on how we measure the distance between K and P and which constraints on P, or even on K, we assume. We refer the reader to the well known surveys of P. M. Gruber [Gru93, Gru94] and E. M. Bronstein [Bro08] for an excellent overview of the huge amount of results and literature about polytopal approximation. In the following 4 sections we prove a few new results which fit the setting of the proofs of Chapter 5.

In our setting, we always assume $K \subset P$, and most of the time fix the number of facets of P. In Section 4.1, we give upper bounds for $d_H(K, P)$. Section 4.2 improves these bounds when K is sufficiently 'elongated', meaning that for some $1 \leq i < j \leq d$ the isoperimeter $V_i(K)^{1/k} V_j(K)^{-1/j}$ is small. In Section 4.3 the convex body K is already a polytope and we approximate it by 'deleting' one facet. There, the distance is measured both with the Hausdorff distance and the difference of Φ-content. Our work on polytopal approximation ends with Section 4.4 where we approximate elongated polytopes by polytopes with less facets.

In the fifth section we generalise tools of integral geometry to the general setting of this manuscript.

4.1 δ-net and polytopal approximation of convex bodies

First, let us set some notation. Assume that M is a *metric space* with distance d_M, i.e. a set M and a function $d_M\colon M \times M \to [0, \infty)$ such that, for any $x, y, z \in M$, $d_M(x, y) = 0$ if and only if $x = y$, $d_M(x, y) = d_M(y, x)$, and $d_M(x, z) \leq d_M(x, y) + d_M(y, z)$. We write

$$B_M(\boldsymbol{x}, r) := \{\boldsymbol{y} \in M \mid d_M(\boldsymbol{x}, \boldsymbol{y}) \leq r\}.$$

Definition 4.1.1. Let M be a metric space and S a discrete subset of M. We say that

- S is a *δ-covering of M* if $\cup_{\boldsymbol{x} \in S} B_M(\boldsymbol{x}, \delta) = M$,
- S is a *δ-packing of M* if $B_M(\boldsymbol{x}, \delta) \cap B_M(\boldsymbol{y}, \delta) = \emptyset$ for any $\boldsymbol{x} \neq \boldsymbol{y} \in S$,
- S is a *δ-net of M* if it is both a *δ-covering of M* and a *$(\delta/2)$-packing of M*.

Note that, in the poset of $(\delta/2)$-packings ordered under inclusion, a maximal element is a δ-net. Zorn's lemma shows that, for any metric space M, there exists a δ-net.

In the following lemma, under some general assumptions, we give bounds for the cardinality of a δ-net. The construction of these bounds is adapted from the proof of the following well known result, see e.g. [Gru07, Prop. 31.1]. If $C \subset \mathbb{R}^d$ is a convex body with non empty interior such that $C = -C$, then there exists a packing of translated copies of C in $\mathbb{R}^d$ of density at least 2^{-d}, where, roughly speaking, density means the proportion of $\mathbb{R}^d$ covered by the translated copies of C.

Lemma 4.1.2. *Let M be a space equipped with a measure ψ and a measurable metric d_M. Assume that $\psi(M) < \infty$. Let $\delta_0 > 0$ and S be a δ-net of M with $\delta \in (0, \delta_0)$. Let $k > 0$.*

1. *Assume there exists a constant $c > 0$ such that, for any $\boldsymbol{x} \in M$ and $r \in (0, \delta_0)$, it holds that $cr^k > \psi(B_M(\boldsymbol{x}, r))$. Then $|S| > c^{-1}\psi(M)\delta^{-k}$.*
2. *Assume there exists a constant $c' > 0$ such that, for any $\boldsymbol{x} \in M$ and $r \in (0, \delta_0)$, it holds that $c'r^k < \psi(B_M(\boldsymbol{x}, r))$. Then $|S| < 2^k c'^{-1}\psi(M)\delta^{-k}$.*

Proof. To prove (1), we only have to observe that since S is a δ-covering, we have that

$$\psi(M) \leq \sum_{\boldsymbol{x}\in S} \psi(B_M(\boldsymbol{x},\delta)) < |S|\, c\delta^k$$

because $M = \cup_{\boldsymbol{x}\in S} B_M(\boldsymbol{x},\delta)$. The proof of (2) is similar. Since S is a $(\delta/2)$-packing, we have that

$$\psi(M) \geq \sum_{\boldsymbol{x}\in S} \psi(B_M(\boldsymbol{x},\delta/2)) > |S| c' \delta^k 2^{-k}$$

because, for any distinct $\boldsymbol{x},\boldsymbol{y} \in S$, we have $B_M(\boldsymbol{x},\delta/2) \cap B_M(\boldsymbol{y},\delta/2) = \emptyset$. □

In Lemma 4.1.4, we will apply the previous lemma to the space $M = \partial(K + B^d)$, where K is an arbitrary convex body and M is equipped with the surface area measure and the restriction of the euclidean distance. In this space the balls are caps on the boundary of the convex body $D = K + B^d$, where a cap is defined as follows. For a convex body D, a point $\boldsymbol{d} \in D$ (usually $\boldsymbol{d} \in \partial D$), and a positive radius $\delta > 0$, we define the *cap of D of center $\boldsymbol{d}$ and radius δ* to be the set

$$\mathrm{cap}\,(D,\boldsymbol{d},\delta) = \{\boldsymbol{y} \in \partial D \mid |\boldsymbol{d} - \boldsymbol{y}| < \delta\}.$$

Note that our definition differs slightly from the more usual one, where a cap is the intersection of the boundary ∂D with a half-space. In the next lemma we give bounds for the surface area of caps of radius $\delta \in (0,\delta_0)$, of bodies of the form $K + B^d$, with $\delta_0 = 1$ independent from K. Precise bounds for spherical caps are known, see e.g. Lemma 2.1 in [BGK+01], Lemmas 2.2 and 2.3 in [Bal97] or Remark 3.1.8 in [AAGM15]. Lemma 6.2 in [RVW08] gives bounds for more general bodies than the sphere, namely those with $\mathcal{C}^2$ boundary of positive curvature, but with a δ_0 depending on K. It does not seem to the author that we can deduce easily Lemma 4.1.3 from these results.

Lemma 4.1.3. *Let $K \in \mathcal{K}$ and $D = K + B^d$. Let $\boldsymbol{d} \in \partial D$ and $\delta \in (0,1)$. Then*

$$\delta^{d-1}\kappa_{d-1}2^{-(d-1)} < \mathcal{H}^{d-1}(\mathrm{cap}\,(D,\boldsymbol{d},\delta)) < \delta^{d-1}\kappa_{d-1}d.$$

Proof. For the lower bound, we approximate the cap by a $(d-1)$-dimensional disc of radius $\delta\sqrt{1-\delta^2/4}$ (see Figure 4.1). Let H be the tangent hyperplane to D at $\boldsymbol{d}$. We have

$$\begin{aligned}\mathcal{H}^{d-1}(\mathrm{cap}\,(D,\boldsymbol{d},\delta)) &\geq \mathcal{H}^{d-1}(H \cap B(\boldsymbol{d}, d(\boldsymbol{d},\boldsymbol{e}))) \\ &= \delta^{d-1}\kappa_{d-1}\left(1 - \frac{\delta^2}{4}\right)^{(d-1)/2} \\ &> \delta^{d-1}\kappa_{d-1}\left(\frac{3}{4}\right)^{(d-1)/2} > \delta^{d-1}\kappa_{d-1}2^{-(d-1)}.\end{aligned}$$

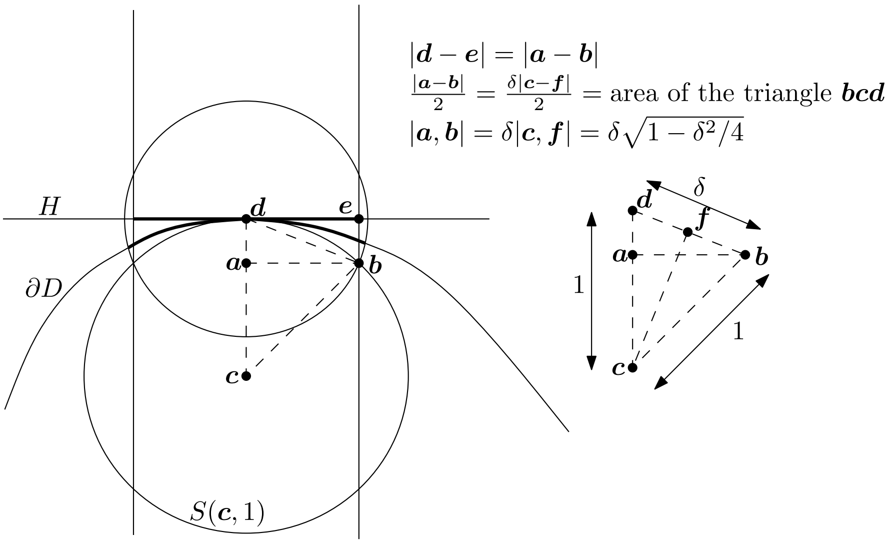

Figure 4.1: $\mathcal{H}^{d-1}(\text{cap}\,(D,\boldsymbol{d},\delta)) \geq \delta^{d-1}\kappa_{d-1}\left(1-\frac{\delta^2}{4}\right)^{(d-1)/2}$

For the upper bound, we approximate the cap by the union of a $(d-1)$-dimensional disc of radius δ and the spherical boundary of a cylinder of radius δ and height δ^2 (see Figure 4.2). Thus

$$\begin{aligned}\mathcal{H}^{d-1}(\text{cap}\,(D,\boldsymbol{d},\delta)) &< \mathcal{H}^{d-1}(H\cap B(\boldsymbol{d},\delta)) + \mathcal{H}^{d-2}(H\cap S(\boldsymbol{d},\delta))\delta^2\\ &= \delta^{d-1}\kappa_{d-1} + \delta^{d-2}\omega_{d-1}\delta^2\\ &= \delta^{d-1}\kappa_{d-1}\,(1+\delta(d-1)) < \delta^{d-1}\kappa_{d-1}d.\end{aligned}$$

□

Set $c_1 := 2d^{-1}\kappa_{d-1}^{-1} = \boldsymbol{\Theta}(d^{1/2})^d$ and $c_2 := 4^d\kappa_{d-1}^{-1} = \boldsymbol{\Theta}(d^{1/2})^d$. As a direct consequence of the two previous lemmas and the fact that $\mathcal{H}^{d-1}(\partial D) = 2V_{d-1}(D)$, we have the following lemma. We omit the proof.

Lemma 4.1.4. *Let $K \in \mathcal{K}$ and $D = K + B^d$, $\delta \in (0,1)$ and S a δ-net of the boundary ∂D. We have that*

$$c_1V_{d-1}(D)\delta^{-(d-1)} < |S| < c_2V_{d-1}(D)\delta^{-(d-1)}.$$

For a convex body K with boundary ∂K of differential class $\mathscr{C}^1$ and $\boldsymbol{x} \in \partial K$, we denote by $\boldsymbol{v}(\boldsymbol{x})$ the outer unit normal vector of K at $\boldsymbol{x}$. Using Lemma 4.1.4, we prove the two following lemmas in a similar way as Propositions 2.4 and Proposition 2.7 of [RSW01].

Lemma 4.1.5. *Let $K \in \mathcal{K}$ with ∂K of class $\mathscr{C}^1$ and $\delta \in (0,1)$. There exists a δ-net of ∂K, with respect to the distance $d_m(\boldsymbol{x},\boldsymbol{y}) = \max(|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y})|)$, of cardinality at most $c_2V_{d-1}(K+B^d)\,\delta^{-(d-1)}$.*

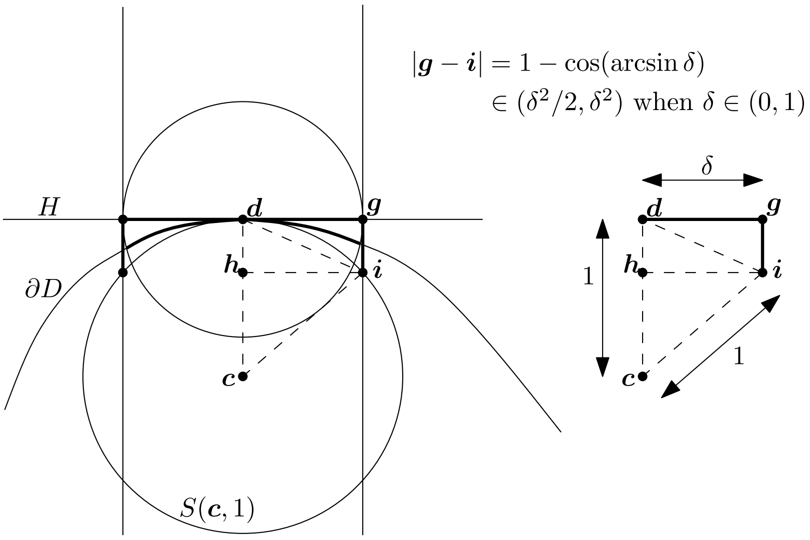

Figure 4.2: $\mathcal{H}^{d-1}(\operatorname{cap}(D,\boldsymbol{d},\delta)) < \delta^{d-1}\kappa_{d-1} + \delta^{d-2}\omega_{d-1}\delta^2$

Note that, in the lemma above, the condition that ∂K is of class $\mathscr{C}^1$ is equivalent to the fact that all boundary points of K have a unique outward normal vector, see e.g. [Sch14, Ch. 2].

Proof. Set $D = K + B^d$. Let S be a δ-net on the boundary ∂D (with respect to the euclidean distance in $\mathbb{R}^d$). Lemma 4.1.4 tells us that $|S| < c_2 V_{d-1}(D)\delta^{-(d-1)}$. Since ∂K is of class $\mathscr{C}^1$, we have that each point of ∂D has a unique representation $\boldsymbol{x}+\boldsymbol{v}(\boldsymbol{x})$, where $\boldsymbol{x} \in \partial K$ and $\boldsymbol{v}(\boldsymbol{x})$ is the outward unit normal vector of K at $\boldsymbol{x}$. Observe that the projections

$$\pi_K\colon \quad \begin{array}{c}\partial D \to \partial K \\ \boldsymbol{x}+\boldsymbol{v}(\boldsymbol{x}) \mapsto \boldsymbol{x}\end{array} \qquad \text{and} \qquad \begin{array}{c}\partial D \to \mathbb{S}^{d-1} \\ \boldsymbol{x}+\boldsymbol{v}(\boldsymbol{x}) \mapsto \boldsymbol{v}(\boldsymbol{x})\end{array}$$

are both 1-Lipschitz. This implies that the set $\pi_K(S)$ is a δ-covering of ∂K, with respect to the distance $d_m(\boldsymbol{x},\boldsymbol{y}) = \max(|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{y})|)$, of cardinality $|\pi_K(S)| = |S| < c_2 V_{d-1}(K+B^d)\,\delta^{-(d-1)}$. Observing that for any δ-covering there exists a sub-δ-covering which is also a $(\delta/2)$-packing (and thus a δ-net) yields the proof. □

Set $c_3 := 3^{(d-1)/4}c_2 = \boldsymbol{\Theta}(d^{1/2})^d$.

Lemma 4.1.6. *Let $K \in \mathcal{K}$ and $0 < \epsilon < 1$. Then, there exists a polytope $P_\epsilon \supset K$ with*

$$d_H(K, P_\epsilon) < \epsilon$$

and with number of facets at most

$$c_3 V_{d-1}(K+B^d)\,\epsilon^{-(d-1)/2}.$$

Proof. Without loss of generalities, we assume that ∂K is of class $\mathscr{C}^1$. Set $\delta = 3^{-1/4}\epsilon^{1/2}$. Consider the δ-net S built in Lemma 4.1.5. We have that $|S| \leq c_2 V_{d-1}(K + B^d)\, \delta^{-(d-1)} = c_3 V_{d-1}(K + B^d)\, \epsilon^{-(d-1)/2}$. Construct the circumscribed polytope $P \supset C$ with one facet tangent to C at each point of S. Elementary geometric estimation gives that $d_H(C, P) < 3^{1/2}\delta^2 = \epsilon$. □

Set $c_4 := c_3^{2/(d-1)} = \mathbf{\Theta}(d)$. From the previous lemma, we can now prove easily the following theorem.

Theorem 4.1.7. *There exist absolute constants* $\boldsymbol{c}_1$ *and* $\boldsymbol{c}_2$*, independent of* d*, such that the following holds. Let* K *be a convex body. Then, for any integer* $n > \boldsymbol{c}_1^d d^{d/2} V_{d-1}(K + B^d)$*, there exists a polytope* $P \supset K$ *with* n *facets such that*

$$d_H(K, P) < \boldsymbol{c}_2 d V_{d-1}(K + B^d)^{2/(d-1)} n^{-2/(d-1)}.$$

Proof. Let $n > c_3 V_{d-1}(K + B^d)$.
Set $\epsilon = c_4 V_{d-1}(K + B^d)^{2/(d-1)} n^{-2/(d-1)}$. By the assumption made on n, we have $\epsilon < 1$. Hence, we can apply Lemma 4.1.6. There exists a polytope $P_\epsilon \supset K$ with $d_H(K, P_\epsilon) < \epsilon$ and such that its number of facets is at most

$$c_3 V_{d-1}(K + B^d)\, \epsilon^{-(d-1)/2} = n.$$

The approximations of the constants c_i using the Landau notation tells us that there exist absolute constants $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ such that $c_3 < \boldsymbol{c}_1^d d^{d/2}$ and $c_4 < \boldsymbol{c}_2 d$ for any d. This yields the proof. □

Theorem 4.1.7 will be a key ingredient for the proof of Theorem 4.2.1. We also use it to recover the following well known result.

Lemma 4.1.8. *There exist constants* c_5 *and* c_6*, depending on* d*, such that the following holds. For any integer* $n > c_5$ *and any* $K \in \mathcal{K}$*, there exists a polytope* $P \supset K$ *with* n *facets such that*

$$d_H(K, P) < c_6 V_1(K) n^{-\frac{2}{d-1}}.$$

Note that the dependence on d of the constants is not explicit in Lemma 4.1.8. There are two reasons for that. First, it is know that the result holds with c_6 independent from the dimension, see e.g. [RSW01, Cor. 2.6], and our proof gives a weaker result. Second, we will ignore the dependence on d for the remaining of the manuscript.

Proof. Apply Theorem 4.1.7 to $K' = V_1(K)^{-1}K$. It shows that there exists a polytope P such that

$$d_H(K', P) < \boldsymbol{c}_2 d V_{d-1}(K' + B^d)^{2/(d-1)} n^{-2/(d-1)}, \tag{4.1}$$

if $n > \boldsymbol{c}_1^d d^{d/2} V_{d-1}(K' + B^d)$. By a compactness argument, we have that

$$c_7 := \max\left\{V_{d-1}(K' + B^d)^{2/(d-1)} : K' \in \mathcal{K}, V_1(K') = 1\right\} < \infty.$$

Indeed, since V_{d-1} is translation invariant, we can take the maximum above over sets K' with center of mass at the origin and first intrinsic volume equal to 1. Because of the Blaschke Selection Theorem 2.1.1 the set of such sets is compact. Hence $c_7 < \infty$, since the map $K' \mapsto \boldsymbol{c}_2 d V_{d-1}(K' + B^d)^{2/(d-1)}$ is continuous.

Observing that $d_H(K, V_1(K)P) = d_H(K', P)V_1(K)$, equation (4.1) implies the lemma with $c_5 := \boldsymbol{c}_1^d d^{d/2} c_7$ and $c_6 := \boldsymbol{c}_2 d c_7$. □

4.2 Polytopal approximation of elongated convex bodies

Let $1 \leq i < j \leq d$ and $K \in \mathcal{K}$. Recall that the isoperimetric inequality (2.1) says that the (i,j)-isoperimetric ratio $V_j(K)^{1/j} V_i(K)^{-1/i}$ of K is maximized when K is a ball. On the other hand, $V_j(K)^{1/j} V_i(K)^{-1/i} \simeq 0$ precisely when the normalized body $V_i(K)^{-1/i}K$ is close to a $(j-1)$-dimensional convex body. If an isoperimetric ratio of K is close to zero, we say that K is *elongated*. The following theorem gives a bound for the Hausdorff distance between a convex body K and its best approximating polytope. This bound can be arbitrarily small if K is sufficiently elongated.

Theorem 4.2.1. *Assume* $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. *Set* $\alpha = 2\lceil (d-1)/2 \rceil (d-1)d^{-1}$ *and* $\beta = \lceil (d-1)/2 \rceil (d-1)^{-1} d^{-1}$. *There exist constants* $\delta_{i,j}$ *and* $n_{i,j}$, *both depending on d, such that the following holds. For any* $\epsilon > 0$ *and any convex body* K

$$\text{if } \frac{V_j(K)^{1/j}}{V_i(K)^{1/i}} < \epsilon \text{ then } d_H(K, P) < \delta_{i,j} \epsilon^{\beta} \frac{V_1(K)}{n^{2/(d-1)}} \text{ for any } n \geq n_{i,j} \epsilon^{-\alpha},$$

where $P = P_{K,n} \supset K$ *is a circumscribed polytope, with at most n facets, minimizing the Hausdorff distance* $d_H(K, P)$.

This theorem will be prove in Subsection 4.2.2. We will need a shape factor which we will describe in Subsection 4.2.1.

4.2.1 Shape factor

In this subsection we define $\mathfrak{g}_l$, a *shape factor*, i.e. a scale and translation invariant function on $\mathcal{K}$. Lemma 4.2.3 tells us how $\mathfrak{g}_l(K)$ describes the elongation of a given convex body K.

Set $c_8 := c_3 V_{d-1}(B^d)$, where c_3 is the constant defined before Lemma 4.1.6.

Definition 4.2.2. For any fixed parameter $l > c_8$ we define the functions $\mathfrak{b}_l, \mathfrak{f}_l, \mathfrak{g}_l : \mathcal{K} \to (0, \infty)$ by

$$\mathfrak{b}_l(K) = \sup\{t \in (0,\infty) \mid l > c_3 V_{d-1}(tK + B^d)\},$$

$$\mathfrak{f}_l(K) = \inf_{t \in (0, \mathfrak{b}_l(K))} \frac{V_{d-1}(tK + B^d)^{2/(d-1)}}{t},$$

and

$$\mathfrak{g}_l(K) = \frac{\mathfrak{f}_l(K)}{V_1(K)}.$$

It is clear that the three functions are translation invariant. One can check that $\mathfrak{b}_l$ is homogeneous of degree -1, $\mathfrak{f}_l$ is homogeneous of degree 1 and $\mathfrak{g}_l$ is homogeneous of degree 0. Therefore, for any fixed l, $\mathfrak{g}_l$ is a shape factor. The next lemma gives a geometric interpretation of $\mathfrak{g}_l$.

Lemma 4.2.3.

1. *For any $K \in \mathcal{K}$, the function $l \mapsto \mathfrak{g}_l(K)$ is decreasing.*

2. *If $d = 2$ and $l > c_8$ is fixed, then $\mathfrak{g}_l$ is constant on $\mathcal{K}$.*

3. *If $d \geq 3$, $l > c_8$ is fixed, and $K \in \mathcal{K}$ is neither an interval nor a ball, then*
$$\mathfrak{g}_l(I) < \mathfrak{g}_l(K) < \mathfrak{g}_l(B) \text{ for any } l > c_8,$$
where I denotes an interval and B a ball.

4. *Assume that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. There exist constants $\delta_{i,j}$ and $n_{i,j}$, both depending on d, such that the following holds. For any convex body $K \in \mathcal{K}$ and $\epsilon > 0$, we have*
$$\text{if } \frac{V_j(K)^{1/j}}{V_i(K)^{1/i}} < \epsilon \text{ then } \mathfrak{g}_{N_{i,j}(\epsilon)}(K) \leq \delta_{i,j}\epsilon^{\beta}, \tag{4.2}$$
where $N_{i,j}(\epsilon) := n_{i,j}\epsilon^{-\alpha}$ with $\alpha = 2\lceil (d-1)/2 \rceil (d-1)d^{-1}$, and $\beta = 2\lceil (d-1)/2 \rceil (d-1)^{-1} d^{-1}$.

Proof. (1) is a direct consequence of the definition of $\mathfrak{g}_l$. (2) comes from the fact that in this case $V_{d-1} = V_1$ is additive. (3) is implied by (2.3). It only remains to prove (4).

For the rest of the proof we write $v_i := V_i(B^d)^{1/i}$ for $i = 1, \ldots, d$. Thanks to point 3 of the present lemma, we have that $\mathfrak{g}_{N_{i,j}(\epsilon)}(K) \leq \mathfrak{g}_{N_{i,j}(\epsilon)}(B)$. This implies that, without loss of generality, we can assume that $\epsilon < c$, for $c > 0$ as small as one need. We also reduce the proof to the case $i = 1$ and $j = j_0 = \lceil (d-1)/2 \rceil$. Because of the isoperimetric inequality (2.1), we have

$$\frac{V_{j_0}(K)^{1/j_0}}{V_1(K)} \leq c_{i,j} \frac{V_j(K)^{1/j}}{V_i(K)^{1/i}} \text{ where } c_{i,j} := \frac{v_{j_0} v_i}{v_j v_1}. \tag{4.3}$$

Assume that there exist constants δ_{1,j_0} and n_{1,j_0} such that (4.2) holds for $i = 1$ and $j = j_0$. Let $1 \le i < j \le j_0$ and $(i,j) \neq (1, j_0)$. We set $\delta_{i,j} := \delta_{1,j_0} c_{i,j}^{\beta}$ and $n_{i,j} := n_{1,j_0} c_{i,j}^{-\alpha}$. In particular, $N_{i,j}(\epsilon) = n_{i,j}\epsilon^{-\alpha} = n_{1,j_0}(c_{i,j}\epsilon)^{-\alpha} = N_{1,j_0}(c_{i,j}\epsilon)$. Assume that K is such that $V_j(K)^{1/j} V_i(K)^{-1/i} < \epsilon$. By (4.3) we have $V_{j_0}(K)^{1/j_0} V_1(K) < c_{i,j}\epsilon$. This implies that $\mathfrak{g}_{N_{i,j}(\epsilon)}(K) = \mathfrak{g}_{N_{1,j_0}(c_{i,j}\epsilon)}(K) \le \delta_{1,j_0}(c_{i,j}\epsilon)^{\beta} = \delta_{i,j}\epsilon^{\beta}$. This shows that we only have to consider the case $i = 1$ and $j = j_0$.

Since both parts of (4.2) are scale invariant, we also assume without loss of generalities that $V_1(K) = 1$. Let $\epsilon \in (0,1)$ and $l > c_8$. From now on, we assume that

$$V_{j_0}(K)^{1/j_0} < \epsilon.$$

Set

$$p_C(t) := V_{d-1}(tK + B^d) \overset{(2.2)}{=} \sum_{k=0}^{d-1} \frac{(d-k)\kappa_{d-k}}{2d} V_k(K) t^k.$$

Observe that it is a strictly increasing and continuous function and that

$$\mathfrak{g}_l(K) = \mathfrak{f}_l(K) = \left(\inf_{t \in (0, \mathfrak{b}_l(K))} t^{-(d-1)/2} p_C(t) \right)^{2/(d-1)} \tag{4.4}$$

and $\mathfrak{b}_l(K) = p_C^{-1}(c_3^{-1} l)$.

Observe that $j_0 - 1 - (d-1)/2 \le -1/2$. Hence, for $t > 1$,

$$t^{-(d-1)/2} p_C(t) \le S_1(K) t^{-1/2} + S_2(K) t^{(d-1)/2},$$

where

$$S_1(K) := \sum_{k=0}^{j_0-1} \frac{(d-k)\kappa_{d-k}}{2d} V_k(K) \text{ and } S_2(K) := \sum_{k=j_0}^{d-1} \frac{(d-k)\kappa_{d-k}}{2d} V_k(K).$$

The isoperimetric inequalities (2.1) gives that

$$S_1(K) \le \frac{\kappa_d}{2} + \sum_{k=1}^{j_0-1} \frac{(d-k)\kappa_{d-k}}{2d} \left(\frac{v_k}{v_1}\right)^k =: c_9.$$

It also implies that, for $k = j_0, \dots, d-1$, we have $V_k(K) \le (v_k/v_{j_0})^k V_{j_0}(K)^{k/j_0}$. And since $V_{j_0}(K)^{k/j_0} < \epsilon^k \le \epsilon^{j_0}$, it follows that

$$S_2(K) \le \sum_{k=j_0}^{d-1} \frac{(d-k)\kappa_{d-k}}{2d} \left(\frac{v_k}{v_{j_0}}\right)^k \epsilon^{j_0} =: c_{10}\epsilon^{j_0}.$$

Therefore, for $t > 1$,

$$t^{-(d-1)/2} p_C(t) \le c_9 t^{-1/2} + c_{10}\epsilon^{j_0} t^{(d-1)/2} =: q_\epsilon(t). \tag{4.5}$$

Since we want $t^{-(d-1)/2}p_C(t)$ small, we define $t_\epsilon > 0$ such that $q_\epsilon(t_\epsilon)$ is minimal. But it holds that the derivative of q_ϵ is

$$q_\epsilon'(t) = \frac{-c_9}{2}t^{-3/2} + \frac{c_{10}\epsilon^{j_0}(d-1)}{2}t^{(d-3)/2}.$$

Thus,

$$t_\epsilon = \left(\frac{c_{10}\epsilon^{j_0}(d-1)}{c_9}\right)^{-2/d} = c_{11}\epsilon^{-2j_0/d}$$

with $c_{11} := (c_{10}(d-1)/c_9)^{-2/d}$. Now, we observe that

$$t_\epsilon^{-(d-1)/2}p_C(t_\epsilon) \overset{(4.5)}{\leq} q_\epsilon(t_\epsilon) = c_9(c_{11}\epsilon^{-2j_0/d})^{-1/2} + c_{10}\epsilon^{j_0}(c_{11}\epsilon^{-2j_0/d})^{(d-1)/2} = c_{12}\epsilon^{j_0/d}$$

with $c_{12} := c_9c_{11}^{-1/2} + c_{10}c_{11}^{(d-1)/2}$. This implies that if $\mathfrak{b}_{N_{1,j_0}(\epsilon)}(K) > t_\epsilon$ then

$$\mathfrak{g}_{N_{1,j_0}(\epsilon)}(K) \overset{(4.4)}{\leq} \left(t_\epsilon^{-(d-1)/2}p_C(t_\epsilon)\right)^{2/(d-1)} \leq \left(c_{12}\epsilon^{j_0/d}\right)^{2/(d-1)} \leq \delta_{1,j_0}\epsilon^\beta$$

with $\delta_{1,j_0} := c_{12}^{2/(d-1)}$ and $\beta := 2j_0(d-1)^{-1}d^{-1}$.

It remains only to set $N_{1,j_0}(\epsilon)$ such that $\mathfrak{b}_{N_{1,j_0}(\epsilon)}(K) > t_\epsilon$. Set

$$c_{13} := \frac{\kappa_d}{2} + \sum_{k=1}^{d-1}\frac{(d-k)\kappa_{d-k}}{2d}\left(\frac{v_k}{v_1}\right)^k \quad \text{and } \tilde{p}(t) := c_{13}t^{d-1}.$$

Again because of the isoperimetric inequality, we have that $p_C(t) < \tilde{p}(t)$, for any $t > 1$. Hence if $u > \tilde{p}(1) = c_{13}$ then $p_C^{-1}(u) > \tilde{p}^{-1}(u)$. Set

$$N_{1,j_0}(\epsilon) := c_3c_{13}t_\epsilon^{d-1} = n_{1,j_0}\epsilon^{-\alpha}$$

with $n_{1,j_0} := c_3c_{13}c_{11}^{d-1}$ and $\alpha := 2j_0(d-1)d^{-1}$. Thus we have

$$\mathfrak{b}_{N_{1,j_0}(\epsilon)}(K) = p_C^{-1}(c_3^{-1}N_{1,j_0}(\epsilon)) = p_C^{-1}(c_{13}t_\epsilon^{d-1}) > \tilde{p}^{-1}(c_{13}t_\epsilon^{d-1}) = t_\epsilon$$

whenever $t_\epsilon > 1$. But $t_\epsilon > 1$ when $\epsilon < c_{11}^{-1/\alpha}$. This completes the proof. □

4.2.2 Proof of Theorem 4.2.1

Theorem 4.2.1 is a direct consequence of the following lemma and point 4 of Lemma 4.2.3. Let $c_{14} > c_4$.

Lemma 4.2.4. *Let $K \in \mathcal{K}$. For any $n > c_8$, there exists a polytope $P \supset K$ with n facets such that*

$$d_H(K,P) < c_{14}\mathfrak{g}_n(K)\frac{V_1(K)}{n^{2/(d-1)}}.$$

Proof. The condition $n > c_8$ implies that $\mathfrak{b}_n(K)$ and $\mathfrak{g}_n(K)$ are well defined. Let $t \in (0, \mathfrak{b}_n(K))$. We have defined $\mathfrak{b}_n(K)$ precisely such that the convex body tK and the number n satisfy the conditions required to apply Theorem 4.1.7. So there exists a polytope P_t with n facets such that

$$d_H(tK, P_t) < c_4 V_{d-1}(tC + B^d)^{2/(d-1)} n^{-2/(d-1)}.$$

Therefore, for any $t \in (0, \mathfrak{b}_n(K))$, we see that

$$d_H\left(K, \frac{1}{t}P_t\right) < c_4 \frac{V_{d-1}(tK + B^d)^{2/(d-1)}}{t} n^{-2/(d-1)}.$$

Since $c_{14} > c_4$, there exists $t_0 \in (0, \mathfrak{b}_n)$ such that

$$d_H\left(K, \frac{1}{t_0}P_{t_0}\right) < c_{14}\left(\inf_{t\in(0,\mathfrak{b}_n)} \frac{V_{d-1}(tK + B^d)^{2/(d-1)}}{t}\right) n^{-2/(d-1)}.$$

But it holds that

$$\inf_{t\in(0,\mathfrak{b}_n)} \frac{V_{d-1}(tK + B^d)^{2/(d-1)}}{t} = \mathfrak{f}_n(K) = \mathfrak{g}_n(K)V_1(K),$$

which yields the proof. □

4.3 Deleting facets of polytopes

The starting point of this subsection is [RSW01] from Reisner, Schütt and Werner. Our goal is to show that if a polytope P has many facets, then a good proportion of them has only a tiny influence. This will be a key ingredient to obtain a recurrence relation between the probabilities $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n)$ and $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n-1)$ in Theorem 5.1.1 (resp. $\mathbb{P}(f(Z_{\text{typ}}) = n)$ and $\mathbb{P}(f(Z_{\text{typ}}) = n-1)$ in Theorem 5.2.1). More precisely, for $I \subset \mathbb{N}$ and a set of halfspaces $H_i^{\epsilon_i}$, $i \in I$, we define

$$P_I := \cap_{i\in I} H_i^{\epsilon_i}.$$

Throughout the manuscript we use the notation

$$[n] = \{1, \dots, n\}.$$

For $j \leq n$ we have $P_{[n]} \subset P_{[n]\setminus\{j\}}$. We will measure the distance between $P_{[n]}$ and $P_{[n]\setminus\{i\}}$, both with the Hausdorff distance and the ratio $V_1\left(P_{[n]\setminus\{i\}}\right)/V_1\left(P_{[n]}\right) > 1$. We will show in the crucial Lemma 4.3.2 that for a subset $J \subset [n]$ of size at least $n/4$ we have good upper bounds of the distances between $P_{[n]}$ and $P_{[n]\setminus\{j\}}$ for $j \in J$.

In the first lemma we approximate a polytope $P = \cap_{i=1}^n H_i^-$ by the intersection P_I of suitable supporting halfspaces of P. Its proof is similar to the proof of Lemma 4.3 in [RSW01].

