Unimodular Covers and Triangulations of Lattice Polytopes

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Contents

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1 Introduction

1.1 Problem definition

This thesis deals with unimodular covers and triangulations of lattice polytopes. It centers around the following problem definition: **Provide a good bound c_d such that for all d-dimensional lattice polytopes and $c \geq c_d$ all multiples cP admit a unimodular cover.**

In [8] Knudsen and Mumford showed that for every lattice polytope P there exists a number c_P such that $c_P P$ admits a unimodular triangulation. One might ask if this result can be generalized. Is there a number c_d only depending on the dimension of the lattice polytope P such that the multiples cP admit a unimodular triangulation for all d-dimensional lattice polytopes P and all $c \geq c_d$?

In [5] Bruns, Gubeladze and Trung showed in a very elegant and simple way that – if we restrict ourselves to unimodular covers – there exists a number c'_P for every lattice polytope $P \subset \mathbb{R}^d$ such that $c'_P P$ admits a unimodular cover for all $c \geq c'_P$. In [3] Bruns and Gubeladze even proved that this number only depends on the dimension d. More precisely, they deduced that there exists a number c_d^{pol} such that for all lattice polytopes $P \subset \mathbb{R}^d$ the multiples cP admit a unimodular cover for all $c \geq c_d^{pol}$. Furthermore, they even provided an upper bound for c_d^{pol}. But this bound is superexponential.

In the second chapter we will improve the bound by modifying a crucial step (with respect to the numerical result) in the proof of the above statement. This crucial step is based on a simple procedure for covering simplicial cones by unimodular cones. We will provide a new procedure for covering simplicial cones, which is better than the original one in a sense that the new procedure gives us a cover with unimodular cones whose generators are relatively short.

In the third chapter we will provide a similar procedure for the unimodular triangulation of simplicial cones which has no consequences on the bounds c_d^{cone} and c_d^{pol}, but might be of interest itself.

In the fourth chapter we turn away from multiples of polytopes and focus on stellar subdivisions of lattice polytopes. This turn is motivated by the fact that all procedures and results in the previous chapters are, roughly speaking, due to the successive application of stellar subdivision. Therefore, we speculate about the importance of this tool for the triangulation of lattice polytopes.

Before we go into detail, we will now introduce the basic definitions.

1.2 Basic lemmas and definitions

In this section we will provide the basic notation and, additionally, some lemmas which are implicitly used throughout this work.

Let G be a subset of $\mathbb{R}^d$. Then the cone $C \subset \mathbb{R}^d$ generated by the set G is defined as the set of finite, non-negative and real linear combinations of the vectors $v \in G$. Furthermore, a cone C is called *polyhedral* if it is generated by a finite set $G = \{v_1, \ldots, v_k\}$ of vectors. This means that

$$C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_k \subset \mathbb{R}^d.$$

We call a cone *pointed* if it does not contain any linear subspace except of $\{0\}$. Moreover, we shall call a cone *rational* if G is a subset of $\mathbb{Q}^d$. From now on, when we use the term *cone*, we always mean a polyhedral, pointed and rational cone. Because such cones are generated by a finite set of vectors $v_1, \ldots, v_k \in \mathbb{Z}^d$, the formula

$$C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_k \subset \mathbb{R}^d.$$

will mean that $v_i \in \mathbb{Z}^d \, (i = 1, \ldots, k)$. An f-cone is a cone of dimension f. Moreover, we define – according to Sebö [11]– the set

$$\mathrm{par}(v_1, \ldots, v_n) = \{l_1 v_1 + \cdots + l_n v_n : \, 0 \leq l_j < 1\} \cap \mathbb{Z}^d$$

for $v_1, \ldots, v_n \in \mathbb{R}^d$. A further class of cones which will be mentioned is the class of *simplicial* cones. These are the cones that are generated by a set of linearly independent vectors.

Let us now have a closer look at the set $C \cap \mathbb{Z}^d$. Therefore, we will describe $C \cap \mathbb{Z}^d$ in algebraic terms. A *monoid* M is defined as a set M together with an operation $M \times M \longmapsto M$ that is associative and has a neutral element. Furthermore, let an *affine monoid* be a finitely generated monoid which is isomorphic to a submonoid of a free abelian group $\mathbb{Z}^d$ for some $d \geq 0$. Very often an affine monoid is also called an *affine semigroup*, especially in the commutative algebra literature (see e.g. [2, 5, 6]). Moreover, if M is an affine monoid, then we define $\mathrm{gp}(M)$ as the subgroup of $\mathbb{Z}^d$ generated by M. We call

$$\overline{M} := \{x \in \mathrm{gp}(M) : nx \in M \text{ for some } n\}$$

the *normalization* of M. And we say that M is normal if $M = \overline{M}$. Then Gordan's Lemma tells us that $C \cap \mathbb{Z}^d$ is an affine monoid.

Lemma 1.2.1. (Gordan's Lemma) *Let $C \subset \mathbb{R}^d$ be a cone. Then $S(C) := C \cap \mathbb{Z}^d$ is an affine monoid.*

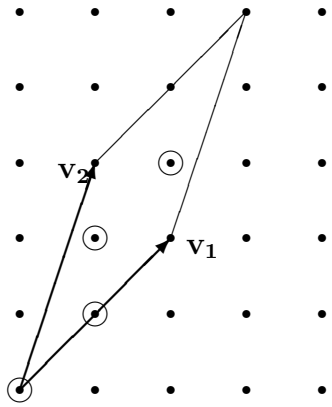

FIGURE 1.1 The set $\mathrm{par}(v_1, v_2)$

Proof. Let $C \subset \mathbb{R}^d$ be a cone. By definition, C is generated by finitely many vectors $v_1, \ldots, v_r$. Furthermore, let $w \in S(C)$. This implies that $w = \sum_{i=1}^r a_i v_i$ with $a_i \in \mathbb{R}_+$ for all i. We can rewrite this formula as follows:

$$w = \sum_{i=1}^r \lfloor a_i \rfloor v_i + \sum_{i=1}^r b_i v_i,$$

where $b_i := a_i - \lfloor a_i \rfloor$. But $x := \sum_{i=1}^r \lfloor a_i \rfloor v_i \in S(C)$. On the other hand, $y := \sum_{i=1}^r b_i v_i$ is an element of the bounded and finite set

$$\mathrm{par}(v_1, \ldots, v_r) = \left\{ \sum_{i=1}^r l_i v_i : \, 0 \leq l_i < 1 \right\} \cap \mathbb{Z}^d.$$

Hence $S(C)$ is finitely generated. More precisely, $S(C)$ is generated by the set $G := \mathrm{par}(v_1, \ldots, v_r) \cup \{v_1, \ldots, v_r\}$. Therefore, $S(C)$ is an affine monoid. $\square$

Now let us call an element $x \in M$ of a monoid M a *unit* if x has an inverse in M. Moreover, we say that x is *irreducible* if in every decomposition $x = y + z$ one of the summands $y \in M$ or $z \in M$ must be a unit.

Then we have the following results for affine monoids.

Lemma 1.2.2. *Let M be an affine monoid. Then every element $x \in M$ has a presentation $x = u + y_1 + \cdots + y_m$ in which u is a unit and $y_1, \ldots, y_m$ are irreducible. Furthermore, up to differences by units, there exist only finitely many irreducible elements in M.*

Proof. See Proposition 2.12(b) and (c) in [4], p. 53. □

In analogy to the positivity of cones a monoid is called positive if 0 is its only unit. Then Lemma 1.2.2 immediately implies that a positive affine monoid M has only finitely many irreducible elements and that these irreducible elements constitute a generating set of M. But, of course, the irreducible elements must also be contained in any other generating set of M. Therefore, it is justified to define the *Hilbert basis* $\mathrm{Hilb}(M)$ of a positive affine monoid M as the unique minimal set of generators of M.

Consequently, we have the following result if the positive affine monoid M is given as the set $S(C) = C \cap \mathbb{Z}^d$.

Lemma 1.2.3. *Let $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_r \subset \mathbb{R}^d$ be a cone and let the Hilbert basis $\mathrm{Hilb}(C)$ of the cone C be defined as the Hilbert basis of the positive affine monoid $C \cap \mathbb{Z}^d$. Then we have*

$$\mathrm{Hilb}(C) \subset \mathrm{par}(v_1, \ldots, v_r) \cup \{v_1, \ldots, v_r\}.$$

Proof. As we have seen in the proof of Gordan's Lemma, the set $G := \mathrm{par}(v_1, \ldots, v_r) \cup \{v_1, \ldots, v_r\}$ is a generating set of $S(C)$. Because the Hilbert basis $\mathrm{Hilb}(C)$ is a subset of every generating set of the positive monoid $S(C)$, the statement of Lemma 1.2.3 is true. □

A polytope P is defined as the convex hull of a finite set of points of $\mathbb{R}^d$. The *dimension* of a polytope $P \subset \mathbb{R}^d$ is given as the dimension of the affine hull of P, the smallest affine subspace containing P. Furthermore, a polytope is called an *e-polytope* if it has dimension e. And a *lattice polytope* is a polytope whose vertices belong to $\mathbb{Z}^d$. Moreover, a *simplex* is defined as a polytope whose vertices are affinely independent. A simplex that is also a lattice polytope will be called a *lattice simplex*. We say that a lattice simplex is *empty* if it contains no elements from $\mathbb{Z}^d$ other than its vertices.

The *multiplicity* $\mu(\Delta)$ of a lattice simplex Δ with the vertices $v_0, \ldots, v_g$ is given as the index of the subgroup U generated by the vectors $v_1 - v_0, \ldots, v_g - v_0$ in the smallest direct summand of $\mathbb{Z}^d$ containing U. For the multiplicity $\mu(\Delta)$ we have the following identity.

Lemma 1.2.4. *If $\Delta \subset \mathbb{R}^d$ is a lattice simplex with the vertices $v_0, \ldots, v_d$, then we have*

$$\mu(\Delta) = |\,\mathrm{par}(v_1 - v_0, \ldots, v_d - v_0)| = |\det(v_1 - v_0, \ldots, v_d - v_0)|.$$

Proof. Obviously, the set $\mathrm{par}(v_1 - v_0, \ldots, v_d - v_0)$ contains exactly one representative from each residue class of $\mathbb{Z}^d$ modulo $U := \mathbb{Z}(v_1 - v_0) + \ldots +$

$\mathbb{Z}(v_d - v_0)$. This implies the first equality. Due to the elementary divisor theorem there exists a basis $e_1, \ldots, e_d$ of $\mathbb{Z}^d$ and natural numbers $n_1, \ldots, n_d$ such that $n_1e_1, \ldots, n_de_d$ is a basis of U. But if $n_1e_1, \ldots, n_de_d$ is a basis of U, then $\mu(\Delta) = n_1 \cdots n_d$. On the other hand, $|\det(v_1 - v_0, \ldots, v_d - v_0)| = |\det(n_1e_1, \ldots, n_de_d)| = n_1 \cdots n_d$ if $n_1e_1, \ldots, n_de_d$ is a basis of U. $\square$

A subclass of the lattice simplices is the class of *unimodular* simplices Δ. These are specified by the property $\mu(\Delta) = 1$. Furthermore, we define for a d-polytope P the set $\mathrm{UC}(P)$ as the union of all unimodular d-simplices contained in P. A *unimodular cover* of a polytope P is defined as a finite system of unimodular lattice simplices, contained in P, which covers the polytope P. And a *unimodular triangulation* of a polytope P is defined as a unimodular cover of P which additionally is a triangulation of P. This means that any pair of simplices Δ', Δ'' from the cover of P intersects in a common face (possibly empty).

Now we can provide one of the central definitions of this thesis. Let c_d^{pol} denote the infimum of the natural numbers c such that

$$c'P = \mathrm{UC}(c'P)$$

for all lattice d-polytopes and all natural numbers $c' \geq c$.

Moreover, if $P \subset \mathbb{Z}^d$ is a lattice d-polytope, we can associate with such a polytope P the submonoid S_P of $\mathbb{Z}^{d+1}$ generated by the elements $(x, 1)$, $x \in P \cap \mathbb{Z}^d$. Then we say that P is a *normal polytope* if the monoid S_P is normal. (S_P is obviously an affine monoid, because the set $P \cap \mathbb{Z}^d$ is finite and, hence, S_P is finitely generated.) A normal polytope P for which, additionally, $\mathrm{gp}(S_P) = \mathbb{Z}^{d+1}$ will be called *integrally closed*.

Let us now come back to the field of cones. The *extreme (integral) generators* of a cone $C \subset \mathbb{R}^d$ are defined as the generators of the monoids $l \cap \mathbb{Z}^d$ where l runs through the edges of C. Then we define Δ_C as the convex hull of these extreme generators and 0. Consequently, a cone is simplicial if and only if Δ_C is a simplex. And a simplicial cone C for which the simplex Δ_C is empty is called an *empty simplicial cone*. Additionally, we define the *multiplicity* $\mu(C)$ of a simplicial cone as the multiplicity $\mu(\Delta_C)$ of the simplex Δ_C.

Analogously to the unimodularity of simplices, we will now define unimodular cones and unimodular covers of cones. A *unimodular cone* $C \subset \mathbb{R}^d$ is defined as a simplicial cone for which Δ_C is a unimodular simplex, and a *unimodular cover* of a cone C is defined as a finite system of unimodular cones whose union is equal to C. A *unimodular triangulation* of a cone C

is given as a unimodular cover which, additionally, is a triangulation, i.e. the covering cones just coincide along faces. Now we can provide the second central definition of this thesis.

We define c_d^{cone} to be the infimum of all natural numbers c such that every d-dimensional cone $C \subset \mathbb{R}^d$ admits a unimodular cover $C = C_1 \cup \ldots \cup C_k$ for which

$$\mathrm{Hilb}(C_j) \subset c\Delta_C, \quad j \in \{1, \ldots, k\}.$$

To construct covers and triangulations of cones, we will often apply *stellar subdivision* by a vector $x \in C$ to a cone $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$. Let $x = \alpha_1 v_1 + \cdots + \alpha_d v_d$. Then this simply means that the cone C is triangulated by the subcones

$$D_i = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_{i-1} + \mathbb{R}_+ x + \mathbb{R}_+ v_{i+1} + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$$

for which $\alpha_i \neq 0$ $(1 \leq i \leq d)$. In the same manner we can apply *stellar subdivision* with respect to a point $x \in \Delta$ to a lattice simplex $\Delta = \mathrm{conv}(v_0, \ldots, v_r)$ (see Figure 1.2).

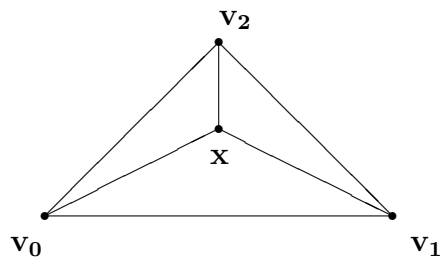

FIGURE 1.2 Stellar subdivision with respect to x

Remark 1.2.5. Lemma 1.2.4 will always be implicitly used if we apply stellar subdivision to a simplex $\Delta = \mathrm{conv}(0, v_1, \ldots, v_d)$ or a cone $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$ by $x \in \Delta$ respectively $x \in C$. Then the lemma provides us with the multiplicities of the cones respectively simplices that triangulate the cone C respectively the simplex Δ. Let me illustrate this in the case of a cone $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$ and a vector $x \in C$. Let $x = \alpha_1 v_1 + \cdots + \alpha_d v_d$ such that $\alpha_i \geq 0$ for all i. If we apply stellar subdivision by x to the cone C, then we end up with a triangulation $C = D_{i_1} \cup \ldots \cup D_{i_k}$, where $\alpha_{i_j} \neq 0$ for all j. Consequently, it follows by Lemma 1.2.4 that

$$\mu(D_{i_j}) = \mu(\Delta_{D_{i_j}}) = |\det(v_1, \ldots, v_{i_j-1}, x, v_{i_j+1}, \ldots, v_d)|.$$

But we also have $|\det(v_1, \ldots, v_{i_j-1}, x, v_{i_j+1}, \ldots, v_d)| = \alpha_{i_j} |\det(v_1, \ldots, v_d)| = \alpha_{i_j} \mu(C)$. Therefore, $\mu(D_{i_j}) = \alpha_{i_j} \mu(C)$.

Remark 1.2.6. As we have mentioned before, the set $\mathrm{par}(v_1, \ldots, v_d)$ contains exactly one representative from each residue class of $\mathbb{Z}^d$ modulo $U = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_d$. Consequently, for $x \in \mathrm{par}(v_1, \ldots, v_d)$, the term $\langle x \rangle$ shall simply denote the subgroup of $\mathbb{Z}^d/U$ generated by the element x.

Moreover, we will implicitly use the following lemma in Chapters 2 and 3. It shows that for every simplicial d-cone $C = \mathbb{R}_+v_1 + \cdots + \mathbb{R}_+v_d \subset \mathbb{R}^d$ there always exist vectors $x \in \mathrm{par}(v_1, \ldots, v_d)$ of a very special form. In the further chapters it will become clear that the existence of such vectors is essential to the new procedures for the unimodular cover and triangulation of simplicial cones.