Lemma 4.3.1. *There exist constants c_{15} and $c_{16} > 0$, such that the following holds. For any integer $k > c_{15}$ and any simple polytope P with n facets, there exists a subset $I \subset [n]$ with $|I| \leq k$ such that*

$$d_H(P, P_I) < c_{16} c_\Phi \Phi(P)^{\frac{1}{r}} k^{-\frac{2}{d-1}}.$$

Proof. We set $c_{15} := d(c_5 + 1)$ and $c_{16} := c_6 \left(d(c_5+1)c_5^{-1}\right)^{2/(d-1)}$, where c_5 and c_6 are the constants of Lemma 4.1.8. We apply Lemma 4.1.8 to P and $m = \lfloor k/d \rfloor > c_5$. We obtain a polytope $Q \supset P$ with $\lfloor k/d \rfloor$ facets and

$$d_H(P, Q) < c_6 V_1(K) \left\lfloor \frac{k}{d} \right\rfloor^{-\frac{2}{d-1}} < c_{16} c_\Phi \Phi(K)^{\frac{1}{r}} k^{-\frac{2}{d-1}},$$

where the second inequality is a direct consequence of the definition of c_Φ, see (3.1), and the simple following computation

$$\frac{k}{\left\lfloor \frac{k}{d} \right\rfloor} < \frac{k}{k-d} d < \frac{c_{15}}{c_{15} - d} d = \frac{c_5 + 1}{c_5} d.$$

By eventually shifting and rotating the facets of Q slightly, we can assume that each of the facets of Q meets exactly one vertex of P in its interior. Let I be the set of indices of facets of P with one vertex in a facet of Q. Since P is simple, we have

$$|I| \leq d\, f(Q) = d \left\lfloor \frac{k}{d} \right\rfloor \leq k.$$

Finally, we observe that $P \subset P_I \subset Q$, which implies $d_H(P, P_I) \leq d_H(P, Q)$. □

The crucial step is to prove that also the Φ-content of P and $P_{[n]\setminus\{j\}}$ are almost the same for $j \in I$.

Lemma 4.3.2. *Assume that $r \geq 1$. There exist constants c_{17}, C_1 and C_2 such that the following holds. For any $n > c_{17}$ and any simple polytope $P = \cap_{i=1}^n H_i^- \in \mathcal{P}_{n,\boldsymbol{o}}$, there exists a subset $J \subset [n]$ of cardinality at least $n/4$ such that for any $j \in J$ we have*

$$d_H\left(P, P_{[n]\setminus\{j\}}\right) < C_1 \Phi(P)^{\frac{1}{r}} n^{-\frac{2}{d-1}}, \tag{4.6}$$

and

$$\Phi\left(P_{[n]\setminus\{j\}}\right) < \exp\left\{C_2 n^{-\frac{d+1}{d-1}}\right\} \Phi(P). \tag{4.7}$$

Moreover c_{17} depends only on d, and $C_1 = c_{18} c_\Phi$ and $C_2 = c_{19}^r c_\Phi^r$, with c_{18} and c_{19} depending only on d.

Note that in the stationary case the lemma extends to any simple polytope $P \in \mathcal{P}_n$, not only the ones containing the origin.

Proof. The first part is just a suitable reformulation of Lemma 4.3.1. Set $c'_{17} := 2c_{15}+4$, where c_{15} is constant of Lemma 4.3.1, and put $k = n-2\lceil n/4\rceil$ which implies $k \geq c_{15}$ when $n > c'_{17}$. Note that we will set the value of $c_{17} \geq c'_{17}$ later in the proof. By Lemma 4.3.1 there is a set $I \subset [n]$ of cardinality k such that

$$d_H(P, P_I) < c_{16} c_\Phi \Phi(P) k^{-\frac{2}{d-1}} \leq (4d)^{-1} c_{18} c_\Phi \Phi(P)^{\frac{1}{r}} n^{-\frac{2}{d-1}},$$

where $c_{18} := 4dc_{16}\left(\max_{n>c_{17}}(n/(n-2\lceil n/4\rceil))^{2/(d-1)}\right)$. Hence for any $j \notin I$,

$$d_H\left(P, P_{[n]\setminus\{j\}}\right) \leq d_H\left(P, P_I\right) < (4d)^{-1} c_{18} c_\Phi \Phi(P)^{\frac{1}{r}} n^{-\frac{2}{d-1}}$$

which gives (4.6). It remains to show that, for at least half of the j not in I, equation (4.7) holds as well. Set

$$\delta' := (4d)^{-1} c_{18} n^{-\frac{2}{d-1}}$$

and

$$U_j = \operatorname{cl}\left\{\boldsymbol{u} \in \mathbb{S}^{d-1} : h\left(P_{[n]\setminus\{j\}}, \boldsymbol{u}\right) \neq h(P, \boldsymbol{u})\right\}.$$

Also, set $c_{17} = \max\left\{c'_{17}, \left((4d)^{-1}c_{18}\right)^{(d-1)/2}\right\}$, so we have $c_{17} \geq c'_{17}$ and $\delta' < 1$ for $n > c_{17}$. Applying Lemma 3.1.1 with $K = P$, $L = P_{[n]\setminus\{j\}}$, $\delta = \delta' c_\Phi \Phi(P)^{\frac{1}{r}}$ and $U = U_j$, gives

$$\Phi\left(P_{[n]\setminus\{j\}}\right) - \Phi(P) < \delta'(c_h+1)^{r-1} c_\Phi^r \Phi(P) \varphi\left(U_j\right). \tag{4.8}$$

We need to estimate the φ-measure of the set U_j. Denote by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ the vertices of the polytope P. Since the polytope is simple, each vertex is the intersection of precisely d hyperplanes. Denote by $N(\boldsymbol{v}_l)$ the unit vectors in the normal cone of P at $\boldsymbol{v}_l$, i.e.

$$N(\boldsymbol{v}_l) = \left\{\boldsymbol{u} \in \mathbb{S}^{d-1} : h(P, \boldsymbol{u}) = \boldsymbol{v}_l \cdot \boldsymbol{u}\right\}.$$

The essential observation is that

$$U_j = \bigcup_{\boldsymbol{v}_l \in H_j} N(\boldsymbol{v}_l).$$

Observe that the sets $N(\boldsymbol{v}_l)$ have pairwise disjoint interiors and cover $\mathbb{S}^{d-1}$. Thus for almost all $\boldsymbol{u} \in \mathbb{S}^{d-1}$ we have

$$\begin{aligned}\sum_{j=1}^n \mathbb{1}(\boldsymbol{u} \in U_j) &= \sum_{j=1}^n \sum_{l=1}^m \mathbb{1}(\boldsymbol{v}_l \in H_j)\mathbb{1}(\boldsymbol{u} \in N(\boldsymbol{v}_l)) \\ &= \underbrace{\sum_{l=1}^m \mathbb{1}(\boldsymbol{u} \in N(\boldsymbol{v}_l))}_{=1} \underbrace{\sum_{j=1}^n \mathbb{1}(\boldsymbol{v}_l \in H_j)}_{=d} = d.\end{aligned}$$

This yields $\sum_{j=1}^{n} \varphi(U_j) = d$ and in particular

$$\sum_{j \notin I} \varphi(U_j) \leq d.$$

This implies that, for at least half of the $j \notin I$, we have

$$\varphi(U_j) \leq d \left(\frac{n-k}{2} \right)^{-1} = d \left\lceil \frac{n}{4} \right\rceil^{-1} \leq 4dn^{-1}.$$

Otherwise we would have at least half of the $j \notin I$ with the reverse inequality and, because $|I| = k = n - 2\lceil n/4 \rceil$, that would imply

$$d \geq \sum_{j \notin I}^{n} \varphi(U_j) > \frac{1}{2}(n-k)\frac{2d}{n-k} = d.$$

Combined with equation (4.8), it shows that there exists a set $J \subset [n] \setminus I$ of cardinality $(n-k)/2 = \lceil n/4 \rceil$ such that, for any $j \in J$, we have

$$\begin{aligned} \Phi\left(P_{[n]\setminus\{j\}}\right) - \Phi(P) &< \delta'(c_h+1)^{r-1} c_\Phi^r \Phi(P) 4dn^{-1} \\ &= c_{18}(c_h+1)^{r-1} c_\Phi^r \Phi(P) n^{-\frac{d+1}{d-1}} \\ &\leq c_{19}^r c_\Phi^r \Phi(P) n^{-\frac{d+1}{d-1}}, \end{aligned}$$

where $c_{19} = \max(c_{18}, 1 + c_h)$. This implies equation (4.7). □

4.4 Deleting facets of elongated polytopes

When a polytope is sufficiently elongated the approximation results of the previous section can be improved. This is a consequence of Theorem 4.2.1.

Lemma 4.4.1. *Assume that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. There exist positive constants C_3 and C_4, both depending on i, j and d, such that the following holds. For any $\epsilon > 0$, any integer $k \geq \lfloor C_3 \epsilon^{-(d-2)} \rfloor$ and any simple polytope $P = \cap_{i=1}^{n} H_i^{\epsilon_i} \in \mathcal{P}_n$ with $n \geq k$ facets and $V_j(P)^{1/j} V_i(P)^{-1/i} < \epsilon$, there exists a subset $J \subset [n]$ with $|J| \leq k$, such that*

$$d_H(P, P_J) < C_4 \epsilon^{\frac{1}{2d}} V_1(P) k^{-\frac{2}{d-1}}.$$

Proof. Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. Because of Theorem 4.2.1, there exist constants $c_{i,j}$ and $n_{i,j}$ (both depending on d), such that the following holds. For any $\epsilon > 0$, any $m \geq n_{i,j} \epsilon^{-(d-2)}$, and any convex body K with

$$\frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} < \epsilon,$$

there exists a polytope $Q \supset K$ with at most m facets satisfying

$$d_H(K,Q) < c_{i,j}\, \epsilon^{\frac{1}{2d}}\, V_1(K)\, m^{-\frac{2}{d-1}}.$$

Assume that P is a simple polytope with isoperimetric ratio $V_j(P)^{1/j}V_i(P)^{-1/i} < \epsilon$ and $f(P) = n > k \geq dm$ facets with $m = \lfloor k/d \rfloor > n_{i,j}\epsilon^{-(d-2)}$. Then there exists a polytope $Q \supset P$ with $m+1$ facets and

$$d_H(P,Q) < c_{i,j}\epsilon^{\frac{1}{2d}}\, V_1(P)\,(m+1)^{-\frac{2}{d-1}} < d^{\frac{2}{d-1}} c_{i,j}\epsilon^{\frac{1}{2d}}\, V_1(P)\, k^{-\frac{2}{d-1}}.$$

We can assume that each of the facets of Q meets exactly one vertex of P in its interior. Let J be the set of indices of facets of P with one vertex in a facet of Q. Since P is simple, we have

$$|J| \leq d\, f(Q) \leq k.$$

And $P_J \subset Q$ implies

$$d_H(P, P_J) \leq d_H(P,Q).$$

□

In the following lemma we prove the uniform continuity of the isoperimetric ratio. To our surprise we could not find any results in this direction, this seems to be an open problem. We state the partial solution to this problem which we need for our purposes.

Lemma 4.4.2. *Let $1 \leq i < j \leq d$. There exists a constant c_{20} such that for any $\delta \in (0,1)$ and for any $K, L \in \mathcal{K}$ with $K \subset L$ and $d_H(K,L) < \delta V_1(K)$, we have*

$$\frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} < \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{20}\delta^{\frac{j-i}{ij(j-1)}}.$$

Proof. A first easy bound is obtained using $V_i(L)^{j-1} \geq c_{ij} V_1(L)^{j-i} V_j(L)^{i-1}$ which is a consequence of the Alexandrov-Fenchel inequality, see [Sch14], p.401, (7.66) therein.

$$\frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} \leq c_{21}\left(\frac{V_i(L)^{\frac{1}{i}}}{V_1(L)}\right)^{\frac{j-i}{j(i-1)}} < \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{21}\left(\frac{V_i(L)^{\frac{1}{i}}}{V_1(L)}\right)^{\frac{j-i}{j(i-1)}} \tag{4.9}$$

A more precise bound uses Steiner's formula. Due to the isoperimetric inequality (2.1), $V_i(K)^{1/i} \leq c_{1,i} V_1(K)$ with $c_{1,i} := V_i(B^d)^{1/i} V_1(B^d)$. Since

$$d_H(K,L) < \delta V_1(K),$$

we have that $L \subset K + \delta V_1(K) B^d$. The monotonicity of the intrinsic volumes and Steiner's formula shows for $\delta \leq 1$

$$\begin{aligned} V_j(L) &< V_j\left(K + \delta V_1(K) B^d\right) \\ &\leq V_j(K) + \sum_{i=1}^{j} \binom{d-j+i}{i} \frac{\kappa_{d-j+i}}{\kappa_{d-j}} c_{1,j-i}^{j-i} V_1(K)^{j-i} (\delta V_1(K))^i \\ &\leq V_j(K) + \delta V_1(K)^j \sum_{i=1}^{j} \binom{d-j+i}{i} \frac{\kappa_{d-j+i}}{\kappa_{d-j}} c_{1,j-i}^{j-i} \\ &\leq V_j(K) + c_{22}\ \delta V_1(L)^j. \end{aligned}$$

Because $a + b \leq \left(a^{\frac{1}{j}} + b^{\frac{1}{j}}\right)^j$, for $a, b > 0$, and because of the monotonicity of the intrinsic volumes this yields

$$\frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} \leq \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{22}^{\frac{1}{j}} \delta^{\frac{1}{j}} \frac{V_1(L)}{V_i(L)^{\frac{1}{i}}}. \tag{4.10}$$

Note that $\min\left\{x, x^{-(j-i)/(j(i-1))}\right\} \leq 1$ for all $x > 0$. We define $c_{20} = \max\{c_{21}, c_{22}^{1/j}\}$ and combine (4.9) and (4.10).

$$\begin{aligned} &\frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} \\ &\leq \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + \delta^{\frac{j-i}{ij(j-1)}} \min\left\{ c_{22}^{\frac{1}{j}} \delta^{\frac{i-1}{i(j-1)}} \frac{V_1(L)}{V_i(L)^{\frac{1}{i}}},\ c_{21} \left(\delta^{\frac{i-1}{i(j-1)}} \frac{V_1(L)}{V_i(L)^{\frac{1}{i}}}\right)^{-\frac{j-i}{j(i-1)}} \right\} \\ &\leq \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{20} \delta^{\frac{j-i}{ij(j-1)}}. \end{aligned}$$

□

Recall that we use the notation $P_I = \cap_{i \in I} H_i^{\epsilon_i}$, for any set of integers I. For integers $k \leq n$ and a permutation $\sigma \in \mathfrak{S}_n$ we write $\sigma[k] = \{\sigma(i) : i \in [k]\}$. In particular $P_{\sigma[k]} = \cap_{i \in I} H_{\sigma(i)}^{\epsilon_{\sigma(i)}}$. We call hyperplanes H_i in generic position, if the intersection of any $d+2$ of them is empty. The constants C_3 and C_4 have been defined in Lemma 4.4.1.

Lemma 4.4.3. *Assume that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$ and let C_3 and C_4 be the constants, depending on i, j and d, of Lemma 4.4.1. There is a constant C_5, depending on i, j, and d, such that for any $\epsilon < C_3^{2/(d-1)} C_4^{-1} c_\Phi^{-1}$ the following holds. For any polytope $P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}$ with $n > m = \lfloor C_3 \epsilon^{-(d-2)} \rfloor$ facets in generic position and $V_j(P_{[n]})^{1/j} V_i(P_{[n]})^{-1/i} < \epsilon$ there exist at least $2^{-n}(n-2m)!$ permutations $\sigma \in \mathfrak{S}_n$ such that*

(1) $d_H\left(P_{\sigma[k]}, P_{\sigma[k-1]}\right) < C_5 c_\Phi \epsilon^{\frac{1}{2d^4}} \Phi\left(P_{\sigma[m]}\right)^{\frac{1}{r}} k^{-\frac{2}{d-1}}$ *for all* $k \in [n] \setminus [2m]$,

(2) $\left\|\mathfrak{c}\left(P_{\sigma[n]}\right) - \mathfrak{c}\left(P_{\sigma[m]}\right)\right\| < \Phi\left(P_{\sigma[n]}\right)^{\frac{1}{r}}$,

(3) $\Phi\left(P_{\sigma[m]}\right) < \left(1 + (1 + c_\Phi)^{r-1}\right) \Phi\left(P_{\sigma[n]}\right)$.

Note that, similarly as in Lemma 4.3.2, in the stationary case, the theorem extends to any polytope $P \in \mathcal{P}_n$, not only the ones containing the origin.

Proof. We set

$$m = \left\lfloor C_3 \epsilon^{-(d-2)} \right\rfloor.$$

By Lemma 4.4.1 there exists a subset $I \subset [n]$ with $|I| = m$, such that for all subsets J with $I \subset J \subset [n]$ we have

$$d_H(P_{[n]}, P_J) < C_4 \epsilon^{\frac{1}{2d}} V_1(P_{[n]}) m^{-\frac{2}{d-1}} < C_3^{-\frac{2}{d-1}} C_4 \epsilon V_1(P_{[n]}).$$

By Lemma 4.4.2 this implies for all such sets J that

$$\frac{V_j(P_J)^{\frac{1}{j}}}{V_i(P_J)^{\frac{1}{i}}} < \epsilon + c_{20}\left(C_3^{-\frac{2}{d-1}} C_4 \epsilon\right)^{\frac{j-i}{ij(j-1)}} < c_{23} \epsilon^{\frac{1}{d^3}}. \tag{4.11}$$

We denote by $S(P_n) \subset \mathfrak{S}_n$ the set of those permutations σ such that

(a) $\sigma[m] = I$, and

(b) $d_H\left(P_{\sigma[k]}, P_{\sigma[k-1]}\right) < 2^{\frac{2}{d-1}} C_4 c_{23}^{\frac{1}{2d}} \epsilon^{\frac{1}{2d^4}} V_1\left(P_{\sigma[k]}\right) k^{-\frac{2}{d-1}}$ for all $k \in [n] \setminus [2m]$.

To estimate $|S(P_n)|$ note first that there are $m!$ possibilities such that $\sigma[m] = I$. Second, assume that $\sigma(n), \dots, \sigma(k+1) \in [n] \setminus I$ are already chosen satisfying Condition (b). Then by (4.11) and by Lemma 4.4.1 applied to the polytope $P' = P_{\sigma[k]}$, the integer $k' = \frac{k}{2} \geq m$ and $\epsilon' = c_{23}\epsilon^{\frac{1}{d^3}}$, there is a set $J_k \subset \sigma[k]$ of size $|J_k| \leq k/2$ such that

$$d_H\left(P_{\sigma[k]}, P_{J_k}\right) < C_4 c_{23}^{\frac{1}{2d}} \epsilon^{\frac{1}{2d^4}} V_1\left(P_{\sigma[k]}\right) \left(\frac{k}{2}\right)^{-\frac{2}{d-1}}.$$

If we choose $\sigma(k) \notin J_k$, Condition (b) is thus satisfied. Because we need in addition $\sigma(k) \notin I$ there are at least $k/2 - m$ possibilities to choose $\sigma(k)$, and thus to determine $\sigma[k-1]$. Continuing until $k = 2m+1$ gives at least $\prod_{2m+1}^{n}(k/2 - m)$ possibilities to choose $\sigma(n), \dots, \sigma(2m+1)$. We obtain

$$|S(P_n)| \geq m! \prod_{k=2m+1}^{n} \left(\frac{k}{2} - m\right) = m! 2^{-n+2m}(n-2m)! > 2^{-n}(n-2m)!$$

Using (3.1), we observe that Condition (1) of our lemma is satisfied by choosing $C_5 = 2^{\frac{2}{d-1}} C_4 c_{23}^{\frac{1}{2d}}$ in Condition (b). Condition (2) follows from the Lipschitz continuity of $\mathfrak{c}$ and

$$d_H(P_{[n]}, P_{\sigma[m]}) < C_3^{-\frac{2}{d-1}} C_4 c_\Phi \Phi\left(P_{[n]}\right)^{\frac{1}{r}} \epsilon < \Phi\left(P_{[n]}\right)^{\frac{1}{r}},$$

since $C_3^{-\frac{2}{d-1}} C_4 c_\Phi \epsilon < 1$. To check Condition (3) we apply the second part of Lemma 3.1.1 with $K = P_{\sigma[n]} = P_{[n]}$, $L = P_{\sigma[m]}$ and $d_H(K, L) < \Phi\left(P_{[n]}\right)^{\frac{1}{r}}$,

$$\begin{aligned}\Phi\left(P_{\sigma[m]}\right) - \Phi\left(P_{[n]}\right) &< \Phi\left(P_{[n]}\right)^{\frac{1}{r}} \left(\Phi\left(P_{[n]}\right)^{\frac{1}{r}} + c_\Phi \Phi\left(P_{[n]}\right)^{\frac{1}{r}}\right)^{r-1} \\ &= \Phi\left(P_{[n]}\right)(1 + c_\Phi)^{r-1}.\end{aligned}$$

□

4.5 Integral transformation formulae

4.5.1 Integration over the simplices of $\mathbb{R}^d$

In this subsection we present with Theorem 4.5.2 a direct construction of a random polytope equivalent to Z_{typ}. It is a generalisation to the non isotropic case of Theorem 10.4.6 of [SW08], which itself generalises results from Miles, Ambartzumian, Mecke, and Calka. See the second note of Section 10.4 in [SW08] for precise references. First, we need to show an easy extension of a theorem due to Calka.

We denote by P the set of $(d+1)$-tuples of unit vectors which are not all in one closed hemisphere of $\mathbb{S}^{d-1}$, and by $\Delta_d(\bar{\boldsymbol{u}})$ the volume of the convex hull of the unit vectors $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_d$. In the following theorem the measure μ on $\mathcal{H}$ is of the form

$$\mu(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(\boldsymbol{u}, t) \in \cdot\right) \mathrm{d}t \, \mathrm{d}\varphi(\boldsymbol{u}).$$

Theorem 4.5.1. *Let $\varphi \in \aleph_{e,c}$. If $f\colon \mathcal{H}^{d+1} \to \mathbb{R}$ is a non negative measurable function, then*

$$\begin{aligned}\int_{\mathcal{H}^{d+1}} f(\boldsymbol{H}) \, \mathrm{d}\mu^{d+1}(\boldsymbol{H}) = d! \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathsf{P}} & f(H(\boldsymbol{u}_0, \langle \boldsymbol{z}, \boldsymbol{u}_0 \rangle + r), \ldots, H(\boldsymbol{u}_d, \langle \boldsymbol{z}, \boldsymbol{u}_d \rangle + r)) \\ & \times \Delta_d(\bar{\boldsymbol{u}}) \, \mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}) \, \mathrm{d}r \, \mathrm{d}\lambda_d(\boldsymbol{z}).\end{aligned}$$

Proof. When φ is the Haar measure, Theorem 4.5.1 is Theorem 7.3.2 of [SW08], which is itself taken from the doctoral thesis of Pierre Calka [Cal02] (in French).

The result extends easily to the more general case where φ is absolutely continuous. We only need to insert the density function of σ into the formula. We write $\varphi = g\sigma$, where $g\colon \mathbb{S}^{d-1} \to \mathbb{R}$ is a measurable function and σ the spherical Lebesgue measure. We have

$$\begin{aligned}
&\int_{\mathcal{H}^{d+1}} f(\boldsymbol{H})\,\mathrm{d}\mu^{d+1}(\boldsymbol{H}) \\
&= \int_{(\mathbb{S}d-1)^{d+1}} \int_{\mathbb{R}_+^{d+1}} f(H(\boldsymbol{u}_0, t_0), \ldots, H(\boldsymbol{u}_d, t_d)) g(\boldsymbol{u}_0) \cdots g(\boldsymbol{u}_d)\,\mathrm{d}\bar{t}\,\mathrm{d}\sigma^{d+1}(\bar{\boldsymbol{u}}) \\
&= d! \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathsf{P}} f(H(\boldsymbol{u}_0, \langle \boldsymbol{z}, \boldsymbol{u}_0\rangle + r), \ldots, H(\boldsymbol{u}_d, \langle \boldsymbol{z}, \boldsymbol{u}_d\rangle + r)) \\
&\qquad \times \Delta_d(\boldsymbol{u})\, g(\boldsymbol{u}_0) \cdots g(\boldsymbol{u}_d)\,\mathrm{d}\sigma^{d+1}(\boldsymbol{u})\,\mathrm{d}r\,\mathrm{d}\lambda_d(z) \\
&= d! \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathsf{P}} f(H(\boldsymbol{u}_0, \langle \boldsymbol{z}, \boldsymbol{u}_0\rangle + r), \ldots, H(\boldsymbol{u}_d, \langle \boldsymbol{z}, \boldsymbol{u}_d\rangle + r)) \\
&\qquad \times \Delta_d(\boldsymbol{u})\,\mathrm{d}\varphi^{d+1}(\boldsymbol{u})\,\mathrm{d}r\,\mathrm{d}\lambda_d(\boldsymbol{z}).
\end{aligned}$$

□

For a $(d+1)$-tuple of unit vectors $\bar{\boldsymbol{u}} \in \mathsf{P}$ we denote

$$\Delta(\bar{\boldsymbol{u}}) := \cap_{i=0}^{d} H(\boldsymbol{u}_i, 1)^-,$$

the simplex circumscribed to $\mathbb{S}^{d-1}$ and with facets with outward normal vectors $\boldsymbol{u}_i$. In the following theorem, η is a stationary Poisson hyperplane process of intensity γ and directional distribution $\varphi \in \aleph_{e,c}$, and Z_{typ} is the typical cell, with respect to the inball center, of the mosaic induced by η.

Theorem 4.5.2. *For Borel sets $A \subset \mathcal{K}$,*

$$\mathbb{P}(Z_{\text{typ}} \in A) = \frac{\gamma^{d+1}}{(d+1)\gamma^{(d)}} \int_{\mathsf{P}} \int_0^\infty e^{-\gamma r} \mathbb{P}\left(\bigcap_{H \in \eta \cap \mathcal{F}^{rB^d}} H^- \cap (r\Delta(\bar{\boldsymbol{u}})) \in A \right) \mathrm{d}r \\ \times \Delta_d(\bar{\boldsymbol{u}})\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}),$$

where $\eta \cap \mathcal{F}^{rB^d}$ denotes the set of hyperplanes of the process which do not hit the ball rB^d.

Proof. In the isotropic case Theorem 4.5.2 is Theorem 10.4.6 of [SW08] (note that the quantity $\widehat{\gamma}$ in [SW08] is half the quantity γ in the present thesis). To prove the non isotropic case, one only needs to follow precisely the same lines of the proof in [SW08], except for the transformation at the end of the proof. One should use the transformation of Theorem 4.5.1 instead of Theorem 7.3.2 in [SW08]. □

4.5.2 Integration over a d-tuple of points on a sphere

We present here a theorem due to Miles, see [Mil71a, Thm 4] where he gives a sketch of a proof. It is mentioned in the book of Schneider and Weil [SW08, note 6 page 286] as a spherical counterpart to the affine Blaschke-Petkanschin formula. We will give a proof of the formula since we did not find in the literature a complete proof of it.

In the following, we denote by $H(\boldsymbol{v},t) = \{\boldsymbol{x} \in \mathbb{R}^d \colon \langle \boldsymbol{x}, \boldsymbol{v}\rangle = t\}$ the hyperplane orthogonal to $\boldsymbol{v}$ at distance t from the origin, by $\sigma'_{\boldsymbol{v},t}$ the surface area measure on the $(d-2)$-dimensional sphere $(H(\boldsymbol{v},t) \cap \mathbb{S}^{d-1})$, and by $\Delta_{d-1}(\bar{\boldsymbol{u}})$ the $(d-1)$-dimensional volume of the convex hull of $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$.

Theorem 4.5.3 (Miles,1971)**.** *Let $d \geq 2$. If $f \colon (\mathbb{S}^{d-1})^d \to \mathbb{R}$ is a measurable function, then*

$$\int_{(\mathbb{S}^{d-1})^d} f(\bar{\boldsymbol{u}})\,\mathrm{d}\sigma^d(\bar{\boldsymbol{u}})$$

$$= (d-1)! \int_{\mathbb{S}^{d-1}} \int_0^1 \int_{(H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1})^d} f(\bar{\boldsymbol{u}})\,\Delta_{d-1}(\bar{\boldsymbol{u}})\,\mathrm{d}\sigma'_{\boldsymbol{v},t}(\bar{\boldsymbol{u}})\,\frac{\mathrm{d}t}{(1-t^2)^{\frac{d}{2}}}\,\mathrm{d}\sigma(\boldsymbol{v}).$$

Proof. As suggested in [SW08, note 6 page 286], our proof will follow the lines of the proof of Theorem 8.2.3 in [SW08]. Without loss of generalities we assume that f is continuous. Set the function $\tilde{f} \colon (\mathbb{R}^d \setminus \boldsymbol{o})^d \to \mathbb{R}$ defined by

$$\tilde{f}(\bar{\boldsymbol{x}}) = f\left(\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}, \dots, \frac{\boldsymbol{x}_d}{\|\boldsymbol{x}_d\|}\right).$$

For $0 \leq \rho < 1$ we denote $B^d_\rho := \{\boldsymbol{x} \in \mathbb{R}^d \colon \rho \leq \|x\| \leq 1\}$ and

$$J_\rho := \int_{(B^d_\rho)^d} \tilde{f}(\bar{\boldsymbol{x}})\,\mathrm{d}\lambda^d_d(\bar{\boldsymbol{x}}). \tag{4.12}$$

The affine Blaschke-Petkanschin formula [SW08, Thm. 7.2.7], applied to $\tilde{f} \cdot \mathbb{1}_{(B^d_\rho)^d}$ and with $q = d-1$, gives

$$J_\rho = (d-1)! \int_{\mathbb{S}^{d-1}} \int_0^1 \int_{(H(\boldsymbol{v},t)\cap B^d_\rho)^d} \tilde{f}(\bar{\boldsymbol{x}})\Delta_{d-1}(\bar{\boldsymbol{x}})\,\mathrm{d}\lambda^d_{H(\boldsymbol{v},t)}(\bar{\boldsymbol{x}})\,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{v}),$$

where $\lambda_{H(\boldsymbol{v},t)}$ denotes the $(d-1)$-dimensional Lebesgue measure on the hyperplane $H(\boldsymbol{v},t)$. By introducing spherical coordinate in (4.12) we also get

$$J_\rho = \left(\frac{1-\rho^d}{d}\right)^d \int_{(\mathbb{S}^{d-1})^d} f(\bar{\boldsymbol{u}})\mathrm{d}\sigma^d(\bar{\boldsymbol{u}}).$$

From the two last equations, we get

$$\int_{(\mathbb{S}^{d-1})^d} f(\bar{\boldsymbol{u}})\mathrm{d}\sigma^d(\bar{\boldsymbol{u}}) = (d-1)! \int_{\mathbb{S}^{d-1}} \int_0^1 I_\rho(\boldsymbol{v},t)\ \mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{v}),$$

with

$$I_\rho(\boldsymbol{v},t) = \left(\frac{d}{1-\rho^d}\right)^d \int_{(H(\boldsymbol{v},t)\cap B_\rho^d)^d} \tilde{f}(\bar{\boldsymbol{x}})\Delta_{d-1}(\bar{\boldsymbol{x}})\ \mathrm{d}\lambda^d_{H(\boldsymbol{v},t)}(\bar{\boldsymbol{x}}).$$

Observe that $H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1}$ is a $(d-2)$-dimensional sphere of radius $\sqrt{1-t^2}$. Using spherical coordinates in $H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1}$ and the mean value theorem of integral calculus (and denoting by a_+ the positive part of a), we get

$$\begin{aligned}
&I_\rho(\boldsymbol{v},t)\\
&= \left(\frac{d}{1-\rho^d}\right)^d \int_{(H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1})^d} \int_{\mathbb{R}^d_+} \mathbb{1}\left(\sqrt{(\rho^2-t^2)_+} \le s_i \le \sqrt{1-t^2}, \forall i\in[d]\right)\\
&\qquad \tilde{f}(t\boldsymbol{v}+\bar{s}\cdot\bar{\boldsymbol{u}})\Delta_{d-1}(t\boldsymbol{v}+\bar{s}\cdot\bar{\boldsymbol{u}})\ s_1^{d-2}\cdots s_d^{d-2}\mathrm{d}\bar{s}\,\mathrm{d}(\sigma'_{\boldsymbol{v},0})^d(\bar{\boldsymbol{u}})\\
&= \left(\frac{d}{1-\rho^d}\right)^d \left(\frac{(1-t^2)^{\frac{d-1}{2}} - (\rho^2-t^2)_+^{\frac{d-1}{2}}}{d-1}\right)^d \int_{(H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1})^d} \tilde{f}(t\boldsymbol{v}+\bar{s}\cdot\bar{\boldsymbol{u}})\\
&\qquad \Delta_{d-1}(t\boldsymbol{v}+\bar{s}\cdot\bar{\boldsymbol{u}})\ \mathrm{d}(\sigma'_{\boldsymbol{v},0})^d(\bar{\boldsymbol{u}})\\
&= \left(\frac{d}{d-1}\right)^d \left(\frac{(1-t^2)^{\frac{d-1}{2}} - (\rho^2-t^2)_+^{\frac{d-1}{2}}}{1-\rho^d}\right)^d \int_{(H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1})^d} \tilde{f}(t\boldsymbol{v}+\bar{s}\cdot\bar{\boldsymbol{u}})\\
&\qquad \Delta_{d-1}(t\boldsymbol{v}+\bar{s}\cdot\bar{\boldsymbol{u}})\ \mathrm{d}(\sigma'_{\boldsymbol{v},0})^d(\bar{\boldsymbol{u}}),
\end{aligned}$$

for a suitable choice of $\bar{s} \in \left[\sqrt{(\rho^2-t^2)_+}\,,\ \sqrt{1-t^2}\right]^d$. Here we denoted by $t\boldsymbol{v}+\bar{s}\cdot\bar{\boldsymbol{u}}$ the d-tuple of vectors $(t\boldsymbol{v}+s_1\boldsymbol{u}_1,\ldots,t\boldsymbol{v}+s_d\boldsymbol{u}_d)$. But

$$\frac{(1-t^2)^{\frac{d-1}{2}} - (\rho^2-t^2)_+^{\frac{d-1}{2}}}{1-\rho^d} \xrightarrow{\rho\to1} \frac{d-1}{d}\left(1-t^2\right)^{\frac{d-3}{2}}.$$

Hence, with $\rho\to1$ and the dominated convergence theorem, we now deduce that

$$\begin{aligned}
I_\rho(\boldsymbol{v},t) \xrightarrow{\rho\to1} &\left(1-t^2\right)^{\frac{d(d-3)}{2}} \int_{(H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1})^d} \tilde{f}(t\boldsymbol{v}+\sqrt{1-t^2}\bar{\boldsymbol{u}})\\
&\qquad \Delta_{d-1}(t\boldsymbol{v}+\sqrt{1-t^2}\bar{\boldsymbol{u}})\ \mathrm{d}(\sigma'_{\boldsymbol{v},0})^d(\bar{\boldsymbol{u}})\\
&= \left(1-t^2\right)^{-\frac{d}{2}} \int_{(H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1})^d} \tilde{f}(\bar{\boldsymbol{u}})\Delta_{d-1}(\bar{\boldsymbol{u}})\ \mathrm{d}(\sigma'_{\boldsymbol{v},t})^d(\bar{\boldsymbol{u}}),
\end{aligned}$$

and therefore

$$\int\limits_{(\mathbb{S}^{d-1})^d} f(\bar{\boldsymbol{u}})\,\mathrm{d}\sigma^d(\bar{\boldsymbol{u}})$$

$$= (d-1)! \int\limits_{\mathbb{S}^{d-1}} \int\limits_0^1 \left(1-t^2\right)^{-\frac{d}{2}} \int\limits_{(H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1})^d} \tilde{f}(\bar{\boldsymbol{u}})\Delta_{d-1}(\bar{\boldsymbol{u}})\ \mathrm{d}(\sigma'_{\boldsymbol{v},t})^d(\bar{\boldsymbol{u}}).$$

□

Chapter 5

Cells with many facets

Contents

Combining the Complementary Theorems of Chapter 3 with the Polytopal Approximation Theory developed in Chapter 4 leads to one of our main results: bounds for the tail distribution of the number of facets of $Z_{\boldsymbol{o}}$ and Z_{typ}.

We organized this chapter in two sections, the first focuses on $Z_{\boldsymbol{o}}$ and the second on Z_{typ}. The structure and strategies for the proofs are the same in both cases and both sections are divided in similar subsections. Therefore Subsections 5.1.X can be compared with Subsection 5.2.X. Many differences between the two cases appear at a technical level because of the different settings. We believe that keeping things together about $Z_{\boldsymbol{o}}$ (resp. Z_{typ}) in one section makes it is easier for the reader to keep in mind the specificities of the setting.

Each section has the following structure. In the first subsection we show a recurrence relation for the probability that $Z_{\boldsymbol{o}}$ (resp. Z_{typ}) has n facets, and this implies a general upper bound. In the second subsection we prove an upper bound for the probability that $Z_{\boldsymbol{o}}$ (resp. Z_{typ}) has n facets and is elongated. In the last subsection we establish a lower bound for the probability that $Z_{\boldsymbol{o}}$ (resp. Z_{typ}) has n facets.

5.1 Bounds for the zero cell

5.1.1 Upper bound for the zero cell

The Complementary Theorem 3.2.1 combined with Lemma 4.3.2 about polytopal approximation implies our main upper bounds. By seeing a polytope with n facets as a polytope with $n-1$ facets cut 'a little bit' by one halfspace, we obtain the following recurrence relation.

Theorem 5.1.1. *Assume that $r \geq 1$. There exist constants C_6 and C_7 such that for $n > C_6$,*

$$\mathbb{P}(f(Z_{\boldsymbol{o}}) = n) \leq C_7 n^{-\frac{2}{d-1}} \mathbb{P}(f(Z_{\boldsymbol{o}}) = n-1).$$

Moreover, $C_6 := \max\left(c_{17}, c_{24}^r c_\Phi^{r(d-1)/2}\right)$ and $C_7 := c_{25} c_h^r c_\Phi^r$, where c_{24} and c_{25} are constants depending only on d.

We prove Theorem 5.1.1 at the end of the subsection. Iterating the recurrence relation of Theorem 5.1.1 gives us the following general upper bound.

Theorem 5.1.2. *Assume that $r \geq 1$. There exists a constant $C_8 > 0$, dependent on r and φ, such that*

$$\mathbb{P}(f(Z_{\boldsymbol{o}}) = n) < C_8^n \, n^{-\frac{2n}{d-1}}$$

for any n. Furthermore, there exists an integer n_Φ such that $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n)$ is either vanishing or strictly decreasing for $n \geq n_\Phi$.

Proof. Set $n_0 := \lceil C_6 \rceil$. Iterating Theorem 5.1.1, gives us that for any $n \geq n_0$,

$$\mathbb{P}(f(Z_{\boldsymbol{o}}) = n) \leq C_7^{n-n_0} \left(\frac{n!}{n_0!}\right)^{-\frac{2}{d-1}}.$$

And Stirling's approximation $n! > n^n e^{-n}$ implies for any $n \geq n_0$,

$$\mathbb{P}(f(Z_{\boldsymbol{o}}) = n) < C_7^{-n_0} (n_0!)^{\frac{2}{d-1}} \left(e^{\frac{2}{d-1}} C_7\right)^n n^{-\frac{2n}{d-1}},$$

from which the first part of the theorem follows.

For the second part of the theorem, we only have to observe that for $n > C_7^{2/(d-1)}$, Theorem 5.1.1 gives $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n) < \mathbb{P}(f(Z_{\boldsymbol{o}}) = n-1)$. $\square$

We need to state the following elementary but useful lemma. We denote by $\mathfrak{S}_n$ the set of permutations of $[n]$. For $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\sigma \in \mathfrak{S}_n$, we write $\boldsymbol{x}_\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. It is clear that the following holds.

Lemma 5.1.3. *Let (X, Σ, ψ) be a measured space, $m, n > 0$ be integers, $f\colon X^n \to [0, \infty)$ be a measurable function and $S, T \subset X^n$ measurable sets. Assume that*

- *f is symmetric: for any $\sigma \in \mathfrak{S}_n$ and any $\boldsymbol{x} \in X^n$, we have $f(\boldsymbol{x}_\sigma) = f(\boldsymbol{x})$;*
- *S is symmetric: for any $\sigma \in \mathfrak{S}_n$, and any $\boldsymbol{x} \in X^n$ we have $\mathbb{1}(\boldsymbol{x}_\sigma \in S) = \mathbb{1}(\boldsymbol{x} \in S)$;*
- *for any $\boldsymbol{x} \in S$, there exist at least p permutations $\sigma \in \mathfrak{S}_n$ such that $\boldsymbol{x}_\sigma \in T$.*

Then

$$\frac{p}{n!} \int\limits_{X^n} \mathbb{1}(\boldsymbol{x} \in S) f(\boldsymbol{x}) \mathrm{d}\psi^n(\boldsymbol{x}) \leq \int\limits_{X^n} \mathbb{1}(\boldsymbol{x} \in T) f(\boldsymbol{x}) \mathrm{d}\psi^n(\boldsymbol{x}).$$

In the next two lemmata we investigate the influence of changes of $P_{[n]} = \cap_{i=1}^n H_i^-$. The first lemma deals with the measure of those polytopes $P_{[n]}$ which are close to $P_{[n-1]}$ in the Hausdorff distance. The second lemma clarifies the dependence of the measure of polytopes $P_{[n]}$ on the Φ-content.

Lemma 5.1.4. *Assume that $r \geq 1$. For any $\alpha > 0$ and any measurable function $f : \mathcal{H}^{n-1} \to [0, \infty)$, it holds that*

$$\int\limits_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha\right) f(H_1, \ldots, H_{n-1}) \mathrm{d}\mu^n(\boldsymbol{H})$$
$$< \alpha c_h^{r-1} c_\Phi^{r-1} \int\limits_{\mathcal{H}^{n-1}} \Phi\left(P_{[n-1]}\right)^{\frac{r-1}{r}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1,\boldsymbol{o}}\right) f(\boldsymbol{H}) \mathrm{d}\mu^{n-1}(\boldsymbol{H}).$$

Proof. The essential part of the proof is to bound

$$I_K = \int\limits_{\mathcal{H}} \mathbb{1}\left(K \cap H_n^- \in \mathcal{K}_{\boldsymbol{o}}\right) \mathbb{1}\left(K \cap H_n^- \neq \emptyset\right) \mathbb{1}\left(d_H(K, K \cap H_n^-) < \alpha\right) \mathrm{d}\mu(H_n),$$

for any $K \in \mathcal{K}_{\boldsymbol{o}}$. We will then apply it in the case $K = P_{[n-1]} = \cap_{i=1}^{n-1} H_i^- \in \mathcal{P}_{n-1,\boldsymbol{o}}$.

By definition of μ

$$I_K = \int\limits_{\mathbb{S}^{d-1}} \int\limits_0^\infty \mathbb{1}\left(K \cap H(\boldsymbol{u}, t)^- \neq \emptyset\right) \mathbb{1}\left(d_H(K, K \cap H(\boldsymbol{u}, t)^-) < \alpha\right) t^{r-1} \mathrm{d}t\, \mathrm{d}\varphi(\boldsymbol{u})$$
$$= \int\limits_{\mathbb{S}^{d-1}} \int\limits_{\max(0, h(K,\boldsymbol{u}) - \alpha)}^{h(K,\boldsymbol{u})} t^{r-1} \mathrm{d}t\, \mathrm{d}\varphi(\boldsymbol{u}).$$

But since $r \geq 1$, the integrand is increasing with t and hence bounded by $h(K, \boldsymbol{u})^{r-1}$. Therefore

$$I_K < \alpha \int\limits_{\mathbb{S}^{d-1}} h(K, \boldsymbol{u})^{r-1} \mathrm{d}\varphi(\boldsymbol{u}).$$

With (3.1) this gives

$$I_K < \alpha \int\limits_{\mathbb{S}^{d-1}} \left(c_h c_\Phi \Phi(K)^{\frac{1}{r}}\right)^{r-1} \mathrm{d}\varphi(\boldsymbol{u}) = \alpha c_h^{r-1} c_\Phi^{r-1} \Phi(K)^{\frac{r-1}{r}}. \tag{5.1}$$

Let us fix now $(H_1, \ldots, H_{n-1}) \in \mathcal{H}^{n-1}$. We observe that for every $H_n \in \mathcal{H}$,

$$\begin{aligned} &\mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha\right) \\ &\leq \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1,\boldsymbol{o}}\right) \mathbb{1}\left(P_{[n-1]} \cap H_n^- \in \mathcal{K}_{\boldsymbol{o}}\right) \mathbb{1}\left(P_{[n-1]} \cap H_n \neq \emptyset\right) \\ &\quad \times \mathbb{1}\left(d_H(P_{[n-1]}, P_{[n-1]} \cap H_n^-) < \alpha\right). \end{aligned} \tag{5.2}$$

Integrating (5.2) over $H_n \in \mathcal{H}$ and combining it with (5.1) applied to $K = P_{[n-1]}$, we obtain

$$\begin{aligned} &\int\limits_{\mathcal{H}} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(d_H(P_{[n]}, P_{[n-1]}) < \alpha\right) \mathrm{d}\mu(H_n) \\ &\leq \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right) \alpha c_h^{r-1} c_\Phi^{r-1} \Phi\left(P_{[n-1]}\right)^{\frac{r-1}{r}}. \end{aligned}$$

We end the proof by multiplying the previous inequality by $f(H_1, \ldots, H_{n-1})$ and integrating it with respect to $(H_1, \ldots, H_{n-1}) \in \mathrm{d}\mu^{n-1}(\boldsymbol{H})$. □

Lemma 5.1.5. *For any $\beta > 0$ we have*

$$\begin{aligned} &\int\limits_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi(P_{[n]}) < \beta\right) \mathrm{d}\mu^n(\boldsymbol{H}) \\ &= \beta^n \int\limits_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi(P_{[n]}) < 1\right) \mathrm{d}\mu^n(\boldsymbol{H}). \end{aligned}$$

Proof. The proof consists of simple computations. By definition of μ_n, see (3.4), we have

$$\int\limits_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi(P_{[n]}) < \beta\right) \mathrm{d}\mu^n(\boldsymbol{H}) = n! \int\limits_{\mathcal{P}_{n,\boldsymbol{o}}} \mathbb{1}\left(\Phi(P) < \beta\right) \mathrm{d}\mu_n(P).$$

But equation (3.5) tells us that μ_n can be decomposed into a product $\lambda_1^{(n)} \otimes \mu_{n,\Phi}$. This gives

$$\begin{aligned} &\int\limits_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi(P_{[n]}) < \beta\right) \mathrm{d}\mu^n(\boldsymbol{H}) \\ &= n! \int\limits_{\mathcal{P}_{n,\boldsymbol{o},\Phi}} \int\limits_0^\beta n t^{n-1} \mathrm{d}t \, \mathrm{d}\mu_{n,\Phi}(P) = \beta^n n! \int\limits_{\mathcal{P}_{n,\boldsymbol{o},\Phi}} \int\limits_0^1 n t^{n-1} \mathrm{d}t \, \mathrm{d}\mu_{n,\Phi}(P) \end{aligned}$$

where the last equality follows from trivial computation. Using (3.5) and (3.4) again yields the proof. □

We are now in the position to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Set $\alpha = c_{18} c_\Phi n^{-2/(d-1)}$ and $\beta = c_{19}^r c_\Phi^r n^{-(d+1)/(d-1)}$, where c_{18} and c_{19} are the constants of Lemma 4.3.2. Setting $S' = \mathcal{P}_{n,\boldsymbol{o},\Phi}$ in (3.7) gives

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) = n! \int\limits_{\mathcal{P}_{n,\boldsymbol{o}}} \mathbb{1}\left(\Phi(P) < 1\right) \mathrm{d}\mu_n(P).$$

By (3.4) this can also be written

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) = \int\limits_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right),$$

where $P_{[n]} = \cap_{i=1}^n H_i^-$. We want to use now Lemma 4.3.2 which, roughly speaking, tells us that the variable H_n has a 'small influence'. Set

$$S = \left\{\boldsymbol{H} \in \mathcal{H}^n : \cap_{i=1}^n H_i^- \in \mathcal{P}_{n,\boldsymbol{o}} \text{ and } \cap_{i=1}^n H_i^- \text{ is a simple polytope}\right\},$$

and

$$T = \left\{\boldsymbol{H} \in S : d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha \Phi\left(P_{[n]}\right)^{\frac{1}{r}},\ \Phi\left(P_{[n-1]}\right) < \exp\left(\beta\right) \Phi\left(P_{[n]}\right)\right\}.$$

Lemma 4.3.2 tells us that, when $n > c_{17}$, for any $\boldsymbol{H} \in S$, there exists at least $n!/4$ permutations $\sigma \in \mathfrak{S}_n$ such that $\boldsymbol{H}_\sigma \in T$. Hence, Lemma 5.1.3 implies

$$\begin{aligned}\frac{1}{4}\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) \leq \int\limits_{\mathcal{H}^n} &\mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha\right) \\ &\times \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(\beta\right)\right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right).\end{aligned}$$

Using Lemma 5.1.4 with $f(H_1, \ldots, H_{n-1}) = \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(\beta\right)\right)$, we have

$$\begin{aligned}\frac{1}{4}\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) &\leq \alpha c_h^{r-1} c_\Phi^{r-1} \int\limits_{\mathcal{H}^{n-1}} \Phi\left(P_{[n-1]}\right)^{\frac{r-1}{r}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1,\boldsymbol{o}}\right) \\ &\qquad \times \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(\beta\right)\right) \mathrm{d}\mu^{n-1}\left(\boldsymbol{H}\right) \\ &\leq \alpha c_h^{r-1} c_\Phi^{r-1} \exp\left(\frac{r-1}{r}\beta\right) \int\limits_{\mathcal{H}^{n-1}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1,\boldsymbol{o}}\right) \\ &\qquad \times \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(\beta\right)\right) \mathrm{d}\mu^{n-1}\left(\boldsymbol{H}\right).\end{aligned}$$

Applying now Lemma 5.1.5 with $n' = n-1$ and $\beta' = \exp(\beta)$, we get

$$\frac{1}{4}\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) \le \alpha c_h^{r-1} c_\Phi^{r-1} \exp\left(\left(\frac{r-1}{r} + n - 1\right)\beta\right) \times \int\limits_{\mathcal{H}^{n-1}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < 1\right) \mathrm{d}\mu^{n-1}(\boldsymbol{H})$$

$$= \alpha c_h^{r-1} c_\Phi^{r-1} \exp\left(\left(\frac{r-1}{r} + n - 1\right)\beta\right) \mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n-1\right).$$

And since $\left(\frac{r-1}{r} + n - 1\right)\beta < n\beta = c_{19}^r c_\Phi^r n^{-\frac{2}{d-1}} < 1$, for $n > (c_{19}^r c_\Phi^r)^{\frac{d-1}{2}}$, we have

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) \le 4ec_{18} c_h^{r-1} c_\Phi^r n^{-\frac{2}{d-1}} \mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n-1\right),$$

for $n > \max\left(c_{17}, (c_{19}^r c_\Phi^r)^{(d-1)/2}\right)$. Hence the theorem holds with $c_{24} = c_{19}^{(d-1)/2}$ and $c_{25} := 4ec_{18}c_h^{-1}$. □

5.1.2 Upper bound for the elongated zero cell

Theorem 5.1.6. *Assume $r \ge 1$ and $1 \le i < j \le \lceil (d-1)/2 \rceil$. For any $\delta > 0$, there exist ϵ and C_9, dependent on φ, r, i, j, and δ, such that*

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n,\ \frac{V_j(Z_{\boldsymbol{o}})^{\frac{1}{j}}}{V_i(Z_{\boldsymbol{o}})^{\frac{1}{i}}} < \epsilon\right) < \delta^n n^{-\frac{2n}{d-1}},$$

for any $n > C_9$.

Proof. We will proceed in a similar way as in the proof of Theorem 5.1.1 with one main difference. In order to take into account the elongation condition, we will use Lemma 4.4.3 instead of Lemma 4.3.2. This explains why we have directly a general upper bound without passing through the intermediate step of a recurrence relation similar as the one of Theorem 5.1.1.

Let $\delta' = \delta/(4e^{2/(d-1)}C_{11})$ where C_{11} is a positive constant dependent on r and φ which will be set at the end of the proof. Let C_3, C_4 and C_5 be the constants dependent on i, j and d, used in Lemma 4.4.3. Set $\epsilon = \epsilon(\delta', \varphi, i, j)$ such that

$$\delta' = C_5 c_\Phi \epsilon^{\frac{1}{2d^4}}.$$

Without loss of generality we can assume that δ' is small enough such that $\epsilon < C_3^{2/(d-1)} C_4^{-1} c_\Phi^{-1}$. Set $m = m(\delta, \varphi, i, j) = \lfloor C_3 \epsilon^{-(d-2)} \rfloor$. By (3.7), we have

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n, \frac{V_j(Z_{\boldsymbol{o}})^{\frac{1}{j}}}{V_i(Z_{\boldsymbol{o}})^{\frac{1}{i}}} < \epsilon\right) = n! \int\limits_{\mathcal{P}_{n,\boldsymbol{o}}} \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(\frac{V_j\left(P_{[n]}\right)^{\frac{1}{j}}}{V_i\left(P_{[n]}\right)^{\frac{1}{i}}} < \epsilon\right) \mathrm{d}\mu_n(P).$$

By (3.4) this can also be written

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}})=n,\frac{V_j(Z_{\boldsymbol{o}})^{\frac{1}{j}}}{V_i(Z_{\boldsymbol{o}})^{\frac{1}{i}}}<\epsilon\right)$$
$$=\int\limits_{\mathcal{H}^n}\mathbb{1}\left(P_{[n]}\in\mathcal{P}_{n,\boldsymbol{o}}\right)\mathbb{1}\left(\Phi\left(P_{[n]}\right)<1\right)\mathbb{1}\left(\frac{V_j\left(P_{[n]}\right)^{\frac{1}{j}}}{V_i\left(P_{[n]}\right)^{\frac{1}{i}}}<\epsilon\right)\mathrm{d}\mu^n\left(\boldsymbol{H}\right),$$

where $P_{[n]}=\cap_{i=1}^n H_i^-$. Roughly speaking, we will now use Lemmata 5.1.3 and 4.4.3 to order the halfspaces such that integrating step by step, starting by $H_n^{\epsilon_n}$, the integrals can well be bounded. Recall that a collection of hyperplanes is said to be in generic position when the intersection of any $d+2$ of them is empty. Set

$$S=\left\{\boldsymbol{H}\in\mathcal{H}^n:\begin{array}{l}H_1,\dots,H_n\text{ are in generic position,}\\ P_{[n]}\in\mathcal{P}_{n,\boldsymbol{o}},\\ \dfrac{V_j\left(P_{[n]}\right)^{\frac{1}{j}}}{V_i\left(P_{[n]}\right)^{\frac{1}{i}}}<\epsilon\end{array}\right\},$$

and

$$T=\left\{\boldsymbol{H}\in\mathcal{H}^n:\begin{array}{l}P_{[n]}\in\mathcal{P}_{n,\boldsymbol{o}},\\ \Phi\left(P_{[m]}\right)<\left(1+(1+c_\Phi)^{r-1}\right)\Phi\left(P_{[n]}\right),\\ d_H\left(P_{[k]},P_{[k-1]}\right)<\delta'\Phi\left(P_{[m]}\right)^{\frac{1}{r}}k^{-\frac{2}{d-1}}\text{ for }2m<k\leq n\end{array}\right\}.$$

Lemma 4.4.3 tells us that, for any $\boldsymbol{H}\in S$, there exist at least $2^{-n}(n-2m)!$ permutations $\sigma\in\mathfrak{S}_n$ such that $\boldsymbol{H}_\sigma\in T$. Hence, Lemma 5.1.3 implies

$$\frac{2^{-n}(n-2m)!}{n!}\mathbb{P}\left(f(Z_{\boldsymbol{o}})=n,\frac{V_j(Z_{\boldsymbol{o}})^{\frac{1}{j}}}{V_i(Z_{\boldsymbol{o}})^{\frac{1}{i}}}<\epsilon\right)$$
$$\leq\int\limits_{\mathcal{H}^n}\mathbb{1}\left(P_{[n]}\in\mathcal{P}_{n,\boldsymbol{o}}\right)\mathbb{1}\left(\Phi\left(P_{[n]}\right)<1\right)\mathbb{1}\left(\Phi\left(P_{[m]}\right)<\left(1+(1+c_\Phi)^{r-1}\right)\Phi\left(P_{[n]}\right)\right)$$
$$\times\mathbb{1}\left(d_H\left(P_{[k]},P_{[k-1]}\right)<\delta'\Phi\left(P_{[m]}\right)k^{-\frac{2}{d-1}}\text{ for }2m<k\leq n\right)\mathrm{d}\mu^n\left(\boldsymbol{H}\right)$$
$$\leq\int\limits_{\mathcal{H}^n}\mathbb{1}\left(P_{[n]}\in\mathcal{P}_{n,\boldsymbol{o}}\right)\mathbb{1}\left(\Phi\left(P_{[m]}\right)<1+(1+c_\Phi)^{r-1}\right)$$
$$\times\mathbb{1}\left(d_H(P_{[k]},P_{[k-1]})<\left(1+(1+c_\Phi)^{r-1}\right)\delta'k^{-\frac{2}{d-1}}\text{ for }2m<k\leq n\right)\mathrm{d}\mu^n\left(\boldsymbol{H}\right).$$

Now, observing that for any $k=n,n-1,\dots,2m+1$, we have $\Phi\left(P_{[k]}\right)<\Phi\left(P_{[2m]}\right)$, and using $n-2m$ times Lemma 5.1.4, we have

$$\frac{2^{-n}(n-2m)!}{n!}\mathbb{P}\left(f(Z_{\boldsymbol{o}})=n,\frac{V_j(Z_{\boldsymbol{o}})^{\frac{1}{j}}}{V_i(Z_{\boldsymbol{o}})^{\frac{1}{i}}}<\epsilon\right)<C_{10}(C_{11}\delta')^{n-2m}\left(\frac{n!}{(2m)!}\right)^{-\frac{2}{d-1}}.$$

where $C_{10} = C_{10}(\delta, \varphi, i, j, r)$ and $C_{11} = C_{11}(\varphi, r)$ are defined by

$$C_{10} := \int_{\mathcal{H}^{2m}} \mathbb{1}\left(P_{[2m]} \in \mathcal{P}_{2m,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi(P_{[m]}) < \left(1 + (1 + c_\Phi)^{r-1}\right)\right) \mathrm{d}\mu^{2m}(\boldsymbol{H})$$

and

$$C_{11} := c_h^{r-1} c_\Phi^{r-1} \left(1 + (1 + c_\Phi)^{r-1}\right)^{1 + \frac{r-1}{r}} .$$

Simple computations end the proof. Using Stirling approximation $n! \geq n^n e^{-n}$, the trivial inequality $\frac{n!}{(n-2m)!} \leq n^{2m}$ and the fact that we defined $\delta' = \delta/(4e^{2/(d-1)} C_{11})$, we have

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n,\ \frac{V_j(Z_{\boldsymbol{o}})^{\frac{1}{j}}}{V_i(Z_{\boldsymbol{o}})^{\frac{1}{i}}} < \epsilon \right) < \left(\frac{C_{10}[(2m)!]^{\frac{2}{d-1}}}{(C_{11}\delta')^2 m} n^{2m} 2^{-n} \right) \delta^n n^{-\frac{2n}{d-1}},$$

which implies the theorem. □

5.1.3 Lower bound for the zero cell

Let us recall that we call cap the intersection of a small ball with the boundary of a convex body. In this subsection we deal with caps on the unit sphere and use the notation

$$C = C(\boldsymbol{x}, R) = \mathbb{S}^{d-1} \cap B(\boldsymbol{x}, R)$$

for the cap $C \subset \mathbb{S}^{d-1}$ of center $\boldsymbol{x} \in \mathbb{S}^{d-1}$ and radius $R > 0$.

In Theorem 5.1.7 we give a lower bound for the probability that $Z_{\boldsymbol{o}}$ has n facets. The argument relies on the particular assumption that the directional distribution φ is well spread, i.e. there exists a constant $c_\varphi > 0$ and a cap $C \subset \mathbb{S}^{d-1}$ such that

$$\varphi(\cdot) > c_\varphi \mathscr{H}^{d-1}(\cdot) \text{ on } C.$$

In the following theorem, c_{27} is constant which depends only on d and which will be defined in Lemma 5.1.8.

Theorem 5.1.7. *Assume that φ is well spread. Then there exists a constant $C_{12} > 0$, such that*

$$\mathbb{P}(f(Z_{\boldsymbol{o}}) = n) > C_{12}^n n^{-\frac{2n}{d-1}},$$

for any $n > c_{27}$. Moreover, if $\varphi(\cdot) > c_\varphi \mathcal{H}^{d-1}(\cdot)$ on a cap of radius R_φ, then $C_{12} = c_\varphi c_{26}^r R_\varphi^{d+r+1}$, where c_{26} is a constant depending only on d.

The proof of Theorem 5.1.7 is based on the following strategy: we construct a set of polytopes with n facets and with bounded Φ-content which we obtain by slightly perturbating a deterministic polytope which is as *regular* as possible. We do so in a way which ensures that $Z_{\boldsymbol{o}}$ is one of these polytopes with a high enough probability. In Lemma 5.1.8, we proceed with the construction of the deterministic polytope and in Lemma 5.1.9, we estimate the probability that $Z_{\boldsymbol{o}}$ is a perturbation of this deterministic polytope.

Lemma 5.1.8. *Let $R \in (0,1)$ and $C \subset \mathbb{S}^{d-1}$ be a cap of radius R. There exist a constant $c_{27} = c_{27}(d)$, and $m = m(d,R) < c_{27}$ points $\boldsymbol{y}_i \in C \cup -C$, $i = 1, \dots, m$ such that the caps $C(\boldsymbol{y}_i, r/12)$ are pairwise disjoint and*

$$\bigcap_{i=1}^{m} H(\boldsymbol{v}_i, 1)^- \subset B(\boldsymbol{o}, 4r^{-1})$$

for any $\boldsymbol{v}_i \in C(\boldsymbol{y}_i, R/12) \cap (C \cup -C)$, $i = 1, \dots, m$.

Proof. Without loss of generalities, we assume that C is centered at the point $\boldsymbol{e}_d = (0, \dots, 0, 1)$. We choose a saturated packing of caps $C(\boldsymbol{y}_i, R_\varphi/12)$ with $\boldsymbol{y_i} \in C \cup -C$, $i = 1, \dots, m$. Here we call a packing saturated if there is no possibility for adding another ball of radius $R_\varphi/12$. Since the curvature of the sphere become negligible when $R_\varphi \to 0$, we have that m is of the same order as a saturated packing of $(d-1)$-dimensional balls of radius $R_\varphi/12$ in $R_\varphi B^{d-1}$. Clearly this is independent from R_φ and therefore $m < c_{27}$ for some constant c_{27} depending only on d.

This implies first that $\bigcup C(\boldsymbol{y}_i, R_\varphi/6)$ is a covering of $C \cup -C$, and second, that each cap $C(\boldsymbol{z}, R_\varphi/4)$, $\boldsymbol{z} \in C$ contains one of the caps $C(\boldsymbol{y}_i, R_\varphi/12)$, because $\boldsymbol{z} \in C(\boldsymbol{y}_i, R_\varphi/6)$ for some $i = 1, \dots, m$.

The rest of the proof follows from explicit geometric calculations. Assume in the contrary that there are $\boldsymbol{v}_i \in C(\boldsymbol{y}_i, R_\varphi/12) \cap (C \cup -C)$ such that

$$\bigcap_{i=1}^{m} H(\boldsymbol{v}_i, 1)^- \not\subseteq B(\boldsymbol{o}, 4R_\varphi^{-1}).$$

This in particular implies that either

$$\boldsymbol{e}_d^\perp \cap \bigcap_{\boldsymbol{v}_i \in C} H(\boldsymbol{v}_i, 1)^- \not\subseteq B(\boldsymbol{o}, 4R_\varphi^{-1}) \quad \text{or} \quad \boldsymbol{e}_d^\perp \cap \bigcap_{\boldsymbol{v}_i \in -C} H(\boldsymbol{v}_i, 1)^- \not\subseteq B(\boldsymbol{o}, 4R_\varphi^{-1}).$$

Recall that C is a cap with center $\boldsymbol{e}_d$. Without loss of generality assume that $\boldsymbol{x} = (4R_\varphi^{-1}, 0, \dots, 0)$ is a point with $\|\boldsymbol{x}\| = 4R_\varphi^{-1}$ which is contained in $\bigcap_{\boldsymbol{v}_i \in C} H(\boldsymbol{v}_i, 1)^-$. Let us define $\boldsymbol{x}_0 = (R_\varphi/4, 0, \dots, 0, \sqrt{1 - R_\varphi^2/16})$. By elementary trigonometric calculations the line through $\boldsymbol{x}$ and $\boldsymbol{x}_0$ is tangent to the sphere at $\boldsymbol{x}_0$. Because $\boldsymbol{x}$ is contained in $\bigcap H(\boldsymbol{v}_i, 1)^-$, none of the points $\boldsymbol{v}_i$ may be contained the cap $C_{\boldsymbol{x}} = C(\boldsymbol{e}_1, \|\boldsymbol{e}_1 - \boldsymbol{x}_0\|)$.

Next observe that the point $\boldsymbol{x}_C = (\sqrt{1-h^2}, 0, \dots, 0, h)$ with $h = 1 - R_\varphi^2/2$ is on the relative boundary of C and in $C_{\boldsymbol{x}}$, and

$$\|\boldsymbol{x}_C - \boldsymbol{x}_0\| \geq \sqrt{1-h^2} - \frac{1}{4}R_\varphi \geq \frac{3}{4}R_\varphi - \frac{1}{4}R_\varphi \geq \frac{1}{2}R_\varphi.$$

Hence $C \cap C_{\boldsymbol{x}}$ contains a cap of radius $R_\varphi/4$. Yet this cap must contain one of the caps $B(\boldsymbol{y}_i, R_\varphi/12)$ and thus one of the points $\boldsymbol{v}_i$, a contradiction. □

In the following lemma, c_{27} is the constant depending only on d of Lemma 5.1.8.

Lemma 5.1.9. *Assume that φ is well spread. There exists a constant C_{13} such that for any $n > c_{27}$ there are subsets $S_1, \dots, S_n \subset \mathcal{H}$ with*

$$\mu(S_i) > C_{13} n^{-\frac{d+1}{d-1}}$$

and for $H_1 \in S_1, \dots, H_n \in S_n$ we have

$$\bigcap_i H_i^- \in \mathcal{P}_n$$

and

$$\bigcap_i H_i^- \subset B^d.$$

Moreover if $\varphi(\cdot) > c_\varphi \mathscr{H}^{d-1}(\cdot)$ on a cap of radius R_φ, then $C_{13} = c_\varphi c_{28}^r R_\varphi^{d+r+1}$, where c_{28} depends only on d.

Proof. Consider the $m < c_{27}$ caps $C(\boldsymbol{y}_i, R_\varphi/12)$ which have been constructed in Lemma 5.1.8, and fix $n > c_{27}$. In each of the sets $C(\boldsymbol{y}_i, R_\varphi/12) \cap (C \cup -C)$ we produce an optimal packing of $\lceil n/m \rceil$ smaller caps $C(\boldsymbol{z}_j, \rho)$ where we can choose ρ such that it satisfies $C(\boldsymbol{z}_j, \rho)$

$$c_{29} n^{-\frac{1}{d-1}} R_\varphi \le \rho \le \frac{R_\varphi}{12}$$

with a constant c_{29} depending only on d. Observe that the number of caps constructed in this way is between n and $n+m$. We choose precisely n of these caps $C(\boldsymbol{z}_i, \rho)$ in such a way that in each set $C(\boldsymbol{y}_i, R_\varphi/12) \cap (C \cup -C)$ there is at least one cap $C(\boldsymbol{z}_i, \rho)$.

As already used above, a cap of radius t has height $t^2/2$. Let $\boldsymbol{v}_i$ be arbitrary points in $C(\boldsymbol{z}_i, \rho/2) \cap (C \cup -C)$, $i = 1, \dots, n$. Since each cap

$$C\left(\boldsymbol{v}_i, \frac{\rho}{2}\right) = H\left(\boldsymbol{v}_i, 1 - \frac{1}{2}\left(\frac{\rho}{2}\right)^2\right)^+ \cap \mathbb{S}^{d-1}$$

is contained in the cap $C(\boldsymbol{z}_i, \rho)$, it is disjoint from all other caps $C(\boldsymbol{z}_j, \rho)$, and thus also disjoint from all other caps $C(\boldsymbol{v}_j, \rho/2)$. Hence for arbitrary R_i with $0 \le R_i \le \rho/2$, all points $(1 - R_i^2/2)\boldsymbol{v}_i$ are on the boundary of $\cap_{i=1}^n H(\boldsymbol{v}_i, 1 - R_i^2/2)^-$ and thus this intersection has n facets.

Since each set $C(\boldsymbol{y}_i, R_\varphi/12)$ contains a cap $C(\boldsymbol{z}_i, \rho)$, there are m points $\boldsymbol{v}_i$, $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$ say, which belong to $C(\boldsymbol{y}_1, R_\varphi/12), \cdots, C(\boldsymbol{y}_m, R_\varphi/12)$ respectively. Combining Lemma 5.1.8 applied to $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$ and the considerations above, we obtain: there are pairwise disjoint sets

$$T_i = \left\{ H(\boldsymbol{v}, t) : \boldsymbol{v} \in C\left(\boldsymbol{z}_i, \frac{\rho}{2}\right),\ t \in \left[1 - \frac{1}{2}\left(\frac{\rho}{2}\right)^2, 1\right] \right\} \subset \mathcal{H},\ i = 1, \dots, n,$$

such that for an arbitrary n-tuple $H(\boldsymbol{v}_i, t_i) \in T_i$, $i = 1, \ldots, n$, we have

$$\bigcap_{i=1}^{n} H(\boldsymbol{v}_i, t_i)^- \subset \bigcap_{i=1}^{m} H(\boldsymbol{v}_i, t_i)^- \subset B\left(\boldsymbol{o}, 4R_\varphi^{-1}\right) \quad \text{and} \quad \bigcap_{i=1}^{n} H(\boldsymbol{v}_i, t_i)^- \in \mathcal{P}_n.$$

We normalize such that $B\left(\boldsymbol{o}, 4R_\varphi^{-1}\right)$ is replaced by the unit ball and define

$$S_i = \left\{ H(\boldsymbol{v}, t) : \ \boldsymbol{v} \in C\left(\boldsymbol{z}_i, \frac{\rho}{2}\right), t \in \frac{R_\varphi}{4}\left[1 - \frac{1}{2}\left(\frac{\rho}{2}\right)^2, 1\right] \right\} = \frac{R_\varphi}{4} T_i \subset \mathcal{H}$$

for $i = 1, \ldots, n$.

It only remains to get lower bounds for the measures of the sets S_i. By definition of the sets, we have

$$\mu(S_i) = \int\limits_{C\left(\boldsymbol{z}_i, \frac{\rho}{2}\right)} \int\limits_{\frac{R_\varphi}{4}\left(1-\frac{1}{2}\left(\frac{\rho}{2}\right)^2\right)}^{\frac{R_\varphi}{4}} t^{r-1} \mathrm{d}t \, \varphi(\mathrm{d}\boldsymbol{u})$$

$$\geq \varphi\left(C\left(\boldsymbol{z}_i, \frac{\rho}{2}\right)\right) \int\limits_{\frac{R_\varphi}{4}\left(1-\frac{1}{2}\left(\frac{\rho}{2}\right)^2\right)}^{\frac{R_\varphi}{4}} t^{r-1} \mathrm{d}t.$$

Using the the assumption $\varphi(\cdot) > c_\varphi \mathscr{H}^{d-1}(\cdot)$, we bound the first factor of the expression above. There exists a constant c_{30}, depending only on d, such that

$$\varphi\left(C\left(\boldsymbol{z}_i, \frac{\rho}{2}\right)\right) \geq c_\varphi \mathscr{H}^{d-1}\left(C\left(\boldsymbol{z}_i, \frac{\rho}{2}\right)\right) > c_\varphi c_{30} \rho^{d-1},$$

where the second inequality comes from the fact that the Hausdorff measure of a cap of radius $\rho/2$ can be approximated by $(\rho/2)^{d-1}\kappa_{d-1}$, see e.g. Lemma 4.1.3. Simple computation give us a bound of the integral factor

$$\int\limits_{\frac{R_\varphi}{4}\left(1-\frac{1}{2}\left(\frac{\rho}{2}\right)^2\right)}^{\frac{R_\varphi}{4}} t^{r-1} \mathrm{d}t > \int\limits_{\frac{R_\varphi}{4}\left(1-\frac{1}{2}\left(\frac{\rho}{2}\right)^2\right)}^{\frac{R_\varphi}{4}} \left[\frac{R_\varphi}{4}\left(1 - \frac{1}{2}\left(\frac{\rho}{2}\right)^2\right)\right]^{r-1} \mathrm{d}t$$

$$= \left(\frac{R_\varphi}{32}\right)^r \rho^2 \left(8 - \rho^2\right)^{r-1}.$$

And since $\rho \leq R_\varphi/12 \leq 1/12$, we have $\left(8 - \rho^2\right)^{r-1} > 1$, and therefore the last equations imply

$$\mu(S_i) > c_\varphi c_{30} \rho^{d+1} \left(\frac{R_\varphi}{32}\right)^r$$

which gives

$$\mu(S_i) > c_\varphi c_{30} c_{29}^{d+1} R_\varphi^{d+1} \left(\frac{R_\varphi}{32}\right)^r n^{-\frac{d+1}{d-1}},$$

because $\rho > c_{29} n^{-\frac{1}{d-1}} R_\varphi$. Setting $c_{28} = \frac{\min(1,c_{30})}{32}$ gives

$$\mu(S_i) > c_\varphi c_{28}^r R_\varphi^{d+r+1} n^{-\frac{d+1}{d-1}}.$$

This yields n sets S_i with the desired properties. □

Proof of Theorem 5.1.7. As we have seen in the beginning of the proof of Theorem 5.1.1, (3.7) and (3.4) gives

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) = \int_{\mathcal{H}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right),$$

where $P_{[n]} = \cap_{i=1}^n H_i^-$. Let $S_1, \ldots, S_n$ be as in Lemma 5.1.9, we then have

$$\begin{aligned}\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) &\geq n! \int_{\mathcal{H}^n} \mathbb{1}\left(H_1 \in S_1\right) \cdots \mathbb{1}\left(H_n \in S_n\right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right)\\ &= n! \prod_{i=1}^n \mu(S_i)\\ &\geq n! \left(c_\varphi c_{28}^r R_\varphi^{d+r+1} n^{-\frac{d+1}{d-1}}\right)^n.\end{aligned}$$

Therefore, with Stirling approximation $n! \geq n^n e^{-n}$, we have

$$\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right) \geq \left(e^{-1} c_\varphi c_{28}^r R^{d+r+1}\right)^n n^{-\frac{2n}{d-1}}.$$

Setting $c_{26} = e^{-1} c_{28}^r$ yields the proof. □

5.2 Bounds for the typical cell

We assume in this section that we are in the stationary case ($r = 1$ and $\varphi \in \aleph_e$). As already explained at the beginning of the chapter, this section follows the same structure as the previous one, with really similar subsections and theorems. The proofs follow analogous strategies but many differences appear at a technical level and therefore we write completely the main proofs.