Lemma 1.2.7. *Let $C = \mathbb{R}_+v_1 + \cdots + \mathbb{R}_+v_d \subset \mathbb{R}^d$ be a simplicial d-cone with $\mu(C) = \prod_{i=1}^m p_i^{a_i}$, p_i prime and $a_i \neq 0$ for all i. Then for every i there exists a vector $x \neq 0$ such that*

$$x = \sum_{j=1}^{d} \frac{l_j}{p_i} v_j \in \mathbb{Z}^d, \quad 0 \leq l_j < p_i, \quad l_j \in \mathbb{N}. \tag{$*$}$$

Proof. By the theorems of Sylow we know that every abelian group G with $|G| = \prod_{i=1}^m p_i^{a_i}$ (p_i prime and $a_i \neq 0$ for all i) contains an element of order p_i for all $i = 1, \ldots, m$. Let $U := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d$. Then $\mathbb{Z}^d/U$ is an abelian group with $\mu(C) = |\mathbb{Z}^d/U| = \mathrm{par}(v_1, \ldots, v_d)$ due to Lemma 1.2.4. Because a representative of an element of $\mathbb{Z}^d/U$ in the set $\mathrm{par}(v_1, \ldots, v_d)$, which has order p_i, is of the form $(*)$, it follows that we find the desired vector for every i . $\square$

1.3 Former results concerning c_d^{cone} and c_d^{pol}

Now that we have developed the necessary notation let us come back to the very beginning of this introduction and review in detail and in the words of our new notation the most important results, which we have outlined in the problem definition. Furthermore, we will also give some additional results concerning the values c_d^{cone} and c_d^{pol}.

We already mentioned in the problem definition that Bruns and Gubeladze showed in [3] that c_d^{cone} and c_d^{pol} are bounded by superexponential functions. In [4] they provide even better bounds. More precisely, they proved that

$$c_d^{cone} \leq \frac{(d+1)d}{2} \left(\left\lceil \sqrt{d-1} \right\rceil (d-1) \right)^{(2\ln 2)d+1}$$

for all $d \geq 2$ and that

$$c_d^{pol} \leq \frac{(d+1)^2 \cdot d^{1.5}}{2} \left(\left\lceil \sqrt{d-1} \right\rceil (d-1) \right)^{(2\ln 2)d+1}$$

for all $d \geq 2$.

The obvious similarity between these two upper bounds is owed to the following inequality, which was also proven by Bruns and Gubeladze in [3].

$$c_d^{cone} \leq c_d^{pol} \leq \sqrt{d}(d+1)c_d^{cone}, \quad d \in \mathbb{N}.$$

In fact, they first established the upper bound for c_d^{cone} and then derived the upper bound for c_d^{pol} from the last inequality.

Before the work of Bruns and Gubeladze no other general results about c_d^{cone} and c_d^{pol} were known. Only in dimensions $d = 1, 2, 3$ we have concrete results.

For c_d^{cone} we have the trivial result that $c_1^{cone} = 1$. Furthermore, we know that $c_2^{cone} = 1$, because the empty simplicial cones in dimension $d = 2$ are unimodular (as we will prove later on). In dimension $d = 3$ we have $c_3^{cone} = 2$. This result is due to Sebö, who showed in [11] that every cone C in dimension $d = 3$ admits a unimodular triangulation by cones which are generated by vectors from the Hilbert basis $\mathrm{Hilb}(C)$ of C. Together with the fact that $\mathrm{Hilb}(C) \subset (d-1)\Delta_C$ for all d, this gives us the desired result.

Sebö also conjectured in [11] that in all dimensions d every cone $C \subset \mathbb{R}^d$ admits a unimodular cover by cones which are exclusively generated by elements of the Hilbert basis $\mathrm{Hilb}(C)$ of C. The correctness of this conjecture would have implied that $c_d^{cone} \leq d-1$ for all d. Taking into account the result of Ewald and Wessels [7], who showed that there exist cones in all dimensions $d \geq 3$ such that $\mathrm{Hilb}(C)$ is not contained in $(d-2)\Delta_C$, it would have followed that $c_d^{cone} = d-1$ for all $d \geq 3$. But in [1] Bruns and Gubeladze provided a counterexample to the conjecture of Sebö. Apart from the above, no other general or concrete results concerning c_d^{cone} were known before the work of Bruns and Gubeladze.

For c_d^{pol} we have $c_1^{pol} = 1$. It is obvious that in dimension $d = 1$ every lattice polytope admits a unique unimodular triangulation and, hence, a unimodular cover. In dimension $d = 2$ we can triangulate every lattice polytope into lattice simplices. Furthermore, one can (by the successive application of stellar subdivision) triangulate these lattice simplices into empty lattice simplices. Then we are done, because every empty lattice simplex is unimodular in dimension $d = 2$ (thus every empty simplicial cone is unimodular in dimension $d = 2$).

To illustrate this, let $\Delta = \mathrm{conv}(0, v_1, v_2)$ with $v_1, v_2 \in \mathbb{Z}^2$ be an arbitrary empty lattice simplex shifted to the origin. Furthermore, let $x \in \mathrm{par}(v_1, v_2)$.

This means that $x = \alpha_1 v_1 + \alpha_2 v_2$ such that $0 \leq \alpha_i < 1$. It follows that either $\alpha_1 + \alpha_2 < 1$ or $\beta_1 + \beta_2 < 1$ with $\beta_i := 1 - \alpha_i$. This implies that either $x \in \Delta$ or $y := \beta_1 v_1 + \beta_2 v_2 \in \Delta$. Since Δ is empty, it follows that either $x = 0$ ($\alpha_i = 0$ for $i = 1, 2$) or $y = 0$ ($\beta_i = 0$ for $i = 1, 2$). But the latter case can be excluded, since otherwise $x = v_1 + v_1 \notin \mathrm{par}(v_1, v_2)$, which is a contradiction. Therefore, $x = 0$, which implies that $\mathrm{par}(v_1, v_2) = \{0\}$. Hence Δ is unimodular.

That every polygon admits a unimodular triangulation leads to the equation $c_2^{pol} = 1$. Moreover, due to Kantor and Sakaria [9] we know that $c_3^{pol} = 2$.

Coming back to the work of Bruns and Gubeladze in which they established the upper bounds for c_d^{cone} and c_d^{pol}, we have to mention that one of the most critical and important theorems in the proof is the following one.

Theorem 1.3.1. *Every simplicial d-cone $C \subset \mathbb{R}^d$, $d \geq 3$, admits a unimodular cover $C = D_1 \cup \ldots \cup D_T$ such that*

$$\mathrm{Hilb}(D_t) \subset \left(\frac{d}{2} \left(\frac{3}{2} \right)^{\mu(\Delta_C) - 2} \right) \Delta_C, \quad t \in [1, T].$$

This theorem is derived from a quite simple procedure to cover simplicial cones by unimodular cones using successive stellar subdivisions. Any improvement of this theorem directly affects the quality of the upper bound for c_d^{cone} and, hence, also the one for c_d^{pol}. Most of this thesis is about providing a better procedure to cover simplicial cones, where "better" means that the unimodular cones that are derived by the new procedure should have generators as short as possible with respect to Δ_C.

2 A new covering procedure

In Chapter 2 we will provide a new procedure for covering simplicial cones. Furthermore, we will prove that the inclusion of this procedure into the proof of Bruns and Gubeladze in [3] results into polynomial upper bounds for c_d^{cone} and c_d^{pol}.

2.1 The procedure

Right at the very beginning of this chapter, we will present the algorithm which provides us for every simplicial d-cone with a good unimodular cover of cones in a sense that the generating vectors of the covering cones are short with respect to Δ_C. So let $C := \mathbb{R}_+v_1 + \cdots + \mathbb{R}_+v_d \subset \mathbb{R}^d$ be an arbitrary simplicial d-cone. Then the call $\mathrm{UC}((v_1, \ldots, v_d), \emptyset)$ will return a unimodular cover of C given as the set CO of covering cones.

Procedure 1 Unimodular Cover – UC

$C := \mathbb{R}_+v_1 + \cdots + \mathbb{R}_+v_d \subset \mathbb{R}^d$
if $\mu(C) = 1$ **then**
  return $CO \cup \{C\}$
else
  $p := \min\{q \in \mathbb{P} : q|\mu(C)\}$
  Determine a vector $x = \frac{l_1}{p}v_1 + \cdots + \frac{l_d}{p}v_d \in \mathrm{par}(v_1, \ldots, v_d) \setminus \{0\}$
  for all $i = 1, \ldots, d$ **do**
    if $l_i \neq 0$ **then**
      Determine the vector $x^i = \frac{p-1}{p}v_i + \sum_{j \neq i} \frac{l_j^i}{p}v_j \in \langle x \rangle$
      $r_i := \gcd(p-1, l_1^i, \ldots, l_{i-1}^i, l_{i+1}^i, \ldots, l_d^i)$
      $y^i := \frac{x^i}{r_i}$
      $D_i := \mathbb{R}_+v_1 + \cdots + \mathbb{R}_+v_{i-1} + \mathbb{R}_+y^i + \mathbb{R}_+v_{i+1} + \cdots + \mathbb{R}_+v_d$
      $CO := \mathrm{UC}((v_1, \ldots, v_{i-1}, y^i, v_{i+1}, \ldots, v_d), CO)$
    end if
  end for
  return CO
end if

So what does this procedure do? First of all, it checks if the given cone C is unimodular. If this is the case, then it just returns $\{C\}$. Otherwise Procedure 1 covers C by certain cones D_i with smaller multiplicities $\mu(D_i) \leq \frac{p-1}{p}\mu(C)$

(as we will prove in the following) and recursively calls itself on the cones D_i. Finally, this will provide us with the desired unimodular cover. Furthermore, we have to mention that line 12 of Procedure 1 is only included, because this will help us to address the covering cones D_i in the following.

Before we go on, we will give a short example to illustrate how Procedure 1 works.

Example 2.1.1. Let $C := \mathbb{R}_+v_1 + \mathbb{R}_+v_2 \subset \mathbb{R}^2$ with $v_1 := 6e_1 + e_2$ and $v_2 := e_1 + e_2$. Then $\mu(C) = 5 \in \mathbb{P}$ and $\mathrm{par}(v_1, v_2) = \langle x \rangle$ for $x := \frac{1}{5}v_1 + \frac{4}{5}v_2 = 2e_1 + e_2$. Therefore, Procedure 1 comes up with the vectors

$$y^1 = \frac{4}{5}v_1 + \frac{1}{5}v_2 = 5e_1 + e_2$$

and

$$y^2 = \frac{1}{5}v_1 + \frac{4}{5}v_2 = 2e_1 + e_2.$$

As a result, $D_1 = \mathbb{R}_+y^1 + \mathbb{R}_+v_2$ and $D_2 = \mathbb{R}_+v_1 + \mathbb{R}_+y^2$ (see Figure 2.1). Furthermore, $\mu(D_1) = \mu(D_2) = 4$.

Now let us see how the cone D_1 is covered by Procedure 1. Because $\mu(D_1) = 2^2$ and $\mathrm{par}(y^1, v_2) = \langle z \rangle$ with $z = \frac{1}{4}y^1 + \frac{3}{4}v_2$, Procedure 1 comes up with the vectors

$$z^1 = \frac{1}{2}y^1 + \frac{1}{2}v_2 = 3e_1 + e_2$$

and

$$z^2 = \frac{1}{2}y^1 + \frac{1}{2}v_2 = z^1.$$

So, Procedure 1 covers cone D_1 by the cones $E_1 = \mathbb{R}_+z^1 + \mathbb{R}_+v_2$ and $E_2 = \mathbb{R}_+y^1 + \mathbb{R}_+z^1$. This means that the cone D_1 is not only covered but also triangulated by the cones E_1 and E_2. Finally, Procedure 1 triangulates the cones E_1 and E_2 by stellar subdivision by the vectors

$$u = 2e_1 + e_2 = y^2$$

respectively

$$w = 4e_1 + e_2$$

into unimodular subcones (see Figure 2.1).

2.2 Results

At first, we prove that the procedure above does really provide us with a unimodular cover of an arbitrary simplicial d-cone. And second, we will show that the generating vectors of the covering cones are short.

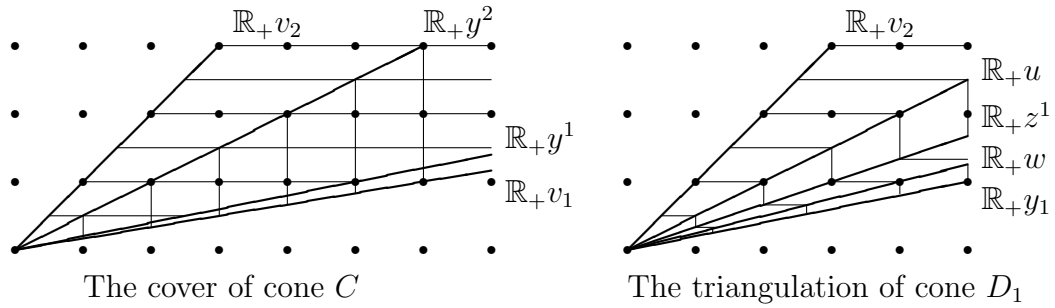

FIGURE 2.1 Example 2.1.1

Theorem 2.2.1. *For all simplicial d-cones $C \subset \mathbb{R}^d$ the algorithm* UC *provides us with a unimodular cover of cone C.*

Proof. For the proof of this theorem it is just necessary to show that in every recursive call the algorithm provides us with a cover of the current cone C by cones D_i. That this procedure will lead to a unimodular cover of the initial cone then follows by the fact that the multiplicities of the covering cones D_i are smaller than the multiplicity of the covered cone C in every recursive call (here we exclude the trivial case that C is already unimodular). We have

$$\mu(D_i) = \frac{p-1}{pr_i}\mu(C) < \mu(C).$$

To show that

$$\bigcup_i D_i = C,$$

we have to prove that for every $w \in C$ there exists an i such that $w \in D_i$. So let $w \in C$, i.e.

$$w = \sum_{j=1}^{d} \lambda_j v_j$$

with $\lambda_j \geq 0$. Furthermore, let λ_k be defined as the infimum of all λ_j ($j = 1, \ldots, d$) for which there exists a vector

$$z = \sum_{l=1}^{d} \gamma_l v_l \in \operatorname{par}(v_1, \ldots, v_d)$$

with $\gamma_j \neq 0$. (If there is no such index j, then C was already unimodular.) Then

$$w = \frac{pr_k\lambda_k}{p-1}y^k + \sum_{j \neq k}\left(\lambda_j - \frac{l_j^k\lambda_k}{p-1}\right)v_j.$$

Therefore, we have $w \in D_k$, because both

$$\frac{pr_k \lambda_k}{p-1} \geq 0$$

and also

$$\lambda_j - \frac{l_j^k \lambda_k}{p-1} \geq 0$$

for all $j \neq k$. The first statement is obvious, the second one shall be discussed in more detail. Here we have to distinguish between two cases.

Case 1. It exists a vector $x = \theta_1 v_1 + \cdots + \theta_d v_d \in \mathrm{par}(v_1, \ldots, v_d)$ such that $\theta_j \neq 0$. By definition $\lambda_k \leq \lambda_j$. Because $x^k \in \mathrm{par}(v_1, \ldots, v_d)$, we have $l_j^k \leq p-1$. It follows that $\lambda_j(p-1) \geq l_j^k \lambda_k$.

Case 2. For all $x = \theta_1 v_1 + \cdots + \theta_d v_d \in \mathrm{par}(v_1, \ldots, v_d)$ we have $\theta_j = 0$. Hence $l_j^k = 0$, because $x^k \in \mathrm{par}(v_1, \ldots, v_d)$. This implies $\lambda_j - \frac{l_j^k \lambda_k}{p-1} = \lambda_j \geq 0$.

So the theorem is proven. □

In the next lemmas we will mention the generations of the cones given by the algorithm UC. So, when we mention the cone of the 0-th generation, we mean the initial cone C. Moreover, the cones of the first generation are defined as the cones $D_i = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_{i-1} + \mathbb{R}_+ y^i + \mathbb{R}_+ v_{i+1} + \cdots + \mathbb{R}_+ v_d$ if the initial cone C is generated by the vectors $v_1, \ldots, v_d$. Consequently, the cones of the first generation are just those generated by the algorithm in the first recursive call. Then the cones of the second generation are defined as the ones which are constructed by the algorithm to cover the cones of the first generation and so on.

When we mention the vectors of the first generation, we mean the vectors y^i generated in the first recursive call of the algorithm. Furthermore, the vectors of the second generation are the vectors (except of the ones generating the cones of the first generation) needed to generate the cones of the second generation and so on.

From now on the notation y_k shall always indicate that we face a vector of the k-th generation, whereas the vector y^k is a vector from the first recursive call, which has the form

$$y^k = \frac{p-1}{pr_k} v_k + \sum_{j \neq k} \frac{l_j^k}{pr_k} v_j.$$

Theorem 2.2.2. *All unimodular cones F which – given by Procedure 1 – constitute a unimodular cover of the initial cone C are out of generations*

$$g_F \leq 2\,\mathrm{ld}(\mu(C)).$$

Proof. We will prove this theorem by induction on the multiplicity $\mu(C)$. If $\mu(C) = 1$, then the cone C itself constitutes a unimodular cover of C. Let now $\mu(C) > 1$. Then we distinguish between two cases.