5.2.1 Upper bound for the typical cell

Theorem 3.3.1, combined with the geometric arguments developed in Chapter 4, implies our main results. By seeing a polytope with n facets as a polytope with $n-1$ facets cut 'a little bit' by one halfspace, we obtain the following recurrence relation.

Theorem 5.2.1. *There exist constants C_{14} and C_{15} depending on φ, such that for $n > C_{14}$,*

$$\mathbb{P}(f(Z_{\text{typ}}) = n) \leq C_{15} n^{-\frac{2}{d-1}} \mathbb{P}(f(Z_{\text{typ}}) = n-1).$$

Moreover $C_{14} = \max\left(c_{17}, c_{31} c_{\Phi}^{2/(d-1)}\right)$ and $C_{15} = c_{32} c_{\Phi}$, where c_{17}, c_{31} and c_{32} are constants depending only on d.

We prove Theorem 5.2.1 at the end of the subsection. Iterating the recurrence relation of Theorem 5.2.1 gives us the following general upper bound.

Theorem 5.2.2. *There exists a constant $C_{16} > 0$, dependent on φ, such that*

$$\mathbb{P}(f(Z_{\text{typ}}) = n) < C_{16}^n \, n^{-\frac{2n}{d-1}}$$

for any n. Furthermore, there exists an integer n_Φ such that $\mathbb{P}(f(Z_{\text{typ}}) = n)$ is either vanishing or strictly decreasing for $n \geq n_\Phi$.

Proof. We omit the proof which follows exactly the same lines as the proof of 5.1.2. □

We need the following two lemmata. They are analogous of Lemmata 5.1.4 and 5.1.5, and their proofs are similar. The main difference is that the polytopes, convex bodies and half spaces considered do not need to contain the origin $\boldsymbol{o}$, so in particular we now use the notation $P_{[n]} = \cap_{i=1}^n H_i^{\epsilon_i}$. Also in Lemma 5.2.4 we take into account the center $\mathfrak{c}\left(P_{[n]}\right)$. These differences make some notation a bit more heavy. On the other hand we have now $r = 1$ and $\varphi \in \aleph_e$, which simplifies some part of the proofs. For completeness we include both proofs.

Lemma 5.2.3. *For any $\alpha > 0$ and any measurable function $f : \widetilde{\mathcal{H}}^{n-1} \to [0, \infty)$, it holds that*

$$\int\limits_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha\right) f(H_1^{\epsilon_1}, \ldots, H_{n-1}^{\epsilon_n}) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right)$$

$$< \alpha \int\limits_{\widetilde{\mathcal{H}}^{n-1}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right) f(\boldsymbol{H}^{\boldsymbol{\epsilon}}) \mathrm{d}\widetilde{\mu}^{n-1}\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right).$$

Proof. The essential part of the proof is to bound

$$I_K = \int\limits_{\widetilde{\mathcal{H}}} \mathbb{1}\left(K \cap H_n^{\epsilon_n} \in \mathcal{K}\right) \mathbb{1}\left(K \cap H_n^{\epsilon_n} \neq \emptyset\right) \mathbb{1}\left(d_H(K, K \cap H_n^{\epsilon_n}) < \alpha\right) \mathrm{d}\widetilde{\mu}(H_n^{\epsilon_n}),$$

for any $K \in \mathcal{K}$. We will then apply it in the case $K = P_{[n-1]} = \cap_{i=1}^{n-1} H_i^{\epsilon_i} \in \mathcal{P}_{n-1}$.

To get an upper bound of I_K, it is more convenient to use the following representation of half spaces: $\widetilde{H}(\boldsymbol{u}, t) = \{\boldsymbol{x} \in \mathbb{R}^{d-1} : \langle \boldsymbol{x}, \boldsymbol{u} \rangle \leq t\}$. By definition (3.2) of $\widetilde{\mu}$

$$\begin{aligned} I_K &= \int\limits_{\mathbb{S}^{d-1}} \int\limits_{-\infty}^{\infty} \mathbb{1}\left(K \cap \widetilde{H}(\boldsymbol{u}, t) \neq \emptyset\right) \mathbb{1}\left(d_H(K, K \cap \widetilde{H}(\boldsymbol{u}, t) < \alpha\right) \mathrm{d}t \, \mathrm{d}\varphi(\boldsymbol{u}) \\ &\leq \int\limits_{\mathbb{S}^{d-1}} \int\limits_{h(K,\boldsymbol{u})-\alpha}^{h(K,\boldsymbol{u})} \mathrm{d}t \, \mathrm{d}\varphi(\boldsymbol{u}) = \alpha. \end{aligned} \tag{5.3}$$

Let us fix now $\left(H_1^{\epsilon_1}, \ldots, H_{n-1}^{\epsilon_n}\right) \in \widetilde{\mathcal{H}}^{n-1}$. We observe that for every $H_n^{\epsilon_n} \in \widetilde{\mathcal{H}}$,

$$\begin{aligned} &\mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha\right) \\ &\leq \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right) \mathbb{1}\left(P_{[n-1]} \cap H_n^{\epsilon_n} \in \mathcal{K}\right) \mathbb{1}\left(P_{[n-1]} \cap H_n \neq \emptyset\right) \\ &\quad \times \mathbb{1}\left(d_H(P_{[n-1]}, P_{[n-1]} \cap H_n^{\epsilon_n}) < \alpha\right). \end{aligned} \tag{5.4}$$

Integrating (5.4) over $H_n^{\epsilon_n} \in \widetilde{\mathcal{H}}$ and combining it with (5.3) applied to $K = P_{[n-1]}$, we obtain

$$\begin{aligned} &\int\limits_{\widetilde{\mathcal{H}}} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(d_H(P_{[n]}, P_{[n-1]}) < \alpha\right) \mathrm{d}\widetilde{\mu}(H_n^{\epsilon_n}) \\ &\leq \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right) \alpha. \end{aligned}$$

We conclude by multiplicating the previous inequality by $f(H_1^{\epsilon_1}, \ldots, H_{n-1}^{\epsilon_{n-1}})$ and integrating it with respect to $(H_1^{\epsilon_1}, \ldots, H_{n-1}^{\epsilon_n}) \in \mathrm{d}\widetilde{\mu}^{n-1}\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right)$. □

Lemma 5.2.4. *For any $\beta > 0$ and any Borel set A, we have*

$$\begin{aligned} &\int\limits_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in A\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < \beta\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right) \\ &= \lambda_d(A) \beta^{n-d} \int\limits_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right). \end{aligned}$$

Proof. The proof consists of simple computations. By definition of μ_n, see (3.3), we have

$$\int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in A\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < \beta\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right)$$
$$= n! \int_{\mathcal{P}_n} \mathbb{1}\left(\mathfrak{c}\left(P\right) \in A\right) \mathbb{1}\left(\Phi\left(P\right) < \beta\right) \mathrm{d}\mu_n(P)$$

But equation (3.6) tells us that μ_n can be decomposed into a product $\lambda_d \otimes \lambda_1^{(n-d)} \otimes \mu_{n,\mathfrak{c},\Phi}$. This gives

$$\int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in A\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < \beta\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right)$$
$$= n! \int_{\mathcal{P}_{n,\Phi}} \int_0^\beta \int_A (n-d)t^{n-d-1} \mathrm{d}\boldsymbol{c}\, \mathrm{d}t\, \mathrm{d}\mu_{n,\Phi}(P)$$
$$= \lambda_d(A)\beta^{n-d} n! \int_{\mathcal{P}_{n,\Phi}} \int_0^1 \int_{[0,1]^d} (n-d)t^{n-d-1} \mathrm{d}\boldsymbol{c}\, \mathrm{d}t\, \mathrm{d}\mu_{n,\Phi}(P)$$

where the last equality follows from trivial computation. Using (3.6) and (3.3) again yields the proof. □

We are now in the position to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Set $\alpha = c_{18}c_\Phi n^{-2/(d-1)}$ and $\beta = c_{19}c_\Phi n^{-(d+1)/(d-1)}$, where c_{18} and c_{19} are the constants of Lemma 4.3.2. Let

$$I_n = \frac{\gamma^{(d)}}{\gamma^d} \frac{n!}{(n-d)!} \mathbb{P}(f(Z_{\mathrm{typ}}) = n).$$

Setting $S' = \mathcal{P}_{n,\Phi}$ in (3.10) gives

$$I_n = n! \int_{\mathcal{P}_n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathrm{d}\mu_n(P).$$

By (3.3) this can also be written

$$I_n = \int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right),$$

where $P_{[n]} = \cap_{i=1}^n H_i^{\epsilon_i}$.

We want to use now Lemma 4.3.2 which, roughly speaking, tells us that the variable $H_n^{\epsilon_n}$ has a 'small influence'. For that we define sets S and T similar to the ones of the proof of Theorem 5.1.1, with the only difference that the built polytopes do not have to contain the origin now. Set

$$S = \left\{(\boldsymbol{H}, \boldsymbol{\epsilon}) \in \widetilde{\mathcal{H}}^n : \cap_{i=1}^n H_i^{\epsilon_i} \in \mathcal{P}_n \text{ and } \cap_{i=1}^n H_i^{\epsilon_i} \text{ is a simple polytope}\right\},$$

and

$$T = \left\{(\boldsymbol{H}, \boldsymbol{\epsilon}) \in S : d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha \Phi\left(P_{[n]}\right),\ \Phi\left(P_{[n-1]}\right) < \exp(\beta)\, \Phi\left(P_{[n]}\right)\right\}.$$

Lemma 4.3.2 tells us that, when $n > c_{17}$, for any $(\boldsymbol{H}, \boldsymbol{\epsilon}) \in S$, there exists at least $n!/4$ permutations $\sigma \in \mathfrak{S}_n$ such that $(\boldsymbol{H}, \boldsymbol{\epsilon})_\sigma \in T$. Hence, Lemma 5.1.3 and the Lipschitz continuity of $\mathfrak{c}$ imply

$$\begin{aligned}
\frac{I_n}{4} &\leq \int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in [0,1]^d\right) \mathbb{1}\left(d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha\right) \\
&\qquad \times \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp(\beta)\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right) \\
&\leq \int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n-1]}\right) \in [-\alpha, 1+\alpha]^d\right) \\
&\qquad \times \mathbb{1}\left(d_H\left(P_{[n]}, P_{[n-1]}\right) < \alpha\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp(\beta)\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right).
\end{aligned}$$

Using Lemma 5.2.3 applied with

$$f(H_1^{\epsilon_1}, \ldots, H_{n-1}^{\epsilon_n}) = \mathbb{1}\left(\mathfrak{c}\left(P_{[n-1]}\right) \in [-\alpha, 1+\alpha]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp(\beta)\right),$$

gives

$$\begin{aligned}
\frac{I_n}{4} &\leq \alpha \int_{\widetilde{\mathcal{H}}^{n-1}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n-1]}\right) \in [-\alpha, 1+\alpha]^d\right) \\
&\qquad \times \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp(\beta)\right) \mathrm{d}\widetilde{\mu}^{n-1}\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right).
\end{aligned}$$

Applying now Lemma 5.2.4 with $n' = n-1$ and $\beta' = \exp(\beta)$, we get

$$\begin{aligned}
\frac{I_n}{4} &\leq \alpha(1+2\alpha)^d \exp\left\{\beta(n-d-1)\right\} \int_{\widetilde{\mathcal{H}}^{n-1}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right) \\
&\qquad \times \mathbb{1}\left(\mathfrak{c}\left(P_{[n-1]}\right) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < 1\right) \mathrm{d}\widetilde{\mu}^{n-1}\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right) \\
&= \alpha(1+2\alpha)^d \exp\left\{\beta(n-d-1)\right\} I_{n-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(f(Z_{\mathrm{typ}}) = n) &\leq 4\alpha(1+2\alpha)^d \exp\left\{\beta(n-d-1)\right\} \frac{(n-d)}{n} \mathbb{P}(f(Z_{\mathrm{typ}}) = n-1) \\
&\leq 4\alpha \exp\left\{2d\alpha + \beta(n-d-1)\right\} \mathbb{P}(f(Z_{\mathrm{typ}}) = n-1).
\end{aligned}$$

And since $\alpha = c_{18}c_\Phi n^{-2/(d-1)}$ and

$$2d\alpha + \beta(n-d-1) = \left(2dc_{18} + c_{19}\frac{n-d-1}{n}\right) c_\Phi n^{-\frac{2}{d-1}} < 1,$$

for $n > [(2dc_{18} + c_{19})c_\Phi]^{2/(d-1)}$, the theorem holds with $c_{31} = (2dc_{18} + c_{19})^{2/(d-1)}$ and $c_{32} = 4ec_{18}$. □

5.2.2 Upper bound for the elongated typical cell

Theorem 5.2.5. *Assume $1 \le i < j \le \lceil (d-1)/2 \rceil$. For any $\delta > 0$, there exist ϵ and C_{17}, dependent on φ, i, j, and δ, such that*

$$\mathbb{P}\left(f(Z_{\text{typ}}) = n,\ \frac{V_j(Z_{\text{typ}})^{\frac{1}{j}}}{V_i(Z_{\text{typ}})^{\frac{1}{i}}} < \epsilon \right) < \delta^n n^{-\frac{2n}{d-1}},$$

for any $n > C_{17}$.

Proof. We will proceed in a similar way as in the proof of Theorem 5.2.1 with one main difference. In order to take into account the elongation condition, we will use Lemma 4.4.3 instead of Lemma 4.3.2. This explains why we have directly a general upper bound without passing through the intermediate step of a recurrence relation similar as the one of Theorem 5.2.1.

Let C_3, C_4 and C_5 be the constants dependent on i, j and d, used in Lemma 4.4.3. Set $\delta' = \delta/(8e^{2/(d-1)})$ and $\epsilon = \epsilon(\delta, \varphi, i, j)$ such that

$$\delta' = C_5 c_\Phi \epsilon^{\frac{1}{2d^4}}.$$

Without loss of generality we can assume that δ is small enough such that $\epsilon < C_3^{2/(d-1)} C_4^{-1} c_\Phi^{-1}$. Set $m = m(\delta, \varphi, i, j) = \lfloor C_3 \epsilon^{-(d-2)} \rfloor$. Set

$$I_n = \frac{\gamma^{(d)}}{\gamma^d} \frac{n!}{(n-d)!} \mathbb{P}\left(f(Z_{\text{typ}}) = n,\ \frac{V_j(Z_{\text{typ}})^{\frac{1}{j}}}{V_i(Z_{\text{typ}})^{\frac{1}{i}}} < \epsilon \right)$$

By (3.10), we have

$$I_n = n! \int_{\mathcal{P}_n} \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathbb{1}\left(\frac{V_j\left(P_{[n]}\right)^{\frac{1}{j}}}{V_i\left(P_{[n]}\right)^{\frac{1}{i}}} < \epsilon \right) \mathrm{d}\mu_n(P).$$

By (3.3) this can also be written

$$I_n = \int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right)$$
$$\times \mathbb{1}\left(\frac{V_j\left(P_{[n]}\right)^{\frac{1}{j}}}{V_i\left(P_{[n]}\right)^{\frac{1}{i}}} < \epsilon \right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^\epsilon\right),$$

where $P_{[n]} = \cap_{i=1}^{n} H_i^{\epsilon_i}$. Roughly speaking, we will now use Lemmata 5.1.3 and 4.4.3 to order the halfspaces such that integrating step by step, starting by $H_n^{\epsilon_n}$, the integrals can be well bounded. Recall that a collection of hyperplanes is said to be in generic position when the intersection of any $d+2$ of them is empty. Set

$$S = \left\{ (\boldsymbol{H}, \boldsymbol{\epsilon}) \in \widetilde{\mathcal{H}}^n : \begin{array}{l} H_1, \dots, H_n \text{ are in generic position,} \\ P_{[n]} \in \mathcal{P}_n, \\ \dfrac{V_j\left(P_{[n]}\right)^{\frac{1}{j}}}{V_i\left(P_{[n]}\right)^{\frac{1}{i}}} < \epsilon \end{array} \right\},$$

and

$$T = \left\{ (\boldsymbol{H}, \boldsymbol{\epsilon}) \in \widetilde{\mathcal{H}}^n : \begin{array}{l} P_{[n]} \in \mathcal{P}_n, \\ \left\| \mathfrak{c}\left(P_{[n]}\right) - \mathfrak{c}\left(P_{[m]}\right) \right\| < \Phi\left(P_{[n]}\right), \\ \Phi\left(P_{[m]}\right) < 2\Phi\left(P_{[n]}\right), \\ d_H\left(P_{[k]}, P_{[k-1]}\right) < \delta' \Phi\left(P_{[m]}\right) k^{-\frac{2}{d-1}} \text{ for } 2m < k \le n \end{array} \right\}.$$

Lemma 4.4.3 tells us that, for any $(\boldsymbol{H}, \boldsymbol{\epsilon}) \in S$, there exist at least $2^{-n}(n-2m)!$ permutations $\sigma \in \mathfrak{S}_n$ such that $(\boldsymbol{H}, \boldsymbol{\epsilon})_\sigma \in T$. Hence, Lemma 5.1.3 implies

$$\begin{aligned}
&\frac{2^{-n}(n-2m)!}{n!} I_n \\
&\le \int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[n]}\right) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \\
&\quad \times \mathbb{1}\left(\left\|\mathfrak{c}\left(P_{[n]}\right) - \mathfrak{c}\left(P_{[m]}\right)\right\| < \Phi\left(P_{[n]}\right)\right) \mathbb{1}\left(\Phi\left(P_{[m]}\right) < 2\Phi\left(P_{[n]}\right)\right) \\
&\quad \times \mathbb{1}\left(d_H\left(P_{[k]}, P_{[k-1]}\right) < \delta' \Phi\left(P_{[m]}\right) k^{-\frac{2}{d-1}} \text{ for } 2m < k \le n\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right) \\
&\le \int_{\widetilde{\mathcal{H}}^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[m]}\right) \in [-1,2]^d\right) \mathbb{1}\left(\Phi\left(P_{[m]}\right) < 2\right) \\
&\quad \times \mathbb{1}\left(d_H\left(P_{[k]}, P_{[k-1]}\right) < 2\delta' k^{-\frac{2}{d-1}} \text{ for } 2m < k \le n\right) \mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right).
\end{aligned}$$

Now, using $n-2m$ times Lemma 5.1.4, we have

$$\frac{2^{-n}(n-2m)!}{n!} I_n < C_{18} (2\delta')^{n-2m} \left(\frac{n!}{(2m)!}\right)^{-\frac{2}{d-1}}.$$

where $C_{18} = C_{18}(\delta, \varphi, i, j)$ is defined by

$$C_{18} := \int_{\widetilde{\mathcal{H}}^{2m}} \mathbb{1}\left(P_{[2m]} \in \mathcal{P}_{2m}\right) \mathbb{1}\left(\mathfrak{c}\left(P_{[m]}\right) \in [-1,2]^d\right) \mathbb{1}\left(\Phi(P_{[m]}) < 2\right) \mathrm{d}\widetilde{\mu}^{2m}\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right).$$

This gives

$$\begin{aligned}&\mathbb{P}\left(f(Z_{\mathrm{typ}})=n,\ \frac{V_j(Z_{\mathrm{typ}})^{\frac{1}{j}}}{V_i(Z_{\mathrm{typ}})^{\frac{1}{i}}}<\epsilon\right)\\&<\frac{\gamma^d}{\gamma^{(d)}}\frac{(n-d)!}{n!}\frac{n!}{2^{-n}(n-2m)!}C_{18}(2\delta')^{n-2m}\left(\frac{n!}{(2m)!}\right)^{-\frac{2}{d-1}}\\&<C_{19}n^{2m-d}(4\delta')^n(n!)^{-\frac{2}{d-1}},\end{aligned}$$

where

$$C_{19}:=C_{19}(\delta,\varphi,i,j)=\frac{\gamma^d}{\gamma^{(d)}}C_{18}(2\delta')^{-2m}((2m)!)^{\frac{2}{d-1}}.$$

Hence, with Stirling approximation $n!\geq n^ne^{-n}$ and because $\delta'=\delta/(8e^{2/(d-1)})$, we get

$$\mathbb{P}\left(f(Z_{\mathrm{typ}})=n,\ \frac{V_j(Z_{\mathrm{typ}})^{\frac{1}{j}}}{V_i(Z_{\mathrm{typ}})^{\frac{1}{i}}}<\epsilon\right)<\left(C_{19}n^{2m-d}2^{-n}\right)\delta^n n^{-\frac{2n}{d-1}},$$

which implies the theorem. □

5.2.3 Lower bound for the typical cell

In this subsection, we will show the following lower bound. Note that c_{27} below is the constant depending only on d of Lemma 5.1.8.

Theorem 5.2.6. *Assume that φ is well spread. Then there exists a constant $C_{20}>0$, such that*

$$\mathbb{P}(f(Z_{\mathrm{typ}})=n)>C_{20}^n n^{-\frac{2n}{d-1}},$$

for $n>c_{27}$. Moreover, if $\varphi(\cdot)>c_\varphi\mathscr{H}^{d-1}(\cdot)$ on a cap of radius R_φ, then $C_{20}=c_\varphi c_{33}R_\varphi^{d+2}$, where c_{33} is a constant depending only on d.

Proof. As we have seen in the beginning of the proof of Theorem 5.2.1, (3.10) and (3.3) gives

$$\begin{aligned}&\mathbb{P}\left(f(Z_{\mathrm{typ}})=n\right)\\&=\frac{\gamma^d}{\gamma^{(d)}}\frac{(n-d)!}{n!}\int\limits_{\widetilde{\mathcal{H}}^n}\mathbb{1}\left(P_{[n]}\in\mathcal{P}_n\right)\mathbb{1}\left(\mathfrak{c}(P)\in[0,1]^d\right)\mathbb{1}\left(\Phi\left(P_{[n]}\right)<1\right)\mathrm{d}\widetilde{\mu}^n\left(\boldsymbol{H}^{\boldsymbol{\epsilon}}\right)\end{aligned}$$

where $P_{[n]}=\cap_{i=1}^n H_i^{\epsilon_i}$. Let $S_1,\dots,S_n$ be as in Lemma 5.1.9, we then have

$$\begin{aligned}\mathbb{P}\left(f(Z_{\boldsymbol{o}})=n\right)&\geq\frac{\gamma^d}{\gamma^{(d)}}(n-d)!\int\limits_{\mathcal{H}^n}\mathbb{1}\left(H_1\in S_1\right)\cdots\mathbb{1}\left(H_n\in S_n\right)\mathrm{d}\mu^n\left(\boldsymbol{H}\right)\\&=\frac{\gamma^d}{\gamma^{(d)}}(n-d)!\prod_{i=1}^n\mu(S_i),\end{aligned}$$

But Lemma 5.1.9 also says that $\mu(S_i) > c_\varphi c_{28}^r R_\varphi^{d+r+1} n^{-\frac{d+1}{d-1}}$. And since here $r = 1$, we have

$$\begin{aligned}\mathbb{P}\left(f(Z_{\mathrm{typ}}) = n\right) &\geq \frac{\gamma^d}{\gamma^{(d)}}(n-d)!\left(c_\varphi c_{28} R_\varphi^{d+2} n^{-\frac{d+1}{d-1}}\right)^n \\ &= \frac{\gamma^d}{\gamma^{(d)}}\frac{(n-d)!}{n^n}\left(c_\varphi c_{28} R_\varphi^{d+2}\right)^n n^{-\frac{2n}{d-1}}.\end{aligned}$$

But, because of Stirling approximation $n! \geq n^n e^{-n}$ and the inequality $n^{1/n} \leq e^{1/e}$, we have

$$\frac{(n-d)!}{n^n} = \frac{n!}{n^n}\cdot\frac{(n-d)!}{n!} \geq e^{-n} n^{-d} \geq \left(e^{-1-\frac{d}{e}}\right)^n.$$

Thus

$$\mathbb{P}\left(f(Z_{\mathrm{typ}}) = n\right) = \frac{\gamma^d}{\gamma^{(d)}}\left(e^{-1-\frac{d}{e}} c_\varphi c_{28} R_\varphi^{d+2}\right)^n n^{-\frac{2n}{d-1}},$$

which implies the theorem with $c_{33} = e^{-1-\frac{d}{e}} c_{28} \min\left(1, \frac{\gamma^d}{\gamma^{(d)}}\right)$. □

Chapter 6

D.G. Kendall's problem and related questions

Contents

D.G. Kendall conjectured that, in the stationary and isotropic planar case, the conditional law of the shape $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\boldsymbol{o}})$, given the area $V_2(Z_{\boldsymbol{o}})$, converges weakly, as $V_2(Z_{\boldsymbol{o}}) \to \infty$, to the degenerate law concentrated at the circular shape. A short history and an exhaustive list of references for this problem and its very general variants can be found in the book of Schneider and Weil [SW08, Note 9 of Sec. 10.4]. We also refer to the more recent lectures notes [Spo13] and in particular the chapters [Hug13, Sec. 7.2.1] from Hug and [Cal13, Sec. 6.2] from Calka.

In the present chapter we consider D.G. Kendall problem in the same general setting as in the previous chapters, namely $d \geq 2$, $r \geq 1$ and $\varphi \in \aleph$. We will also consider both the cases of $Z_{\boldsymbol{o}}$ and Z_{typ}. Since most of the statements and their proofs are either identical or really similar in both cases, we introduce the following convention.

Convention 6.0.1. One can use consistently in the full chapter either

$$Z = Z_{\boldsymbol{o}},\ \mathfrak{s}(Z) = \mathfrak{s}_{\Phi}(Z_{\boldsymbol{o}}),\ \mathcal{K}_{\mathfrak{s}} = \mathcal{K}_{\boldsymbol{o},\Phi},\ r \geq 1,\ \varphi \in \aleph \text{ and } \widetilde{d} = 0$$

or

$$Z = Z_{\text{typ}},\ \mathfrak{s}(Z) = \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}),\ \mathcal{K}_{\mathfrak{s}} = \mathcal{K}_{\mathfrak{c},\Phi},\ r = 1,\ \varphi \in \aleph_e \text{ and } \widetilde{d} = d.$$

Nevertheless the main results of the chapter are stated in an explicit way such that the reader do not need to refer to this convention.

Following the setting of [HS07] we will consider arbitrary size functional, i.e. a functional $\Sigma\colon \mathcal{K} \to \mathbb{R}$ satisfying the following four axioms

1. increasing under set inclusion: $K \subset L \Rightarrow \Sigma(K) \leq \Sigma(L)$,
2. homogeneous of some degree $k > 0$: $\Sigma(tK) = t^k \Sigma(K)$,
3. continuous,
4. not identically zero.

Most of the functions commonly used to measure convex bodies are size functionals, including the intrinsic volumes V_i, $i > 0$ and Φ. A notable exception is f, the number of facets, which only satisfies the last axiom.

We will characterise the shapes of cells with big Σ-content by the isoperimetric ratio $\Phi(Z)\Sigma(Z)^{-\frac{r}{k}}$. It is easy to see that $\tau = \inf_{K\in\mathcal{K}} \Phi(K)\Sigma(K)^{-\frac{r}{k}} > 0$. Hug and Schneider considered the problem in the case of the zero cell and proved in [HS07] that the conditional law for this isoperimetric ratio, given the size $\Sigma(Z_{\boldsymbol{o}})$, converges weakly, as $\Sigma(Z_{\boldsymbol{o}}) \to \infty$, to the Dirac measure concentrated at τ. They also gave an exponential upper bound for the rate of convergence. At the really end of the chapter, we will recover their result with a slight improvement about the rate of convergence: we prove an equivalence rather then only an upper bound. Also, our result apply both for $Z_{\boldsymbol{o}}$ and Z_{typ}. In order to reach this goal we study various tail distributions and conditional laws. Some of them are not directly linked to Kendall's problem, but because of their similarities, it is appropriate to include them here.

In the first section we consider cells with many facets. We recall the bounds on the tail distribution of $f(Z)$, provide upper and lower bounds for probabilities of the form $\mathbb{P}(\mathfrak{s}(Z) \in S \mid f(Z) = n)$, recall the Gamma distribution of $\Phi(Z)$ when Z is conditioned on the event $\{f(Z) = n\}$, and finally study the tail distribution of $\Sigma(Z)$ when Z is conditioned on the event $\{f(Z) = n\}$.

In the second section we are interested in cells with a big Φ-content. We give bounds for the tail distribution of $\Phi(Z)$ and give partial result describing the shape distribution of Z conditioned on the event $\{\Phi(Z) > a\}$.

In the last section we study cells with big Σ-content. We give precise estimation $\mathbb{P}\left(\Sigma(Z) > a, \mathfrak{s}(Z) \in S\right)$, for specific sets of shapes S and when $a \to \infty$. From these bounds we derive easily our result answering D.G. Kendall's problem.

6.1 Big number of facets

6.1.1 Tail distribution of the number of facets

The tail distribution of the number of facets has been investigated in Chapter 5. In this subsection we summarize the results obtained in the previous chapter by writing bounds which hold for all n and both $Z_{\boldsymbol{o}}$ and Z_{typ}. The price of making these bound holding in this general setting is that we lose the explicit dependency of the constants on φ and r.

We say that φ is *well spread* if there exists a cap on the unit sphere where φ is bounded bellow by a multiple of the spherical Lebesgue measure. Under the condition that φ is well spread, Theorems 5.1.7 and 5.2.6 give lower bounds of $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n)$ and $\mathbb{P}(f(Z_{\text{typ}}) = n)$, respectively. As a corollary of both theorems, we have that if φ is well spread there exists a constant C_{21}, depending on φ and r, such that for any n,

$$C_{21}^n n^{-\frac{2n}{d-1}} < \mathbb{P}(f(Z) = n), \tag{6.1}$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}. Note that the lower bounds in Theorems 5.1.7 and 5.2.6 do not hold for small n. By setting the constant C_{21} small enough, the lower bound holds for small values of n as well.

Similarly Theorems 5.1.2 and 5.2.2 give upper bounds of $\mathbb{P}(f(Z_{\boldsymbol{o}}) = n)$ and $\mathbb{P}(f(Z_{\text{typ}}) = n)$, respectively. As a corollary of both theorems, there exists a constant C_{22}, depending on φ and r, such that for any n,

$$\mathbb{P}(f(Z) = n) < C_{22}^n n^{-\frac{2n}{d-1}}, \tag{6.2}$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}. Note that the upper bounds in Theorems 5.1.2 and 5.2.2 do not hold for small n. By setting the constant C_{22} big enough, the lower bound holds for small values of n as well.

6.1.2 Shape of cells with many facets

Similarly as in Kendall's problem a natural question is to describe the asymptotic shape distribution of cells with many facets. We conjecture that when φ is well spread, it concentrates on a deterministic shape. We present, with Theorem 6.1.1 below, a partial result in that direction.

Upper bound for the elongated cells with many facets

We say that a convex body K is $(\epsilon\colon i,j)$-elongated if $V_j(K)^{1/j} V_i(K)^{-1/i} < \epsilon$, with $i < j$. For small ϵ, the geometrical interpretation is the following: K is $(\epsilon\colon i,j)$-elongated if the rescaled body $V_j(K)^{1/j} K$ is close, with respect to the Hausdorff distance, to a $(i-1)$ dimensional convex body. Theorem 6.1.1 says that cells with many facets are not $(\epsilon\colon i,j)$-elongated if ϵ is small.

Theorem 6.1.1. *Assume that φ is well spread and that $1 \le i < j \le \lceil (d-1)/2 \rceil$. For any $\delta > 0$, there exists $\epsilon > 0$, depending on δ, φ and r, such that*

$$\mathbb{P}\left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \,\middle|\, f(Z) = n \right) < \delta^n$$

for all n, where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

Proof. Recall that Theorems 5.1.6 and 5.2.5 tells us that, for $1 \le i < j \le \lceil (d-1)/2 \rceil$ and any $\delta' > 0$, there exist $\epsilon > 0$ and C_{23}, depending on δ', φ and r, such that

$$\mathbb{P}\left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon, f(Z) = n \right) < (\delta')^n n^{-\frac{2n}{d-1}}, \tag{6.3}$$

for any $n > C_{23}$. By setting $\epsilon \in (0, \epsilon')$ small enough, the inequality holds for all n. Thus, with the lower bound (6.1), we have

$$\mathbb{P}\left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \mid f(Z) = n \right) < \left(\frac{\delta'}{C_{21}} \right)^n.$$

Therefore setting $\delta' = C_{21}\delta$ yields the proof. □

For $i \in [d]$, we denote by $\mathcal{K}_i$ the set of i-dimensional convex bodies in $\mathbb{R}^d$, meaning that $K \in \mathcal{K}_i$ if there exists a i-dimensional flat F, such that $K \subset F$ is a convex and compact set with non empty interior (with respect to the topology of F). We extend the definition of the Hausdorff distance in the following way:

$$d_H(K, \mathcal{K}_i) := \inf_{L \in \mathcal{K}_i} d_H(K, L).$$

We have the following corollary, which has a clearer geometric meaning.

Corollary 6.1.2. *Assume that φ is well spread and that $1 \le i < \lceil (d-1)/2 \rceil$. For any $\delta > 0$, there exists $\epsilon' > 0$, depending on δ, φ and r, such that*

$$\mathbb{P}\left(d_H(\Phi(Z)^{-\frac{1}{r}} Z, \mathcal{K}_i) < \epsilon' \,\middle|\, f(Z) = n \right) < \delta^n$$

for all n, where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

Proof. Set $j = i + 1$, so we have $1 \le i < j \le \lceil (d-1)/2 \rceil$. Because of Theorem 6.1.1, there exists $\epsilon > 0$ such that

$$\mathbb{P}\left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \,\middle|\, f(Z) = n \right) < \delta^n,$$

for all n. One just need to observe that (using continuity on a compact) there exists $\epsilon' > 0$ such that, for any $K \in \mathcal{K}$,

$$d_H\left(\Phi(Z)^{-\frac{1}{r}}Z, \mathcal{K}_i\right) < \epsilon' \Rightarrow \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon.$$

□

Lower bound for cells with many facets and shape in a given set

In the previous theorem we gave an exponential upper bound for probabilities of the form

$$\mathbb{P}\left(\mathfrak{s}(Z) \in S \mid f(Z) = n\right) \tag{6.4}$$

with

$$S = \left\{K \in \mathcal{K}_{\mathfrak{s}} : \frac{V_j(K)^{1/j}}{V_i(K)^{1/i}} < \epsilon\right\} \subset \mathcal{K}_{\mathfrak{s}}.$$

We will now consider more general set of shapes S. We cannot give a non trivial general upper bound of the probabilities (6.4), but the following theorem and corollary give a lower bounds exponential in n.

Theorem 6.1.3. *Assume that φ is strongly well spread, i.e. there exists a constant C such that $\varphi > C\mathscr{H}^{d-1}$.*

$Z_{\boldsymbol{o}}$: *For any shape $K \in \mathcal{K}_{\boldsymbol{o},\Phi}$ and $\epsilon > 0$, there exist constants $C_{K,\epsilon} \in (0,1)$ and $N_{K,\epsilon}$ such that, for $n > N_{K,\epsilon}$,*

$$\mathbb{P}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K\right) < \epsilon \mid f(Z_{\boldsymbol{o}}) = n\right) > (C_{K,\epsilon})^n.$$

Z_{typ} : *For any shape $K \in \mathcal{K}_{\mathfrak{c},\Phi}$ and $\epsilon > 0$, there exist constants $C_{K,\epsilon} \in (0,1)$ and $N_{K,\epsilon}$ such that, for $n > N_{K,\epsilon}$,*

$$\mathbb{P}\left(d_H(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}), K) < \epsilon \mid f(Z_{\text{typ}}) = n\right) > (C_{K,\epsilon})^n.$$

We will prove Theorem 6.1.3 at the end of the current subsubsection, but first let us present a straightforward corollary.

Corollary 6.1.4. *Assume that φ is strongly well spread, i.e. there exists a constant C such that $\varphi > C\mathscr{H}^{d-1}$.*

$Z_{\boldsymbol{o}}$: *Let $S \subset \mathcal{K}_{\boldsymbol{o},\Phi}$ be an open set. There exist constants C_S and N_S such that, for $n > N_S$,*

$$\mathbb{P}\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}) \in S \mid f(Z_{\boldsymbol{o}}) = n\right) > (C_S)^n.$$

Z_{typ} : *Let $S \subset \mathcal{K}_{\mathfrak{c},\Phi}$ be an open set. There exist constants C_S and N_S such that, for $n > N_S$,*

$$\mathbb{P}\left(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in S \mid f(Z_{\text{typ}}) = n\right) > (C_S)^n.$$

Proof. The proof of both cases $Z_{\boldsymbol{o}}$ and Z_{typ} are almost identical. Thus we will only consider the case of the zero cell. We only have to observe that there exist $K \in \mathcal{K}_{\boldsymbol{o},\Phi}$ and $\epsilon > 0$ such that $\{L \in \mathcal{K}_{\boldsymbol{o},\Phi} : d_H(K, L) < \epsilon\} \subset S$. The corollary follows directly from Theorem 6.1.3. □

In order to prove Theorem 6.1.3, we need first to establish the next four lemmas.

The first considers a convex body K in which a ball rolls. It gives a lower bound for the distance between a point $\boldsymbol{x} \in \partial K$ and the hyperplane tangent to K at a point $\boldsymbol{y}$. The lower bound is given in term of the distance between the outward normal vectors at $\boldsymbol{x}$ and $\boldsymbol{y}$.

In the second lemma we consider a convex body K which rolls into a ball. From a δ'-covering $V = \{\boldsymbol{v}_i\}_i$ of $\mathbb{S}^{d-1}$, we build the polytope $P = \cap_i H(\boldsymbol{v}_i, h(K, \boldsymbol{v}_i))^-$ with facets supported by the tangent hyperplanes of K with directions in V. Lemma 6.1.6 gives a bound for the distance between K and P.

The third lemma provides a set $U \subset \mathbb{S}^{d-1}$ of cardinality precisely n which is both a covering and a packing of the unit sphere.

The last lemma is the key point in the proof of Theorem 6.1.3 and is analogous to Lemma 5.1.9. It uses the three first lemmas and provides n sets of hyperplanes satisfying suitable properties.

Lemma 6.1.5. *Let $R > 0$ and K a convex body of the form $K = L + RB^d$ where $L \in \mathcal{K}$ is strictly convex. Let $\delta \in (0, R)$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^{d-1}$ with $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| > \delta$. Let $\boldsymbol{x}_{K,\boldsymbol{v}_1} \in \partial K$ be the point on the boundary of K with outward normal vector $\boldsymbol{v}_1$. It holds that*

$$h(K, \boldsymbol{v}_2) - \langle \boldsymbol{x}_{K,\boldsymbol{v}_1}, \boldsymbol{v}_2 \rangle \geq \frac{R}{2}\delta^2$$

Proof. The lemma holds with an equality when K is the unit ball. It extends to the more general case since

$$\begin{aligned} &h(L + RB^d, \boldsymbol{v}_2) - \langle \boldsymbol{x}_{L+RB^d,\boldsymbol{v}_1}, \boldsymbol{v}_2 \rangle \\ &= \underbrace{h(L, \boldsymbol{v}_2) - \langle \boldsymbol{x}_{L,\boldsymbol{v}_1}, \boldsymbol{v}_2 \rangle}_{\geq 0} + R \underbrace{\left(h(B^d, \boldsymbol{v}_2) - \langle \boldsymbol{x}_{B^d,\boldsymbol{v}_1}, \boldsymbol{v}_2 \rangle \right)}_{= \frac{\delta^2}{2}}. \end{aligned}$$

□

Lemma 6.1.6. *Let $R' > 0$ and $K \in \mathcal{K}_{\boldsymbol{o}}$ a convex body of the form $R'B^d = K + L'$ where $L' \in \mathcal{K}$ is strictly convex. Let $\delta' \in (0, R')$. For any δ'-covering $\{\boldsymbol{v}_i\}_i \subset \mathbb{S}^{d-1}$ of $\mathbb{S}^{d-1}$, we have*

$$K + \sqrt{\delta' R'} B^d \supset \bigcap_i H(\boldsymbol{v}_i, h(K, \boldsymbol{v}_i))^- \supset K.$$

Proof. We only have to show the first inclusion since the second is trivial. For any unit vector $\boldsymbol{u}$, we denote $\boldsymbol{x}_{K,\boldsymbol{u}}$ the point on the boundary of K with outward normal vector $\boldsymbol{u}$. And reciprocally, for any point $\boldsymbol{y} \in \partial K$, we denote by $\boldsymbol{u}_{K,\boldsymbol{y}}$ the outward vector of K at $\boldsymbol{y}$.

Simple geometric computations, similar as those of Figures 4.1 and 4.2, show that the lemma is true if, for any $\boldsymbol{y} \in \partial K$, there exists i such that

$$\max \{\|\boldsymbol{x}_{K,\boldsymbol{u}_i} - \boldsymbol{y}\| \ , \ \|\boldsymbol{u}_i - \boldsymbol{u}_{K,\boldsymbol{y}}\|\} < \delta' R'.$$

When K is a ball, this is trivially true. It extends to the more general setting by observing that the map

$$\begin{array}{rcl} \partial(R'B^d) & \to & \partial K \\ R'\boldsymbol{u} & \mapsto & \boldsymbol{x}_{K,\boldsymbol{u}} \end{array}$$

is 1-Lipschitz because K is a summand of $R'B^d$. □

Lemma 6.1.7. *There exists constants c_{34}, c_{35} and c_{36} such that for any $n > c_{34}$, there exists a set $U \subset \mathbb{S}^{d-1}$ with $|U| = n$ which is both a $(c_{35} n^{-\frac{1}{d-1}})$-packing and $(c_{36} n^{-\frac{1}{d-1}})$-covering of $\mathbb{S}^{d-1}$.*

Proof. Let $c_1 < c_2$ be the constant of Lemma 4.1.4. Set $c_{34} = c_2 V_{d-1}(B^d)$. Apply Lemma 4.1.4 with $D = B^d$ and

$$\delta_1 = \left(\frac{c_2 V_{d-1}(B^d)}{n} \right)^{\frac{1}{d-1}} < 1.$$

We get a set U_1 with

$$|U_1| < c_2 V_{d-1}(B^d) \delta_1^{-(d-1)} = n,$$

which is a δ_1-net of $\mathbb{S}^{d-1}$ and in particular a δ_1-covering of $\mathbb{S}^{d-1}$. We apply again Lemma 4.1.4, but this time with $D = B^d$ and

$$\delta_2 = \left(\frac{c_1 V_{d-1}(B^d)}{n} \right)^{\frac{1}{d-1}} < 1.$$

We get a set U_2 with

$$|U_2| > c_1 V_{d-1}(B^d) \delta_1^{-(d-1)} = n,$$

which is a δ_2-net of $\mathbb{S}^{d-1}$, i.e. a δ_2-covering and $\frac{\delta_2}{2}$-packing of $\mathbb{S}^{d-1}$. In particular for each $\boldsymbol{x} \in U_1$ there exists $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}) \in U_2$ with $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta_2$. And since $|U_1| < n < |U_2|$, we can set $U_3 \subset \mathbb{S}^{d-1}$ such that $|U_3| = n$ and $\{\boldsymbol{y}(\boldsymbol{x}) : \boldsymbol{x} \in U_1\} \subset U_3 \subset U_2$.

Since $U_3 \subset U_2$ it is a $\frac{\delta_2}{2}$-packing of $\mathbb{S}^{d-1}$. And by construction, U_3 is also a $(\delta_1 + \delta_2)$-covering of $\mathbb{S}^{d-1}$. Therefore the lemma holds with $c_{35} = \frac{1}{2}(c_2 V_{d-1}(B^d))^{-\frac{1}{d-1}}$ and $c_{36} = (c_1 V_{d-1}(B^d))^{-\frac{1}{d-1}} + (c_2 V_{d-1}(B^d))^{-\frac{1}{d-1}}$ □

Lemma 6.1.8. *Assume that φ is strongly well spread. Let $K \in \mathcal{K}_{\boldsymbol{o}}$ smooth and strictly convex and $\epsilon > 0$. There exist positive constants $C'_{K,\epsilon}$ and $N_{K,\epsilon}$ such that for any $n > N_{K,\epsilon}$ there exist disjoints subsets $S_1, \dots, S_n \subset \mathcal{H}$ with*

$$\mu(S_i) > C'_{K,\epsilon} n^{-\frac{d+1}{d-1}}$$

and for $H_1 \in S_1, \dots, H_n \in S_n$, we have

$$\bigcap_i H_i^- \in \mathcal{P}_n, \tag{6.5}$$

and

$$d_H\left(K\,,\bigcap_i H_i^-\right) < \epsilon. \tag{6.6}$$

Proof. Since K is smooth and strictly convex there exist $0 < R < R'$ and $L, L' \in \mathcal{K}$ strictly convex such that $K = L + RB^d$ and $R'B^d = K + L'$. Let c_{34}, c_{35} and c_{36} be the constants of Lemma 6.1.7, and set

$$N_{K,\epsilon} = \max\left\{c_{34}, \left(\frac{R(c_{35})^2}{8\min_{\boldsymbol{u}\in\mathbb{S}^{d-1}} h(K,\boldsymbol{u})}\right)^{\frac{d-1}{2}}, \left(\frac{c_{35}}{2R}\right)^{d-1}, \left(\frac{R(c_{35})^2}{8\epsilon}\right)^{\frac{d-1}{2}},\right.$$
$$\left.\left(\frac{\left(c_{36} + \frac{c_{35}}{4}\right)R'}{\epsilon^2}\right)^{d-1}\right\}.$$

For $n > N_{K,\epsilon} \geq c_{34}$, we apply Lemma 6.1.7 and get a set $\{\boldsymbol{u}_i\}_{i\in[n]} \subset \mathbb{S}^{d-1}$ which is both a $(c_{35} n^{-\frac{1}{d-1}})$-packing and $(c_{36} n^{-\frac{1}{d-1}})$-covering of $\mathbb{S}^{d-1}$. Set $\rho = c_{35} n^{-\frac{1}{d-1}}$, and for $i \in [n]$, set

$$S_i = \left\{H(\boldsymbol{u}, t)\ :\ \boldsymbol{u} \in C\left(\boldsymbol{u}_i, \frac{\rho}{4}\right),\ t \in \left[h(K,\boldsymbol{u}) - \frac{R\rho^2}{8}, h(K,\boldsymbol{u})\right]\right\}.$$

Note that, since $n > N_{K,\epsilon} \geq \left(\frac{R(c_{35})^2}{8\min_{\boldsymbol{u}\in\mathbb{S}^{d-1}} h(K,\boldsymbol{u})}\right)^{\frac{d-1}{2}}$ and $\rho = c_{35} n^{-\frac{1}{d-1}}$, we have $\frac{R\rho^2}{8} < \min_{\boldsymbol{u}\in\mathbb{S}^{d-1}} h(K,\boldsymbol{u})$, and thus $t_i > 0$ for any $H(\boldsymbol{u}_i, t_i) \in S_i$. We have

$$\mu(S_i) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(\boldsymbol{u} \in C\left(\boldsymbol{u}_i, \frac{\rho}{4}\right)\right) \mathbb{1}\left(h(K,\boldsymbol{u}) - \frac{R\rho^2}{8} \le t \le h(K,\boldsymbol{u})\right) t^{r-1} \mathrm{d}t \varphi(\mathrm{d}\boldsymbol{u})$$
$$\ge \mathscr{H}^{d-1}\left(C\left(\boldsymbol{u}_i, \frac{\rho}{4}\right)\right) \frac{R\rho^2}{8} \min_{\boldsymbol{u}\in\mathbb{S}^{d-1}} \left(h(K,\boldsymbol{u}) - \frac{R\rho^2}{8}\right)^{r-1}$$
$$\ge C'_{K,\epsilon} n^{-\frac{d+1}{d-1}},$$

for an appropriate constant $C'_{K,\epsilon}$ independent from n.

Let $H_i = H(\boldsymbol{v}_i, t_i) \in S_i$, for $i = 1, \ldots, n$. It remains to show that (6.5) and (6.6) hold.

For $i \in [n]$, let $\boldsymbol{x}_{K,\boldsymbol{v}_i} \in \partial K$ be the point of the boundary of K with outward normal vector $\boldsymbol{v}_i$. Observe the following implications,

$$(6.5) \Leftrightarrow H_i \bigcap \cap_{j\neq i} H_j^- \neq \emptyset, \text{ for any } i,$$
$$\Leftarrow H(\boldsymbol{v}_i, h(K,\boldsymbol{v}_i)) \bigcap \cap_{j\neq i} H\left(\boldsymbol{v}_j, h(K,\boldsymbol{v}_j) - \frac{R\rho^2}{8}\right)^- \neq \emptyset, \text{ for any } i,$$
$$\Leftarrow \boldsymbol{x}_{K,\boldsymbol{v}_i} \in H\left(\boldsymbol{v}_j, h(K,\boldsymbol{v}_j) - \frac{R}{8}\rho^2\right)^-, \text{ for any } i \neq j, \tag{6.7}$$

But, by construction of S_i and with the triangular inequality,

$$\|\boldsymbol{v}_i - \boldsymbol{v}_j\| > \|\boldsymbol{u}_i - \boldsymbol{u}_j\| - \|\boldsymbol{u}_i - \boldsymbol{v}_i\| - \|\boldsymbol{u}_j - \boldsymbol{v}_j\| > \frac{\rho}{2}.$$

Therefore, since $\frac{\rho}{2} < \frac{c_{35}}{2} N_{K,\epsilon}^{-\frac{1}{d-1}} < R$, we can apply Lemma 6.1.5 with $\delta = \rho/2$, which gives (6.7), and therefore (6.5).

By construction we have that

$$K \subset \bigcap_i H_i^- + \frac{R\rho^2}{8} B^d \subset \bigcap_i H_i^- + \epsilon B^d,$$

since $\frac{R\rho^2}{8} < \frac{Rc_{35}^2}{8} N_{K,\epsilon}^{-\frac{2}{d-1}} \le \epsilon$. Observe that since $\{\boldsymbol{u}_i\}_{i\in[n]}$ is a $(c_{36} n^{-\frac{1}{d-1}})$-covering of $\mathbb{S}^{d-1}$ and that for any $i \in [n]$, $\|\boldsymbol{v}_i - \boldsymbol{u}_i\| < \frac{\rho}{4} = \frac{c_{35}}{4} n^{-\frac{1}{d-1}}$, we have that $\{\boldsymbol{v}_i\}_{i\in[n]}$ is a $\left(\left(c_{36} + \frac{c_{35}}{4}\right) n^{-\frac{1}{d-1}}\right)$-covering of $\mathbb{S}^{d-1}$. Thus, applying Lemma 6.1.6 with

$$\delta' = \left(c_{36} + \frac{c_{35}}{4}\right) n^{-\frac{1}{d-1}} < \left(c_{36} + \frac{c_{35}}{4}\right) N_{K,\epsilon}^{-\frac{1}{d-1}} \le \frac{\epsilon^2}{R'}$$

gives

$$K + \epsilon B^d \supset K + \sqrt{\delta' R'} B^d \supset \bigcap_i H(\boldsymbol{v}_i, h(K,\boldsymbol{v}_i))^- \supset \bigcap_i H_i^-.$$

Therefore (6.6) holds. □

Proof of Theorem 6.1.3. Here, we prove Theorem 6.1.3 in the case of the zero cell. We omit the proof in the case of the typical cell since it follows the same lines with the same adaptations as the one made between the proofs of Theorem 5.1.7 and 5.2.6.

Let $K' \in \mathcal{K}_{\boldsymbol{o}}$ smooth and strictly convex such that $d_H(K, K') < \epsilon/2$. Set $\epsilon' > 0$ such that for any convex body $L \in \mathcal{K}_{\boldsymbol{o}}$ with $d_H(K', L) < \epsilon'$ we have $d_H(K', \mathfrak{s}_\Phi(L)) < \epsilon/2$. With this setting, we only have to prove that there exist constants $C_{K,\epsilon} > 0$ and $N_{K,\epsilon}$ such that

$$\mathbb{P}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K'\right) < \epsilon' \mid f(Z_{\boldsymbol{o}}) = n\right) > (C_{K,\epsilon})^n,$$

for $n > N_{K,\epsilon}$.

Similarly as in the beginning of the proof of Theorem 5.1.1, (3.7) and (3.4) gives

$$\begin{aligned}
&\mathbb{P}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K'\right) < \epsilon',\ f(Z_{\boldsymbol{o}}) = n\right) \\
&= \int\limits_{\mathcal{H}^n} \mathbb{1}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K'\right) < \epsilon'\right) \mathbb{1}\left(P_{[n]} \in \mathcal{P}_{n,\boldsymbol{o}}\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right),
\end{aligned}$$

where $P_{[n]} = \cap_{i=1}^n H_i^-$. Let $C'_{K,\epsilon}$, and $N_{K,\epsilon}$, and $S_1, \ldots, S_n$ be as in Lemma 6.1.8, we then have

$$\begin{aligned}
&\mathbb{P}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K'\right) < \epsilon',\ f(Z_{\boldsymbol{o}}) = n\right) \\
&\geq n! \int\limits_{\mathcal{H}^n} \mathbb{1}\left(H_1 \in S_1\right) \cdots \mathbb{1}\left(H_n \in S_n\right) \mathrm{d}\mu^n\left(\boldsymbol{H}\right) \\
&= n! \prod_{i=1}^n \mu(S_i) \\
&\geq n! \left(C'_{K,\epsilon} n^{-\frac{d+1}{d-1}}\right)^n.
\end{aligned}$$

Therefore, with Stirling approximation $n! \geq n^n e^{-n}$, we have

$$\mathbb{P}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K'\right) < \epsilon',\ f(Z_{\boldsymbol{o}}) = n\right) \geq \left(e^{-1} C'_{K,\epsilon}\right)^n n^{-\frac{2n}{d-1}}. \tag{6.8}$$

Thus, with the upper bound (6.2), we have

$$\begin{aligned}
&\mathbb{P}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K'\right) < \epsilon' \mid f(Z_{\boldsymbol{o}}) = n\right) \\
&= \frac{\mathbb{P}\left(d_H\left(\mathfrak{s}_\Phi(Z_{\boldsymbol{o}}), K'\right) < \epsilon',\ f(Z_{\boldsymbol{o}}) = n\right)}{\mathbb{P}\left(f(Z_{\boldsymbol{o}}) = n\right)} > \left(\frac{e^{-1} C'_{K,\epsilon}}{C_{22}}\right)^n.
\end{aligned}$$

Setting $C_{K,\epsilon} = e^{-1} C'_{K,\epsilon} C_{22}^{-1}$ yields the proof. ☐

6.1.3 Φ-content of the cells with many facets

The Complementary Theorem 3.2.1 (resp. 3.3.1) tells us that, under the condition that $Z_{\boldsymbol{o}}$ (resp. Z_{typ}) has n facets, its Φ-content is $\Gamma_{\gamma,n}$ (resp. $\Gamma_{\gamma,n-d}$) distributed. With respect to the Convention 6.0.1 it says that, under the condition that Z has n facets, its Φ-content is $\Gamma_{\gamma,n-\widetilde{d}}$ distributed.

6.1.4 Σ-content of the cells with many facets

Let

$$\tau = \tau(\Phi, \Sigma) := \inf_{K \in \mathcal{K}} \frac{\Phi(K)}{\Sigma(K)^{\frac{r}{k}}},$$

so that $\Phi(K) \geq \tau \Sigma(K)^{r/k}$ for any $K \in \mathcal{K}$. A convex body K_{ext} for which $\Phi(K_{\text{ext}}) = \tau \Sigma(K_{\text{ext}})^{r/k}$ is call *extremal*. Thanks to the Blaschke selection Theorem 2.1.1, there exists extremal bodies. We give now two examples to make the reader familiar with the notion. In the isotropic case, Φ is proportional to V_1, hence if $\Sigma = V_k$ with $k \geq 2$, the Isoperimetric Inequality (2.1) says that the balls are the only extremal bodies. If, in the situation we just described, we have $\Sigma = V_1$ instead, then $\Phi(K)\Sigma(K)^{-\frac{r}{k}}$ is constant and every convex body is extremal.

Now we give bounds for the distribution of the Σ-content under condition on the number facets.

Theorem 6.1.9. *For $a > (\gamma\tau)^{-k/r}$,*

$$\mathbb{P}(\Sigma(Z_{\boldsymbol{o}}) > a \mid f(Z_{\boldsymbol{o}}) = n) < \exp\left(-\gamma\tau a^{\frac{r}{k}} + 1\right) \left(\gamma\tau a^{\frac{r}{k}}\right)^{n-1},$$

and

$$\mathbb{P}(\Sigma(Z_{\text{typ}}) > a \mid f(Z_{\text{typ}}) = n) < \exp\left(-\gamma\tau a^{\frac{1}{k}} + 1\right) \left(\gamma\tau a^{\frac{1}{k}}\right)^{n-d-1}.$$

For any $\epsilon > 0$, there exists constants C_ϵ and N_ϵ such that for $n > N_\epsilon$ and $a > 0$,

$$\begin{aligned}&\mathbb{P}(\Sigma(Z_{\boldsymbol{o}}) > a \mid f(Z_{\boldsymbol{o}}) = n)\\ &> \frac{(C_\epsilon)^n}{n!} \exp\left(-\gamma(\tau+\epsilon) a^{\frac{r}{k}}\right) \left(\gamma(\tau+\epsilon) a^{\frac{r}{k}}\right)^{n-1},\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}(\Sigma(Z_{\text{typ}}) > a \mid f(Z_{\text{typ}}) = n)\\ &> \frac{(C_\epsilon)^n}{n!} \exp\left(-\gamma(\tau+\epsilon) a^{\frac{1}{k}}\right) \left(\gamma(\tau+\epsilon) a^{\frac{1}{k}}\right)^{n-d-1}.\end{aligned}$$

Proof. This proof covers both $Z_{\boldsymbol{o}}$ and Z_{typ} by using the Convention 6.0.1. We start by proving the upper bound. By definition of τ we trivially have

$$\mathbb{P}(\Sigma(Z) > a \mid f(Z) = n) < \mathbb{P}\left(\Phi(Z) > \tau a^{\frac{r}{k}} \mid f(Z) = n\right),$$

which can be written

$$\mathbb{P}(\Sigma(Z) > a \mid f(Z) = n) < \frac{\gamma^{n-\widetilde{d}}}{(n-\widetilde{d}-1)!} \int\limits_{\tau a^{r/k}}^{\infty} e^{-\gamma t} t^{n-\widetilde{d}-1} \mathrm{d}t,$$

because of the Complementary Theorems 3.2.1 and 3.3.1. Now observe that, iterated integrations by part give, for any $a' \in \mathbb{R}$,

$$\frac{\gamma^{n-\widetilde{d}}}{(n-\widetilde{d}-1)!} \int\limits_{a'}^{\infty} e^{-\gamma t} t^{n-\widetilde{d}-1} \mathrm{d}t = e^{-\gamma a'} \sum_{m=0}^{n-\widetilde{d}-1} \frac{(\gamma a')^m}{m!}. \tag{6.9}$$

Thus, for $a > (\gamma\tau)^{-k/r}$,

$$\begin{aligned}\mathbb{P}(\Sigma(Z) > a \mid f(Z) = n) &< \exp\left(-\gamma\tau a^{\frac{r}{k}}\right) \left(\sum_{m=0}^{n-\widetilde{d}-1} \frac{1}{m!}\right) \left(\gamma\tau a^{\frac{r}{k}}\right)^{n-\widetilde{d}-1} \\ &< \exp\left(-\gamma\tau a^{\frac{r}{k}} + 1\right) \left(\gamma\tau a^{\frac{r}{k}}\right)^{n-\widetilde{d}-1},\end{aligned}$$

which is the upper bound of the theorem.

The proof of the lower bound is similar. Set $K_{\text{ext}} \in \mathcal{K}_{\mathfrak{s}}$ to be an extremal body, i.e. such that $\Phi(K_{\text{ext}}) = \tau\Sigma(K_{\text{ext}})^{r/k}$. Set $\epsilon' > 0$ small enough such that

$$\sup\left\{\frac{\Phi(K)}{\Sigma(K)^{\frac{r}{k}}} \; : \; K \in \mathcal{K} \; , \; d_H(\mathfrak{s}(K), K_{\text{ext}}) < \epsilon'\right\} < \tau + \epsilon.$$

With this setting we have

$$\begin{aligned}&\mathbb{P}(\Sigma(Z) > a \mid f(Z) = n) \\ &> \mathbb{P}\left(\Sigma(Z) > a, \; d_H(\mathfrak{s}(Z), K_{\text{ext}}) < \epsilon' \mid f(Z) = n\right) \\ &> \mathbb{P}\left(\Phi(Z) > (\tau+\epsilon)a^{r/k}, \; d_H(\mathfrak{s}(Z), K_{\text{ext}}) < \epsilon' \mid f(Z) = n\right),\end{aligned}$$

which can be written

$$\begin{aligned}&\mathbb{P}(\Sigma(Z) > a \mid f(Z) = n) \\ &> \mathbb{P}\left(d_H(\mathfrak{s}(Z), K_{\text{ext}}) < \epsilon' \mid f(Z) = n\right) \frac{\gamma^{n-\widetilde{d}}}{(n-\widetilde{d}-1)!} \int\limits_{(\tau+\epsilon)a^{r/k}}^{\infty} e^{-\gamma t} t^{n-\widetilde{d}-1} \mathrm{d}t.\end{aligned}$$

because of the Complementary Theorems 3.2.1 and 3.3.1. But Theorem 6.1.3 tells us that there exist positive constants $C_{K_{\text{ext}},\epsilon'}$ and $N_{K_{\text{ext}},\epsilon'}$ such that, for $n > N_{K_{\text{ext}},\epsilon'}$,

$$\mathbb{P}\left(d_H(\mathfrak{s}(Z), K_{\text{ext}}) < \epsilon' \mid f(Z) = n\right) > (C_{K_{\text{ext}},\epsilon'})^n.$$

And with (6.9) we have that

$$\frac{\gamma^{n-\widetilde{d}}}{(n-\widetilde{d}-1)!}\int\limits_{(\tau+\epsilon)a^{r/k}}^{\infty} e^{-\gamma t}t^{n-\widetilde{d}-1}\mathrm{d}t > e^{-\gamma(\tau+\epsilon)a^{r/k}}\frac{\left(\gamma(\tau+\epsilon)a^{r/k}\right)^{n-\widetilde{d}-1}}{(n-\widetilde{d}-1)!}.$$

Combining the three last inequalities implies the theorem since $n-\widetilde{d} \leq n$. □

6.2 Big Φ-content

6.2.1 Tail of the distribution of the Φ-content

In this subsection we will prove the following bounds for the tail distribution of the Φ-content of $Z_{\boldsymbol{o}}$ and Z_{typ}.

Theorem 6.2.1. *There exist constants $C_{24} > C_{25} > 0$ and $C_{26} > 0$ depending on φ, such that the following holds. For $a > 0$, we have*

$$\mathbb{P}(\Phi(Z) > a) < \exp\left\{-\gamma a + C_{24}(\gamma a)^{\frac{d-1}{d+1}}\right\},$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}. Assume that φ is well spread. Then, for $a > \gamma^{-1}C_{26}$, we also have

$$\mathbb{P}(\Phi(Z) > a) > \exp\left\{-\gamma a + C_{25}(\gamma a)^{\frac{d-1}{d+1}}\right\},$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

In the rest of the subsection, in order to treat at once both the case of $Z_{\boldsymbol{o}}$ and Z_{typ} we use the Convention 6.0.1.

We start with three intermediary lemmas: Lemma 6.2.2 builds upon the Complementary Theorems to get a rewriting of the distribution tail of $\Phi(Z)$ as a function of the distribution tail of $f(Z)$. In Lemma 6.2.3 we deduce upper and lower-bounds for the tail distribution of $f(Z)$ from (6.1) and (6.2). Finally, Lemma 6.2.4 contains analytical estimates for some subexponential power series.

In the sequel we use the abbreviations

$$q_n := \mathbb{P}(f(Z) = n) \text{ and } r_n := \sum_{k\geq n} q_k.$$

Note that for $n \leq d$, $q_n = 0$.

In the following lemma, we rewrite the probability $\mathbb{P}(\Phi(Z) > a)$ as a power series in a. Recall that $\widetilde{d} = 0$ when $Z = Z_{\boldsymbol{o}}$ and $\widetilde{d} = d$ when $Z = Z_{\text{typ}}$.

Lemma 6.2.2. *For every $a > 0$, we have*

$$\mathbb{P}(\Phi(Z) > a) = e^{-\gamma a} \sum_{n \geq 0} r_{n+\widetilde{d}+1} \frac{(\gamma a)^n}{n!}.$$

Proof. Because of the Complementary Theorems 3.2.1 and 3.3.1 we have for every $a > 0$

$$\begin{aligned}\mathbb{P}(\Phi(Z) > a) &= \sum_{n \geq d+1} q_n \mathbb{P}(\Phi(Z) > a \mid f(Z) = n) \\ &= \sum_{n \geq d+1} q_n \int_a^\infty e^{-\gamma t} \frac{\gamma^{n-\widetilde{d}} t^{n-\widetilde{d}-1}}{(n-\widetilde{d}-1)!} \mathrm{d}t.\end{aligned}$$

Now we recall that iterated integrations by parts show that for every $n \geq d+1$,

$$\int_a^\infty e^{-\gamma t} \frac{\gamma^{n-\widetilde{d}} t^{n-\widetilde{d}-1}}{(n-\widetilde{d}-1)!} \mathrm{d}t = e^{-\gamma a} \sum_{m=0}^{n-\widetilde{d}-1} \frac{(\gamma a)^m}{m!}.$$

Consequently, we obtain that

$$\begin{aligned}\mathbb{P}(\Phi(Z) > a) &= e^{-\gamma a} \sum_{n \geq d+1} \sum_{m=0}^{n-\widetilde{d}-1} q_n \frac{(\gamma a)^m}{m!} \\ &= e^{-\gamma a} \sum_{m \geq 0} \sum_{n \geq \max(d+1, m+\widetilde{d}+1)} q_n \frac{(\gamma a)^m}{m!}.\end{aligned}$$

But since $q_n = 0$ for $n \leq d$, it holds that $\sum_{n \geq \max(d+1, m+\widetilde{d}+1)} q_n = r_{m+\widetilde{d}+1}$ even when $m + \widetilde{d} + 1 \leq d$. Therefore

$$\mathbb{P}(\Phi(Z) > a) = e^{-\gamma a} \sum_{m \geq 0} r_{m+\widetilde{d}+1} \frac{(\gamma a)^m}{m!}$$

which yields the proof. □

The relation from Lemma 6.2.2 indicates that in order to bound $\mathbb{P}(\Phi(Z) > a)$, we need to find bounds for r_{n+d+1}. This is done in the next lemma.

Lemma 6.2.3. *There exists a constant C_{27} depending on φ such that for $n \geq 0$ we have*

$$r_{n+\widetilde{d}+1} < C_{27}^n (n!)^{-\frac{2}{d-1}}.$$

Assume that φ is well spread. Then there exists a constant $C_{28} > 0$ depending on φ such that for $n \geq 0$ we have

$$r_{n+\widetilde{d}+1} \geq C_{28}^n (n!)^{-\frac{2}{d-1}}.$$

Proof. We start with the upper-bound. Recall the upper bound (6.2)

$$q_n < C_{22}^n n^{-\frac{2n}{d-1}}.$$

This gives

$$\begin{aligned} r_{n+\widetilde{d}+1} &< \sum_{k \geq n+\widetilde{d}+1} C_{22}^k\, k^{-\frac{2}{d-1}k} \\ &< C_{22}^n n^{-\frac{2}{d-1}n} \sum_{k \geq \widetilde{d}+1} C_{22}^k\, k^{-\frac{2}{d-1}k}. \end{aligned}$$

We use $n^{-n} \leq (n!)^{-1}$, and observe that the remaining sum is convergent and independent of n. Hence in order to get the upper-bound, it suffices to set

$$C_{27} := C_{22} \max \left\{ 1, \sum_{k \geq d+1} C_{22}^k\, k^{-\frac{2}{d-1}k} \right\}.$$

We assume now that φ is well spread and prove the lower-bound for $r_{n+\widetilde{d}+1}$. The lower bound (6.1) tells us that when φ is well spread, for every $n \geq 0$,

$$q_{n+d+1} > C_{21}^{n+d+1}\,(n+d+1)^{-\frac{2(n+d+1)}{d-1}}$$

Consequently, using Stirling's approximation $n^{-n} > e^{-n}(n!)^{-1}$ and the simple inequality $r_{n+\widetilde{d}+1} \geq r_{n+d+1} > q_{n+d+1}$, we get

$$\begin{aligned} r_{n+\widetilde{d}+1} &> \left(C_{21} e^{-\frac{2}{d-1}}\right)^{n+d+1} [(n+d+1)!]^{-\frac{2}{d-1}} \\ &> \left(C_{21} e^{-\frac{2}{d-1}}\right)^{n+d+1} [(n+d+1)^{d+1} \cdot n!]^{-\frac{2}{d-1}} \\ &> \left(C_{21} (d+1)^{-\frac{2}{d-1}} e^{-\frac{2}{d-1}}\right)^{n+d+1} (n!)^{-\frac{2}{d-1}} \end{aligned}$$

because $(n+d+1)^{d+1} < (d+1)^{n+d+1}$ for $n+d+1 \geq d+1 \geq 3$.

Taking $C_{28} = C_{21}(d+1)^{-\frac{2}{d-1}} e^{-\frac{2}{d-1}} \min(1, (C_{21}(d+1)^{-\frac{2}{d-1}} e^{-\frac{2}{d-1}})^{d+1})$, we get the required result. □

The combination of the two previous lemmas implies that $e^{\gamma a}\mathbb{P}(\Phi(Z) > a)$ is well approximated by subexponential power series of type $\sum_{n \geq 0} \frac{x^n}{(n!)^\alpha}$. The next lemma, which is purely analytical, investigates the behaviour of such power series.

Lemma 6.2.4. *For any $\alpha > 1$, we have*

$$\exp\left(\frac{1}{2}\alpha x^{\frac{1}{\alpha}}\right) < \sum_{n \geq d+1} \frac{x^n}{(n!)^\alpha} < \sum_{n \geq 0} \frac{x^n}{(n!)^\alpha} < \exp\left(\alpha x^{\frac{1}{\alpha}}\right)$$

where the first inequality holds for $x \geq (2(3d+5))^\alpha$ and the second for all $x > 0$.

Proof. The right-hand side inequality follows immediately from the following simple computations.

$$\sum_{n\geq d+1} \frac{x^n}{(n!)^\alpha} < \sum_{n\geq 0} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha < \left(\sum_{n\geq 0} \frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha = \exp\left(\alpha x^{\frac{1}{\alpha}}\right).$$

For the left-hand side inequality, Hölder's inequality gives for any finite $I \subset \mathbb{N} \setminus [d+1]$

$$\sum_{n\geq d+1} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha \geq \sum_{n\in I} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha \geq |I|^{-(\alpha-1)} \left(\sum_{n\in I} \frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha. \tag{6.10}$$

For Y a Poisson distributed random variable with mean λ it is well known, and can be proved e.g. by Chebishev's inequality, that for $I = (\lambda - \sqrt{2\lambda}, \lambda + \sqrt{2\lambda}) \cap \mathbb{N}$, we have

$$\sum_{n\in I} e^{-\lambda} \frac{\lambda^n}{n!} = 1 - \mathbb{P}\left(|Y-\lambda| \geq \sqrt{2\lambda}\right) \geq \frac{1}{2}.$$

I has at most $2\sqrt{2\lambda}+1 < 4\sqrt{\lambda}$ elements, when $\lambda \geq 1$. Putting this for $\lambda = x^{1/\alpha}$ into (6.10) yields

$$\sum_{n\geq d+1} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha \geq \left(4x^{\frac{1}{2\alpha}}\right)^{-(\alpha-1)} \left(e^{x^{\frac{1}{\alpha}}} \frac{1}{2}\right)^\alpha \geq \left(8^{-\alpha} x^{-\frac{1}{2}}\right) e^{\alpha x^{\frac{1}{\alpha}}}$$

as long as the condition $x^{1/\alpha} - \sqrt{2x^{1/\alpha}} \geq d+1$ is fulfilled. Observe that $x \geq (3d+5)^\alpha$ implies $x^{1/(2\alpha)} \geq \sqrt{d+2}+1$ which in turn implies $x^{1/\alpha} - 2x^{1/(2\alpha)} + 1 \geq d+2$ which gives the required condition.