Case 1. $\mu(C)$ is even. In this case it follows that $p = 2$ is the smallest prime divisor of $\mu(C)$. Hence each covering cone D_i has multiplicity $\mu(D_i) = \frac{\mu(C)}{2}$. By induction the application of Procedure 1 to the cone D_i results into a cover of each cone D_i by unimodular cones H^i which are all out of generations $g_{H^i} \leq 2\,\mathrm{ld}(\mu(D_i)) \leq 2\,\mathrm{ld}(\mu(C)) - 2$ (regarding the cone D_i as being of the 0-th generation.). This proves that the unimodular cones H^i covering cone C are out of generations $g_{H^i} \leq 2\,\mathrm{ld}(\mu(C)) - 1$ with respect to cone C.

Case 2. $\mu(C)$ is odd. In this case $x^i = \frac{p-1}{p} v_i + \sum_{j \neq i} \frac{l^i_j}{p} v_j$ where p is an odd prime number.

Now we again distinguish between two cases. Either the numbers l^i_j are coprime, i.e. $r_i = 1$, or not. In the first case the resulting cone D_i has multiplicity $\mu(D_i) = \frac{p-1}{p}\mu(D)$. Hence $\mu(D_i)$ is even. This means we are in Case 1. We have seen there that all unimodular cones H^i resulting from an application of Procedure 1 to the cone D_i are out of generations $g_{H^i} \leq 2\,\mathrm{ld}(\mu(D_i)) - 1$. It follows that the unimodular cones H^i are out of generations $g_{H^i} \leq 2\,\mathrm{ld}(\mu(C))$ with respect to cone C.

If the numbers l^i_j are not coprime, i.e. $r_i > 1$, then the resulting cones have multiplicities $\mu(D_i) = \frac{p-1}{r_i p}\mu(C) \leq \frac{\mu(C)}{2}$. Therefore the unimodular cones H^i resulting from the application of Procedure 1 to the cones D_i are by induction out of generations $g_{H^i} \leq 2\,\mathrm{ld}(\mu(D_i)) \leq 2\,\mathrm{ld}(\mu(C)) - 2$, which proves the claim. □

In the following we will take a closer look at the vectors y^i which are generated by the algorithm UC. When we apply the procedure to a cone $D = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_k$, then all vectors y^i generated in the first recursive call are elements of the parallelepiped spanned by the generators w_j of D, i.e. $y^i \in \mathrm{par}(w_1, \ldots, w_d)$. Therefore, $y^i \in d\Delta_D$.

The next lemma shows that we can say a bit more about the form of these vectors. It says that when we do not only restrict ourselves to vectors of the first generation but have a look at vectors $y_1, \ldots, y_f$ of the first, $\ldots, f$-

th generation given by the algorithm UC to construct a unimodular cone F of the f-th generation, then at least half of them have a very special form. More precisely, it states that they are of the form

$$y_i = \sum_{j=1}^{d} \lambda_j u_j$$

such that $\lambda_j \leq \frac{1}{2}$ if $E = \mathbb{R}_+ u_1 + \cdots + \mathbb{R}_+ u_k$ is the cone of the $(i-1)$-th generation to which we apply stellar subdivision by the vector y_i. This also means that we have a better upper bound for the lengths of these vectors with respect to Δ_E. It implies that

$$y_i \in \frac{d}{2} \Delta_E.$$

This will be important when we provide an upper bound for the lengths of the vectors which generate the unimodular cones covering the underlying cone C.

Lemma 2.2.3. *Let F be a unimodular cone which is generated by the algorithm* UC. *And let $y_1, y_2, \ldots, y_f$ (with $f \leq 2\,\mathrm{ld}(\mu(C))$ due to Theorem 2.2.2) be the sequence of vectors of the first, second, ..., f-th generation provided by the algorithm to construct the cone F. Then at least $\frac{f}{2}$ of these vectors y_k are of the form*

$$y_k = \sum_{j=1}^{d} \lambda_j w_j$$

where $\lambda_j \leq \frac{1}{2}$ for all j and $G = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_k$ is the cone of the $(k-1)$-th generation that contains F.

Proof. We will prove this theorem in the same manner as Theorem 2.2.2 by induction on the multiplicity $\mu(C)$. If $\mu(C) = 1$, then $f = 0$ and $\frac{f}{2} = 0$ vectors y_i are of the desired form. Now let $\mu(C) > 1$. Then we distinguish between two cases.

Case 1. $\mu(C)$ is even. Let F be a unimodular cone which is generated by the algorithm UC. And let $y_1, y_2, \ldots, y_f$ be the sequence of vectors of the first, second, ..., f-th generation provided by the algorithm to construct the cone F. Because $p = 2$ is the smallest prime divisor of $\mu(C)$, the vector y_1 is of the form $y_1 = \sum_{j=1}^{d} \frac{l_j}{2} v_j$ where $C := \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$ and $l_j \in \{0, 1\}$.

Furthermore, it follows by induction that at least $\frac{f-1}{2}$ of the vectors $y_2, \ldots, y_f$ are of the desired form. Hence at least $\frac{f+1}{2}$ of the vectors $y_1, \ldots, y_f$ are of the desired form.

Case 2. $\mu(C)$ is odd. Let F be a unimodular cone which is generated by the algorithm UC. And let $y_1, y_2, \ldots, y_f$ be the sequence of vectors of the first, second, ..., f-th generation provided by the algorithm to construct the cone F. In this case $x^i = \frac{p-1}{p} v_i + \sum_{j \neq i} \frac{l_j^i}{p} v_j$ where p is an odd prime number.

Now we again distinguish between two cases. Either the numbers l_j^i are coprime, i.e. $r_i = 1$, or not. In the first case the vector y_1 might not be of the above form, but the resulting cone D_i has multiplicity $\mu(D_i) = \frac{p-1}{p}\mu(D)$. Hence $\mu(D_i)$ is even. This means y_2 is of the desired form. Furthermore, it follows by induction that at least $\frac{f-2}{2}$ of the vectors $y_3, \ldots, y_f$ are of the above form. Finally, at least $\frac{f}{2}$ of the vectors $y_1, \ldots, y_f$ are of the desired form.

If the numbers l_j^i are not coprime, i.e. $r_i > 1$, then y_1 is of the desired form. Furthermore, we have by induction that at least $\frac{f-1}{2}$ of the vectors $y_2, \ldots, y_d$ are of the above form. Finally, it follows that at least $\frac{f+1}{2}$ of the vectors $y_1, \ldots, y_d$ are of the desired form. □

The next lemma will provide us with the essential tool for showing that that the vectors y_i in Procedure 1 are short.

Lemma 2.2.4. *Let $g \in \mathbb{N}$ and $\nu \in \{1,2\}^g$. Furthermore, let $a_1^k := |\{i : 1 \leq i \leq k \text{ and } \nu_i = 1\}|$, $a_2^k := |\{i : 1 \leq i \leq k \text{ and } \nu_i = 2\}|$. We define the increasing sequence h_k, $k \geq -d$, of numbers as follows:*

$$h_k = 1, \quad k \leq 0,$$

$$h_k = \left(\frac{d}{2}\right) \cdot 2^{a_1^k} \left(\frac{3}{2}\right)^{a_2^k - \nu_k + 1}, \quad 1 \leq k \leq g.$$

Then we have

$$h_k \geq \frac{1}{\nu_k}(h_{k-1} + \cdots + h_{k-d}), \quad 1 \leq k \leq g.$$

Proof. Let $g \in \mathbb{N}$ and $\nu \in \{1,2\}^g$. That the sequence h_k is increasing is due to the facts that $a_i^k \geq a_i^{k-1}$ for $i = 1, 2$ and that $a_2^k > a_2^{k-1}$ if $2 = \nu_k > \nu_{k-1} = 1$.

We will now prove the second statement by induction on k. So let $k = 1$. If $\nu_1 = 1$, then we have

$$h_1 = d \geq \frac{1}{\nu_k}(h_0 + \cdots + h_{-d+1}) = d.$$

If $\nu_1 = 2$, then we have

$$h_1 = \frac{d}{2} \geq \frac{1}{\nu_k}(h_0 + \cdots + h_{-d+1}) = \frac{d}{2}.$$

Therefore, the statement is true for $k = 1$.

Let now $k \geq 2$. In this situation we will distinguish between four cases.

Case 1. $\nu_k = \nu_{k-1} = 1$. Due to the definition, $h_k = 2h_{k-1}$. Furthermore, by induction it follows that

$$2h_{k-1} = h_{k-1} + h_{k-1} \geq h_{k-1} + (h_{k-2} + \cdots + h_{k-d}).$$

Therefore, we are done.

Case 2. $\nu_k = 1$ and $\nu_{k-1} = 2$. Due to the definition, $h_k = 3h_{k-1}$. Furthermore, by induction it follows that

$$3h_{k-1} = h_{k-1} + 2h_{k-1} \geq h_{k-1} + (h_{k-2} + \cdots + h_{k-d}).$$

Therefore, we are done.

Case 3. $\nu_k = 2$ and $\nu_{k-1} = 1$. Due to the definition, $h_k = h_{k-1}$. Furthermore, by induction it follows that

$$h_{k-1} = \frac{1}{2}h_{k-1} + \frac{1}{2}h_{k-1} \geq \frac{1}{2}h_{k-1} + \frac{1}{2}(h_{k-2} + \cdots + h_{k-d}).$$

Therefore, we are done.

Case 4. $\nu_k = \nu_{k-1} = 2$. Due to the definition, $h_k = \frac{3}{2}h_{k-1}$. Furthermore, by induction it follows that

$$\frac{3}{2}h_{k-1} = \frac{1}{2}h_{k-1} + h_{k-1} \geq \frac{1}{2}h_{k-1} + \frac{1}{2}(h_{k-2} + \cdots + h_{k-d}).$$

Therefore, we are done.

Finally, Lemma 2.2.4 is proven. $\square$

Now we are in the position to prove the central theorem of this chapter.

Theorem 2.2.5. *For all unimodular cones $F = \mathbb{R}_+u_1 + \cdots + \mathbb{R}_+u_d \subset \mathbb{R}^d$ which the algorithm* UC *returns as elements of the unimodular cover of $C = \mathbb{R}_+v_1 + \cdots + \mathbb{R}_+v_d \subset \mathbb{R}^d$ we have*

$$u_i \in \left(\frac{d}{2}\right) \cdot \mu(C)^{\mathrm{ld}(3)}\Delta_C, \qquad i = 1, \ldots, d.$$

Proof. Let F be an arbitrary unimodular cone generated by the algorithm UC and let $y_1, y_2, \ldots, y_f$ (with $f \leq 2\,\mathrm{ld}(\mu(C))$ due to Theorem 2.2.2) be the sequence of vectors of the first, second, $\ldots f$-th generation given by the algorithm to construct the cone F. Furthermore, let $D_0 = C, D_1, D_2, \ldots, D_f = F$ be the corresponding cones of the 0-th, first, second, $\ldots, f$-th generation which are generated by these vectors. This means if $D_l = \mathbb{R}_+w_1 + \cdots + \mathbb{R}_+w_d$ for $0 \leq l \leq f-1$, then D_{l+1} is given as the cone

$$D_{l+1} = \mathbb{R}_+w_1 + \cdots + \mathbb{R}_+w_{j-1} + \mathbb{R}_+y_{l+1} + \mathbb{R}_+w_{j+1} + \cdots + \mathbb{R}_+w_d$$

for a certain j. Furthermore, let $\nu \in \{1,2\}^f$ indicate the form of each vector y_l. Thus if $D_{l-1} = \mathbb{R}_+w_1 + \cdots + \mathbb{R}_+w_d$ and $\nu_l = 2$, then we have

$$y_l = t_1w_1 + \cdots + t_dw_d$$

such that $t_j \leq \frac{1}{2}$ for all j. Otherwise if $\nu_l = 1$, then we have

$$y_l = \frac{t_1}{p}w_1 + \cdots + \frac{t_d}{p}w_d$$

with $p \in \mathbb{P} \setminus \{2\}$ and $0 \leq \ t_j \leq p-1$.

Now we claim that (for h_l defined as in Lemma 2.2.4 by the above given vector ν)

$$y_l \in h_l\Delta_C \tag{$*$}$$

for all l with $1 \leq l \leq f$. We simply prove this claim by induction on l. If $l = 1$ and $\nu_l = 1$, then

$$y_1 = \frac{t_1}{p}v_1 + \cdots + \frac{t_d}{p}v_d$$

such that $p \in \mathbb{P} \setminus \{2\}$ and $0 \leq \ t_j \leq p-1$. Therefore,

$$y_1 \in \left(\sum_{j=1}^{d} \frac{t_j}{p}\right)\Delta_C \subset d\Delta_C.$$

If $l = 1$ and $\nu_l = 2$, then

$$y_1 = t_1 w_1 + \cdots + t_d w_d$$

such that $t_j \leq \frac{1}{2}$. So

$$y_1 \in \left(\sum_{j=1}^{d} t_j \right) \Delta_C \subset \frac{d}{2} \Delta_C.$$

It follows that the claim is true for $l = 1$. Let now $1 < l \leq f$ and let $D_{l-1} = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d$. Then we distinguish between two cases.

Case 1. $\nu_l = 1$. This implies

$$y_l = \frac{t_1}{p} w_1 + \cdots + \frac{t_d}{p} w_d$$

such that $p \in \mathbb{P} \setminus \{2\}$ and $0 \leq \ t_j \leq p - 1$. Because the $w_j \notin \{v_1, \ldots, v_d\}$ are all out of different generations and since the sequence (h_k) is increasing, we can assume that $w_1 \in h_{l-1}\Delta_C, w_2 \in h_{l-2}\Delta_C, \ldots, w_d \in h_{l-d}\Delta_C$. Therefore, it follows by induction and by Lemma 2.2.4 that

$$y_l \in \left(\sum_{j=1}^{d} \frac{t_j}{p} h_{l-j} \right) \Delta_C \subset \left(\sum_{j=1}^{d} h_{l-j} \right) \Delta_C \subset h_l \Delta_C.$$

Case 2. $\nu_l = 2$. Then we have

$$y_l = t_1 w_1 + \cdots + t_d w_d$$

such that $t_j \leq \frac{1}{2}$. Therefore, we have by induction and due to Lemma 2.2.4 that

$$y_l \in \left(\sum_{j=1}^{d} t_j h_{l-j} \right) \Delta_C \subset \left(\frac{1}{2} \left(\sum_{j=1}^{d} h_{l-j} \right) \right) \Delta_C \subset h_l \Delta_C.$$

So the claim is true.

Let now a_1^f and a_2^f be defined as in Lemma 2.2.4. It follows by Lemma 2.2.3 that

$$a_2^f \geq \frac{f}{2}\ , \quad a_1^f \leq \frac{f}{2}.$$

Hence we have

$$h_f = \left(\frac{d}{2}\right) \cdot 2^{a_1^f} \left(\frac{3}{2}\right)^{a_2^f - \nu_f + 1} \leq \left(\frac{d}{2}\right) \cdot 2^{\frac{f}{2}} \left(\frac{3}{2}\right)^{\frac{f}{2}},$$

whereas the inequality is also due to $\nu_f \geq 1$. Moreover, by Lemma 2.2.2 it follows that

$$h_f \leq \left(\frac{d}{2}\right) \cdot 2^{\mathrm{ld}(\mu(C))} \left(\frac{3}{2}\right)^{\mathrm{ld}(\mu(C))} = \left(\frac{d}{2}\right) \cdot \mu(C)^{\mathrm{ld}(3)}.$$

Finally, by $(*)$ and the inequality above we arrive at

$$y_l \in \left(\frac{d}{2}\right) \cdot \mu(C)^{\mathrm{ld}(3)} \Delta_C$$

for all l. This, of course, implies for all u_i (recall that $F = \mathbb{R}_+ u_1 + \cdots + \mathbb{R}_+ u_d$ and, hence, $u_i \in \{y_1, \ldots, y_f\} \cup \{v_1, \ldots, v_d\}$) the desired result that

$$u_i \in \left(\frac{d}{2}\right) \cdot \mu(C)^{\mathrm{ld}(3)} \Delta_C.$$

□

It immediately follows from Theorem 2.2.5:

Corollary 2.2.6. *Every simplicial d-cone $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$, $d \geq 3$, has a unimodular cover $C = D_1 \cup \ldots \cup D_t$ such that*

$$\mathrm{Hilb}(D_i) \subset \left(\frac{d}{2}\right) \cdot \mu(C)^{\mathrm{ld}(3)} \Delta_C, \qquad i = 1, \ldots, t.$$

If we follow the argumentation of Bruns and Gubeladze presented in [3] and, additionally, take into account Corollary 2.2.6, then we obtain a better upper bound for the value c_d^{cone}. How is this result derived? Let us outline the proof of Bruns and Gubeladze in [3].

First of all, we provide the following lemma, which allows us to restrict the argumentation to empty simplicial cones.