For $t \geq 3$ we have $2\ln 8 + t \leq 1 + t + t^2/2 \leq e^t$, or equivalently

$$-\alpha \ln 8 - \frac{1}{2}\ln x \geq -\frac{1}{2}\alpha x^{\frac{1}{\alpha}}, \quad \text{i.e.} \quad 8^{-\alpha} x^{-\frac{1}{2}} \geq e^{-\frac{1}{2}\alpha x^{1/\alpha}}$$

for $x^{1/\alpha} \geq e^3$. The inequality $2(3d+5) > e^3$ concludes the proof. □

We are now ready to prove Theorem 6.2.1.

Proof of Theorem 6.2.1. We start by proving the upper bound. Combining Lemma 6.2.2 and the upper bound of Lemma 6.2.3, we get

$$\mathbb{P}(\Phi(Z) > a) < e^{-\gamma a} \sum_{n\geq 0} C_{27}^n \frac{(\gamma a)^n}{(n!)^{\frac{d+1}{d-1}}}.$$

Applying now Lemma 6.2.4 to $x = C_{27}\gamma a$ and $\alpha = \frac{d+1}{d-1}$, we obtain that

$$\mathbb{P}(\Phi(Z) > a) < \exp\left(-\gamma a + \frac{d+1}{d-1}(C_{27}\gamma a)^{\frac{d-1}{d+1}}\right).$$

Setting $C_{24} = \frac{d+1}{d-1} C_{27}^{\frac{d-1}{d+1}}$ yields the proof of the upper bound.

The proof of the lower bound is nearly identical. Combining Lemma 6.2.2 and the lower bound of Lemma 6.2.3, we get

$$\mathbb{P}(\Phi(Z) > a) > e^{-\gamma a} \sum_{n \geq 0} C_{28}^{n} \frac{(\gamma a)^n}{(n!)^{\frac{d+1}{d-1}}}.$$

Applying now Lemma 6.2.4 to $x = C_{28}\gamma a$ and $\alpha = \frac{d+1}{d-1}$, we obtain that

$$\mathbb{P}(\Phi(Z) > a) > \exp\left(-\gamma a + \frac{d+1}{2(d-1)}(C_{28}\gamma a)^{\frac{d-1}{d+1}}\right)$$

for $C_{28}\gamma a > (2(3d+5))^{(d+1)/(d-1)}$. Setting $C_{25} = \frac{d+1}{2(d-1)} C_{28}^{\frac{d-1}{d+1}}$ and $C_{26} = C_{28}^{-1}(2(3d+5))^{(d+1)/(d-1)}$ yields the proof of the lower bound. □

6.2.2 Shape of cells with big Φ-content

In this subsection we show

Theorem 6.2.5. *Assume that φ is well spread and that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. Then for any $\epsilon > 0$ sufficiently small we have*

$$\lim_{a \to \infty} \mathbb{P}\left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \mid \Phi(Z) > a\right) = 0,$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

It is a direct corollary of the lower bound

$$\mathbb{P}(\Phi(Z) > a) > \exp\left\{-\gamma a + C_{25}(\gamma a)^{\frac{d-1}{d+1}}\right\},$$

of Theorem 6.2.1, and the following theorem, which must be applied with $\delta < C_{25}$.

Theorem 6.2.6. *Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. For any $\delta > 0$, there exists $\epsilon > 0$ such that for any $a > 0$*

$$\mathbb{P}\left(\Phi(Z) > a,\ \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon\right) \leq \exp\left(-\gamma a + \delta(\gamma a)^{\frac{d-1}{d+1}}\right),$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

In the rest of the subsection, in order to treat at once both the case of $Z_{\boldsymbol{o}}$ and Z_{typ} we use the Convention 6.0.1.

Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. In the sequel, we use the notation $q_n^\epsilon := \mathbb{P}\left(f(Z) = n,\ \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \right)$ and $r_n^\epsilon := \sum_{k \geq n} q_k^\epsilon$, for every $n \geq d+1$ and $\epsilon > 0$. The proof follows along the same lines as the upper bound of Theorem 6.2.1 with minor adaptations. Indeed, we need some analogues to the statements of Lemmas 6.2.2 and 6.2.3 when q_n is replaced by q_n^ϵ, i.e. when the extra-condition that Z is $(\epsilon\colon i, j)$-elongated is added.

The lemma below is a rewriting of the joint distribution of $(\mathfrak{s}(Z), \Phi(Z))$ as a power series.

Lemma 6.2.7. *For any measurable set of shapes S and $a > 0$, we have*

$$\begin{aligned} \mathbb{P}(\mathfrak{s}(Z) \in S,\ \Phi(Z) > a) &= e^{-\gamma a} \sum_{k \geq d+1} \mathbb{P}(\mathfrak{s}(Z) \in S,\ f(Z) = k) \sum_{n=0}^{k-\widetilde{d}-1} \frac{(\gamma a)^n}{n!} \\ &= e^{-\gamma a} \sum_{n \geq 0} \mathbb{P}(\mathfrak{s}(Z) \in S,\ f(Z) \geq n + \widetilde{d} + 1) \frac{(\gamma a)^n}{n!}. \end{aligned}$$

The proof of this result is fully analogous to that of Lemma 6.2.2 and is therefore omitted.

As in Lemma 6.2.3, we require now an upper-bound for r_{n+d+1}^ϵ.

Lemma 6.2.8. *Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. For any $\delta > 0$, there exists $\epsilon > 0$ such that*

$$r_{n+\widetilde{d}+1}^\epsilon < \delta^n (n!)^{-\frac{2}{d-1}}$$

for $n \geq 0$.

Proof. Set ϵ such that (6.3) holds, that is

$$q_n^\epsilon < \delta^n n^{-\frac{2n}{d-1}}$$

for n bigger than a constant C_{23}. Since for any n, $q_n^\epsilon \to 0$ when $\epsilon \to 0$, we can assume that it holds for any n. Hence

$$\begin{aligned} r_{n+\widetilde{d}+1}^\epsilon &< \sum_{k \geq n+\widetilde{d}+1} \delta^k\ k^{-\frac{2k}{d-1}} \\ &\leq \delta^n\ n^{-\frac{2n}{d-1}} \sum_{k \geq \widetilde{d}+1} \delta^k k^{-\frac{2k}{d-1}}. \end{aligned}$$

But without loss of generality we can assume that δ is small enough such that the series is smaller than 1. Thus, the trivial bound $n^{-n} \leq (n!)^{-1}$ concludes the proof. $\square$

Let us now proceed with the proof of Theorem 6.2.6. Applying Lemma 6.2.7 to the set $S = \{K \in \mathcal{K}_{\mathfrak{c},\Phi} \colon V_j(Z)^{1/j} V_i(Z)^{-1/i} < \epsilon\}$, we get

$$\mathbb{P}\left(\Phi(Z) > a,\ \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon\right) = e^{-\gamma a} \sum_{n \geq 0} r^{\epsilon}_{n+\tilde{d}+1} \frac{(\gamma a)^n}{n!}.$$

We combine this with Lemma 6.2.8 to deduce

$$\mathbb{P}\left(\Phi(Z) > a,\ \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon\right) \leq e^{-\gamma a} \sum_{n \geq 0} \frac{(\delta\gamma a)^n}{(n!)^{\frac{d+1}{d-1}}}.$$

Lemma 6.2.4 applied to $x = \delta\gamma a$ and $\alpha = \frac{d+1}{d-1}$ ends the proof:

$$\mathbb{P}\left(\Phi(Z) > a,\ \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon\right) \leq \exp\left(-\gamma a + \frac{d+1}{d-1}(\delta\gamma a)^{\frac{d-1}{d+1}}\right).$$

6.3 Big Σ-content

6.3.1 Tail distribution of the Σ-content.

Hug and Schneider proved in [HS07, Thm. 2] that

$$\lim_{a \to \infty} a^{-\frac{r}{k}} \ln\left(\mathbb{P}\left(\Sigma(Z_{\boldsymbol{o}}) > a\right)\right) = -\tau\gamma.$$

In this subsection, we prove the following theorem, which is a slightly stronger result (see Corollary 6.3.2 for an easy comparison with the result of Hug and Schneider) and which also cover the case of Z_{typ}.

Theorem 6.3.1. *For any $\epsilon \geq 0$ the following holds.*

Upper bound. *For any $a > 0$, we have*

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right) < \exp\left(-(\tau+\epsilon)\gamma a^{\frac{r}{k}} + C_{24}\left((\tau+\epsilon)\gamma a^{\frac{r}{k}}\right)^{\frac{d-1}{d+1}}\right),$$

where C_{24} is the constant from Theorem 6.2.1, and where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

Lower bound. *Assume there exists $K \in \mathcal{K}$ with $\Phi(K)\Sigma(K)^{-r/k} > \tau + \epsilon$. For any $\epsilon' > \epsilon$ there exists $A_{\epsilon,\epsilon'}$ such that for any $a > A_{\epsilon,\epsilon'}$,*

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right) > \exp\left(-(\tau+\epsilon')\gamma a^{\frac{r}{k}}\right),$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

From this theorem we get the following corollary.

Corollary 6.3.2. *For any $\epsilon \geq 0$, if there exists $K \in \mathcal{K}$ such that $\Phi(K)\Sigma(K)^{-r/k} > \tau + \epsilon$, then we have*

$$\lim_{a\to\infty} a^{-\frac{r}{k}} \ln\left(\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right)\right) = -(\tau+\epsilon)\gamma, \qquad (6.11)$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}. In particular, it always holds that

$$\lim_{a\to\infty} a^{-\frac{r}{k}} \ln\left(\mathbb{P}\left(\Sigma(Z) > a\right)\right) = -\tau\gamma, \qquad (6.12)$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

Proof. From the lower bound of Theorem 6.3.1, we get

$$\lim_{a\to\infty} a^{-\frac{r}{k}} \ln\left(\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right)\right) \geq -(\tau+\epsilon')\gamma,$$

for any $\epsilon' > \epsilon$. Thus

$$\lim_{a\to\infty} a^{-\frac{r}{k}} \ln\left(\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right)\right) \geq -(\tau+\epsilon)\gamma. \qquad (6.13)$$

And using the upper bound of the same theorem, we see that (6.13) holds as well if we replace the lower bound by an upper bound. Thus we get the first part of the corollary.

To prove the second part we need to distinguish two cases. In the first case, there exists $K \in \mathcal{K}$ with $\Phi(K)\Sigma(K)^{-r/k} > \tau$, and thus by setting $\epsilon = 0$, (6.11) implies (6.12).

In the second case $\Phi(K) = \Sigma(K)^{\frac{r}{k}}$ for any $K \in \mathcal{K}$. Thus, Theorem 6.2.1 implies (6.12). □

In the rest of the subsection, in order to treat at once both the case of $Z_{\boldsymbol{o}}$ and Z_{typ} we use the Convention 6.0.1.

Upper bound

Observe that $\Sigma(Z) > a$ and $\Phi(Z)\Sigma(Z)^{-r/k} \geq \tau+\epsilon$ imply $\Phi(Z) > (\tau+\epsilon)a^{r/k}$. Thus

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right) < \mathbb{P}\left(\Phi(Z) > (\tau+\epsilon)a^{r/k}\right).$$

Therefore, with the upper bound of the tail distribution of the Φ-content in Theorem 6.2.1, we have

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right)$$
$$< \exp\left(-\gamma(\tau+\epsilon)a^{r/k} + C_{24}\left(\gamma(\tau+\epsilon)a^{r/k}\right)^{\frac{d-1}{d+1}}\right),$$

which proves the upper bound of Theorem 6.3.1.

Lower bound

The proof of the lower bound is more complicate to get because the ratio $\Phi(K)\Sigma(K)^{-r/k}$ has a lower bound but no upper bound. The first part of the proof consists in restricting the problem to cells with n facets and for which this ratio is upper bounded by $(\tau + \epsilon')$. The rests of the proof has many similarities with the proof of the lower bound of the tail distribution of $\Phi(Z)$ in Theorem 6.2.1.

Set $K_{\epsilon,\epsilon'} \in \mathcal{K}_{\mathfrak{s}}$ such that

$$\tau + \epsilon < \frac{\Phi(K_{\epsilon,\epsilon'})}{\Sigma(K_{\epsilon,\epsilon'})^{\frac{r}{k}}} < \tau + \epsilon',$$

and let $\epsilon'' > 0$ such that, for any $K \in \mathcal{K}$,

$$d_H(\mathfrak{s}(K), K_{\epsilon,\epsilon'}) < \epsilon'' \ \Rightarrow\ \tau + \epsilon < \frac{\Phi(K)}{\Sigma(K)^{\frac{r}{k}}} < \tau + \epsilon'.$$

Thus,

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon,\ f(Z) = n\right)$$
$$\geq \mathbb{P}\left(\Sigma(Z) > a,\ d_H(\mathfrak{s}(Z), K_{\epsilon,\epsilon'}) < \epsilon'',\ f(Z) = n\right)$$
$$\geq \mathbb{P}\left(\Phi(Z) > (\tau+\epsilon')a^{r/k},\ d_H(\mathfrak{s}(Z), K_{\epsilon,\epsilon'}) < \epsilon'',\ f(Z) = n\right).$$

This implies

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right)$$
$$= \sum_{n \geq d+1} \mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon,\ f(Z) = n\right)$$
$$\geq \sum_{n \geq d+1} \mathbb{P}\left(\Phi(Z) > (\tau+\epsilon')a^{r/k},\ d_H(\mathfrak{s}(Z), K_{\epsilon,\epsilon'}) < \epsilon'',\ f(Z) = n\right). \quad (6.14)$$

In order to bound the terms of this sum we use the Complementary Theorems 3.2.1 and 3.3.1, which tells us that for a fixed number of facets, the Φ-content and the shape of Z are independent, and that $\Phi(Z)$ is $\Gamma_{\gamma,n-\widetilde{d}}$ distributed. Hence by setting $A=(\tau+\epsilon')a^{r/k}$ we have

$$\begin{aligned}
&\mathbb{P}\left(\Phi(Z)>A,\ d_H(\mathfrak{s}(Z),K_{\epsilon,\epsilon'})<\epsilon'',\ f(Z)=n\right)\\
&\geq \mathbb{P}\left(d_H(\mathfrak{s}(Z),K_{\epsilon,\epsilon'})<\epsilon'',\ f(Z)=n\right)\mathbb{P}\left(\Phi(Z)>A \mid f(Z)=n\right)\\
&=\mathbb{P}\left(d_H(\mathfrak{s}(Z),K_{\epsilon,\epsilon'})<\epsilon'',\ f(Z)=n\right)e^{-A}\sum_{l=0}^{n-\widetilde{d}-1}\frac{A^l}{l!}.
\end{aligned}$$

Therefore with (6.8) from the proof of Theorem 6.1.3, there exists constants $C_{\epsilon,\epsilon'}$ and $N_{\epsilon,\epsilon'}$ such that, for $n\geq N_{\epsilon,\epsilon'}$, we have

$$\begin{aligned}
&\mathbb{P}\left(\Phi(Z)>A,\ d_H(\mathfrak{s}(Z),K_{\epsilon,\epsilon'})<\epsilon'',\ f(Z)=n\right)\\
&\geq \left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}}e^{-A}\sum_{l=0}^{n-\widetilde{d}-1}\frac{A^l}{l!},
\end{aligned}$$

and thus (6.14) implies

$$\begin{aligned}
\mathbb{P}\left(\Sigma(Z)>a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}}\geq\tau+\epsilon\right) &> \sum_{n\geq N_{\epsilon,\epsilon'}}\left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}}e^{-A}\sum_{l=0}^{n-\widetilde{d}-1}\frac{A^l}{l!}\\
&= e^{-A}\sum_{l\geq 0}\ \sum_{n\geq\max\left(N_{\epsilon,\epsilon'},l+\widetilde{d}+1\right)}\left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}}\frac{A^l}{l!}\\
&\overset{a\to\infty}{\sim} e^{-A}\sum_{l\geq 0}\ \sum_{n\geq l+\widetilde{d}+1}\left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}}\frac{A^l}{l!}.
\end{aligned}\tag{6.15}$$

We now bound $\sum_{n\geq l+\widetilde{d}+1}\left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}}$ in the same way as we bounded $r_{n+\widetilde{d}+1}$ in Lemma 6.2.3. First we observe the trivial inequality $l+d+1\geq l+\widetilde{d}+1$ implies

$$\sum_{n\geq l+\widetilde{d}+1}\left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}}>\left(C_{\epsilon,\epsilon'}\right)^{(l+d+1)}(l+d+1)^{-\frac{2(l+d+1)}{d-1}},$$

which, with Stirling approximation $n^{-n}\geq e^{-n}(n!)^{-1}$, gives

$$\begin{aligned}
\sum_{n\geq l+\widetilde{d}+1}\left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}} &> \left(C_{\epsilon,\epsilon'}e^{-\frac{2}{d-1}}\right)^{(l+d+1)}\left[(l+d+1)!\right]^{-\frac{2}{d-1}}\\
&> \left(C_{\epsilon,\epsilon'}e^{-\frac{2}{d-1}}\right)^{(l+d+1)}\left[(l+d+1)^{d+1}l!\right]^{-\frac{2}{d-1}}\\
&> \left(C_{\epsilon,\epsilon'}e^{-\frac{2}{d-1}}(d+1)^{-\frac{2}{d-1}}\right)^{(l+d+1)}(l!)^{-\frac{2}{d-1}}
\end{aligned}$$

because $(l+d+1)^{d+1} < (d+1)^{l+d+1}$. By setting

$$C'_{\epsilon,\epsilon'} = C_{\epsilon,\epsilon'} e^{-\frac{2}{d-1}} (d+1)^{-\frac{2}{d-1}} \min\left(1, \left(C_{\epsilon,\epsilon'} e^{-\frac{2}{d-1}} (d+1)^{-\frac{2}{d-1}}\right)^{d+1}\right)$$

we have

$$\sum_{n \geq \max l + \widetilde{d} + 1} \left(C_{\epsilon,\epsilon'}\right)^n n^{-\frac{2n}{d-1}} > (C'_{\epsilon,\epsilon'})^l (l!)^{-\frac{2}{d-1}}.$$

Therefore (6.15) gives that for a big enough we have

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right) > e^{-A} \sum_{l \geq 0} \frac{\left(C'_{\epsilon,\epsilon'} A\right)^l}{(l!)^{\frac{d+1}{d-1}}}.$$

Applying Lemma 6.2.4 with $x = C'_{\epsilon,\epsilon'} A$ and $\alpha = \frac{d+1}{d-1}$ we get

$$\mathbb{P}\left(\Sigma(Z) > a,\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right) > \exp\left(-A + \frac{d+1}{2(d-1)} \left(C'_{\epsilon,\epsilon'} A\right)^{\frac{d-1}{d+1}}\right).$$

Set $C_{29} = \frac{d+1}{2(d-1)} \left(C'_{\epsilon,\epsilon'}\right)^{\frac{d-1}{d+1}}$ and recall that $A = (\tau + \epsilon') a^{r/k}$. We have shown that, for any $\epsilon' > \epsilon''' > \epsilon \geq 0$ and a big enough,

$$\begin{aligned} \mathbb{P}\left(\Sigma > a),\ \frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} \geq \tau + \epsilon\right) &> \exp\left(-\gamma(\tau+\epsilon') a^{r/k} + C_{29} \left(\gamma(\tau+\epsilon') a^{r/k}\right)^{\frac{d-1}{d+1}}\right) \\ &> \exp\left(-\gamma(\tau+\epsilon''') a^{r/k}\right) \end{aligned}$$

which is the lower bound of Theorem 6.3.1.

6.3.2 Shape of the cells with big Σ-content

In this section we derive similar results as in the papers on big cells from Hug, Reitzner and Schneider [HRS04] and Hug and Schneider [HS07]. Note that our result is stronger since it is an equivalence and not only an upper bound, and that it holds both for $Z_{\boldsymbol{o}}$ and Z_{typ}.

Theorem 6.3.3. *Let $\epsilon \geq 0$. Assume that there exists $K \in \mathcal{K}$ such that $\Phi(K)\Sigma(K)^{-r/k} > \tau + \epsilon$. It holds that*

$$\lim_{a\to\infty} a^{-\frac{r}{k}} \ln\left(\mathbb{P}\left(\frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} > \tau + \epsilon \mid \Sigma(Z) > a\right)\right) = -\epsilon\gamma,$$

where Z can be replaced by $Z_{\boldsymbol{o}}$ or, in the stationary case, by Z_{typ}.

Proof. It is a corollary of Corollary 6.3.2. We have

$$
\begin{aligned}
&a^{-\frac{r}{k}} \ln \left(\mathbb{P}\left(\frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} > \tau + \epsilon \mid \Sigma(Z) > a \right) \right) \\
&= a^{-\frac{r}{k}} \ln \mathbb{P}\left(\frac{\Phi(Z)}{\Sigma(Z)^{\frac{r}{k}}} > \tau + \epsilon,\ \Sigma(Z) > a \right) - a^{-\frac{r}{k}} \ln \mathbb{P}\left(\Sigma(Z) > a\right) \\
&\xrightarrow{a \to \infty} -(\tau + \epsilon)\gamma + \tau\gamma = -\epsilon\gamma.
\end{aligned}
$$

□

Chapter 7

Small cells

Contents

We restrict the scope of this chapter to the typical cell, which implies that we are in the stationary case: $r = 1$ and $\varphi \in \aleph_e$. In contrast with the previous chapter, we are now interested in small cells of the tessellation. More precisely, we are interested in the conditional law of the shape $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}})$, given that $\Sigma(Z_{\text{typ}}) < a$ and $a \to 0$. Here Σ is a size functional satisfying the same four axioms as the one described at the beginning of Chapter 6. This question can be divided into two sub-questions:

(Q1) What is the number of facets of small cells?

(Q2) Let n be such that $\mathbb{P}(f(Z_{\text{typ}}) = n) > 0$. What is the shape distribution of small cells conditioned on the event $\{f(Z_{\text{typ}}) = n\}$?

We will give answer to both questions in many cases, covering the isotropic case or the case when $\Sigma^{\frac{1}{k}}$ is of the same order as Φ, i.e. there exist positive constants C and C' such that $C\Sigma(K)^{\frac{1}{k}} \leq \Phi(K) \leq C'\Sigma(K)^{\frac{1}{k}}$ for any $K \in \mathcal{K}$.

In the first section we characterise the set $\mathcal{N} = \{n \in \mathbb{N} : \mathbb{P}(f(Z_{\text{typ}}) = n) > 0\}$. Indeed we will need later the fact that if $|\mathcal{N}| > 1$, then $n_{\min}+1 \in \mathcal{N}$, where $n_{\min} = \min \mathcal{N}$. In the second section we cover the case of cells with small Φ-content. The complementary theorem makes this case easy to deal with. In the third and fourth sections we consider the case where $\varphi \in \aleph_{e,c}$, i.e. is absolutely continuous, and where Σ can be any size functional. First we show that the shape of small cells are random simplices and we describe the asymptotic shape distribution. Second we give a rate of convergence for $\mathbb{P}(f(Z_{\text{typ}}) > d+1 \mid \Sigma < a) \to 0$, as $a \to 0$. Finally, in the last section we consider the most general case.

Before starting with Section 1, we want to give a few references to the literature. Miles gave heuristic arguments in [Mil95] that small cells in a stationary isotropic line tessellation are triangles with a random shape, and that the random shape depends on Σ.

Beermann, Redenbach and Thäle in [BRT14] considered the planar case where φ is concentrated in two couples of antipodal points. Thus after a linear transformation, this reduces to the case where all the lines are either horizontal or vertical and the intensity of the subproceses of horizontal lines and of vertical lines are the same. In that case all the cells are rectangles. They consider two size measurements, the perimeter and the area. It turns out that the perimeter is proportional to the Φ-content, so conditioning on it does not affect the shape distribution, because of the complementary theorem and the fact that all cells have the same number of facets. The case of the area is more interesting. They show that the shape of cells with small area tends a degenerated shape, i.e. the shape of a line segment. They also provide a rate of convergence, which is incorrect and was corrected in Beermann doctoral thesis [Bee15]. At the really end of the chapter we recover their result as a corollary applying to a much broader setting. We also slightly improve the approximation of the rate of convergence.

Finally we want to mention that Schulte and Thäle in [ST12, Ex. 6 of Sec. 2], and Chenavier and Hemsley in [CH15, remark above Thm. 2], both considered a related problem. They gave some results about the smallest cell in a window of increasing size.

7.1 Possible number of facets

In this section we characterise the set $\{n \in \mathbb{N} : \mathbb{P}(f(Z_{\text{typ}}) = n) > 0\}$ for general directional distribution $\varphi \in \aleph_e$. Let $\operatorname{supp}(\varphi) \subset \mathbb{S}^{d-1}$ denotes the support of φ and

$$V_\varphi := \{\boldsymbol{v} \in \operatorname{supp}(\varphi) : \operatorname{supp}(\varphi) \setminus \{\boldsymbol{v}, -\boldsymbol{v}\} \text{ is concentrated in a great circle}\}.$$

Theorem 7.1.1. *With respect to the notation above,*

$$\left\{ \frac{1}{2}|V_\varphi| : \varphi \in \aleph_e \right\} = \{0, 1, \ldots, d-2, d\},$$

and for any $\varphi \in \aleph_e$,

$$\{n \in \mathbb{N} : \mathbb{P}(f(Z_{\text{typ}}) = n) > 0\}$$
$$= \begin{cases} \{2d\} & \text{if } |V_\varphi| = 2d \\ \left[d+1+\frac{1}{2}|V_\varphi|\,,\ |\operatorname{supp}(\varphi)|\right] \cap \mathbb{N} & \text{if } |V_\varphi| < 2d \end{cases}.$$

Proof. Clearly $V_\varphi = -V_\varphi$, i.e. if $\boldsymbol{v} \in V_\varphi$, then $-\boldsymbol{v} \in V_\varphi$. For any $\boldsymbol{v} \in V_\varphi$, the linear space $H_{\boldsymbol{v}}$ spanned by $\mathrm{supp}(\varphi)\setminus\{\boldsymbol{v}, -\boldsymbol{v}\}$ is a hyperplane which does not contain $\boldsymbol{v}$. It does not contain $\boldsymbol{v}$ because $\mathrm{supp}(\varphi)\setminus\{\boldsymbol{v}, -\boldsymbol{v}\}$ is concentrated in a great circle. It is a hyperplane since otherwise φ would be concentrated in a great circle. From this we get that $\bigcap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}}$ is a $\big(d - \frac{1}{2}|V_\varphi|\big)$-dimensional space. This implies that $\frac{1}{2}|V_\varphi| \in \{0, \dots, d\}$. Set

$$\varphi_0 := \varphi - \sum_{\boldsymbol{v}\in V_\varphi} \varphi(\{\boldsymbol{v}\})\delta_{\boldsymbol{v}},$$

where $\delta_{\boldsymbol{v}}$ is the Dirac measure concentrated at $\boldsymbol{v}$. It is the restriction of φ to the $\big(d - \frac{1}{2}|V_\varphi| - 1\big)$-dimensional sphere $\mathbb{S}^{d-1} \cap \bigcap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}}$. In the last sentence, we used the convention $\dim \emptyset = -1$.

We will show by contradiction that $\frac{1}{2}|V_\varphi| \neq d-1$. Assume the contrary. Then φ_0 is concentrated in the $(d-(d-1)-1) = 0$-dimensional sphere $\mathbb{S}^{d-1} \cap \bigcap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}}$. This 0-dimensional sphere is of the form $\{\boldsymbol{u}, -\boldsymbol{u}\}$, with $\boldsymbol{u} \notin V_\varphi$. But then, $\mathrm{supp}(\varphi) \setminus \{\boldsymbol{u}, -\boldsymbol{u}\} = V_\varphi$. And since, in this situation, V_φ spans a hyperplane, we have that $\mathrm{supp}(\varphi) \setminus \{\boldsymbol{u}, -\boldsymbol{u}\}$ is concentrated in a great circle. Thus $\boldsymbol{u} \in V_\varphi$, which is a contradiction.

Thus we proved

$$\left\{\frac{1}{2}|V_\varphi| : \varphi \in \aleph_e\right\} \subset \{0, 1, \dots, d-2, d\}.$$

We prove the reverse inclusion by construction. Let $k \in \{0, 1, \dots, d-2, d\}$. Consider $S_{d-k} := \{\boldsymbol{u} = (u_1, \dots, u_d) \in \mathbb{S}^{d-1} : u_1 = \dots = u_k = 0\} \simeq \mathbb{S}^{d-k-1}$, and σ_k the $(d-k-1)$-dimensional spherical Lebesgue measure concentrated on S_{d-k}. Set the measure $\widetilde{\varphi} := \sigma_k + \sum_{i=1}^{k}(\delta_{\boldsymbol{e}_i} + \delta_{-\boldsymbol{e}_i})$, and $\varphi := \widetilde{\varphi}\left(\mathbb{S}^{d-1}\right)^{-1}\widetilde{\varphi}$ its normalization. Here $\delta_{\boldsymbol{e}_i}$ denotes the Dirac measure concentrated at the canonical unit vector $\boldsymbol{e}_i$. In this case $V_\varphi = \{\boldsymbol{e}_i, -\boldsymbol{e}_i : i \in [k]\}$, and therefore $\frac{1}{2}|V_\varphi| = k$.

Now we will prove the second part of the theorem. If $|V_\varphi| = 2d$, then V_φ is the support of φ and all the cells of the the tessellation are parallelepipeds, and thus have $2d$ facets. We assume from now that $|V_\varphi| < 2d$. Observe that

$$\begin{aligned}
\mathbb{P}(f(Z_{\mathrm{typ}}) = n) > 0 \Leftrightarrow \mu\left(\left\{\boldsymbol{H} \in \mathcal{H}^n : \bigcap_{i=1}^{n} H_i^- \in \mathcal{P}_n\right\}\right) > 0& \\
\Leftrightarrow \exists \boldsymbol{u}_1, \dots, \boldsymbol{u}_n \in \mathrm{supp}(\varphi) \text{ and } \exists t_1, \dots, t_n \in \mathbb{R}_+,& \\
\text{such that } \bigcap_{i=1}^{n} H(\boldsymbol{u}_i, t_i)^- \in \mathcal{P}_n& \\
\Leftrightarrow \exists \boldsymbol{u}_1, \dots, \boldsymbol{u}_n \in \mathrm{supp}(\varphi),& \\
\text{such that } \bigcap_{i=1}^{n} H(\boldsymbol{u}_i, 1)^- \in \mathcal{P}_n.&
\end{aligned}$$

But the intersection $\bigcap_{i=1}^n H(\boldsymbol{u}_i,1)^-$ is bounded if and only if the vectors $\boldsymbol{v}_i$ do not all belong to one half sphere, which implies that $V_\varphi \subset \{\boldsymbol{u}_i : i \in [n]\}$ (and that $n \geq |V_\varphi|$). With this observation we can reduce the problem to the $\left(d - \frac{1}{2}|V_\varphi|\right)$-dimensional space $\bigcap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}}$,

$$\mathbb{P}(f(Z_{\text{typ}}) = n) > 0 \Leftrightarrow n \geq |V_\varphi| \text{ and } \exists \boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-|V_\varphi|} \in \operatorname{supp}(\varphi_0), \text{ such that}$$
$$\bigcap_{i=1}^n H(\boldsymbol{u}_i,1)^- \cap \bigcap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}} \in \mathcal{P}_{n-|V_\varphi|}\left(\cap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}}\right),$$

where $\mathcal{P}_{n-|V_\varphi|}\left(\cap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}}\right)$ denotes the set of polytopes with $n - |V_\varphi|$ facets in the linear space $\cap_{\boldsymbol{v}\in V_\varphi} H_{\boldsymbol{v}}$. It gives,

$$\mathbb{P}(f(Z_{\text{typ}}) = n) > 0 \Leftrightarrow n \geq |V_\varphi| \text{ and } n - |V_\varphi| \in \left[d - \frac{1}{2}|V_\varphi| + 1\,,\, |\operatorname{supp}(\varphi_0)|\right],$$

which yields the proof. □

7.2 Cells with small Φ-content are $n_{\min}$-topes with random shape

Because of the Complementary Theorem 3.3.1, it is easy to study cells with small Φ-content. Let $n_{\min} := \min\{n \in \mathbb{N} : \mathbb{P}(f(Z_{\text{typ}}) = n) > 0\}$. The next theorem gives an equivalence for the asymptotic of $\mathbb{P}(\Phi(Z_{\text{typ}}) < a)$, as $a \to 0$, and describes the conditional law of the shape $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}})$, given $\Phi(Z_{\text{typ}})$. In particular $f(Z_{\text{typ}}) = n_{\min}$ with high probability when $\Phi(Z_{\text{typ}})$ is small.

Theorem 7.2.1. *When $a \to 0$, it holds that*

$$\mathbb{P}(\Phi(Z_{\text{typ}}) < a) \sim \frac{\mathbb{P}(f(Z_{\text{typ}}) = n_{\min})}{(n_{\min} - d)!} (\gamma a)^{n_{\min} - d},$$

and

$$(\gamma a)^{-1} \mathbb{P}(f(Z_{\text{typ}}) > n_{\min} \mid \Phi(Z_{\text{typ}}) < a)$$
$$\to \left(\frac{\mathbb{P}(f(Z_{\text{typ}}) = n_{\min} + 1)}{\mathbb{P}(f(Z_{\text{typ}}) = n_{\min})(n_{\min} + 1 - d)} \right),$$

and, for any open set of shapes $A \subset \mathcal{K}_{\mathfrak{c},\Phi}$,

$$\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A \mid \Phi(Z_{\text{typ}}) < a) \to \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A \mid f(Z_{\text{typ}}) = n_{\min}).$$

Proof. With the Complementary Theorem, we have for any n,

$$\begin{aligned}
&\mathbb{P}(f(Z_{\text{typ}}) = n\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\, \Phi(Z_{\text{typ}}) < a) \\
&= \mathbb{P}(f(Z_{\text{typ}}) = n\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A)\frac{\gamma^{n-d}}{(n-d-1)!}\int_0^a e^{-\gamma t}t^{n-d-1}\mathrm{d}t \\
&\sim_< \frac{\mathbb{P}(f(Z_{\text{typ}}) = n\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A)}{(n-d)!}(\gamma a)^{n-d}, \qquad (7.1)
\end{aligned}$$

when $a \to 0$, where $f \sim_< g$, means $f \sim g$ and $f < g$. This implies that, for any n such that $\mathbb{P}(f(Z_{\text{typ}}) = n) > 0$, and any open set of shape $A \in \mathcal{K}_{\mathfrak{c},\Phi}$, we have

$$\begin{aligned}
&\mathbb{P}(f(Z_{\text{typ}}) \geq n\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\, \Phi(Z_{\text{typ}}) < a) \\
&\sim \mathbb{P}(f(Z_{\text{typ}}) = n\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\, \Phi(Z_{\text{typ}}) < a). \qquad (7.2)
\end{aligned}$$

In particular, by setting $n = n_{\min}$ and $A = \mathcal{K}_{\mathfrak{c},\Phi}$, (7.1) and (7.2) give the first equivalence of the Theorem.

For the second point we need first to consider the set $\mathcal{N} := \{n \in \mathbb{N} : \mathbb{P}(f(Z_{\text{typ}}) = n) > 0\}$. If $\mathcal{N} = \{n_{\min}\}$, then we trivially have $0 \to 0$. Otherwise, Theorem 7.1.1 tells us that $\mathbb{P}(f(Z_{\text{typ}}) = n_{\min}+1) > 0$. Therefore applying (7.2) and then (7.1), both with $n = n_{\min} + 1$ and $A = \mathcal{K}_{\mathfrak{c},\Phi}$, and using the first equivalence of the theorem gives

$$\begin{aligned}
&\mathbb{P}(f(Z_{\text{typ}}) > n_{\min} \mid \Phi(Z_{\text{typ}}) < a) \\
&= \frac{\mathbb{P}(f(Z_{\text{typ}}) \geq n_{\min} + 1\,,\, \Phi(Z_{\text{typ}}) < a)}{\Phi(Z_{\text{typ}}) < a} \\
&\sim \left(\frac{\mathbb{P}(f(Z_{\text{typ}}) = n_{\min} + 1)}{(n_{\min} - d + 1)!}(\gamma a)^{n_{\min}+1-d}\right)\left(\frac{\mathbb{P}(f(Z_{\text{typ}}) = n_{\min})}{(n_{\min} - d)!}(\gamma a)^{n_{\min}-d}\right)^{-1},
\end{aligned}$$

which gives immediately the second point of the theorem.

For the last point of the theorem, observe that

$$\begin{aligned}
&\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A \mid \Phi(Z_{\text{typ}}) < a) \\
&= \frac{\mathbb{P}(f(Z_{\text{typ}}) \geq n_{\min}\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\, \Phi(Z_{\text{typ}}) < a)}{\mathbb{P}(f(Z_{\text{typ}}) \geq n_{\min}\,,\, \Phi(Z_{\text{typ}}) < a)}.
\end{aligned}$$

With (7.2), this gives

$$\begin{aligned}
&\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A \mid \Phi(Z_{\text{typ}}) < a) \\
&\sim \frac{\mathbb{P}(f(Z_{\text{typ}}) = n_{\min}\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\, \Phi(Z_{\text{typ}}) < a)}{\mathbb{P}(f(Z_{\text{typ}}) = n_{\min}\,,\, \Phi(Z_{\text{typ}}) < a)} \\
&= \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A \mid f(Z_{\text{typ}}) = n_{\min}\,,\, \Phi(Z_{\text{typ}}) < a).
\end{aligned}$$

This yields the proof since the Complementary Theorem tells us that, when we condition on $f(Z_{\text{typ}}) = n_{\min}$, the random variables $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}})$ and $\Phi(Z_{\text{typ}})$ are independent. □

7.3 Absolute continuous case: Cells with small Σ-content are simplices with random shape

Now we are interested by the more general situation where the cells are measured by an arbitrary size measurement Σ instead of Φ. In order to give precise answers to the questions (Q1) and (Q2), we assume in this section and the following one that $\varphi \in \aleph_{e,c}$, i.e. is even and absolutely continuous. Also, we consider that Z_{typ} is defined with respect to the inball center.

Recall that Theorem 4.5.2 tells us that for any Borel $A \subset \mathcal{K}$,

$$\mathbb{P}(Z_{\text{typ}} \in A) = \frac{\gamma^{d+1}}{(d+1)\gamma^{(d)}} \int_{\mathsf{P}} \int_0^\infty e^{-\gamma r} \mathbb{P}\left(\bigcap_{H \in \eta \cap \mathcal{F}^{rB^d}} H^- \cap (r\Delta(\bar{\boldsymbol{u}})) \in A \right) \mathrm{d}r \times \Delta_d(\bar{\boldsymbol{u}})\, \mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}),$$

where $\eta \cap \mathcal{F}^{rB^d}$ denotes the set of hyperplanes of the process η with empty intersection with the ball rB^d, and

$$\mathsf{P} = \left\{ \bar{\boldsymbol{u}} \in (\mathbb{S}^{d-1})^d + 1 : \boldsymbol{u}_0, \dots \boldsymbol{u}_d \text{ are not all in one closed half sphere} \right\},$$

and

$$\Delta(\bar{\boldsymbol{u}}) = \bigcap_{i=0}^{d} H(\boldsymbol{u}_i, 1)^-, \quad \text{and} \quad \Delta_d(\bar{\boldsymbol{u}}) = \lambda_d(\text{ConvexHull}(\boldsymbol{u}_0, \dots, \boldsymbol{u}_d)).$$

This can be rewritten in the following form,

$$\frac{(d+1)\gamma^{(d)}}{\gamma^{d+1}} \mathbb{P}(Z_{\text{typ}} \in A) = \int_{\mathsf{P}} \int_0^\infty e^{-\gamma r} \mathbb{P}\left(r \left(\bigcap_{H \in r^{-1}\eta \cap \mathcal{F}^{B^d}} H^- \cap \Delta(\bar{\boldsymbol{u}}) \right) \in A \right) \mathrm{d}r \Delta_d(\bar{\boldsymbol{u}})\, \mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}), \tag{7.3}$$

where $r^{-1}\eta := \{r^{-1}H : H \in \eta\}$ is the dilatation of ratio r^{-1} of η and therefore has intensity $r\gamma$. Observe that, for any $\bar{\boldsymbol{u}} \in \mathsf{P}$,

$$\Sigma(r\Delta(\bar{\boldsymbol{u}})) < a \Leftrightarrow r < \left(\frac{a}{\Sigma(\Delta(\bar{\boldsymbol{u}}))} \right)^{1/k},$$

and

$$f\left(\bigcap_{H \in r^{-1}\eta \cap \mathcal{F}^{B^d}} H^- \cap \Delta(\bar{\boldsymbol{u}}) \right) = d+1 \Leftrightarrow r\eta \cap \left(\Delta(\bar{\boldsymbol{u}}) \setminus B^d \right) = \emptyset. \tag{7.4}$$

Thus, (7.3) applied to the set

$$A = \{P \in \mathcal{P} : f(P) = d+1\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(P) \in S\,,\, \Sigma(P) < a\},$$

where $a > 0$ and $S \subset \mathcal{P}_{\mathfrak{c},\Phi}$ is an open set of shapes, it gives

$$\begin{aligned} &\frac{(d+1)\gamma^{(d)}}{\gamma^{d+1}} \mathbb{P}(f(Z_{\text{typ}}) = d+1\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(P) \in S\,,\, \Sigma(Z_{\text{typ}}) < a) \\ &= \int\limits_{\mathsf{P}} \int\limits_0^{\left(\frac{a}{\Sigma(\Delta(\bar{\boldsymbol{u}}))}\right)^{1/k}} \mathbb{1}\left(\mathfrak{s}_{\mathfrak{c},\Phi}(\Delta(\bar{\boldsymbol{u}})) \in S\right) e^{-\gamma r}\, \mathbb{P}\left(r^{-1}\eta \cap \left(\Delta(\bar{\boldsymbol{u}}) \setminus B^d\right) = \emptyset\right) \\ &\quad \times \mathrm{d}r\, \Delta_d(\bar{\boldsymbol{u}})\, \mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}). \end{aligned} \tag{7.5}$$

Since the intensity of $r^{-1}\eta$ tends to 0 when $r \to 0$, we have that

$$\mathbb{P}\left(r^{-1}\eta \cap \left(\Delta(\bar{\boldsymbol{u}}) \setminus B^d\right) = \emptyset\right) \xrightarrow{r\to 0} 1,$$

for any $\bar{\boldsymbol{u}} \in \mathsf{P}$, where the convergence is monotonically increasing. Hence

$$\frac{1}{a^{\frac{1}{k}}} \int\limits_0^{\left(\frac{a}{\Sigma(\Delta(\bar{\boldsymbol{u}}))}\right)^{\frac{1}{k}}} e^{-\gamma r}\mathbb{P}\left(r^{-1}\eta \cap \left(\Delta(\bar{\boldsymbol{u}}) \setminus B^d\right) = \emptyset\right) \mathrm{d}r \xrightarrow{r\to 0} \frac{1}{\Sigma(\Delta(\bar{\boldsymbol{u}}))^{\frac{1}{k}}},$$

for any $\bar{\boldsymbol{u}} \in \mathsf{P}$, where the convergence is monotonically increasing. Therefore, (7.5) and the monotone convergence theorem give

$$\begin{aligned} &\frac{1}{\gamma a^{\frac{1}{k}}} \mathbb{P}(f(Z_{\text{typ}}) = d+1\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(P) \in S\,,\, \Sigma(Z_{\text{typ}}) < a) \\ &\xrightarrow{a\to 0} \frac{\gamma^d}{(d+1)\gamma^{(d)}} \int\limits_{\mathsf{P}} \mathbb{1}\left(\mathfrak{s}_{\mathfrak{c},\Phi}(\Delta(\bar{\boldsymbol{u}})) \in S\right) \frac{\Delta_d(\bar{\boldsymbol{u}})}{\Sigma(\Delta(\bar{\boldsymbol{u}}))^{\frac{1}{k}}}\, \mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}). \end{aligned} \tag{7.6}$$

Now, we will consider the probability $\mathbb{P}(f(Z_{\text{typ}}) > d+1\,,\, \Sigma(Z_{\text{typ}}) < a)$. Observe that the trivial inclusion

$$rB^d \subset r\left(\bigcap_{H \in r^{-1}\eta \cap \mathcal{F}_{B^d}} H^- \cap \Delta(\bar{\boldsymbol{u}})\right),$$

gives the following implication

$$\Sigma\left(r\left(\bigcap_{H \in r^{-1}\eta \cap \mathcal{F}_{B^d}} H^- \cap \Delta(\bar{\boldsymbol{u}})\right)\right) < a \Rightarrow r < \left(\frac{a}{\Sigma(B^d)}\right)^{\frac{1}{k}}. \tag{7.7}$$

Therefore (7.3) applied to the set $A = \{P \in \mathcal{P} : \Sigma(P) < a\}$ implies

$$\begin{aligned}\frac{(d+1)\gamma^{(d)}}{\gamma^{d+1}}\mathbb{P}(\Sigma(Z_{\text{typ}}) < a) &\leq \int\limits_{\mathsf{P}} \int\limits_{0}^{\left(\frac{a}{\Sigma(B^d)}\right)^{\frac{1}{k}}} e^{-\gamma r} \mathrm{d}r \Delta_d(\bar{\boldsymbol{u}}) \,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}) \\ &\leq a^{\frac{1}{k}} \Sigma(B^d)^{-\frac{1}{k}} \int\limits_{\mathsf{P}} \Delta_d(\bar{\boldsymbol{u}}) \,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}) \\ &\leq a^{\frac{1}{k}} \left(\Sigma(B^d)^{-\frac{1}{k}} \kappa_d\right).\end{aligned}$$

We will use this bound to apply the dominated convergence theorem to the integral representations of $a^{-1/k}\mathbb{P}(f(Z_{\text{typ}}) > d+1\,,\, \Sigma(Z_{\text{typ}}) < a)$, when $a \to 0$. Equation (7.3) applied to $A = \{P \in \mathcal{P} : f(Z_{\text{typ}}) > d+1\,,\, \Sigma(Z_{\text{typ}}) < a\}$, (7.4) and (7.7) give

$$\begin{aligned}&\frac{(d+1)\gamma^{(d)}}{\gamma^{d+1}}\mathbb{P}(f(Z_{\text{typ}}) > d+1\,,\, \Sigma(Z_{\text{typ}}) < a) \\ &\leq \int\limits_{\mathsf{P}} \int\limits_{0}^{\left(\frac{a}{\Sigma(B^d)}\right)^{1/k}} e^{-\gamma r} \mathbb{P}\left(r^{-1}\eta \cap \left(\Delta(\bar{\boldsymbol{u}}) \setminus B^d\right) \neq \emptyset\right) \mathrm{d}r \Delta_d(\bar{\boldsymbol{u}}) \,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}).\end{aligned}$$

Since the intensity of $r^{-1}\eta$ tends to 0 when $r \to 0$, we have that

$$\mathbb{P}\left(r^{-1}\eta \cap \left(\Delta(\bar{\boldsymbol{u}}) \setminus B^d\right) \neq \emptyset\right) \xrightarrow{r \to 0} 0,$$

for any $\bar{\boldsymbol{u}} \in \mathsf{P}$. Hence

$$\frac{1}{a^{\frac{1}{k}}} \int\limits_{0}^{\left(\frac{a}{\Sigma(\Delta(\bar{\boldsymbol{u}}))}\right)^{\frac{1}{k}}} e^{-\gamma r} \mathbb{P}\left(r^{-1}\eta \cap \left(\Delta(\bar{\boldsymbol{u}}) \setminus B^d\right) \neq \emptyset\right) \mathrm{d}r \xrightarrow{a \to 0} 0,$$

for any $\bar{\boldsymbol{u}} \in \mathsf{P}$. Therefore, (7.5) and the dominated convergence theorem give

$$\frac{1}{\gamma a^{\frac{1}{k}}}\mathbb{P}(f(Z_{\text{typ}}) > d+1\,,\, \Sigma(Z_{\text{typ}}) < a) \xrightarrow{a \to 0} 0. \tag{7.8}$$

Thus, with (7.8) and (7.6) we proved the following theorem.

Theorem 7.3.1. *Let $\varphi \in \aleph_{e,c}$. For any open set of shapes $S \subset \mathcal{K}_{\mathfrak{c},\Phi}$,*

$$\mathbb{P}(f(Z_{\text{typ}}) = d+1\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(P) \in S\,,\, \Sigma(Z_{\text{typ}}) < a) \sim c_\varphi(S)\gamma a^{\frac{1}{k}},$$

when $a \to 0$*, and*

$$\mathbb{P}(f(Z_{\mathrm{typ}}) > d+1,\, \Sigma(Z_{\mathrm{typ}}) < a) = o\left(\gamma a^{\frac{1}{k}}\right),$$

where

$$c_\varphi(S) := \frac{\gamma^d}{(d+1)\gamma^{(d)}} \int_{\mathsf{P}} \mathbb{1}\left(\mathfrak{s}_{\mathfrak{c},\Phi}(\Delta(\bar{\boldsymbol{u}})) \in S\right) \Sigma(\Delta(\bar{\boldsymbol{u}}))^{-\frac{1}{k}} \Delta_d(\bar{\boldsymbol{u}})\, \mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}).$$

As a direct corollary, we get that the conditional law of the shape $\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}})$, given $\Sigma(Z_{\mathrm{typ}})$, converges weakly to a random simplex.

Corollary 7.3.2. *Let* $\varphi \in \aleph_{e,c}$*. For any open set of shapes* $S \subset \mathcal{K}_{\mathfrak{c},\Phi}$*,*

$$\mathbb{P}(f(Z_{\mathrm{typ}}) = d+1\,,\, \mathfrak{s}_{\mathfrak{c},\Phi}(P) \in S \mid \Sigma(Z_{\mathrm{typ}}) < a) \to \frac{c_\varphi(S)}{c_\varphi(\mathcal{P}_{d+1,\mathfrak{c},\Phi})},$$

when $a \to 0$*.*

7.4 Absolute continuous case: Speed of convergence

In this section, as in the previous one, we assume that $\varphi \in \aleph_{e,c}$, i.e. is even and absolutely continuous. Our goal is to find how fast $\mathbb{P}(f(Z_{\mathrm{typ}}) = d+1 \mid \Sigma(Z_{\mathrm{typ}}) < a)$ tends to 1 when $a \to 0$. Since Theorem 7.3.1 gives us a precise estimation of $\mathbb{P}(f(Z_{\mathrm{typ}}) = d+1\,,\, \Sigma(Z_{\mathrm{typ}}) < a)$, the question is reduced to study how fast does the joint distribution $\mathbb{P}(f(Z_{\mathrm{typ}}) > d+1\,,\, \Sigma(Z_{\mathrm{typ}}) < a)$ tends to 0.

We denote by $\mathtt{r}(Z_{\mathrm{typ}})$ the *inradius* of Z_{typ}, that is the maximal radius of a ball $B = B(Z_{\mathrm{typ}})$ inscribed in Z_{typ}. The inclusion $B(Z_{\mathrm{typ}}) \subset Z_{\mathrm{typ}}$ implies $\Sigma(Z_{\mathrm{typ}}) \geq \Sigma(B(Z_{\mathrm{typ}})) = \mathtt{r}(Z_{\mathrm{typ}})^k \Sigma(B^d)$. Therefore

$$\begin{aligned} &\mathbb{P}(f(Z_{\mathrm{typ}}) > d+1,\, \Sigma(Z_{\mathrm{typ}}) < a) \\ &\leq \mathbb{P}\left(f(Z_{\mathrm{typ}}) > d+1,\, \mathtt{r}(Z_{\mathrm{typ}}) < \Sigma(B^d)^{-\frac{1}{k}} a^{\frac{1}{k}}\right). \end{aligned} \tag{7.9}$$

This shows that it is essential to study the case of the inradius. By (7.3), we have

$$\begin{aligned}
&\frac{(d+1)}{\gamma}\frac{\gamma^{(d)}}{\gamma^d}\,\mathbb{P}(f(Z_{\mathrm{typ}})>d+1,\,\mathtt{r}(Z_{\mathrm{typ}})<a)\\
&=\int\limits_{\mathsf{P}}\int\limits_0^a e^{-\gamma r}\mathbb{P}\left(r^{-1}\eta\cap\left(\Delta(\bar{\boldsymbol{u}})\setminus B^d\right)\neq\emptyset\right)\mathrm{d}r\,\Delta_d(\bar{\boldsymbol{u}})\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}})\\
&=\int\limits_{\mathsf{P}}\int\limits_0^a e^{-\gamma r}\left(1-e^{-\gamma r(\Phi(\Delta(\bar{\boldsymbol{u}}))-1)}\right)\mathrm{d}r\,\Delta_d(\bar{\boldsymbol{u}})\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}),\\
&<\int\limits_{\mathsf{P}}\int\limits_0^a 1-e^{-\gamma r\Phi(\Delta(\bar{\boldsymbol{u}}))}\mathrm{d}r\,\Delta_d(\bar{\boldsymbol{u}})\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}).
\end{aligned}$$

It is tempting to upper bound the integrand $1-e^{-\gamma r\Phi(\Delta(\bar{\boldsymbol{u}}))}$ by $\gamma r\Phi(\Delta(\bar{\boldsymbol{u}}))$, since for $r\to 0$ these quantities are equivalent. This would lead us to

$$\begin{aligned}
&\mathbb{P}(f(Z_{\mathrm{typ}})>d+1,\,\mathtt{r}(Z_{\mathrm{typ}})<a)\\
&<(\gamma a)^2\frac{1}{2(d+1)}\frac{\gamma^d}{\gamma^{(d)}}\int\limits_{\mathsf{P}}\Phi(\Delta(\bar{\boldsymbol{u}}))\Delta_d(\bar{\boldsymbol{u}})\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}),
\end{aligned}$$

but this is useless since the integral $\int_{\mathsf{P}}\Phi(\Delta(\bar{\boldsymbol{u}}))\Delta_d(\bar{\boldsymbol{u}})\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}})$ turns out to diverge. We need to consider more carefully the contribution of simplices $\Delta(\bar{\boldsymbol{u}})$ which have a big Φ-content. The factor $\Delta_d(\bar{\boldsymbol{u}})$ is not important to get the order of the integral. We upper bound it by $\Delta_{\max}=\max_{\bar{\boldsymbol{u}}\in\mathsf{P}}\Delta_d(\bar{\boldsymbol{u}})$. So we have

$$\begin{aligned}
&\frac{(d+1)}{\gamma\Delta_{\max}}\frac{\gamma^{(d)}}{\gamma^d}\,\mathbb{P}(f(Z_{\mathrm{typ}})>d+1,\,\mathtt{r}(Z_{\mathrm{typ}})<a)\\
&<\int\limits_{\mathsf{P}}\int\limits_0^a 1-e^{-\gamma r\Phi(\Delta(\bar{\boldsymbol{u}}))}\mathrm{d}r\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}) \qquad (7.10)
\end{aligned}$$

In order to go further, we need to prove the following essential lemma.

Lemma 7.4.1. *Let $\varphi\in\aleph_{e,c}$. There exists a constant C_{30}, depending on φ, such that for any increasing function $f\colon\mathbb{R}\to\mathbb{R}$, we have*

$$\int\limits_{\mathsf{P}} f(\Phi(\Delta(\bar{\boldsymbol{u}})))\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}})<C_{30}\int\limits_1^\infty f(t)\frac{1}{t^2}\mathrm{d}t.$$

Proof. In this proof we set

$$\mathsf{P}_k:=\{\bar{\boldsymbol{u}}\in\mathsf{P}:\Phi(\Delta(\bar{\boldsymbol{u}}))\in[k-1,k]\}.$$

Observe that, since f is increasing,

$$\int_{\mathsf{P}} f(\Phi(\Delta(\bar{\boldsymbol{u}})))\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}) \leq \sum_{k\geq 1}\int_{\mathsf{P}_k} f(k)\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}) = \sum_{k\geq 1} f(k)\varphi^{d+1}(\mathsf{P}_k),$$

and

$$\sum_{k\geq 1}\frac{f(k)}{k^2} \leq 4\sum_{k\geq 1}\frac{f(k)}{(k+1)^2} \leq 4\sum_{k\geq 1}\int_k^{k+1} f(t)\frac{1}{t^2}\mathrm{d}t \leq 4\int_0^{\infty}\frac{f(t)}{t^2}\mathrm{d}t.$$

Thus we only have to show that there exists a constant $C'_{30} > 0$, such that $\varphi^{d+1}(\mathsf{P}_k) < C'_{30}k^{-2}$, for any $k \geq 1$.

Since φ is absolutely continuous with respect to the spherical Lebesgue measure σ, there exists a constant c_φ such that $\varphi < c_\varphi\sigma$. Thus we can reduce the problem one step further. We only have to show that there exists a constant $C''_{30} > 0$ such that $\sigma^{d+1}(\mathsf{P}_k) < C''_{30}k^{-2}$.

For any $\bar{\boldsymbol{u}} \in \mathsf{P}$ and $i \in \{0,\dots,d\}$, we consider

$$\boldsymbol{v}(\bar{\boldsymbol{u}}, i) := \bigcap_{j\in\{0,\dots d\}\setminus\{i\}} H(\boldsymbol{u}_j, 1),$$

the vertex of the simplex $\Delta(\bar{\boldsymbol{u}})$ which is not contained in the face with outward normal vector $\boldsymbol{u}_i$, see Figure 7.4. With respect to this notation we define

$$\mathsf{P}'_k := \{\bar{\boldsymbol{u}} \in \mathsf{P}_k : \|\boldsymbol{v}(\bar{\boldsymbol{u}}, i)\| \leq \|\boldsymbol{v}(\bar{\boldsymbol{u}}, 0)\| \text{ for any } i \in [n]\}.$$

It is easy to see that $\sigma^{d+1}(\mathsf{P}_k) = (d+1)\sigma^{d+1}(\mathsf{P}'_k)$. Let $p\colon (\mathbb{S}^{d-1})^{d+1} \to (\mathbb{S}^{d-1})^{d+1}$ be the permutation defined by $p(\boldsymbol{u}_0,\dots,\boldsymbol{u}_d) = (\boldsymbol{u}_1,\dots,\boldsymbol{u}_d,\boldsymbol{u}_0)$. We have $\mathsf{P}_k = \cup_{i=0}^d p^i(\mathsf{P}'_k)$ and $\varphi(p^i(\mathsf{P}'_k)\cap p^j(\mathsf{P}'_k)) = 0$ for any $i \neq j$. Therefore $\sigma^{d+1}(\mathsf{P}_k) = (d+1)\sigma^{d+1}(\mathsf{P}'_k)$.

Using Theorem 4.5.3, we have

$$\begin{aligned}\sigma^{d+1}\left(\mathsf{P}'_k\right) &= \int_{\mathbb{S}^{d-1}}\int_{(\mathbb{S}^{d-1})^d} \mathbb{1}\left(\bar{\boldsymbol{u}} \in \mathsf{P}'_k\right)\,\mathrm{d}\sigma^d(\boldsymbol{u}_1,\dots,\boldsymbol{u}_d)\,\mathrm{d}\sigma(\boldsymbol{u}_0)\\ &= (d-1)!\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^1\int_{(H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1})^d} \mathbb{1}\left(\bar{\boldsymbol{u}} \in \mathsf{P}'_k\right)\,\Delta_{d-1}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_d)\\ &\qquad \mathrm{d}(\sigma'_{\boldsymbol{v},t})^d(\boldsymbol{u}_1,\dots,\boldsymbol{u}_d)\,\frac{\mathrm{d}t}{(1-t^2)^{\frac{d}{2}}}\,\mathrm{d}\sigma(\boldsymbol{v})\,\mathrm{d}\sigma(\boldsymbol{u}_0),\end{aligned}$$

where $\sigma'_{\boldsymbol{v},t}$ denotes the surface area measure on the $(d-2)$-dimensional sphere $H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1}$. For any $\boldsymbol{u}_0, \boldsymbol{v} \in \mathbb{S}^{d-1}$ and $t \in (0,1)$, considering the

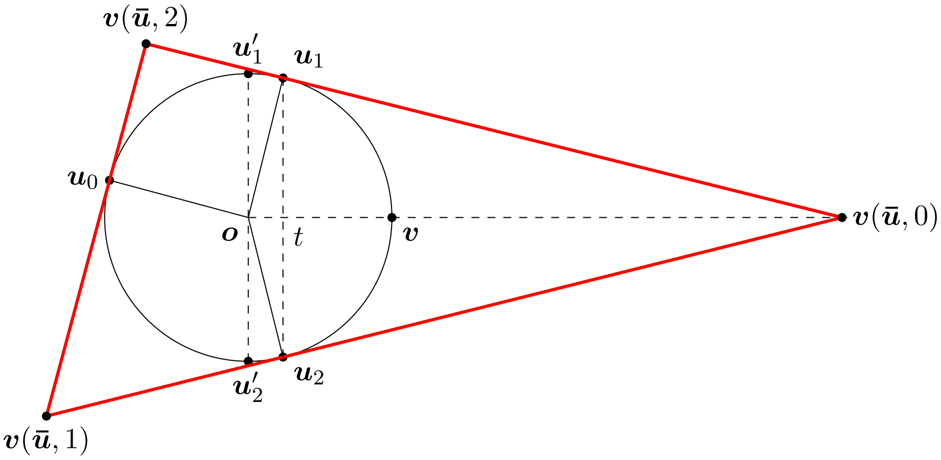

Figure 7.1: Construction of $\Delta(\bar{\boldsymbol{u}})$ (red thick triangle).

diffeomorphism

$$\begin{aligned} H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1} &\to H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1} \\ \boldsymbol{u}' &\mapsto \boldsymbol{u}=t\boldsymbol{v}+\sqrt{1-t^2}\boldsymbol{u}', \end{aligned}$$

we get

$$\begin{aligned} &\int\limits_{(H(\boldsymbol{v},t)\cap\mathbb{S}^{d-1})^d} \mathbb{1}\left(\bar{\boldsymbol{u}}\in\mathsf{P}_k'\right)\,\Delta_{d-1}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_d)\mathrm{d}(\sigma'_{\boldsymbol{v},t})^d(\boldsymbol{u}_1,\dots,\boldsymbol{u}_d) \\ &= \int\limits_{(H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1})^d} \mathbb{1}\left(\bar{\boldsymbol{u}}\in\mathsf{P}_k'\right)\,(1-t^2)^{\frac{d-1}{2}}\Delta_{d-1}(\boldsymbol{u}_1',\dots,\boldsymbol{u}_d') \\ &\qquad \left((1-t^2)^{\frac{d-2}{2}}\right)^d \mathrm{d}(\sigma'_{\boldsymbol{v},0})^d(\boldsymbol{u}_1',\dots,\boldsymbol{u}_d'), \end{aligned}$$

where

$$\bar{\boldsymbol{u}}=(\boldsymbol{u}_0,\dots,\boldsymbol{u}_d)=\left(\boldsymbol{u}_0,t\boldsymbol{v}+\sqrt{1-t^2}\boldsymbol{u}_1',\dots,t\boldsymbol{v}+\sqrt{1-t^2}\boldsymbol{u}_d'\right).$$

Thus

$$\begin{aligned} &\sigma^{d+1}(\mathsf{P}_k') \\ &= (d-1)!\int\limits_{\mathbb{S}^{d-1}}\int\limits_{\mathbb{S}^{d-1}}\int\limits_{(H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1})^d}\Delta_{d-1}(\boldsymbol{u}_1',\dots,\boldsymbol{u}_d') \qquad (7.11) \\ &\quad\times\int_0^1 \mathbb{1}\left(\bar{\boldsymbol{u}}\in\mathsf{P}_k'\right)(1-t^2)^{\frac{d^2-2d-1}{2}}\,\mathrm{d}t\,\mathrm{d}(\sigma'_{\boldsymbol{v},0})^d(\boldsymbol{u}_1',\dots,\boldsymbol{u}_d')\,\mathrm{d}\sigma(\boldsymbol{v})\,\mathrm{d}\sigma(\boldsymbol{u}_0). \end{aligned}$$

Fix $\boldsymbol{u}_0, \boldsymbol{v} \in \mathbb{S}^{d-1}$ and $\boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_d \in H(\boldsymbol{v}, 0) \cap \mathbb{S}^{d-1}$. We want to investigate the asymptotic of

$$\int_0^1 \mathbb{1}\left(\bar{\boldsymbol{u}} \in \mathsf{P}'_k\right)(1-t^2)^{\frac{d^2-2d-1}{2}} \, \mathrm{d}t$$

when $k \to \infty$.

For this, we need to get a good approximation of $\Phi(\Delta(\bar{\boldsymbol{u}}))$, as $t \to 0$. The first step to get this approximation is to describe the vertices of $\Delta(\bar{\boldsymbol{u}})$. Recall that we denote

$$\boldsymbol{v}(\bar{\boldsymbol{u}}, i) := \bigcap_{j \in \{0, \ldots d\} \setminus \{i\}} H(\boldsymbol{u}_j, 1).$$

Simple geometric computations give $\boldsymbol{v}(\bar{\boldsymbol{u}}, 0) = t^{-1}\boldsymbol{v}$. Since $\boldsymbol{u}_j \to \boldsymbol{u}'_j$ as $t \to 0$, we have that

$$\boldsymbol{v}(\bar{\boldsymbol{u}}, i) \to \boldsymbol{v}'_i := H(\boldsymbol{u}_0, 1) \cap \left(\bigcap_{j \in [d] \setminus \{i\}} H(\boldsymbol{u}'_j, 1) \right), \text{ when } t \to \infty.$$

Set $\Delta_{\boldsymbol{o},t} := [\boldsymbol{o}, \boldsymbol{v}(\bar{\boldsymbol{u}}, 1), \ldots, \boldsymbol{v}(\bar{\boldsymbol{u}}, d)]$ and $\Delta_{\boldsymbol{o}} := [\boldsymbol{o}, \boldsymbol{v}'(\bar{\boldsymbol{u}}, 1), \ldots, \boldsymbol{v}'(\bar{\boldsymbol{u}}, d)]$, where $[S]$ denotes the convex hull of the set S. We have

$$\begin{aligned}
\Phi(\Delta(\bar{\boldsymbol{u}})) &= \mu(\{H \in \mathcal{H} : H \cap \Delta(\bar{\boldsymbol{u}}) \neq \emptyset\}) \\
&= \mu(\{H \in \mathcal{H} : H \cap [\boldsymbol{o}, t^{-1}\boldsymbol{v}] \neq \emptyset\}) \\
&\quad + \mu(\{H \in \mathcal{H} : H \cap \Delta_{\boldsymbol{o},t} \neq \emptyset \,,\, H \cap [\boldsymbol{o}, t^{-1}\boldsymbol{v}] = \emptyset\}) \\
&= t^{-1}\Phi([\boldsymbol{o}, \boldsymbol{v}]) + \mu(\{H \in \mathcal{H} : H \cap \Delta_{\boldsymbol{o},t} \neq \emptyset \,,\, H \cap [\boldsymbol{o}, t^{-1}\boldsymbol{v}] = \emptyset\}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\Phi(\Delta(\bar{\boldsymbol{u}})) - t^{-1}\Phi([\boldsymbol{o}, \boldsymbol{v}]) \to C &= C(\boldsymbol{u}_o, \boldsymbol{v}, \boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_d) \\
&= \mu(\{H \in \mathcal{H} : H \cap \Delta_{\boldsymbol{o}} \neq \emptyset \,,\, H \cap \{t\boldsymbol{v} : t > 0\} = \emptyset\}).
\end{aligned}$$

In particular, when t is small enough, we have

$$t^{-1}\Phi([\boldsymbol{o}, \boldsymbol{v}]) + C - 1 < \Phi(\Delta(\boldsymbol{u})) < t^{-1}\Phi([\boldsymbol{o}, \boldsymbol{v}]) + C + 1.$$

Thus, for k big enough,

$$\boldsymbol{u} \in \mathsf{P}'_k \Leftrightarrow \Phi(\Delta(\boldsymbol{u})) \in [k-1, k] \Rightarrow t \in \left[\frac{\Phi([\boldsymbol{o}, \boldsymbol{v}])}{k - C + 1}, \frac{\Phi([\boldsymbol{o}, \boldsymbol{v}])}{k - C - 2} \right].$$

Hence, for k big enough,

$$\begin{aligned}
&\int_0^1 \mathbb{1}\left(\bar{\boldsymbol{u}} \in \mathsf{P}'_k\right)(1-t^2)^{\frac{d^2-2d-1}{2}} \, \mathrm{d}t \\
&\leq \int_0^1 \mathbb{1}\left(t \in \left[\frac{\Phi([\boldsymbol{o}, \boldsymbol{v}])}{k - C + 1}, \frac{\Phi([\boldsymbol{o}, \boldsymbol{v}])}{k - C - 2} \right]\right)(1-t^2)^{\frac{d^2-2d-1}{2}} \, \mathrm{d}t.
\end{aligned}$$

But, for $t \in (0, \frac{1}{2})$,

$$(1-t^2)^{\frac{d^2-2d-1}{2}} \leq (1-t^2)^{-\frac{1}{2}} \leq \left(\frac{3}{4}\right)^{-\frac{1}{2}} < 2.$$

Therefore, for k big enough,

$$\begin{aligned}\int_0^1 \mathbb{1}\left(\bar{\boldsymbol{u}} \in \mathsf{P}'_k\right)(1-t^2)^{\frac{d^2-2d-1}{2}} \mathrm{d}t &\leq \sqrt{2}\int_0^1 \mathbb{1}\left(t \in \left[\frac{\Phi([\boldsymbol{o},\boldsymbol{v}])}{k+C+1}, \frac{\Phi([\boldsymbol{o},\boldsymbol{v}])}{k+C-2}\right]\right)\mathrm{d}t \\ &= 2\Phi([\boldsymbol{o},\boldsymbol{v}])\left(\frac{1}{k-C-2} - \frac{1}{k-C+1}\right) \\ &< 7\Phi([\boldsymbol{o},\boldsymbol{v}])\frac{1}{k^2}\end{aligned}$$

Consequently, with (7.11), and the dominated convergence theorem, we get that for k big enough

$$k^2\sigma^{d+1}(\mathsf{P}'_k) < C,$$

where

$$\begin{aligned}C = (d-1)! &\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_{(H(\boldsymbol{v},0)\cap\mathbb{S}^{d-1})^d} \Delta_{d-1}(\boldsymbol{u}'_1,\dots,\boldsymbol{u}'_d) \\ &\times 8\Phi([\boldsymbol{o},\boldsymbol{v}])\,\mathrm{d}(\sigma'_{\boldsymbol{v},0})^d(\boldsymbol{u}'_1,\dots,\boldsymbol{u}'_d)\,\mathrm{d}\sigma(\boldsymbol{v})\,\mathrm{d}\sigma(\boldsymbol{u}_0) \\ &< \omega_d^2\omega_{d-1}^d \max_{(\boldsymbol{u}'_1,\dots,\boldsymbol{u}'_d)\in(\mathbb{S}^{d-2})^d} \Delta_{d-1}(\boldsymbol{u}'_1,\dots,\boldsymbol{u}'_d) 8 \max_{\boldsymbol{v}\in\mathbb{S}^{d-1}}\Phi([\boldsymbol{o},\boldsymbol{v}]) \\ &< \infty.\end{aligned}$$

This implies that there exists a constant C''_{30} such that $\sigma^{d+1}(\mathsf{P}'_k) < C''_{30}k^{-2}$ for any $k \geq 1$, which ends the proof. □

Now that we proved Lemma 7.4.1, we go back to our original problem which is to get a upper bound for $\mathbb{P}(f(Z_{\text{typ}}) > d+1, \mathtt{r}(Z_{\text{typ}}) < a)$. Equation (7.10) and Lemma 7.4.1 give

$$\begin{aligned}&\frac{(d+1)}{\gamma\Delta_{\max}}\frac{\gamma^{(d)}}{\gamma^d}\,\mathbb{P}(f(Z_{\text{typ}}) > d+1, \mathtt{r}(Z_{\text{typ}}) < a) \\ &< \int_{\mathsf{P}}\int_0^a 1 - e^{-\gamma r\Phi(\Delta(\bar{\boldsymbol{u}}))}\mathrm{d}r\,\mathrm{d}\varphi^{d+1}(\bar{\boldsymbol{u}}) \\ &< \int_1^\infty\int_0^a 1 - e^{-\gamma r t}\mathrm{d}r\,\frac{C_{30}}{t^2}\mathrm{d}t \\ &= C_{30}\left(I_1 + I_2 + I_3\right),\end{aligned}$$

where

$$I_1 = \int_1^{\frac{1}{\gamma a}} \int_0^a \frac{1 - e^{-\gamma r t}}{t^2} \mathrm{d}r\mathrm{d}t < \int_1^{\frac{1}{\gamma a}} \int_0^a \frac{\gamma r t}{t^2} \mathrm{d}r\mathrm{d}t = \frac{\gamma a^2}{2} \ln\left(\frac{1}{\gamma a}\right),$$

$$I_2 = \int_{\frac{1}{\gamma a}}^{\infty} \int_0^{\frac{1}{\gamma t}} \frac{1 - e^{-\gamma r t}}{t^2} \mathrm{d}r\mathrm{d}t < \int_{\frac{1}{\gamma a}}^{\infty} \int_0^{\frac{1}{\gamma t}} \frac{\gamma r t}{t^2} \mathrm{d}r\mathrm{d}t = \frac{\gamma a^2}{4},$$

and

$$I_3 = \int_{\frac{1}{\gamma a}}^{\infty} \int_{\frac{1}{\gamma t}}^{a} \frac{1 - e^{-\gamma r t}}{t^2} \mathrm{d}r\mathrm{d}t = \int_0^a \int_{\frac{1}{\gamma r}}^{\infty} \frac{1 - e^{-\gamma r t}}{t^2} \mathrm{d}t\mathrm{d}r = \left(\int_1^{\infty} \frac{1 - e^{-t}}{t^2} \mathrm{d}t \right) \frac{\gamma a^2}{2} < \frac{\gamma a^2}{2}.$$

Therefore, by setting $C_{31} := C_{30}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2}\right) = \frac{5}{4} C_{30}$, we have

$$\mathbb{P}(f(Z_{\text{typ}}) > d + 1, \, \mathtt{r}(Z_{\text{typ}}) < a) < C_{31} (\gamma a)^2 \ln\left(\frac{1}{\gamma a}\right),$$

for $a < e^{-1}\gamma^{-1}$. And with (7.9), we obtain

$$\mathbb{P}(f(Z_{\text{typ}}) > d + 1, \, \Sigma(Z_{\text{typ}}) < a) \leq C_{31} (\gamma \Sigma(B^d)^{-\frac{1}{k}} a^{\frac{1}{k}})^2 \ln\left(\frac{1}{\gamma \Sigma(B^d)^{-\frac{1}{k}} a^{\frac{1}{k}}}\right),$$

for $a < \Sigma(B^d) e^{-k} \gamma^{-k}$. This implies the existence of a constant C_{32}, depending on φ and Σ, such that

$$\mathbb{P}(f(Z_{\text{typ}}) > d + 1, \, \Sigma(Z_{\text{typ}}) < a) \leq C_{32} (\gamma a^{\frac{1}{k}})^2 \ln\left(\frac{1}{\gamma a^{\frac{1}{k}}}\right), \tag{7.12}$$

for $a < e^{-k}\gamma^{-k}$. Now, we easily get the following theorem.