Lemma 2.2.7. *Every cone $C \subset \mathbb{R}^d$ can be triangulated into empty simplicial cones D such that $\Delta_D \subset \Delta_C$*

Proof. Every cone C can be triangulated into simplicial cones E which are generated by extreme generators of C. Hence $\Delta_E \subset \Delta_C$. Furthermore,

if we triangulate a nonempty simplicial cone E into simplicial cones F by stellar subdivision by a vector $v \in \Delta_E \cap \mathbb{Z}^d$, then it follows that $\Delta_F \subset \Delta_E$ and $|\Delta_F \cap \mathbb{Z}^d| < |\Delta_E \cap \mathbb{Z}^d|$. Hence, by the successive application of stellar subdivisions, we arrive at a triangulation of C into empty simplicial cones D for which $\Delta_D \subset \Delta_C$. $\square$

Moreover, let us define

$$\gamma(d) := \left\lceil \sqrt{d-1} \right\rceil (d-1)$$

and

$$\kappa(d) := \gamma(d) \cdot \frac{d(d+1)}{2} \cdot \gamma(d)^{\mathrm{ld}(3)}.$$

We will prove that $c_d^{cone} \leq \kappa(d)$ for $d \geq 2$ by induction on the dimension d. The inequality holds for $d = 2$, since $c_2^{cone} = 1$ and $\kappa(2) = 3$. Furthermore, because $\kappa(d) \geq \kappa(d-1)+1$, we can assume by induction that $c_{d-1}^{cone} \leq \kappa(d)-1$.

Then Bruns and Gubeladze provided the following lemma.

Let C be a cone and v one of its extreme generators. Then we say that a system $\{C_j\}_{j=1}^k$ of subcones $C_j \subset C$ covers the corner of C at v if $v \in \mathrm{Hilb}(C_j)$ for all j and the union $\bigcup_{j=1}^k C_j$ contains a neighborhood of v in C.

Lemma 2.2.8. Suppose that $c_{d-1}^{cone} < \infty$, and let C be a simplicial d-cone with extreme generators $v_1, \ldots, v_d$.

(a) Then there is a system of unimodular subcones $C_1, \ldots, C_k \subset C$ covering the corner of C at v_1 such that $\mathrm{Hilb}(C_1), \ldots, \mathrm{Hilb}(C_k) \subset (c_{d-1}^{cone} + 1)\Delta_C$.

(b) Moreover, each element $w \neq v_1$ of a Hilbert basis of C_j, $j \in [1, k]$, has a representation $w = \xi_1 v_1 + \cdots + \xi_d v_d$ with $\xi_1 < 1$.

Therefore, because we can assume that $c_{d-1}^{cone} \leq \kappa(d) - 1$, Lemma 2.2.8 provides us for every i with a system of unimodular cones $C_1^i, \ldots, C_{k_i}^i \subset C$ covering the corner of C at v_i such that

$$\mathrm{Hilb}(C_j^i) \subset \kappa(d)\Delta_C, \quad j \in [1, k_i].$$

Of course, the system of cones C_j^i will in general not provide us with a unimodular cover of the whole cone C. This means we have to extend the corner covers far into C. But how far do we have to extend the corner covers? Before we can answer this question, we have to provide some definitions and results. Let Γ_0 be defined as the facet $\mathrm{conv}(v_1, \ldots, v_d)$, let

$$H_i = \mathrm{Aff}(0, v_i+(d-1)v_1, \ldots, v_i+(d-1)v_{i-1}, v_i+(d-1)v_{i+1}, \ldots, v_i+(d-1)v_d)$$

for $i = 1, \ldots, d$ and

$$\Gamma_i = \mathrm{conv}(v_i, \Gamma_0 \cap H_i).$$

Then H_i is the vector subspace of dimension $d-1$ through the barycenter of Γ_0, i.e. $(1/d)(v_1 + \cdots + v_d)$, that is parallel to the facet of Γ_0 opposite to v_i. It follows that

$$\bigcup_{i=1}^{d} \Gamma_i = \Gamma_0.$$

Therefore, to prove that $c_d^{cone} \leq \kappa(d)$ it suffices to show

Claim A. For each $i = 1 \ldots, d$ there exists a system of unimodular cones $D_{i1}, \ldots, D_{ik_i} \subset C$ such that

$$\mathrm{Hilb}(D_{ij}) \subset \kappa(d)\Delta_C$$

for all j, and

$$\Gamma_i \subset \bigcup_{j=1}^{k_i} D_{ij}.$$

This claim answers the above question. It tells us that it suffices to extend the corner cover at v_i beyond H_i. We will now restrict ourselves to the case that $i = 1$. For simplicity of notation, let $C_j := C_j^1$ for all $j = 1, \ldots, k_1$, and $k := k_1$.

Furthermore, let us fix an index $j \in [1, k]$. Then the simplicial d-cone $D \subset \mathbb{R}^d$ shall be defined by the following conditions:

(i) $C_j \subset D$,
(ii) the facets of D contain those facets of C_j that pass through O and v_1,
(iii) the remaining facet of D is in H_1.

Figure 2.2 describes the situation in the cross-section Γ_0.

Because the cones C_j cover the corner at v_1, it follows that to prove Claim A it is enough to show the following

Claim B. There exists a system of unimodular cones $D_1, \ldots, D_T \subset C$ such that

$$\mathrm{Hilb}(D_t) \subset \kappa(d)\Delta_C, \quad t \in [1, T],$$

and

$$D \subset \bigcup_{t=1}^{T} D_t.$$

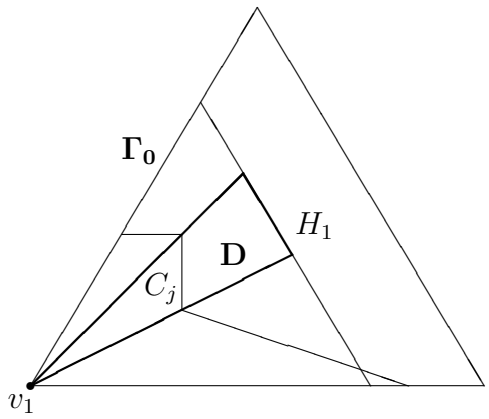

FIGURE 2.2

Claim B precisely tells us what it means to extend the corner cover at v_1: We have to find unimodular cones $D_t \subset C$ covering D. Of course, C_j is a unimodular subcone of D, but in general C_j does not, roughly speaking, reach beyond the subspace H_1.

Finally, Bruns and Gubeladze showed in [3] that it is indeed possible to extend the corner cover far enough into C. This means they provided, again roughly speaking, a system of cones $D_1, \ldots, D_T \subset C$ such that

$$\mathrm{Hilb}(D_t) \subset \gamma(d)(d+1)\Delta_C, \quad t \in [1, T], \tag{$*$}$$

and $D \subset \bigcup_{t=1}^{T} D_t$. But here we loose unimodularity, which means that the cones D_t are in general not unimodular. However, the multiplicities $\mu(D_t)$ of the cones D_t are bounded by $\gamma(d)$.

And at that point of the proof Corollary 2.2.6 comes into play. It ensures that each of the cones D_t admits a unimodular cover $D_t = E_1^t \cup \ldots \cup E_{k_t}^t$ such that

$$\mathrm{Hilb}(E_j^t) \subset \left(\frac{d}{2}\right) \cdot \gamma(d)^{\mathrm{ld}(3)} \Delta_{D_t}, \quad j \in [1, k_t]. \tag{$**$}$$

The statements $(*)$ and $(**)$ together imply the desired result. Hence we have

Theorem 2.2.9. *Let* $\gamma(d) := \lceil \sqrt{d-1} \rceil \, (d-1)$. *Then*

$$c_d^{cone} \leq \frac{d(d+1)}{2} \cdot \gamma(d)^{\mathrm{ld}(3)+1}, \quad d \geq 2.$$

Bruns and Gubeladze also proved the following theorem, which states a connection between c_d^{cone} and c_d^{pol}.

Theorem 2.2.10. *Let d be a natural number. Then c_d^{pol} is finite if and only if c_d^{cone} is finite, and, moreover,*

$$c_d^{cone} \leq c_d^{pol} \leq \sqrt{d}(d+1)c_d^{cone}.$$

Finally, the Theorems 2.2.9 and 2.2.10 directly imply

Theorem 2.2.11. *Let $\gamma(d) := \left\lceil \sqrt{d-1} \right\rceil (d-1)$. Then*

$$c_d^{pol} \leq \frac{d^{1.5}(d+1)^2}{2} \cdot \gamma(d)^{\mathrm{ld}(3)+1}, \quad d \geq 2.$$

3 A new triangulation procedure

In the second chapter we presented a new algorithm for the unimodular cover of simplicial cones. In this chapter we will provide a similar algorithm for the unimodular triangulation of simplicial cones. This algorithm will not improve the upper bounds for c_d^{cone} and c_d^{pol} we derived in Chapter 2, but might be of interest itself.

3.1 The procedure

In Section 1.4 we mentioned Theorem 1.3, which is built on a quite simple way to triangulate simplicial cones. The next theorem shows that we can do much better if the underlying cone C has multiplicity $\mu(C)$ equal to a power of two. This result motivates us to come up with a triangulation procedure which at first triangulates the underlying cone into cones D with $\mu(D) = 2^l$ ($l \in \mathbb{N}$), and subsequently triangulates these cones D.

Theorem 3.1.1. *Let $d \geq 3$ and let $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$ be a simplicial d-cone with $\mu(C) = 2^l$ ($l \in \mathbb{N}$). Then there exists a unimodular triangulation $C = C_1 \cup \ldots \cup C_k$ such that*

$$\mathrm{Hilb}(C_j) \subset \left(\frac{d}{2}\left(\frac{3}{2}\right)^l\right)\Delta_C, \quad 1 \leq j \leq k.$$

Proof. The proof of this theorem is related to the proof of Theorem 4.1 in [3]. The following sequence of numbers plays an important part in this proof. (We already introduced a related sequence of numbers in Lemma 2.2.4.) It will provide us, roughly speaking, with an upper bound for the lengths of the vectors of the covering cones.

$$h_k = 1, \quad k \leq 0, \quad h_1 = \frac{d}{2}, \quad h_k = \frac{1}{2}(h_{k-1} + \cdots + h_{k-d}), \quad k \geq 2.$$

Because we have

$$h_k - h_{k-1} = \frac{1}{2}h_{k-1} - \frac{1}{2}h_{k-d-1}$$

for $k \geq 3$ and $h_1 > h_l$ for $l \leq 0$, it follows by induction that this sequence is increasing. Since for $k \geq 3$

$$h_k = \frac{1}{2}h_{k-1} + \frac{1}{2}(h_{k-2} + \cdots + h_{k-d-1}) - \frac{1}{2}h_{k-d-1} = \frac{3}{2}h_{k-1} - \frac{1}{2}h_{k-d-1} < \frac{3}{2}h_{k-1},$$

and because $h_1 = \frac{d}{2}$, $h_2 < \frac{3d}{4}$, we arrive at

$$h_k \leq \frac{d}{2}\left(\frac{3}{2}\right)^{k-1}$$

for $k \geq 1$. This result will be needed in the following.

Let $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$ be an arbitrary simplicial d-cone with $\mu(C) = 2^l$ ($l \in \mathbb{N}$). If C is already unimodular (i.e. $l = 0$), we are done. If C is not unimodular (i.e. $l \geq 1$), then by Lemma 1.2.7 there exist $i_j \in \{1, \ldots, d\}$ with $1 \leq j \leq m \leq d$ and $i_j \neq i_k$ for $j \neq k$ such that

$$u = \frac{1}{2}(v_{i_1} + \cdots + v_{i_m}) \in \mathbb{Z}^d \setminus \{0\}.$$

Now we apply stellar subdivision to the cone C by the vector u, which will give us the cones

$$C_{i_s} = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_{i_s - 1} + \mathbb{R}_+ u + \mathbb{R}_+ v_{i_s+1} + \cdots + \mathbb{R}_+ v_d, \quad 1 \leq s \leq m \leq d.$$

For these cones of the first generation (we regard the initial cone C as the cone belonging to the 0-th generation) we have

$$\mu(C_{i_s}) = |\det(v_1, \ldots, v_{i_s-1}, \frac{1}{2}(v_{i_1} + \cdots + v_{i_m}), v_{i_s+1}, \ldots, v_d)| = \frac{1}{2}\mu(C) = 2^{l-1}.$$

If $\mu(C_{i_s}) = 1$, then the procedure stops. Otherwise it is continued until we end with a triangulation of the initial cone C by unimodular cones of the l-th generation.

For the vectors w_k which have been used for the stellar subdivisions of the cones of the $(k-1)$-th generation we get

$$w_k \in h_k \Delta_C.$$

We will prove this statement by induction on k. For $k = 1$ this is obvious, because $\frac{1}{2}(v_{i_1} + \cdots + v_{i_m}) \in \frac{d}{2}\Delta_C$. For $k > 1$ we have that all generators $u_1, \ldots, u_d$ of a certain cone $C' = \mathbb{R}_+ u_1 + \cdots + \mathbb{R}_+ u_d$ of the $(k-1)$-th generation either belong to the initial vectors $v_1, \ldots, v_d$ or are vectors which have been used for stellar subdivisions of cones of different generations. So by induction it follows

$$u_i \in h_{n_i} \Delta_C, \quad n_i \leq k - 1,$$

where the n_i are pairwise different. The equality

$$w_k = \frac{1}{2}(u_{j_1} + \cdots + u_{j_v}), \quad 1 \leq v \leq d,$$

immediately leads us to

$$w_k \in \frac{1}{2}(h_{k-1} + \cdots + h_{k-d})\Delta_C = h_k \Delta_C,$$

because the h_i are increasing. Hence we are done. $\square$

The next lemma will be of great importance for the new procedure for the triangulation of an arbitrary simplicial cone. More precisely, it will be essential in the process of triangulating the underlying cone C by cones D for which $\mu(D) = 2^l$ ($l \in \mathbb{N}$). As we have explained before, this will be done by the successive application of stellar subdivisions. The vectors needed for these stellar subdivisions have to fulfill some properties to be of use in this process. And here the next lemma comes into play.

Lemma 3.1.2. *Let m and p be two odd integers with $\frac{p}{2} < m < p$. Then there exist natural numbers $s \leq \mathrm{ld}(p)$ and $t < \frac{p}{2}$ such that*

$$2^s t = (2^{s-1} - 1)p + m.$$

Proof. Let s be defined as the infimum of all natural $i > 0$ such that

$$p - m \not\equiv 0 \pmod{2^i}.$$

Obviously, we have $s > 1$, since both m and p are odd. Furthermore, there exist $x, y \in \mathbb{N}$ and $u \notin 2\mathbb{N}$ such that

$$m = 2^s x + u, \quad p = 2^s y + 2^{s-1} + u,$$

since $p - m \not\equiv 0 \pmod{2^s}$, $p - m \equiv 0 \pmod{2^{s-1}}$ and $p > m$. It follows that

$$m + (2^{s-1} - 1)p = 2^s(x + (2^{s-1} - 1)y) + (2^{s-1} - 1)(2^{s-1} + u) + u$$

is divisible by 2^s, because $(2^{s-1} - 1)(2^{s-1} + u) + u = 2^{s-1}(u - 1 + 2^{s-1})$ and $u - 1 + 2^{s-1}$ is even due to u being odd and $s > 1$. Therefore,

$$t := \frac{(2^{s-1} - 1)p + m}{2^s}$$

is a natural number.

It remains to show that $s \leq \mathrm{ld}(p)$ and $t < \frac{p}{2}$. The first statement follows easily from the fact that

$$p - m \not\equiv 0 \pmod{2^{\lfloor \mathrm{ld}(p) \rfloor}},$$

because $0 < p - m < \frac{p}{2} < 2^{\lfloor \mathrm{ld}(p) \rfloor}$. From this we can conclude that

$$s \leq \lfloor \mathrm{ld}(p) \rfloor.$$

The second statement simply follows from $m < p$, because this implies that

$$\frac{(2^{s-1}-1)p+m}{2^s} = \frac{2^{s-1}p + (m-p)}{2^s} < \frac{2^{s-1}p}{2^s}.$$

□

Remark 3.1.3. Lemma 3.1.2 is perhaps the most critical step in my proof. Improving this result (in a sense that we could find natural numbers $s \ll \mathrm{ld}(p)$, $t < \frac{p}{2}$ and x such that $2^s t = xp + m$ for given odd numbers m and p with $\frac{p}{2} < m < p$), would critically effect the main results.

Before we provide the triangulation procedure, we will at first **describe this procedure in prose.** The aim of this procedure is to triangulate the original cone C by cones D in a way that the multiplicities of all the cones D are powers of two and that the generators of the cones D are relatively short with respect to the simplex Δ_C.

What is the motivation for this? As Theorem 3.1.1 shows, cones whose multiplicities are a power of two admit a unimodular triangulation by cones whose generators are very short with respect to Δ_C. Therefore, it might be a good idea to triangulate a cone first by cones whose multiplicity is a power of two and then take advantage of this good property.