Theorem 7.4.2. *Let $\varphi \in \aleph_{e,c}$. For any size measurement Σ, there exists a constant C_{33}, depending on φ and Σ, such that*

$$\mathbb{P}(f(Z_{\text{typ}}) > d + 1 \mid \Sigma(Z_{\text{typ}}) < a) < C_{33} \left(\gamma a^{\frac{1}{k}}\right) \ln\left(\frac{1}{\gamma a^{\frac{1}{k}}}\right),$$

for $a < e^{-k}\gamma^{-k}$.

Proof. Theorem 7.3.1, says that, when $a \to 0$,

$$\mathbb{P}(f(Z_{\mathrm{typ}}) = d+1 \mid \Sigma(Z_{\mathrm{typ}}) < a) \sim c_\varphi \gamma a^{\frac{1}{k}},$$

where c_φ is a constant. In particular, it exists a constant C_{34}, depending on φ and Σ, such that

$$\mathbb{P}(\Sigma(Z_{\mathrm{typ}}) < a) > \mathbb{P}(f(Z_{\mathrm{typ}}) = d+1 \mid \Sigma(Z_{\mathrm{typ}}) < a) > C_{34}\gamma a^{\frac{1}{k}},$$

for any $a < e^{-k}\gamma^{-k}$. Thus, with (7.12), we get

$$\begin{aligned}
\mathbb{P}(f(Z_{\mathrm{typ}}) > d+1 \mid \Sigma(Z_{\mathrm{typ}}) < a) &= \frac{\mathbb{P}(f(Z_{\mathrm{typ}}) > d+1\,,\, \Sigma(Z_{\mathrm{typ}}) < a)}{\Sigma(Z_{\mathrm{typ}}) < a)} \\
&< \frac{C_{32}\left(\gamma a^{\frac{1}{k}}\right)^2 \ln\left(\frac{1}{\gamma a^{\frac{1}{k}}}\right)}{C_{34}\gamma a^{\frac{1}{k}}} \\
&= \frac{C_{32}}{C_{34}}\left(\gamma a^{\frac{1}{k}}\right) \ln\left(\frac{1}{\gamma a^{\frac{1}{k}}}\right),
\end{aligned}$$

for $a < e^{-k}\gamma^{-k}$. This yields the proof. □

7.5 General case

In the previous chapter, when studying cells with big Σ-content, we have seen the importance of polytopes P for which the isoperimetric ratio $\frac{\Sigma(P)}{\Phi(P)^k}$ is close to the minimum $\tau = \inf_{K\in\mathcal{K}} \frac{\Phi(P)}{\Sigma(P)^{\frac{1}{k}}}$. In contrast, when studying small cells, polytopes with isoperimetric ratio $\frac{\Sigma(P)^{\frac{1}{k}}}{\Phi(P)}$ 'close to 0' are essential.

Set the measure $\mu_{n,\Sigma}$ on the shape space $\mathcal{P}_{n,\mathfrak{c},\Phi}$ defined by

$$\mu_{n,\Sigma}(A) := \frac{\gamma^d}{(n-d)\gamma^{(d)}} \int_A \Sigma(P)^{-\frac{n-d}{k}} \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P), \tag{7.13}$$

for any Borel set of shapes $A \subset \mathcal{P}_{n,\mathfrak{c},\Phi}$. The following lemma will be used in several proofs.

Lemma 7.5.1. *Let n be such that $\mathbb{P}(f(Z) = n) > 0$, and $A \subset \mathcal{P}_{n,\mathfrak{c},\Phi}$ a Borel set.*

1. *If $\inf\{\Sigma(P) : P \in A\} > 0$, then $\mu_{n,\Sigma}(A) < \infty$.*
2. *If $\mu_{n,\Sigma}(A) < \infty$, then when $a \to 0$,*

$$\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\, \Sigma(Z_{\mathrm{typ}}) < a) \sim a^{\frac{n-d}{k}} \mu_{n,\Sigma}(A).$$

Proof. Assume that $C_A = \inf\{\Sigma(P) : P \in A\} > 0$, and recall that

$$\mu_{n,\mathfrak{c},\Phi}(A) \leq \mu_{n,\mathfrak{c},\Phi}(\mathcal{P}_{n,\mathfrak{c},\Phi}) = \mu_n\left(\left\{P \in \mathcal{P}_n : \mathfrak{c}(P) \in [0,1]^d\,,\ \Phi(P) < 1\right\}\right) < \infty.$$

We have

$$\begin{aligned}\mu_{n,\Sigma}(A) &= \frac{\gamma^d}{(n-d)\gamma^{(d)}} \int_A \Sigma(P)^{-\frac{n-d}{k}} \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) \\ &< \frac{\gamma^d}{(n-d)\gamma^{(d)}} \mu_{n,\mathfrak{c},\Phi}(A) C_A^{-\frac{n-d}{k}} < \infty,\end{aligned}$$

which proves the first point.

Now assume that $\mu_{n,\Sigma}(A) < \infty$. By the Complementary Theorem 3.3.1, we have

$$\begin{aligned}&\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \\ &= \frac{\gamma^n}{\gamma^{(d)}} \int_A \int_0^\infty \mathbb{1}(\Sigma(tP) < a) e^{-\gamma t} t^{n-d-1} \mathrm{d}t\, \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P).\end{aligned}$$

But for any $P \in A$,

$$\int_0^\infty \mathbb{1}(\Sigma(tP) < a) e^{-\gamma t} t^{n-d-1} \mathrm{d}t = \int_0^{a^{\frac{1}{k}}\Sigma(P)^{\frac{1}{k}}} e^{-\gamma t} t^{n-d-1} \mathrm{d}t \sim_< \frac{a^{\frac{n-d}{k}} \Sigma(P)^{-\frac{n-d}{k}}}{n-d},$$

where $f \sim_< g$ means $f \sim g$ and $f < g$. Thus, with the dominated convergence theorem, we get

$$\begin{aligned}&a^{-\frac{n-d}{k}} \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \\ &\to \frac{\gamma^n}{(n-d)\gamma^{(d)}} \int_A \Sigma(P)^{-\frac{n-d}{k}} \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) = \mu_{n,\Sigma}(A),\end{aligned}$$

which is the second point of the lemma. □

If $\inf_{K\in\mathcal{K}} \frac{\Sigma(K)^{\frac{1}{k}}}{\Phi(K)} > 0$, then $\Sigma^{\frac{1}{k}}$ is of the same order as Φ and the behaviour of cells with small Σ-content is similar to the one of cells with small Φ-content, which we studied in Section 7.2. Theorem 7.5.2 gives a precise result for this case.

Theorem 7.5.2. *Assume that* $\inf_{K\in\mathcal{K}} \frac{\Sigma(K)^{\frac{1}{k}}}{\Phi(K)} > 0$. *Let* $n_{\min} = \min\{n \in \mathbb{N} : \mathbb{P}(f(Z_{\mathrm{typ}}) = n) > 0\}$. *When* $a \to 0$,

$$a^{-\frac{1}{k}} \mathbb{P}(f(Z) > n_{\min} \mid \Sigma(Z) < a) \to \frac{\mu_{n_{\min}+1,\Sigma}(\mathcal{P}_{n_{\min}+1,\mathfrak{c},\Phi})}{\mu_{n_{\min},\Sigma}(\mathcal{P}_{n_{\min},\mathfrak{c},\Phi})},$$

and for any Borel set of shape $A \subset \mathcal{K}_{\mathfrak{c},\Phi}$,

$$\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z) \in A \mid \Sigma(Z) < a) \to \frac{\mu_{n_{\min},\Sigma}(A \cap \mathcal{P}_{n_{\min}})}{\mu_{n_{\min},\Sigma}(\mathcal{P}_{n_{n_{\min}},\mathfrak{c},\Phi})}.$$

Proof. Let $A \subset \mathcal{P}_{n,\mathfrak{c},\Phi}$ be a Borel set of shapes of polytopes with n facets. By the Complementary Theorem 3.3.1, we have

$$\begin{aligned}
&\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\ f(Z_{\text{typ}}) = n\,,\ \Sigma(Z_{\text{typ}}) < a) \\
&= \frac{\gamma^n}{\gamma^{(d)}} \int_A \int_0^\infty \mathbb{1}(\Sigma(tP) < a) e^{-\gamma t} t^{n-d-1} \mathrm{d}t \, \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) \qquad (7.14) \\
&< \frac{\gamma^n}{\gamma^{(d)}} \int_A \int_0^\infty \mathbb{1}\left(t < a^{\frac{1}{k}} \sup_{K \in \mathcal{K}} \frac{\Phi(K)}{\Sigma(K)^{\frac{1}{k}}} \right) t^{n-d-1} \mathrm{d}t \, \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P).
\end{aligned}$$

Integrating over t gives

$$\begin{aligned}
&a^{-\frac{n-d}{k}} \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\ f(Z_{\text{typ}}) = n\,,\ \Sigma(Z_{\text{typ}}) < a) \\
&< \frac{\gamma^n}{\gamma^{(d)}(n-d)} \left(\sup_{K \in \mathcal{K}} \frac{\Phi(K)}{\Sigma(K)^{\frac{1}{k}}} \right)^{n-d} \mu_{n,\mathfrak{c},\Phi}(A) < \infty.
\end{aligned}$$

Therefore the dominated convergence theorem and (7.14) gives

$$a^{-\frac{n-d}{k}} \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\ f(Z_{\text{typ}}) = n\,,\ \Sigma(Z_{\text{typ}}) < a) \to \mu_{n,\Sigma}(A). \qquad (7.15)$$

In particular,

$$\mathbb{P}(f(Z_{\text{typ}}) = n\,,\ \Sigma(Z_{\text{typ}}) < a) \sim a^{\frac{n-d}{k}} \mu_{n,\Sigma}(\mathcal{P}_{n,\mathfrak{c},\Phi}),$$

which implies the first part of the theorem. The second part of the theorem follows directly from the first part and (7.15). □

In the next theorem, we consider a set $A \subset \mathcal{P}_{n,\mathfrak{c},\Phi}$. The set A is such that there exist polytopes $P \in A$ with $\Sigma(A)$ arbitrarily small. The theorem gives asymptotic lower and upper bounds of the probability $\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A\,,\ f(Z_{\text{typ}}) = n\,,\ \Sigma(Z_{\text{typ}}) < a)$, as $a \to 0$. The bounds depend on the asymptotic of $\mu_{n,\mathfrak{c},\Phi}(\{P \in A : \Sigma(P) < a\})$.

Theorem 7.5.3. *Let* $A \subset \mathcal{P}_{n,\mathfrak{c},\Phi}$ *be a Borel set. Assume that* $\alpha \geq \beta$ *are positive constants such that*

$$C_{35} := \liminf_{a \to 0} a^{-\alpha} \mu_{n,\mathfrak{c},\Phi}(\{P \in A : \Sigma(P) < a\}) \in (0, \infty),$$

and

$$C_{36} := \limsup_{a \to 0} a^{-\beta} \mu_{n,\mathfrak{c},\Phi}(\{P \in A : \Sigma(P) < a\}) \in (0, \infty).$$

1. *If* $\alpha < \frac{n-d}{k}$, *then*

$$\liminf a^{-\alpha}\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \geq C_{35}\frac{\Gamma(n-d-\alpha k)}{\gamma^{n-d-\alpha k}}.$$

2. *If* $\beta < \frac{n-d}{k}$, *then*

$$\limsup a^{-\beta}\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \leq C_{36}\frac{\Gamma(n-d-\alpha k)}{\gamma^{n-d-\alpha k}}.$$

3. *If* $\alpha = \frac{n-d}{k}$, *then*

$$\liminf \left(-\ln(a)a^{\frac{n-d}{k}}\right)^{-1}\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \geq C_{35}k^{-1}.$$

4. *If* $\beta = \frac{n-d}{k}$, *then*

$$\limsup \left(-\ln(a)a^{\frac{n-d}{k}}\right)^{-1}\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \leq C_{36}k^{-1}.$$

5. *If* $\alpha > \frac{n-d}{k}$, *then there exists a constant* C_{37} *such that*

$$\limsup a^{-\frac{n-d}{k}}\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \geq C_{37}.$$

6. *If* $\beta > \frac{n-d}{k}$, *then* $\mu_{n,\Sigma}(A) < \infty$ *and*

$$\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A\,,\ f(Z_{\mathrm{typ}}) = n\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \sim a^{\frac{n-d}{k}}\mu_{n,\Sigma}(A).$$

Proof. For any $a > 0$, set

$$A_a := \{P \in A : \Sigma(P) < a\}.$$

Let $\epsilon > 0$ and $\delta > 0$ such that for any $a < \delta$,

$$(C_{35} - \epsilon)a^{\alpha} \leq \mu_{n,\mathfrak{c},\Phi}\left(A_a\right) \leq (C_{36} + \epsilon)a^{\beta}. \tag{7.16}$$

By Lemma 7.5.1,

$$\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A \setminus A_\delta\,,\ f(Z_{\mathrm{typ}}) = n\,,\ \Sigma(Z_{\mathrm{typ}}) < a) \sim a^{\frac{n-d}{k}}\mu_{n,\Sigma}(A \setminus A_\delta).$$

Therefore we only have to study the asymptotic behaviour of $\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A_\delta\,,\ f(Z_{\mathrm{typ}}) = n\,,\ \Sigma(Z_{\mathrm{typ}}) < a)$. By the Complementary Theorem 3.3.1, we have

$$\begin{aligned}&\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\mathrm{typ}}) \in A_\delta\,,\ f(Z_{\mathrm{typ}}) = n\,,\ \Sigma(Z_{\mathrm{typ}}) < a)\\ &= \frac{\gamma^n}{\gamma^{(d)}}\int_{A_\delta}\int_0^\infty \mathbb{1}(\Sigma(tP) < a)e^{-\gamma t}t^{n-d-1}\mathrm{d}t\,\mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P).\end{aligned}$$

With Fubini, and using the k-homogeneity of Σ,

$$\begin{aligned}&\mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A_\delta\,,\, f(Z_{\text{typ}}) = n\,,\, \Sigma(Z_{\text{typ}}) < a) \\ &= \frac{\gamma^n}{\gamma^{(d)}} \int_0^\infty \left(\int_{A_\delta} \mathbb{1}(\Sigma(P) < at^{-k}) \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) \right) e^{-\gamma t} t^{n-d-1} \mathrm{d}t.\end{aligned}$$

Observe that $\Sigma(P) < at^{-k}$, for any $P \in A_\delta$ and $t < \left(\frac{a}{\delta}\right)^{\frac{1}{k}}$. Therefore

$$\begin{aligned}&\frac{\gamma^{(d)}}{\gamma^n} \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A_\delta\,,\, f(Z_{\text{typ}}) = n\,,\, \Sigma(Z_{\text{typ}}) < a) \\ &= \mu_{n,\mathfrak{c},\Phi}(A_\delta) \int_0^{\left(\frac{a}{\delta}\right)^{\frac{1}{k}}} e^{-\gamma t} t^{n-d-1} \mathrm{d}t \\ &\quad + \int_{\left(\frac{a}{\delta}\right)^{\frac{1}{k}}}^\infty \left(\int_{A_\delta} \mathbb{1}(\Sigma(P) < at^{-k}) \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) \right) e^{-\gamma t} t^{n-d-1} \mathrm{d}t.\end{aligned}$$

Thus, when $a \to 0$,

$$\begin{aligned}&\frac{\gamma^{(d)}}{\gamma^n} \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A_\delta\,,\, f(Z_{\text{typ}}) = n\,,\, \Sigma(Z_{\text{typ}}) < a) \\ &\sim \frac{\mu_{n,\mathfrak{c},\Phi}(A_\delta)}{(n-d)\delta^{\frac{n-d}{k}}} a^{\frac{n-d}{k}} \\ &\quad + \int_{\left(\frac{a}{\delta}\right)^{\frac{1}{k}}}^\infty \left(\int_{A_\delta} \mathbb{1}(\Sigma(P) < at^{-k}) \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) \right) e^{-\gamma t} t^{n-d-1} \mathrm{d}t,\end{aligned}$$

which can be written

$$\begin{aligned}&\frac{\gamma^{(d)}}{\gamma^n} \mathbb{P}(\mathfrak{s}_{\mathfrak{c},\Phi}(Z_{\text{typ}}) \in A_\delta\,,\, f(Z_{\text{typ}}) = n\,,\, \Sigma(Z_{\text{typ}}) < a) \\ &\sim \frac{\mu_{n,\mathfrak{c},\Phi}(A_\delta)}{(n-d)\delta^{\frac{n-d}{k}}} a^{\frac{n-d}{k}} + \int_{\left(\frac{a}{\delta}\right)^{\frac{1}{k}}}^\infty \mu_{n,\mathfrak{c},\Phi}\left(A_{at^{-k}}\right) e^{-\gamma t} t^{n-d-1} \mathrm{d}t,\end{aligned} \tag{7.17}$$

By elementary computations we get that, for $\alpha \in \mathbb{R}$ and $a \to 0$ with $a > 0$,

$$\int_{\left(\frac{a}{\delta}\right)^{\frac{1}{k}}}^\infty \left(at^{-k}\right)^\alpha e^{-\gamma t} t^{n-d-1}\, \mathrm{d}t \sim \begin{cases} a^\alpha \gamma^{-n+d+\alpha k} \Gamma(n-d-\alpha k) & \text{if } \alpha < \frac{n-d}{k} \\ -\ln(a) a^{\frac{n-d}{k}} k^{-1} & \text{if } \alpha = \frac{n-d}{k} \\ a^{\frac{n-d}{k}} \frac{\delta^{\frac{n-d}{k} - \alpha}}{-n+d+\alpha k} & \text{if } \alpha > \frac{n-d}{k} \end{cases}.$$

Thus, with (7.16), we get for a small enough

$$\int_{\left(\frac{a}{\delta}\right)^{\frac{1}{k}}}^{\infty} \mu_{n,\mathfrak{c},\Phi}\left(A_{at^{-k}}\right) e^{-\gamma t} t^{n-d-1} \mathrm{d}t$$

$$\geq (C_{35} - 2\epsilon) \begin{cases} a^{\alpha} \gamma^{-n+d+\alpha k} \Gamma(n-d-\alpha k) & \text{if } \alpha < \frac{n-d}{k} \\ -\ln(a) a^{\frac{n-d}{k}} k^{-1} & \text{if } \alpha = \frac{n-d}{k} \\ a^{\frac{n-d}{k}} \frac{\delta^{\frac{n-d}{k}-\alpha}}{-n+d+\alpha k} & \text{if } \alpha > \frac{n-d}{k} \end{cases},$$

and

$$\int_{\left(\frac{a}{\delta}\right)^{\frac{1}{k}}}^{\infty} \mu_{n,\mathfrak{c},\Phi}\left(A_{at^{-k}}\right) e^{-\gamma t} t^{n-d-1} \mathrm{d}t$$

$$\leq (C_{36} + 2\epsilon) \begin{cases} a^{\beta} \gamma^{-n+d+\beta k} \Gamma(n-d-\beta k) & \text{if } \beta < \frac{n-d}{k} \\ -\ln(a) a^{\frac{n-d}{k}} k^{-1} & \text{if } \beta = \frac{n-d}{k} \\ a^{\frac{n-d}{k}} \frac{\delta^{\frac{n-d}{k}-\beta}}{-n+d+\beta k} & \text{if } \beta > \frac{n-d}{k} \end{cases}.$$

Since these inequalities hold for any $\epsilon > 0$, with (7.17) they imply the inequalities of the theorem.

It remains to prove the point 6. Assume that $\beta > \frac{n-d}{k}$.

$$\begin{aligned} \mu_{n,\Sigma}(A) &= \frac{\gamma^d}{(n-d)\gamma^{(d)}} \int_A \Sigma(P)^{-\frac{n-d}{k}} \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) \\ &= \frac{\gamma^d}{(n-d)^2\gamma^{(d)}} \int_A \int_0^{\Sigma(P)^{-\frac{1}{k}}} t^{n-d-1} \mathrm{d}t \, \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P). \end{aligned}$$

Using Fubini,

$$\begin{aligned} \mu_{n,\Sigma}(A) &= \frac{\gamma^d}{(n-d)^2\gamma^{(d)}} \int_0^{\infty} \int_A \mathbb{1}\left(\Sigma(P) < t^{-k}\right) \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(A) t^{n-d-1} \mathrm{d}t \\ &= \frac{\gamma^d}{(n-d)^2\gamma^{(d)}} \int_0^{\infty} \mu_{n,\mathfrak{c},\Phi}(A_{t^{-k}}) t^{n-d-1} \mathrm{d}t. \end{aligned}$$

But, by assumption, there exists t_0 such that

$$\mu_{n,\mathfrak{c},\Phi}(A_{t^{-k}}) \leq 2C_{36} t^{-k\beta},$$

for $t > t_0$. Therefore

$$\frac{(n-d)^2\gamma^{(d)}}{\gamma^d}\mu_{n,\Sigma}(A) \le \int\limits_0^{t_0} \mu_{n,\mathfrak{c},\Phi}(A_{t^{-k}})t^{n-d-1}\mathrm{d}t + \int\limits_{t_0}^{\infty} 2C_{36}t^{n-d-1-k\beta}\mathrm{d}t.$$

The first integral above is finite because $\mu_{n,\mathfrak{c},\Phi}(A_{t^{-k}}) < \mu_{n,\mathfrak{c},\Phi}(A) < \infty$ and $n \ge d+1$. The second integral is also finite since $n-d-1-k\beta < -1$. Hence $\mu_{n,\Sigma}(A) < \infty$ and we can apply Lemma 7.5.1, which ends the proof. □

We present now two corollaries of the cases 3 and 4 of Theorem 7.5.3. The second is a specific case of the first.

Corollary 7.5.4. *Let n be such that $\mathbb{P}(f(Z_{\text{typ}}) = n) > 0$, and assume that there exists a constant C_{38} such that*

$$\int\limits_{\mathcal{P}_{n,\mathfrak{c},\Phi}} \mathbb{1}\left(\Sigma(P) < a\right) \mathrm{d}\mu_{n,\mathfrak{c},\Phi}(P) \sim C_{38}a^{\frac{n-d}{k}},$$

when $a \to 0$. Then, for any $\epsilon > 0$ and $a \to 0$,

$$\mathbb{P}\left(\frac{\Sigma(Z_{\text{typ}})^{\frac{1}{k}}}{\Phi(Z_{\text{typ}})} > \epsilon \mid f(Z_{\text{typ}}) = n\,,\, \Sigma(Z_{\text{typ}}) < a\right) \sim \frac{C_\epsilon}{-\ln(a)},$$

where

$$C_\epsilon := kC_{38}^{-1}\mu_{n,\Sigma}\left(\left\{P \in \mathcal{P}_{n,\mathfrak{c},\Phi} : \Sigma(P) > \epsilon^k\right\}\right).$$

Proof. Lemma 7.5.1 gives us that

$$C'_\epsilon := \mu_{n,\Sigma}\left(\left\{P \in \mathcal{P}_{n,\mathfrak{c},\Phi} : \Sigma(P) > \epsilon^k\right\}\right) < \infty,$$

and that

$$\mathbb{P}\left(\frac{\Sigma(Z_{\text{typ}})^{\frac{1}{k}}}{\Phi(Z_{\text{typ}})} > \epsilon\,,\, f(Z_{\text{typ}}) = n\,,\, \Sigma(Z_{\text{typ}}) < a\right) \sim C'_\epsilon a^{\frac{n-d}{k}}. \tag{7.18}$$

With the setting of Theorem 7.5.3, we have $\alpha = \beta = \frac{n-d}{k}$ and $C_{35} = C_{36} = C_{38}$. Therefore the cases 3 and 4 of the theorem tells us that

$$\mathbb{P}\left(f(Z_{\text{typ}}) = n\,,\, \Sigma(Z_{\text{typ}}) < a\right) \sim k^{-1}C_{38}(-\ln a)a^{\frac{n-d}{k}}. \tag{7.19}$$

Combining (7.18) and (7.19) ends the proof. □

As announced above the next corollary is a specific case of the previous one. Basically it shows that in a planar line tessellation with only horizontal and vertical lines, cells with small area tend to be degenerated, meaning that their shape converges weakly to the shape of a line segment. It is a result which was proved first by Beermann, Redenbach and Thäle in [BRT14], but with an incorrect rate of convergence. The correct rate was presented in the doctoral thesis of Beermann [Bee15]. Our result is slightly more precise since we provide an explicit constant C_ϵ.

Corollary 7.5.5. *Set $d = 2$, $\gamma = 1$ and $\varphi = \frac{1}{4}(\delta_{\boldsymbol{e}_1} + \delta_{-\boldsymbol{e}_1} + \delta_{\boldsymbol{e}_2} + \delta_{-\boldsymbol{e}_2})$. Then, for any $\epsilon > 0$,*

$$\mathbb{P}\left(\frac{V_2(Z_{\mathrm{typ}})^{\frac{1}{2}}}{V_1(Z_{\mathrm{typ}})} > \epsilon \mid V_2(Z_{\mathrm{typ}}) < a\right) \sim \frac{C_{4\epsilon}}{-\ln(a)},$$

when $a \to 0$, where

$$C_{4\epsilon} = 16\ln\left(\frac{1+\sqrt{1-4\epsilon^2}}{1-\sqrt{1-4\epsilon^2}}\right) \sim -32\ln(\epsilon),$$

when $\epsilon \to 0$.

Proof. Let us first describe the model. We have a planar line mosaic with lines either horizontal or vertical. Each cell is a rectangle with sides supported by lines of equations $x = x_1$, $x = x_2$, $y = y_1$ and $y = y_2$ with $x_1 < x_2$ and $y_1 < y_2$. Therefore we consider the following identification

$$\mathcal{P}_4 = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 : x_1 < x_2\,,\, y_1 < y_2\},$$

and

$$\mu_4(\cdot) = \int_{\mathcal{P}_4} \mathbb{1}((x_1, x_2, y_1, y_2) \in \cdot)\frac{\mathrm{d}x_1}{4}\frac{\mathrm{d}x_2}{4}\frac{\mathrm{d}y_1}{4}\frac{\mathrm{d}y_2}{4}.$$

The Φ-content of such a cell is $\frac{1}{4}[(x_2-x_1)+(y_2-y_1)]$. Thus $\Phi = \frac{1}{4}V_1$ in the setting of the corollary. We set the center to be the lower left corner, i.e. the point of coordinate (x_1, y_1). By definition, the space of shapes $\mathcal{P}_{4,\mathfrak{c},\Phi}$ is

$$\left\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 : 0 = x_1 < x_2\,,\, 0 = y_1 < y_2\,,\, \frac{1}{4}(x_2+y_2) = 1\right\}.$$

To simplify the notation, we do the following identification,

$$\mathcal{P}_{4,\mathfrak{c},\Phi} = \left\{(x, y) \in \mathbb{R}^2 : 0 < x\,,\, 0 < y\,,\, \frac{1}{4}(x+y) = 1\right\}.$$

Then the measure on $\mathcal{P}_{4,\mathfrak{c},\Phi}$ can be written

$$\begin{aligned}\mu_{4,\mathfrak{c},\Phi}(\cdot) = \int_{\mathcal{P}_4} &\mathbb{1}\left((x_1, y_1) \in [0,1]^2\right)\mathbb{1}\left(\frac{1}{4}[(x_2-x_1)+(y_2-y_1)] \le 1\right)\\ &\times \mathbb{1}\left(\left(\frac{4(x_2-x_1)}{(x_2-x_1)+(y_2-y_1)}, \frac{4(y_2-y_1)}{(x_2-x_1)+(y_2-y_1)}\right) \in \cdot\right)\\ &\times \frac{\mathrm{d}x_1}{4}\frac{\mathrm{d}x_2}{4}\frac{\mathrm{d}y_1}{4}\frac{\mathrm{d}y_2}{4},\end{aligned}$$

which can be written in a shorter form

$$\mu_{4,\mathfrak{c},\Phi}(\cdot) = \frac{1}{16} \int_0^4 \int_0^{4-x} \mathbb{1}\left(\left(\frac{4x}{x+y}, \frac{4y}{x+y}\right) \in \cdot\right) \frac{\mathrm{d}x}{4} \frac{\mathrm{d}y}{4}$$

Using the isomorphism

$$\begin{array}{rcl} [0,1] \times [-1,1] & \to & \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 4\} \\ (s,t) & \mapsto & (x,y) = (2s(1+t), 2s(1-t)), \end{array}$$

we get

$$\begin{aligned} \mu_{4,\mathfrak{c},\Phi}(\cdot) &= \frac{1}{16} \int_0^1 \int_{-1}^1 \mathbb{1}\left((2+2t\,,\,2-2t) \in \cdot\right) \mathrm{d}t\, s\mathrm{d}s \\ &= \frac{1}{32} \int_{-1}^1 \mathbb{1}\left((2+2t\,,\,2-2t) \in \cdot\right) \mathrm{d}t. \end{aligned} \tag{7.20}$$

Now that we set up the model and that we presented the measure $\mu_{4,\mathfrak{c},\Phi}$ in a explicit simple form, we want to measure the importance of the quadrilaterals P with a low isoperimetric ratio $V_2(P)^{1/2}\Phi(P)^{-1}$. For any $a > 0$, we have

$$\begin{aligned} \int_{\mathcal{P}_{4,\mathfrak{c},\Phi}} \mathbb{1}\left(V_2(P) < a\right) \mathrm{d}\mu_{4,\mathfrak{c},\Phi}(P) &= \frac{1}{32} \int_{-1}^1 \mathbb{1}\left((2+2t)(2-2t) < a\right) \mathrm{d}t \\ &= \frac{1 - \sqrt{1 - \frac{a}{4}}}{16} \\ &\sim \frac{a}{128}. \end{aligned}$$

Applying Corollary 7.5.4 gives, for $a \to 0$,

$$\begin{aligned} \mathbb{P}\left(\frac{V_2(Z_{\mathrm{typ}})^{\frac{1}{2}}}{V_1(Z_{\mathrm{typ}})} > \epsilon \mid V_2(Z_{\mathrm{typ}}) < a\right) &= \mathbb{P}\left(\frac{V_2(Z_{\mathrm{typ}})^{\frac{1}{2}}}{\Phi(Z_{\mathrm{typ}})} > 4\epsilon \mid V_2(Z_{\mathrm{typ}}) < a\right) \\ &\sim \frac{C_{4\epsilon}}{-\ln(a)}, \end{aligned}$$

with $C_{4\epsilon} = 128\, k\, \mu_{n,\Sigma}\left(\left\{P \in \mathcal{P}_{n,\mathfrak{c},\Phi} : \Sigma(P) > (4\epsilon)^k\right\}\right)$, where $n = 4$, $\Sigma = V_2$ and $k = 2$. It only remains to compute explicitly the constant $C_{4\epsilon}$. By definition of $\mu_{n,\Sigma}$, see (7.13),

$$C_{4\epsilon} = \frac{128\gamma^2}{\gamma^{(d)}} \int_{\mathcal{P}_{4,\mathfrak{c},\Phi}} \mathbb{1}\left(V_2(P) > (4\epsilon)^2\right) V_2(P)^{-1} \mathrm{d}\mu_{4,\mathfrak{c},\Phi}(P).$$

Recall that $\gamma = 1$. It is easy to get $\gamma^{(d)} = \frac{1}{16}$. Thus, with (7.20), we have

$$\begin{aligned}
C_{4\epsilon} &= \frac{128}{32\frac{1}{16}} \int_{-1}^{1} \mathbb{1}\left((2+2t)(2-2t) > (4\epsilon)^2\right)[(2+2t)(2-2t)]^{-1}\mathrm{d}t \\
&= 32 \int_{0}^{\sqrt{1-4\epsilon^2}} (1-t^2)^{-1}\mathrm{d}t \\
&= 32 \left[\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)\right]_{t=0}^{\sqrt{1-4\epsilon^2}} \\
&= 16\ln\left(\frac{1+\sqrt{1-4\epsilon^2}}{1-\sqrt{1-4\epsilon^2}}\right).
\end{aligned}$$

Asymptotically we get, as $\epsilon \to 0$,

$$\begin{aligned}
C_{4\epsilon} &\sim 16\ln\left(\frac{2}{1-\sqrt{1-4\epsilon^2}}\right) \\
&= 16\ln\left(\frac{2}{2\epsilon^2 + O(\epsilon^4)}\right) \\
&\sim -32\ln(\epsilon),
\end{aligned}$$

which yields the proof. □

List of Notations

q_n^ϵ		90
$\mathbb{R}_+$	non negative numbers	10
r	distance exponent	12
$\mathtt{r}$	inradius	105
$\mathbb{R}^d$	euclidean space	9
r_n		85
r_n^ϵ		90
$\mathfrak{s}_{\mathfrak{c},\Phi}$	shape	21
$\mathfrak{s}_\Phi$	shape	21
Σ	size measurement	74
σ	surface area measure	10
$\mathfrak{S}_n$	set of the permutations of n elements	46
$\sigma'_{\boldsymbol{v},t}$		50
$\mathfrak{s}$	shape	74
$\mathbb{S}^{d-1}(\boldsymbol{x}, r)$	sphere of center $\boldsymbol{x}$ and radius r	10
$\mathbb{S}^{d-1}$	unit sphere	10
τ		83
Θ	intensity measure of η	12
$\boldsymbol{\Theta}$		14
V_φ		98
$V_i(K)$	i-th intrinsic volume of K	11
$\frac{V_j(K)^{1/j}}{V_i(K)^{1/i}}$	(i,j)-isoperimetric ratio of K	11
$\boldsymbol{v}(\bar{\boldsymbol{u}}, i)$		107
X	hyperplane tessellation	13
$X(\cdot)$		25
Z		74
$Z_{\boldsymbol{o}}$	zero cell	13
Z_{typ}	typical cell	27

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