So how do we reach a triangulation of an arbitrary cone C by cones D with $\mu(D) = 2^f$ ($f \in \mathbb{N}$) and short generators? We successively apply stellar subdivisions with special vectors $x \in C$ to the cones $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$. Due to Lemma 1.2.7 we know that there exists a vector

$$x = \sum_{j=1}^{d} \frac{z_j}{p} v_j \in \mathrm{par}(v_1, \ldots, v_d) \setminus \{0\}$$

for all prime divisors p of $\mu(C)$. As we will see later on, it makes things easier (in a way that we end up faster with a triangulation of C by cones D with $\mu(D) = 2^f$) if z_j is either a composite number or small (here this means $z_j \leq \frac{p}{2}$). Because in general not all z_j fulfill these properties, we add certain multiples kv_j ($k \in \mathbb{N}$) of v_j to the vector x if z_j is a prime number and $z_j > \frac{p}{2}$. This results in a vector $x' \in C$ with

$$x' = \sum_{j=1}^{d} \frac{z'_j}{p} v_j$$

such that all z'_j are of the form $z'_j = 2^{f_j} t_j$ with natural numbers $f_j \leq \mathrm{ld}(p)$ and $t_j < p$. Additionally (due to Lemma 3.1.2), the numbers t_j do fulfill the desired restrictions, i.e. $t_j \leq \frac{p}{2}$ or t_j is a composite number.

Of course, we wish the vectors x' to be as short as possible. Therefore, we want to avoid the situation that both z'_j is big and v_j is a long vector, because this would also mean that x' would be long. And there a set M_C comes into play. M_C is given as a certain subset of all generators of the cone C which have been used for a stellar subdivision beforehand. So it tells us which vectors of C are possibly long, because vectors used for a stellar subdivision can be much longer than the other generators of a cone C.

Furthermore, we have in general that z'_j is much bigger than z_j. As a result, to keep the vector x' as short as possible, we need to have that z_j is either composite or $z_j \leq \frac{p}{2}$ for all j with $v_j \in M_C$. This is because then we would not have to add multiples of v_j to the vector x to arrive at x'.

After these statements ,we will now present the procedure which provides us for a simplicial d-cone $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$ with a triangulation of C by cones D with $\mu(D) = 2^f$ $(f \in \mathbb{N})$.

Procedure 2 Power two triangulation – PTT

$CO := \{C\}$
$M_C := \emptyset$
while CO contains a cone D such that $\mu(D)$ is not a power of two **do**
  if CO contains a cone $D = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d$ such that
    I) $\mu(D) \neq 2^l$ $(l \in \mathbb{N})$ and such that there exists
    II) $x = \sum_{j=1}^{d} \frac{z_j}{p} w_j \in \mathrm{par}(w_1, \ldots, w_d) \setminus \{0\}$ with
      IIa) $p = \max\{p_i \in \mathbb{P} : p_i \mid \mu(D)\}$ and
      IIb) for all $v_j \in M_D$ we have that
        IIb1) z_j is a composite number or
        IIb2) $z_j \leq \frac{p}{2}$ or
        IIb3) $z_j = 2$ and $p = 3$ **then**
    for all $j = 1, \ldots, d$ **do**
      if $z_j \notin \mathbb{P}$ or $z_j \leq \frac{p}{2}$ or $z_j = 2$ **then**
        $z'_j := z_j$
      else
        $z'_j := z_j + kp$ with $k \in \mathbb{N}$ such that $z_j + kp = 2^s t$ where $s \leq \mathrm{ld}(p)$ and $t < \frac{p}{2}$ (due to Lemma 3.1.2)
      end if
    end for
    $x' := \sum_{j=1}^{d} \frac{z'_j}{p} w_j$

```
13:       for all E ∈ CO with x′ ∈ E do
14:         Apply stellar subdivision to E by x′ (let E_j (j = 1, ..., m) be the
            resulting cones)
15:         CO := (CO \ {E}) ∪ {E_j : j = 1, ..., m}
16:         for all j do
17:           M_{E_j} := M_E ∪ {x′}
18:         end for
19:       end for
20:     else
21:       for all D ∈ CO do
22:         M_D := ∅
23:       end for
24:     end if
25:   end while
```

3.2 Results

The triangulation of the initial cone C resulting from this procedure has some good properties. We will provide them in the following.

First of all, we will gather some properties which all the cones, vectors and sets M_C fulfill. And in the end we will show that these properties ensure that both the multiplicities of the cones $D \in CO$ are relatively small and that the vectors generating the cones $D \in CO$ are short.

Definition 3.2.1. We say that a cone $D \in CO$ is of the k-th generation if it is one of the cones generated by the application of stellar subdivision to a cone of $(k-1)$-th generation. The initial cone C is said to be of the 0-th generation.

Lemma 3.2.2. *Let $D = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d$ be a cone to which we apply a stellar subdivision by a vector x' in the above procedure. Then x' is of the form*

$$x' = \sum_{j=1}^{d} \frac{l_j}{p_{\max}} w_j, \quad p_{\max} := \max\{p \in \mathbb{P} : p \mid \mu(D)\},$$

such that for all j we have (1) $l_j = 2^{g_j} m_j$ ($g_j \in \mathbb{N}$, $g_j \leq \mathrm{ld}(p_{\max})$) and (2) $m_j \leq \frac{2}{3} p_{\max}$ or $m_j < p_{\max}$ is a composite number.

Proof. To prove this lemma, we have to take a look at the lines 4 and

9 of Procedure 2. Then we see in line 4 that we do not change coefficients z_j which are composite numbers or for which $z_j \leq \frac{p_{\max}}{2}$ or $z_j = 2$ (when $p_{\max} = 3$). Of course, the coefficients which were not changed by Procedure 2 do fulfill the above conditions. To see this, we simply set $g_j = 0$. Then $m_j = z_j$, and therefore m_j is a composite number or $m_j \leq \frac{p_{\max}}{2} \leq \frac{2}{3}p_{\max}$ or $m_j = 2$ (when $p_{\max} = 3$).

Otherwise we change the coefficient z_j and come up with a coefficient z'_j due to line 9 of the above procedure such that $z'_j = 2^s t$ with $s \leq \mathrm{ld}(p)$ and $t < \frac{p}{2}$ (due to Lemma 3.1.2). Hence z'_j does also fulfill the conditions above. To see this, we have to set $g_j = s$ and $m_j = t$. So we are done. □

From now on, when we mention the iterations of Procedure 2, we always mean the iterations of the dominant while loop starting in line 3 of the procedure.

Theorem 3.2.3. *For a simplicial d-cone C the above procedure constitutes a triangulation of C.*

Proof. We will prove by induction that after every iteration of the while loop (starting in line 3) of Procedure 2 the set CO constitutes a triangulation of the underlying cone C. Obviously, before the first iteration (after setting $CO := \{C\}$) the set CO gives us a triangulation of the cone C.

Let us now assume that the set CO constitutes a triangulation of C before the k-th iteration. Then in the k-th iteration we choose a certain vector $x' \in C$ and apply stellar subdivision to all cones $D \in CO$ for which $x' \in D$ by x'. This provides us with a new triangulation of the cone C taking into account that CO already constituted a triangulation of C before the k-th generation. □

The following lemma is important. This is because it shows that, roughly speaking, if we choose a vector $x \in D$ for the application of stellar subdivision to a cone D and x is short with respect to Δ_D, then it is also short with respect to Δ_E for all other cones E with $x \in E$.

Lemma 3.2.4. *After every iteration of the above procedure we have for all $D, E \in CO$ that*

$$x \in M_D \cap E \Rightarrow x \in M_E.$$

Proof. We will prove the lemma by induction on the number of iterations. Before the first iteration, the claim is obviously true, because $CO = \{C\}$ and $M_C = \emptyset$. So let the claim be true after the r-th iteration. Then in the $(r+1)$-

th iteration we either do not find a cone D such that it fulfills the conditions given in line 4 of Procedure 2 or we do find one.

In the first case either all cones D have multiplicity $\mu(D) = 2^l$ ($l \in \mathbb{N}$) and the procedure stops which means that we are done. Or we have $M_D = \emptyset$ for all cones $D \in CO$ (see line 22) after the $(r+1)$-th iteration. Obviously, the claim is also true for the latter case.

In the second case we find a cone $D = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d \in CO$ which fulfills the conditions of line 4 of the procedure above for a certain vector $x' \in D$. Then we apply stellar subdivision to all the cones $E \in CO$ with $x' \in E$ by x'. Finally, we substitute the cones $E \in CO$ for which $x' \in E$ by the cones E_j resulting from the stellar subdivision to E by x'. And we set $M_{E_j} := M_E \cup \{x'\}$. So let G, H be two arbitrary cones in the set CO after the $(r+1)$-th iteration and let y be a vector such that $y \in M_G \cap H$. Then we distinguish between five cases

Case 1. $y = x'$. It follows that $H = E_k$ for some k. Therefore, we also have $y \in M_H$.

Case 2. $y \neq x'$ and $G \neq E_j$, $H \neq E_k$ for all j, k. In this case it simply follows by induction that $y \in M_H$, because both the sets M_G and M_H do not change in the $(r+1)$-th iteration.

Case 3. $y \neq x'$, there exists a j such that $G = E_j$ and $H \neq E_k$ for all k. Because $y \neq x'$ and $M_{E_j} = M_E \cup \{x'\}$ (see line 17), it follows that $y \in M_E$. Therefore, it follows by induction that $y \in M_H$, since the set M_H does not change in the $(r+1)$-th iteration.

Case 4. $y \neq x'$, $G \neq E_j$ for all j and there exists a k such that $H = E_k$. Because $E_k \subset E$, it follows that $y \in E$. This implies by induction that $y \in M_E$, since M_G does not change in the $(r+1)$-th iteration. Therefore, $y \in M_H = M_E \cup \{x'\}$.

Case 5. $y \neq x'$ and there exist j, k such that $G = E_j$ and $H = E_k$. Because $M_{E_j} = M_{E_k} = M_E \cup \{x'\}$, it follows that $y \in M_H$. $\square$

So why is Lemma 3.2.4 important for our considerations? To answer this question, we have to focus on the vectors x' which have been chosen in an iteration of Procedure 2 to apply a stellar subdivision. So let $D = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d$ and $x \in D$ with $x = \sum_{j=1}^{d} \frac{z_j}{p} w_j \in \mathbb{Z}^d$ be the pair of cone and vector – chosen in line 4 of Procedure 2 – such that $z_j < p$ is a composite number or $z_j \leq \frac{p}{2}$ or $z_j = 2$ (and $p = 3$) for all $w_j \in M_D$.

Let now CO be given as the set of covering cones before we chose the pair of cone D and vector x. This implies that $D \in CO$. Furthermore, let $E \in CO$ be an arbitrary cone such that $x' \in E$. Then we have $w_j \in E$ for all j with $z_j \neq 0$, because the set of cones CO constitutes a triangulation of the initial cone C and $x' \in D$. Moreover, we have that $z_j < p$ is a composite number or $z_j \leq \frac{p}{2}$ or $z_j = 2$ (and $p = 3$) for all $w_j \in M_E$. Otherwise there would exist a k which does not fulfill these conditions from line 4 of Procedure 2 and for which $w_k \in M_E$. But this is a contradiction, since $w_k \in M_E \cap D \subset M_D$ due to Lemma 3.2.4.

So, when we choose a cone D and a vector $x' \in D$ – due to the procedure above –, then the components of x' corresponding to the vectors $w_j \in M_D$ are relatively small, because for $x' = \sum_{j=1}^{d} \frac{l_j}{p} w_j$ we have $l_j < p$ for all $w_j \in M_D$. Lemma 3.2.4 does now tell us that the components of x' corresponding to the vectors $w_j \in M_E$ (for a cone with $x' \in E$) are also relatively small, simply because $w_j \in M_E$ if $w_j \in M_D \cap E$. Therefore, we are never in danger of choosing a vector x' which might be relatively short with respect to the set of vectors M_D but long with respect to the set M_E.

More precisely, Lemma 3.2.4 ensures that whenever for an arbitrary cone $E \in CO$ there exists a vector $x \in E$ such that x fulfills the conditions from line 4 of Procedure 2 for the set M_E, then E is definitely triangulated by stellar subdivision by a vector y that fulfills the conditions from line 4 for the set M_E. This result will be used implicitly throughout the remainder of this chapter.

The next lemma provides us with an auxiliary function and its nice properties. It will help us to show that the multiplicities of the cones which constitute the final set CO are relatively small and that this also true for the number of generations.

Definition 3.2.5. Let n be a natural number, $n = \prod_{i=1}^{\infty} p_i^{\alpha_i}$ be its prime decomposition. Furthermore, let $p_{\max}(n) := \max\{p_i : \alpha_i \neq 0\}$ for $n > 1$ and $p_{\max}(1) = 1$. Then we define the function $\mathrm{u} : \mathbb{N} \longmapsto \mathbb{R}$ as follows: $\mathrm{u}(n) := \mathrm{ld}(n) - \mathrm{r}(n)$, where $\mathrm{r}(n) := \sum_{i=1}^{\infty} \alpha_i$. Moreover, we define the function $\mathrm{t} : \mathbb{N} \longmapsto \mathbb{N}$ as follows: $\mathrm{t}(n) := n \cdot 2^{-\alpha_1}$ (where $p_1 = 2$).

The function u has some nice properties, which we will need in the following.

Lemma 3.2.6. *We have that*

(1) $\mathrm{u}(ab) = \mathrm{u}(a) + \mathrm{u}(b), \quad a, b \in \mathbb{N}$,

(2) there exists $s \in \mathbb{N}$ such that $n = 2^s$ if and only if $\mathrm{u}(n) = 0$,

(3) $p_{\max}(n) \leq 2^{\mathrm{u}(n)+1}$.

Proof. The first statement is obvious, because both the functions $\mathrm{ld} : \mathbb{R} \longmapsto \mathbb{R}$ and $\mathrm{r} : \mathbb{N} \longmapsto \mathbb{N}$ fulfill property (1). Therefore, $\mathrm{u} = \mathrm{ld} - \mathrm{r}$ also does. The second statement is also obviously correct.

We will prove statement (3) by induction on n. For $n = 1$ we have $p_{\max}(1) = 1 \leq 2^{\mathrm{u}(n)+1} = 2$. So let $n > 1$. We distinguish between two cases. Either n is a prime number or composite. In the first case it follows that $p_{\max}(n) = n$ and $2^{\mathrm{u}(n)+1} = 2^{\mathrm{ld}(n)} = n$, and therefore the statement is true. In the latter case we have $n = st$ with $s, t \in \mathbb{N}$ and $s, t > 1$. This implies that $p_{\max}(n) = \max\{p_{\max}(s), p_{\max}(t)\}$. On the other hand, we have by induction that $p_{\max}(s) \leq 2^{\mathrm{u}(s)+1}$ and $p_{\max}(t) \leq 2^{\mathrm{u}(t)+1}$. From this we can conclude that $p_{\max}(n) \leq \max\{2^{\mathrm{u}(s)+1}, 2^{\mathrm{u}(s)+1}\}$. Because $\mathrm{u}(n) = \mathrm{u}(s) + \mathrm{u}(t)$ and $\mathrm{u}(m) \geq 0$ for all $m \in \mathbb{N}$, it follows that $p_{\max}(n) \leq 2^{\mathrm{u}(n)+1}$. $\square$

Theorem 3.2.7. *For a simplicial d-cone C the above procedure provides a triangulation of C by cones D which are all out of generations $n_D \leq 2\,\mathrm{u}(\mu(C))$.*

Proof. Procedure 2 does not substitute a cone C by other cones – resulting from a stellar subdivision – if its multiplicity $\mu(C)$ is a power of two. Furthermore, the procedure stops if for all cones $D \in CO$ we have $\mu(D) = 2^l$ ($l \in \mathbb{N}$).

Consequently, to prove the lemma we have to show that for all cones D which are element of the final set CO there exists a $k \leq 2\,\mathrm{u}(\mu(C))$ such that D is of the k-th generation. But this means to show that $2\,\mathrm{u}(\mu(C))$ is an upper bound for the generations of all cones in the final set CO. We will prove the last claim by induction on the value $\mathrm{t}(\mu(C))$ (see Definition 3.2.5).

If $\mathrm{t}(\mu(C)) = 1$, the statement is obviously correct, because it follows that $\mu(C) = 2^l$ ($l \in \mathbb{N}$) and $2\,\mathrm{u}(\mu(C)) = 0$. Moreover, the cone C itself constitutes a cover of C of the desired kind.

So let $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d$ be an arbitrary simplicial d-cone with $\mathrm{t}(\mu(C)) > 1$. Then in the first iteration of the above procedure we come up with cones E of the first generation. Let us pick an arbitrary cone E from these. Due to Lemma 3.2.2 the cone E is generated by the stellar subdivision

of the cone C with respect to a vector

$$x' = \sum_{j=1}^{d} \frac{l_j}{p} v_j, \qquad l_j = 2^{g_j} m_j,$$

where $p := p_{\max}(\mu(C))$, $g_j \in \mathbb{N}$, and $m_j < p$ is a composite number or $m_j \leq \frac{2p}{3}$.

Hence there exists a j such that

$$E = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_{j-1} + \mathbb{R}_+ x' + \mathbb{R}_+ v_{j+1} + \cdots + \mathbb{R}_+ v_d.$$

It follows

$$\mu(E) = \frac{l_j}{p} \mu(C).$$

Therefore, $2\,\mathrm{u}(\mu(E)) = 2\,\mathrm{u}(\mu(C)) + 2\,\mathrm{u}(l_j) - 2\,\mathrm{u}(p)$. Now we will distinguish between two cases. (It is to say that the two cases do not exclude each other, but this does not affect our argumentation.)

Case 1. $l_j = 2^{g_j} m_j$ with $m_j \leq \frac{2}{3}p$. In this case it follows by Lemma 3.2.6

$$2\,\mathrm{u}(p) - 2\,\mathrm{u}(l_j) = 2\,\mathrm{ld}(p) - 2 - 2\,\mathrm{u}(m_j) \geq 2\,\mathrm{ld}(p) - 2 - 2\left(\mathrm{ld}\left(\frac{2p}{3}\right) - 1\right) \geq 1.$$

Case 2. $l_j = 2^{g_j} m_j$ with $m_j < p$ being a composite number. This means that $m_j = ab < p$ with natural numbers $a, b > 1$. In this case Lemma 3.2.6 implies

$$2\,\mathrm{u}(p) - 2\,\mathrm{u}(l_j) = 2\,\mathrm{u}(p) - 2\,\mathrm{u}(a) - 2\,\mathrm{u}(b) \geq 1.$$

Altogether, we can conclude that $2\,\mathrm{u}(\mu(E)) \leq 2\,\mathrm{u}(\mu(C)) - 1$ in both cases. On the other hand, we have $\mathrm{t}(\mu(E)) = \mathrm{t}(\mu(C)) - \mathrm{t}(p) + \mathrm{t}(l_j)$ by definition of the function t. But $\mathrm{t}(p) = p$ and $\mathrm{t}(l_j) = \mathrm{t}(m_j) < p$. Hence $\mathrm{t}(\mu(E)) < \mathrm{t}(\mu(C))$. Therefore, by induction $2\,\mathrm{u}(\mu(E))$ is an upper bound for the generations of all cones which are provided by the above procedure as a triangulation of cone E. Because of this and because the cone C is triangulated by the cones E of the first generation, we have that

$$2\,\mathrm{u}(\mu(C)) \geq 2\,\mathrm{u}(\mu(E)) + 1$$

is the corresponding upper bound for the cone C. □

Theorem 3.2.7 immediately implies

Corollary 3.2.8. *For a simplicial d-cone C the above procedure constitutes a triangulation of C by cones D which are all out of generations $n_D \leq \max(2\,\mathrm{ld}(\mu(C)) - 2, 0)$.*

Proof. Due to the definition of the function u we have $2\,\mathrm{u}(\mu(C)) \leq 2(\mathrm{ld}(\mu(C)) - 1)$ if $\mu(C) > 1$. If $\mu(C) = 1$, then C is already unimodular and therefore C (as the cone of the 0-th generation) constitutes already its own unimodular triangulation. □

Theorem 3.2.9. *For a simplicial d-cone C the above procedure constitutes a triangulation of C by cones D such that for every cone D there exists a natural number $w \leq 2\left(\mathrm{ld}\left(\mu(C)\right)\right)^2$ with*

$$\mu(D) = 2^w.$$

Proof. Due to Lemma 3.2.2 every cone D_k of the k-th generation is generated by a stellar subdivision to a cone $D_{k-1} = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d$ of the $(k-1)$-th generation with respect to a vector

$$x' = \sum_{j=1}^{d} \frac{l_j}{p} w_j \quad \text{with } l_j = 2^{g_j} m_j,$$

where $p := p_{\max}(\mu(D_{k-1}))$, $g_j \in \mathbb{N}$ and $m_j \leq p$. Furthermore, due to Lemma 3.1.2 (see also line 9 of Procedure 2) we also have that $g_j \leq \mathrm{ld}(p)$. Because $\mu(D_k) = \frac{l_j}{p}\mu(D_{k-1})$, it follows that

$$\mu(D_k) \leq p \cdot \mu(D_{k-1}) \leq \mu(C) \cdot \mu(D_{k-1}),$$

This implies that for all cones of the k-th generation we have

$$\mu(D_k) \leq \mu(C)^{k+1}, \tag{$*$}$$

taking into account that C is of the 0-th generation. Due to Corollary 3.2.8 the cones D that are elements of the final set CO, and for which $\mu(D) = 2^w$ ($w \in \mathbb{N}$), are out of generations $n_D \leq \max(2\,\mathrm{ld}(\mu(C)) - 2, 0)$. Together with $(*)$ it follows that for all these cones which are elements of the final set CO we have

$$\mu(D) = 2^w \leq \mu(C)^{n_D+1} \leq \mu(C)^{2\,\mathrm{ld}(\mu(C))}.$$

Thus $\mu(D) = 2^w$ such that $w \leq 2(\mathrm{ld}(\mu(C)))^2$. □

In the following we will refer to a special variant of the Prime Number Theorem which J. Rosser and L. Schoenfeld proved in 1962 [10].

Theorem 3.2.10. (Prime Number Theorem) *For $x > 0$ let $\pi(x)$ denote the number of prime numbers p with $p < x$. Then for all $x > 11$ we have*

$$\frac{x}{\ln(x)} < \pi(x) < \left(1 + \frac{3}{2\ln(x)}\right)\frac{x}{\ln(x)}.$$

Now we will show that the vectors generating the cones $D \in CO$ are short. But before we do so, we establish a lemma that will help us in the following.

Lemma 3.2.11. *Let X be a finite set and let $M, N \subset X$. Furthermore, let $(\sigma^i)_{i\in X}$ be a family of permutations $\sigma^i : X \longmapsto X, \quad i \in X$, such that $\sigma^j(x) \neq \sigma^k(x)$ for all $j \neq k$, $x \in X$. Then there exists $l \in X$ such that $\sigma^l(x) \notin N$ for all $x \in M$ if*

$$|M| \cdot |N| < |X|.$$

Proof. Let $M, N \subset X$ such that $|M| \cdot |N| < |X|$. Because $\sigma^j(x) \neq \sigma^k(x)$ for all $j \neq k$ and $x \in X$ we have

$$N = \dot{\bigcup_{i\in X}} \left(\{\sigma^i(x)\} \cap N\right)$$

for all $x \in X$. This implies that

$$|M| \cdot |N| = \sum_{x\in M} \left| \dot{\bigcup_{i\in X}} \left(\{\sigma^i(x)\} \cap N\right) \right|.$$

On the other hand,

$$\begin{aligned} \sum_{x\in M} \left| \dot{\bigcup_{i\in X}} \left(\{\sigma^i(x)\} \cap N\right) \right| &= \sum_{x\in M} \left(\sum_{i\in X} \left|\{\sigma^i(x)\} \cap N\right| \right) \\ &= \sum_{i\in X} \left(\sum_{x\in M} \left|\{\sigma^i(x)\} \cap N\right| \right). \end{aligned}$$

Now, assume that the statement of Lemma 3.2.11 is false. This would mean that

$$\left| \dot{\bigcup_{x\in M}} \left(\{\sigma^i(x)\} \cap N\right) \right| \geq 1 \qquad (*)$$

for all i. Hence we would arrive at

$$|M| \cdot |N| = \sum_{i \in X} \left(\sum_{x \in M} \left| \{\sigma^i(x)\} \cap N \right| \right) \geq |X|,$$

where the last inequality is due to $(*)$. But this contradicts the conditions of the lemma. Therefore, the statement must be true. $\square$

Remark 3.2.12. Let $p \in \mathbb{P}$ be an arbitrary prime number and let $X := \{1, \ldots, p-1\}$. Then the family of permutations $(\sigma^k)_{k \in X}$ is given as $\sigma^k : X \longmapsto X$,

$$\sigma^k(x) := kx \pmod p.$$

Consequently, $\sigma^j(x) \neq \sigma^k(x)$ for all $j \neq k$, $x \in X$.

Lemma 3.2.13. *Let $v_1, \ldots, v_d \in \mathbb{Z}^d$, $p \geq 3$ be a prime number and $x = \sum_{i=1}^d \frac{l_i}{p} v_i \in \mathbb{Z}^d \setminus \{0\}$ with $0 \leq l_i < p$ and $M \subset \{v_1, \ldots, v_d\}$. Then there exists a vector $y = \sum_{i=1}^d \frac{m_i}{p} v_i \in \mathbb{Z}^d \setminus \{0\}$ such that both $0 \leq m_i < p$ and $m_i \notin \mathbb{P}_{>2}$ for all $v_i \in M$ if*

$$|M| \leq \frac{\ln(p)}{2}.$$

Proof. Let $X := \{1, \ldots, p-1\}$ and $(\sigma^k)_{k \in X}$ be defined as in Remark 3.2.12. Then for all elements $y \in \langle x \rangle$ (where $\langle x \rangle$ is meant as the subgroup of $\mathbb{Z}^d$ modulo $U := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d$) with

$$y = \sum_{i=1}^d \frac{m_i}{p} v_i$$

we have that there exists a number $k \in X$ such that

$$\sigma^k(l_j) = m_j$$

for all $j = 1, \ldots, d$. Furthermore, let $N := \mathbb{P}_{>2} \cap \{1, \ldots, p-1\}$ and let $M' := \{l_j : v_j \in M\}$. Due to Lemma 3.2.11 there exists $r \in X$ such that

$$\sigma^r(l_j) \notin N$$

for all l_j if

$$|M'| \cdot |N| < |X|.$$

But this means that there exists a vector y of the desired form if

$$|M'| \cdot |N| = |M'| \cdot (\pi(p) - 1) < p - 1.$$

Furthermore, it follows due to the prime number theorem in the variant of Rosser and Schoenfeld that

$$\pi(p) - 1 < 2 \cdot \frac{p-1}{\ln(p)}$$

for all $p \geq 3$. All in all we have that there exists a vector y of the desired form if $|M'| \leq |M| \leq \frac{\ln(p)}{2}$. Thus the lemma is true. $\square$

The next theorem will provide us with the central numerical consequence resulting from our triangulation procedure.

Due to line 22 of Procedure 2 it could happen that for some cone D the set M_D – which might not have been the empty set beforehand, i.e. $M_D = \{y_1, \ldots, y_r\} \neq \emptyset$ – is reset to the empty set. Afterwards, the set M_D will be the empty set throughout all iterations of Procedure 2. But in fact, from now on we will be interested in the set $M_D = \{y_1, \ldots, y_r\}$ before the reset. Therefore, whenever we mention the set M_D from now on, we always mean the set $M_D = \{y_1, \ldots, y_r\}$ given as the set M_D before it is reset to $\emptyset$.

Theorem 3.2.14. *For all vectors x which are used in the above procedure for a stellar subdivision of a cone C we have*

$$x \in \left(d \cdot (\mu(C))^{log_{\frac{3}{2}}(e^2)} \cdot 2^{2\,\mathrm{ld}(\mu(C))}\right) \Delta_C.$$

Proof. Let C_k again be a cone of the k-th generation with $\mu(C_k) = 2^s$ ($s \in \mathbb{N}$). Furthermore, let $x_1, x_2, \ldots, x_{k-1}$ be the vectors which are used in a stellar subdivision to produce the cone C_k by generating the cones $C_1, C_2, \ldots$ of the first, second, ... generation in which the cone C_k is embedded.

We define $a(j)$ ($0 \leq j \leq f$) to be the generation of the cone $C_{a(j)}$ which is the j-th cone in the sequence $C_1, C_2, \ldots$ of cones for which

$$|M_{C_{a(j)}}| \geq |M_{C_{a(j)+1}}| = 1.$$

This implies that $M_{C_{a(j)+1}} = \{x_{a(j)+1}\}$. Thus the cones $C_{a(0)+1}$, $C_{a(1)+1}, \ldots,$ $C_{a(f)+1}$ are the only cones for which

$$|M_{C_{a(0)+1}}| = |M_{C_{a(1)+1}}| = \ldots = |M_{C_{a(f)+1}}| = 1.$$

We set $a(0) := 0$. Furthermore, let $b(j)$ be the greatest prime factor of the multiplicity $\mu(C_{a(j)})$. Hence

$$b(j) := \max\{p \in \mathbb{P} : \, p \mid \mu(C_{a(j)})\}.$$

Finally, we define $c(j)$ as the cardinality of the set $M_{C_{a(j)}}$, thus

$$c(j) := |M_{C_{a(j)}}|.$$

Moreover, let $w_1, \ldots, w_d$ be the generators of the cone $C_{a(j)}$, i.e.

$$C_{a(j)} = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d.$$

Then – according to the definition of $a(j)$ – there does not exist a vector

$$y = \sum_{i=1}^{d} \frac{m_i}{b(j)} w_i \in \mathbb{Z}^d \setminus \{0\}$$

such that both $0 \leq m_i < b(j)$ and $m_i \notin \mathbb{P}_{>2}$ for all $v_i \in M_{C_{a(j)}}$. (See line 4 of Procedure 2. In fact, we only take into account that the conditions IIb1) and IIb3) are not fulfilled.) Otherwise, if there would be such a vector y, then we would have that $M_{C_{a(j)+1}} = M_{C_{a(j)}} \cup \{y\}$, hence $|M_{C_{a(j)}}| < |M_{C_{a(j)+1}}|$. But this would be a contradiction. Due to Lemma 3.2.13 this implies that

$$c(j) > \frac{\ln(b(j))}{2}.$$

But we also have by means of the procedure that the cardinality $c(j)$ of the set $M_{C_{a(j)}}$ is equal to the difference $a(j) - a(j-1)$. So we arrive at $a(j) - a(j-1) > \frac{\ln(b(j))}{2}$, hence

$$e^{2(a(j)-a(j-1))} > b(j),$$

which directly implies

$$\prod_{i=1}^{t} b(i) < e^{2\sum_{i=1}^{t}(a(i)-a(i-1))} = e^{2(a(t)-a(0))} \qquad (*)$$

for all $t \leq f$. For a cone C_k of the k-th generation with $p := p_{\max}(\mu(C_{k-1}))$ we have

$$\mu(C_k) = \frac{l}{p}\mu(C_{k-1})$$

such that $l = 2^g m$ with $g \in \mathbb{N}$, $g \leq \mathrm{ld}(p)$ and $m \leq p$ is a composite number or $m \leq \frac{2p}{3}$ due to Lemma 3.2.2. Furthermore, with $t : \mathbb{N} \longmapsto \mathbb{N}$ given as in Definition 3.2.5, we have

$$\mathrm{t}(\mu(C_k)) \leq \frac{2}{3} \cdot \mathrm{t}(\mu(C_{k-1})).$$

This implies immediately that

$$\mathrm{t}(\mu(C_{a(i)})) \leq \frac{\mu(C)}{\left(\frac{3}{2}\right)^{a(i)}}.$$

for all i. Therefore, we have

$$b(i) \leq \frac{\mu(C)}{\left(\frac{3}{2}\right)^{a(i)}}. \qquad (**)$$

Furthermore, let the increasing sequence (h_n) be defined (we already used similar sequences in Lemma 2.2.4 and in Theorem 3.1.1) as follows:

$$h_n = 1, \quad n < 1, \quad h_1 = d, \quad h_n = h_{n-1} + \cdots + h_{n-d}, \quad n > 1.$$

Now we will prove the following lemma by induction.

Lemma 3.2.15. *Let x_z – as described above – be a vector of the z-th generation such that $a(s) < z \leq a(s+1)$ for $s+1 \leq f$ respectively $a(s) < z \leq k$ if $s = f$. Then we have*

$$x_z \in \left(\left(\prod_{i=0}^{s} b(i)\right) h_z\right) \Delta_C.$$

Proof. We will prove the lemma for $s = 0$ by induction on z. For $0 = a(0) < z = 1$ the statement is obviously true, because in this case the vector x_1 is of the form

$$x_1 = \sum_{i=1}^{d} q_i v_i \quad \text{with } q_i \leq b(0)$$

for all i. It follows

$$x_1 \in (b(0)h_1)\Delta_C.$$

Now let us suppose that the statement is true for $a(0) < z \leq a(1) - 1$. Then the vector x_{z+1} is used for a stellar subdivision of the cone $C_z = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d$. For these vectors w_i we have $w_i \in \{v_1, \ldots, v_d, x_1, \ldots, x_z\}$. Furthermore, $M_{C_z} = \{x_1, \ldots, x_z\}$, because by definition $z < a(1)$. Hence we have

$$x_{z+1} = \sum_{i=1}^{d} q_i w_i \quad \text{with } q_i < 1 \text{ if } w_i \in \{x_1, \ldots, x_z\}$$
$$\text{and } q_i \leq p_z \leq b(0) \text{ if } w_i \in \{v_1, \ldots, v_d\},$$

where p_z is the greatest prime factor of $\mu(C_z)$. Because by induction $x_i \in (b(0)h_i)\Delta_C$ for $i \leq z$ and $v_i \in \Delta_C$ $(i = 1, \ldots, d)$ we have

$$x_{z+1} \in \left(b(0) \left(\sum_{i=z-d+1}^{z} h_i \right) \right) \Delta_C,$$

which is what we wanted to show.

So let us suppose the lemma is true for $s \geq 0$. Again we will prove that it is also true for $s+1$ by induction on z. For $z = a(s+1)+1$ we have that x_z is of the form $x_z = \sum_{i=1}^{d} q_i w_i$ where $C_{z-1} = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d$ and

$$q_i \leq 2^{\mathrm{ld}(b(s+1))} = b(s+1)$$

due to Lemma 3.2.2. Because the w_i are all out of different generations $f_i \leq z-1 = a(s+1)$ (we also regard the vectors $v_1, \ldots, v_d$ to be out of the different generations $0, \ldots, -d+1$), we have

$$w_i \in \left(\left(\prod_{i=0}^{s} b(i) \right) h_{f_i} \right) \Delta_C,$$

and therefore

$$x_z \in \left(b(s+1) \left(\prod_{i=0}^{s} b(i) \right) \left(\sum_{i=z-d}^{z-1} h_i \right) \right) \Delta_C.$$

Now the induction step is done in the same way as in the case $s = 0$, where it is just to be taken into account that here $M_{C_z} = \{x_{a(s+1)+1}, \ldots, x_z\}$. Therefore, the lemma is proven. $\square$

So we can go on in the proof of Theorem 3.2.14. By $(*)$ and $(**)$ it follows that

$$\prod_{i=1}^{t} b(i) \leq e^{2(a(t)-a(0))} = \left(\frac{3}{2}\right)^{\log_{\frac{3}{2}}(e^2)a(t)} = \left(\left(\frac{3}{2}\right)^{a(t)} \right)^{\log_{\frac{3}{2}}(e^2)}$$

$$\leq \left(\frac{\mu(C)}{b(t)} \right)^{\log_{\frac{3}{2}}(e^2)}$$

for all $t \leq f$. Therefore, we have for all $t \leq f$ that

$$\prod_{i=1}^{t} b(i) \leq (\mu(C))^{\log_{\frac{3}{2}}(e^2)}.$$

Together with Lemma 3.2.15 and Corollary 3.2.8 which says that all cones G arising from the above procedure are out of a generation $n_G \leq 2\,\mathrm{ld}(\mu(C))$ as well as the fact that $h_n \leq d2^n$ for $n \geq 0$ it follows that the theorem is true. $\square$

If we do now collect the results from the Theorems 3.1.1, 3.2.9 and 3.2.14 and additionally keep in mind that the procedure provides us with a triangulation of the cone C by cones D for which $\mu(D)$ is a power of two, then we arrive at the desired result

Theorem 3.2.16. *Let $\epsilon := \log_{\frac{3}{2}}(e^2) + 2$. Then every simplicial d-cone $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$, $d \geq 3$, has a unimodular triangulation $C = D_1 \cup \ldots \cup D_t$ such that*

$$\mathrm{Hilb}(D_i) \subset \left(\frac{d^2}{2} \cdot (\mu(C))^{\epsilon} \cdot \left(\frac{9}{4}\right)^{(\mathrm{ld}(\mu(C)))^2} \right) \Delta_C, \quad i \in [1, t].$$

Proof. After a proper ordering of the above terms it becomes clear that the above statement is equivalent to

$$\mathrm{Hilb}(D_i) \subset \left(\frac{d^2}{2} \cdot (\mu(C))^{\epsilon} \cdot 2^{2\,\mathrm{ld}(\mu(C))} \cdot \left(\frac{3}{2}\right)^{2(\mathrm{ld}(\mu(C)))^2} \right) \Delta_C,$$

which is a direct consequence of the results given by the Theorems 3.1.1, 3.2.9 and 3.2.14. $\square$

4 Stellar subdivisions and triangulations

In this chapter we will pose a lot more questions than we will provide answers. These questions arise from the following observations.

When we look back to the preceding chapters, we see that all results have been achieved by applying successively stellar subdivisions to cones respectively polytopes such that in the end we came up with a unimodular triangulation of a cone or a polytope which is at its best unimodular or fulfills some other properties.

Hence one might ask: Is it just owed to the simplicity of this tool (stellar subdivision) or the narrowness of the author that it was used so often in the preceding chapters? Or are stellar subdivisions absolutely essential when we deal with triangulations of lattice simplices? Are they essential in a sense that a lattice simplex Δ which admits a unimodular triangulation does also admit a unimodular triangulation resulting from a successive stellar subdivision?

More naively, one might even ask if all triangulations of a lattice simplex are just the result of a successive stellar subdivision. In dimension $d = 1$ this is obviously true. But the following example shows that already in dimension $d = 2$ this is not the case.

Example 4.0.1. Let $\Delta = \operatorname{conv}(0, 2e_1, 2e_2)$. Then $\Delta = \operatorname{conv}(0, e_1, e_2) \cup \operatorname{conv}(e_1, e_2, e_1 + e_2) \cup \operatorname{conv}(e_1, 2e_1, e_1 + e_2) \cup \operatorname{conv}(e_2, 2e_2, e_1 + e_2)$ constitutes a unimodular triangulation of Δ which can not be achieved by a successive stellar subdivision (see Figure 4.1).

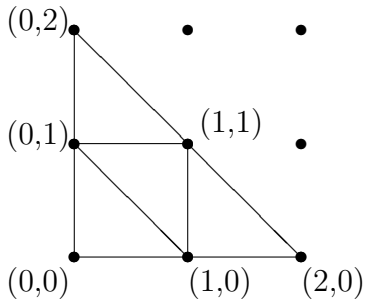

Figure 4.1

So let us come back to the questions we posed initially. At first, it is to say that stellar subdivisions obviously play a crucial part in the field of polytopes and cones, especially when we deal with triangulations. This is

because a stellar subdivision of a cone C respectively a stellar subdivision of a lattice simplex Δ by a vector x respectively a point x does provide us with information about the multiplicities of the resulting cones respectively lattice simplices. But are they essential in a sense that a lattice simplex Δ which admits a unimodular triangulation does also admit a unimodular triangulation which is only achieved by the successive application of stellar subdivisions?

4.1 Some conjectures

The first conjecture directly refers to the question above. So let $\Delta \subset \mathbb{R}^d$ be an arbitrary lattice d-simplex.

Conjecture 4.1.1. *Δ admits a unimodular triangulation if and only if Δ admits a unimodular triangulation which results from a successive stellar subdivision.*

In fact, it is even not obvious that the multiples $k\Delta$ ($k \in \mathbb{N}$) of a unimodular simplex Δ always admit a unimodular triangulation achieved by successive stellar subdivision. But Francisco Santos Leal has communicated in November 2007 that this is true and has delivered a proof.

When we discuss the meaning of stellar subdivisions for triangulations of lattice simplices, we do not have to restrict ourselves to unimodular triangulations. Having in mind that a unimodular triangulation of a lattice d-simplex Δ is simply a triangulation of Δ into $k = \mathrm{vol}(\Delta)$ (where $\mathrm{vol}(\Delta)$ is defined as the standardized volume of Δ) other lattice simplices, we could also ask for the correctness of this more general conjecture.

Conjecture 4.1.2. *There exists a triangulation of Δ by at least k lattice simplices if and only if there exists a triangulation achieved by a successive stellar subdivision of Δ by at least k lattice simplices.*

As we mentioned before, Conjecture 4.1.1 follows from Conjecture 4.1.2 simply by setting $k = \mathrm{vol}(\Delta)$, because a triangulation of a lattice d-simplex by $k = \mathrm{vol}(\Delta)$ other lattice simplices Γ_j means that $\mathrm{vol}(\Gamma_j) = 1$ for all j. Hence the triangulation is unimodular.

A much weaker version of Conjecture 4.1.1 (and thus also of Conjecture 4.1.2) is the following one.

Conjecture 4.1.3. *There exists a map $g : \mathbb{N} \to \mathbb{R}$ such that if $\Delta \subset \mathbb{R}^d$*

admits a unimodular triangulation, then there exists $c \leq \mathrm{g}(d)$ such that $c\Delta$ admits a unimodular triangulation which is achieved by a successive stellar subdivision.

This conjecture follows from Conjecture 4.1.1 by setting $\mathrm{g}(d) = 1$ and $c = 1$ for all $d \in \mathbb{N}$.

Furthermore, the correctness of Conjecture 4.1.3 would also provide us with a very interesting and significant relation between triangulations in general and triangulations which are achieved by successive stellar subdivision as the correctness of Conjecture 4.1.1 would do.

Conjecture 4.1.3 has an analogue in the field of cones. So let C be an arbitrary simplicial d-cone C.

Conjecture 4.1.3'. *There exists a map $\mathrm{g} : \mathbb{N} \to \mathbb{R}$ such that if $C \subset \mathbb{R}^d$ admits a unimodular triangulation $C = G_1 \cup \ldots \cup G_d$ with $\mathrm{Hilb}(G_i) \subset k\Delta_C$ for all i, then there exists $c \leq \mathrm{g}(d)$ such that C admits a unimodular triangulation $C = H_1 \cup \ldots \cup H_d$ with $\mathrm{Hilb}(H_i) \subset ck\Delta_C$ for all i and which is achieved by a successive stellar subdivision.*

Furthermore, we know that the analogue of Conjecture 4.1.1 in the field of cones is obviously true, because every cone C admits a unimodular triangulation which results from a successive stellar subdivision (see e.g. [6], p. 6, Theorem 3.1.2).

Let us now suppose that Conjecture 4.1.1 is true. What would this mean? It would imply that a lattice simplex which does not admit a unimodular triangulation achieved by a successive stellar subdivision would also not admit any other kind of unimodular triangulation.

Moreover, let $c \in \mathbb{N}$ be defined as the smallest number such that the multiple $c\Delta$ of a lattice simplex Δ admits a unimodular triangulation. And let $c' \in \mathbb{N}$ be the smallest number such that the multiple $c'\Delta$ of the same lattice simplex Δ admits a unimodular triangulation resulting from successive stellar subdivisions. Then, if the Conjecture 4.1.1 would be true, we would have that $c = c'$. Therefore, you would just have to deal with stellar subdivisions in these situations.

4.2 Triangulations, stellar subdivisions and lattice simplices

The preceding section should motivate us to have a closer look at unimodular triangulations of lattice simplices which result from successive stellar subdi-

visions. (From now on we will call these special unimodular triangulation shortly SUTs.)

More precisely, it should motivate us to ask the following questions: Which lattice simplices Δ admit a SUT respectively under which conditions does a lattice simplex Δ admit a SUT? Which conditions does a lattice simplex Δ fulfill that admits a SUT?

If $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d)$, then we define $\operatorname{par}(\Delta)$ as follows:

$$\operatorname{par}(\Delta) := \operatorname{par}(v_1, \ldots, v_d).$$

Furthermore, let $\Delta, \Delta' \subset \mathbb{R}^d$ be two lattice d-simplices such that $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d)$ and $\Delta' = \operatorname{conv}(0, w_1, \ldots, w_d)$. Then we say that these two simplices Δ and Δ' are *isomorphic* if there exists a linear map $\phi : \mathbb{R}^d \longmapsto \mathbb{R}^d$ such that $\phi(v_i) = w_i$ for all i, $|\det(\phi)| = 1$, and $\phi(e_i) \in \mathbb{Z}^d$ for all $i = 1, \ldots, d$. Then we have

Lemma 4.2.1. *Let $\Delta, \Delta' \subset \mathbb{R}^d$ be two lattice d-simplices such that $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d)$ and that $\Delta' = \operatorname{conv}(0, w_1, \ldots, w_d)$. Furthermore, let $V := \operatorname{vol}(\Delta) = \operatorname{vol}(\Delta')$ and let there be numbers $a_1, \ldots, a_d \in \mathbb{N}$ such that both $x := \frac{a_1}{V} v_1 + \cdots + \frac{a_d}{V} v_d \in \Delta \cap \mathbb{Z}^d$ and $y := \frac{a_1}{V} w_1 + \cdots + \frac{a_d}{V} w_d \in \Delta' \cap \mathbb{Z}^d$. Moreover, let $\operatorname{par}(\Delta) = \langle x \rangle$ and $\operatorname{par}(\Delta') = \langle y \rangle$. Then the simplices Δ and Δ' are isomorphic.*

Proof. Let $C := \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d$. Because every simplicial cone admits a unimodular triangulation, there exist vectors $z_1, \ldots, z_d$ with

$$z_i = u_i + l_i v_i \in C, \quad l_i \in \mathbb{N},$$

such that

$$u_i = \sum \frac{b_1^i}{V} v_1 + \cdots + \frac{b_d^i}{V} v_d \in \operatorname{par}(\Delta)$$

for all i and such that the simplex $\Gamma := \operatorname{conv}(0, z_1, \ldots, z_d)$ is unimodular. Because $\operatorname{par}(\Delta') = \langle y \rangle$ and $\operatorname{par}(\Delta) = \langle x \rangle$ for $x := \frac{a_1}{V} v_1 + \cdots + \frac{a_d}{V} v_d \in \Delta$ and $y := \frac{a_1}{V} w_1 + \cdots + \frac{a_d}{V} w_d \in \Delta'$, it follows that

$$u_i' = \sum \frac{b_1^i}{V} w_1 + \cdots + \frac{b_d^i}{V} w_d \in \operatorname{par}(\Delta').$$

Let $z_i' := u_i' + l_i w_i$ for all i. Then the lattice simplex $\Gamma' := \operatorname{conv}(0, z_1', \ldots, z_d')$ is also unimodular, because $\operatorname{vol}(\Delta) = \operatorname{vol}(\Delta')$. In particular, we have that Γ and Γ' are isomorphic. But this implies that the simplices Δ and Δ' are also isomorphic. $\square$

Now suppose that the group $\mathbb{Z}^d$ modulo $U := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d$ is cyclic. Hence for $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d)$ there exist $a_i \in \mathbb{N}$ with $0 \leq a_i < V$ such that

$$\operatorname{par}(\Delta) = \left\langle \frac{a_1}{V} v_1 + \cdots + \frac{a_d}{V} v_d \right\rangle, \quad V := \operatorname{vol}(\Delta).$$

Due to Lemma 4.2.1 it is justified to refer to the above notation instead of to concrete lattice simplices Δ, because all lattice simplices $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d)$ for which $\operatorname{par}(\Delta) = \left\langle \frac{a_1}{V} v_1 + \cdots + \frac{a_d}{V} v_d \right\rangle$ are isomorphic.

So let us come back to this question: Which conditions does a lattice simplex $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ fulfill that admits a SUT? For example, we can provide the following results (which are indeed not really satisfying).

Theorem 4.2.2. *Let $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ be a lattice d-simplex which admits a unimodular cover by subsimplices of Δ. Furthermore, let us suppose that there exists $x \in \operatorname{par}(\Delta)$ and an index i with $1 \leq i \leq d$ such that $x = \frac{1}{V} v_i + \sum_{j \neq i} \frac{a_j}{V} v_j$, where $V := \operatorname{vol}(\Delta)$. Then we have*

$$x \in \operatorname{par}(\Delta) \cap \Delta.$$

Proof. Because $x = \frac{1}{V} v_i + \sum_{j \neq i} \frac{a_j}{V} v_j$ is an element of $\operatorname{par}(\Delta)$, it follows that the group $\mathbb{Z}^d$ modulo $U := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d$ is cyclic and generated by x, since

$$|\langle x \rangle| = V = |\operatorname{par}(\Delta)|.$$

Therefore, the face $F = \operatorname{conv}(0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d)$ of Δ is empty. But Δ admits a unimodular cover. This implies that there exists a unimodular subsimplex Δ' of Δ which contains face F. Since F is empty, it follows that Δ' must be of the form

$$\Delta' = \operatorname{conv}(0, v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_d)$$

such that $y \in \Delta \cap \operatorname{par}(\Delta)$. Furthermore, Δ' is unimodular, hence $y = x$, and therefore $x \in \operatorname{par}(\Delta) \cap \Delta$. $\square$

For example, Theorem 4.2.2 implies that all lattice simplices $\Delta = \operatorname{conv}(0, v_1, v_2, v_3) \subset \mathbb{R}^3$ such that

$$\operatorname{par}(\Delta) = \left\langle \frac{1}{4} v_1 + \frac{2}{4} v_2 + \frac{2}{4} v_3 \right\rangle$$

do not admit a unimodular cover, and hence no SUT, since

$$\frac{1}{4} v_1 + \frac{2}{4} v_2 + \frac{2}{4} v_3 \notin \Delta.$$

$\Delta = \mathrm{conv}(0, 4e_1 - 2e_2 - 2e_3, e_2, e_3) \subset \mathbb{R}^3$ gives us an example for such a simplex (see Figure 4.2, where Δ is shifted by the vector $z = -2e_1 + e_2 + e_3$).

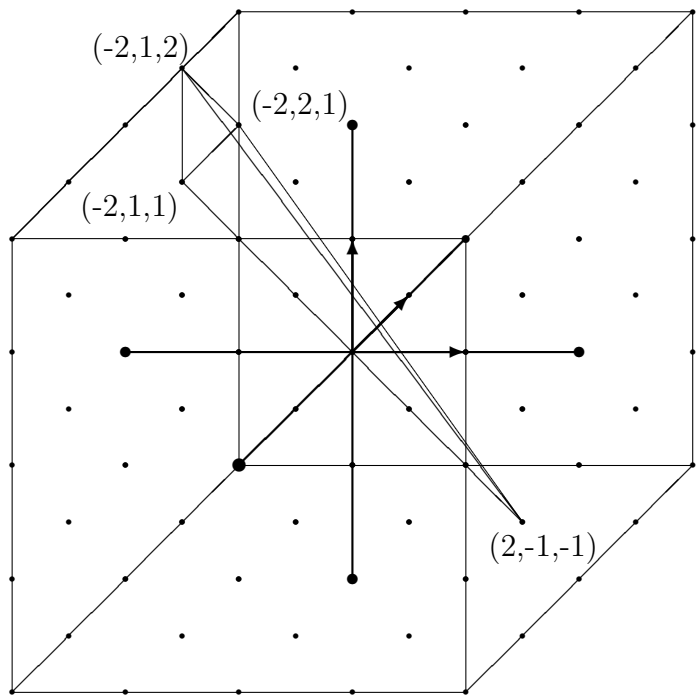

FIGURE 4.2

One might ask now if this simplex $\Delta = \mathrm{conv}(0, 4e_1 - 2e_2 - 2e_2, e_2, e_3)$, which does not admit a unimodular cover, is normal or even integrally closed. In fact, at present no integrally closed lattice polytope which admits no unimodular cover is known in dimension $d = 3$. But there exists an example of an integrally closed lattice polytope which admits no unimodular triangulation in dimension $d = 3$ (see [9]). And we also have an example of an integrally closed polytope which admits no unimodular cover in dimension $d = 5$ (see [1]).

So, let us have a look at Δ. We have

$$\Delta \cap \mathbb{Z}^d = \{0, 4e_1 - 2e_2 - 2e_3, e_2, e_3, 2e_1 - e_2 - e_3\}.$$

Furthermore, S_Δ is defined as the submonoid of $\mathbb{Z}^4$ which is generated by the elements $(x, 1)$, $x \in \Delta \cap \mathbb{Z}^3$. It follows that the subgroup $\mathrm{gp}(S_\Delta)$ of $\mathbb{Z}^4$

generated by $S := S_\Delta$ is given as

$$\mathrm{gp}(S) = 2\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4.$$

Hence the index of $\mathrm{gp}(S)$ in $\mathbb{Z}^4$ equals 2 and S is not integrally closed in $\mathbb{Z}^4$. On the other hand, S is normal, i.e. $\overline{S} := \overline{S}_{\mathrm{gp}(S)} = S$.

Let $x \in \overline{S}$. On the one hand, this means that there exist $z_1, z_2, z_3, z_4 \in \mathbb{Z}$ such that

$$x = 2z_1e_1 + z_2e_2 + z_3e_3 + z_4e_4. \tag{$*$}$$

On the other hand, there exists $m \in \mathbb{N}$ such that

$$mx \in S.$$

But this implies that there exist $n_i \in \mathbb{N}, \quad i = 1, \ldots, 4$, such that

$$mx = (2n_4 + 4n_5)e_1 + (n_3 - n_4 - 2n_5)e_2 +$$

$$(n_2 - n_4 - 2n_5)e_3 + (n_1 + n_2 + n_3 + n_4 + n_5)e_4.$$

Together with $(*)$ it follows that $m \mid n_4 + 2n_5, \quad m \mid n_3, \quad m \mid n_2$, and $m \mid n_1 - n_5$. Now we distinguish between two cases. Either $n_1 \geq n_5$ or $n_1 < n_5$. In the former case we set $r_1 := \frac{n_1 - n_5}{m}$, $r_2 := \frac{n_2}{m}$, $r_3 := \frac{n_3}{m}$, $r_4 := \frac{n_4 + 2n_5}{m}$, $r_5 := 0$. Then $r_i \in \mathbb{N}$ for all i and hence

$$x = (2r_4 + 4r_5)e_1 + (r_3 - r_4 - 2r_5)e_2 +$$

$$(r_2 - r_4 - 2r_5)e_3 + (r_1 + r_2 + r_3 + r_4 + r_5)e_4 \in S.$$

If $n_1 < n_5$, then we set $r_1 := 0$, $r_2 := \frac{n_2}{m}$, $r_3 := \frac{n_3}{m}$, $r_4 := \frac{n_4 + 2n_1}{m}$, $r_5 := \frac{n_5 - n_1}{m}$. Again we have $r_i \in \mathbb{N}$ for all i and hence

$$x = (2r_4 + 4r_5)e_1 + (r_3 - r_4 - 2r_5)e_2 +$$

$$(r_2 - r_4 - 2r_5)e_3 + (r_1 + r_2 + r_3 + r_4 + r_5)e_4 \in S.$$

Besides, Theorem 4.2.2 also implies that there does not exist a unimodular cover of a lattice simplex Δ with

$$\mathrm{par}(\Delta) = \left\langle \frac{1}{V}v_1 + \frac{1}{V}v_2 + \frac{2}{V}v_3 \right\rangle$$

and V odd. This follows from the fact that, due to Theorem 4.2.2, $\Delta' = \mathrm{conv}(0, v_1, v_2, x)$ must be one of the covering simplices, where

$$x = \frac{b}{V}v_1 + \frac{b}{V}v_2 + \frac{1}{V}v_3 \in \mathrm{par}(\Delta) \cap \Delta,$$

if Δ would admit a unimodular cover. Since $\operatorname{par}(\Delta) = \left\langle \frac{1}{V}v_1 + \frac{1}{V}v_2 + \frac{2}{V}v_3 \right\rangle$ and because V is odd, we have $b = \left\lceil \frac{V}{2} \right\rceil$. Therefore, $b + b > V$ and hence $x \notin \Delta$. This is a contradiction. Therefore, Δ admits no unimodular cover.

Let $\Delta = \operatorname{conv}(0, v_1, v_2, v_3)$ be such a simplex with $v_1 = Ve_1 - e_2 - 2e_3$, $v_2 = e_2$, $v_3 = e_3$, and $V > 3$ is an odd number. Again we might ask if this simplex Δ is normal or even integrally closed. Here we have

$$e_1, e_2, e_3 \in \Delta \cap \mathbb{Z}^d,$$

since $e_1 = \frac{1}{V}v_1 + \frac{1}{V}v_2 + \frac{2}{V}v_3$. Therefore, $\operatorname{gp}(S_\Delta) = \mathbb{Z}^4$. But on the other hand, by definition $w_i \in S_\Delta \subset \mathbb{Z}^4$, $\ i = 1, \ldots, 4$, where $w_1 = e_1 + e_4$, $w_2 = e_2 + e_4$, $w_3 = e_4$, $w_4 = Ve_1 - e_2 - 2e_3 + e_4$. This implies that

$$x := (V+1)e_1 - 2e_3 + 4e_4 = w_1 + w_2 + w_3 + w_4 \in S,$$

but, because V is odd, it follows that $\frac{x}{2} \in \mathbb{Z}^4$. Hence Δ is not normal, since $\frac{x}{2} \notin S$, as one might check easily.

Furthermore, we can generalize the result above in the following way.

Remark 4.2.3. A lattice d-simplex Δ with

$$\operatorname{par}(\Delta) = \left\langle \frac{a_1}{V}v_1 + \cdots + \frac{a_d}{V}v_d \right\rangle$$

such that $a_1 = \cdots = a_r = 1$, $a_d = r$ and $\gcd(r, V) = 1$ does not admit a SUT. Otherwise, due to Theorem 4.2.2, we would have that

$$\frac{b_1}{V}v_1 + \cdots + \frac{b_{d-1}}{V}v_{d-1} + \frac{1}{V}v_d \in \operatorname{par}(\Delta) \cap \Delta$$

if Δ would admit a unimodular cover. But it follows that $b_i > \left\lceil \frac{V}{r} \right\rceil$ if $1 \leq i \leq r$. Therefore,

$$\sum_{i=1}^{r} b_i > V$$

and hence $x \notin \Delta$, which is a contradiction. It follows that Δ admits no unimodular cover and hence no unimodular triangulation and, of course, no SUT.

Furthermore, we have

Theorem 4.2.4. *Let $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ be a lattice d-simplex which admits a unimodular triangulation $\Delta = \Gamma_1 \cup \ldots \cup \Gamma_t$. Let Γ_j be a lattice*

simplex with $0 \in \Gamma_j$*, i.e.* $\Gamma_j = \mathrm{conv}(0, w_1, \ldots, w_d)$ *for some* $w_1, \ldots, w_d \in \Delta \cap \mathbb{Z}^d$*. If* $\{w_1, \ldots, w_{d-1}\} \nsubseteq \{v_1, \ldots, v_d\}$*, then there exists* k *such that* $\Gamma_k = \mathrm{conv}(0, w_1, \ldots, w_{d-1}, x)$ *and* $x = -w_d + \sum_{i=1}^{d-1} z_i w_i, \;\; z_i \in \mathbb{Z}$*.*

Proof. Let $\Delta = \Gamma_1 \cup \ldots \cup \Gamma_t$ be a unimodular triangulation of Δ and let Γ_j be a lattice simplex with $0 \in \Gamma_j$, i.e. $\Gamma_j = \mathrm{conv}(0, w_1, \ldots, w_d)$ for some $w_1, \ldots, w_d \in \Delta$. Since $\Delta = \Gamma_1 \cup \ldots \cup \Gamma_t$ constitutes a triangulation of Δ, it follows that there exists k such that Γ_k coincides with Γ_j along face $F := \mathrm{conv}(0, w_1, \ldots, w_{d-1})$. This simply means that we have $\Gamma_k = \mathrm{conv}(0, w_1, \ldots, w_{d-1}, x)$ with $x \in \mathrm{par}(\Delta)$. Because both Γ_j and Γ_k are unimodular, we have that x must be of the form

$$x = -w_d + \sum_{i=1}^{d-1} z_i w_i$$

such that $z_i \in \mathbb{Z}$ for all i. $\square$

All these statements about the properties which lattice simplices must fulfill when they admit a SUT are unsatisfactory. (In fact, these properties do not even take into account that the simplices admit a unimodular triangulation resulting from a successive stellar subdivision, but they just take into account that either the simplex admits a unimodular cover or a unimodular triangulation.) Nothing enlightening can be provided. This and the following shall motivate us to deal with a special kind of SUTs.

Besides, when we look back to the former chapters we see that it was always, roughly speaking, very helpful if we could apply stellar subdivision to a cone (lattice simplex) by a vector x which lies in the interior of this cone (lattice simplex) and not on one of its faces. Why was this so useful? Because when you faced a set of cones (lattice simplices) which constituted a triangulation of an underlying cone C (lattice simplex Δ), then getting another triangulation of this cone by stellar subdivision meant to apply stellar subdivision to all faces F on which vector x was lying. Therefore, you had to, again roughly speaking, take care for all the cones D (lattice simplices Γ) with $F \subset D$ ($F \subset \Gamma$). So applying stellar subdivision to a cone (lattice simplex) by a vector $x \in \mathrm{int}(C)$ ($x \in \mathrm{int}(\Delta)$) made things easier.

So let us have a closer look at unimodular triangulations of lattice simplices $\Delta \subset \mathbb{R}^d$ resulting from successive stellar subdivisions by vectors which always lie in the interior of the corresponding simplices and do not lie on any face F with $\dim(F) < d$. We will call this special kind of unimodular triangulations shortly ISUTs.

As in the case of unimodular triangulations and SUTs it must be said

that not every SUT is achieved by an ISUT, as one might naively conjecture. Concretely, if a lattice simplex Δ admits a unimodular triangulation $\Delta = \Gamma_1 \cup \ldots \cup \Gamma_t$ which results from successive stellar subdivisions, then it might be impossible to achieve this triangulation $\Delta = \Gamma_1 \cup \ldots \cup \Gamma_t$ by successive stellar subdivisions using just inner points in every step. The following example illustrates this statement.

Example 4.2.5. Let $\operatorname{par}(\Delta) = \left\langle \frac{1}{7}v_1 + \frac{2}{7}v_2 \right\rangle$. Then Δ admits a SUT. An instance of Δ with $v_1 = e_1 - 2e_2$ and $v_2 = 3e_1 + e_2$ is depicted in Figure 4.3.

In fact, the unimodular triangulation of Δ, which is illustrated in Figure 4.3, is the only one that Δ admits. Therefore, this example also shows that there exist lattice simplices which do admit a SUT but no ISUT.

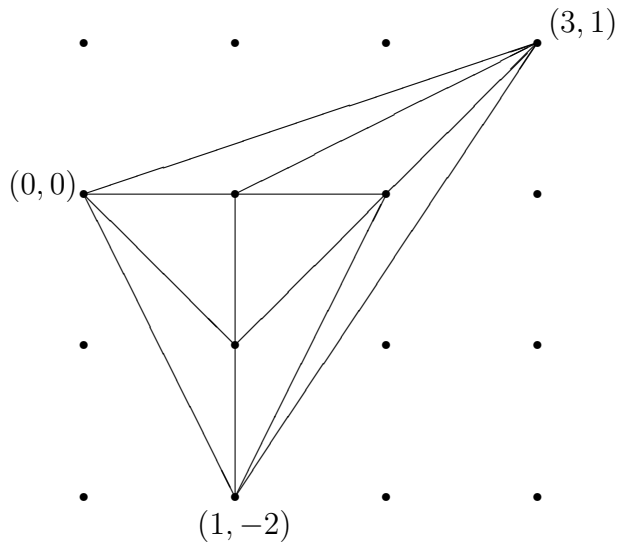

FIGURE 4.3

Furthermore, we have the following examples for ISUTs.

Examples 4.2.6. **(1)** $\operatorname{par}(\Delta) = \left\langle \frac{1}{7}v_1 + \frac{1}{7}v_2 + \frac{4}{7}v_3 \right\rangle$. At first, apply stellar subdivision by $x = \frac{1}{7}v_1 + \frac{1}{7}v_2 + \frac{4}{7}v_3$ and then by $y = \frac{2}{7}v_1 + \frac{2}{7}v_2 + \frac{1}{7}v_3$.
(2) Every lattice simplex $\Delta \subset \mathbb{R}^d$ with $\operatorname{par}(\Delta) = \left\langle \frac{1}{n}v_1 + \cdots + \frac{1}{n}v_d \right\rangle$, where $n := |\operatorname{par}(\Delta)|$ and $n - 1 \equiv 0 \pmod{d}$, admits an ISUT. Apply stellar subdivision by $x_1 = \frac{1}{d}v_1 + \cdots + \frac{1}{d}v_d$, $x_2 = \frac{2}{d}v_1 + \cdots + \frac{2}{d}v_d$, $\ldots$, $x_r = \frac{r}{d}v_1 + \cdots + \frac{r}{d}v_d$ to the simplex Δ, where $r := \frac{n-1}{d}$.

So what can be stated about lattice simplices which admit an ISUT?

What properties do they have? The Examples 4.2.5 and 4.2.6 give us a hint. In Example 4.2.5 we had $|\operatorname{par}(\Delta) \cap \Delta| = 4$, in Example 4.2.6 (1) we had $|\operatorname{par}(\Delta) \cap \Delta| = 3$ and in Example 4.2.6 (2) we had $|\operatorname{par}(\Delta) \cap \Delta| = d^{-1}(|\operatorname{par}(\Delta)| + d - 1)$. (Recall that $0 \in \operatorname{par}(\Delta) \cap \Delta$.) Is this true in general? Is

$$|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)| + d - 1}{d}$$

a necessary condition for the existence of an ISUT?

The following theorem gives us an answer to this question.

Theorem 4.2.7. *If the lattice d-simplex $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d)$ admits an* ISUT*, then we have*

$$|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)| + d - 1}{d}.$$

Proof. Let Δ be a lattice d-simplex which admits an ISUT. This means that there is an order $x_1, x_2, \ldots, x_f$ with $f = |\operatorname{par}(\Delta) \cap \Delta| - 1$ of all points $x_i \in (\operatorname{par}(\Delta) \cap \Delta) \setminus \{0\}$ such that the successive application of stellar subdivision to Δ by $x_1, x_2, \ldots, x_f$ provides us with a unimodular triangulation of Δ.

Because the triangulation is unimodular, it consists of exactly $n := |\operatorname{par}(\Delta)|$ lattice simplices.

Furthermore, this means that in every step (when we apply stellar subdivision by x_i) a simplex Γ which is part of the current triangulation is substituted by exactly $d + 1$ other simplices $\Gamma_1, \ldots, \Gamma_{d+1}$. This is, because x_i is lying in the interior of one of the triangulating simplices (here: Γ). It follows that in every step the number of simplices which triangulate the underlying simplex Δ increases by d. Therefore, after the successive stellar subdivision with all points $x_1, x_2, \ldots, x_f$, we end up with a triangulation of Δ with $fd + 1$ simplices.

It follows that $n = fd + 1$, hence $f = \frac{n-1}{d}$. And finally, this implies

$$|\operatorname{par}(\Delta) \cap \Delta| = f + 1 = \frac{n + d - 1}{d} = \frac{|\operatorname{par}(\Delta)| + d - 1}{d}.$$

□

Now we know that the condition $|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)|-1}{d}$ is necessary for the existence of an ISUT. But in fact, this condition is also sufficient in dimension $d > 2$ as the next statements will show.

Let $\Delta = \operatorname{conv}(v_0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ be a lattice d-simplex. Then we define F_x^Δ as the minimal face of Δ containing x. This means that $x \in F_x^\Delta$ and

that there exists no other face G of Δ with $G \subset F_x^\Delta$ and $x \in G$. For $x \notin \Delta$ we set $\dim(F_x^\Delta) := 0$.

Lemma 4.2.8. *Let $\Delta = \operatorname{conv}(v_0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ be a lattice d-simplex and let $x, y \in \Delta \cap \mathbb{Z}^d$. Furthermore, let $\Delta = \Gamma_0 \cup \ldots \cup \Gamma_t$ be the triangulation of Δ resulting from a stellar subdivision of Δ by x. Then we have*

$$\sum_{i=0}^{t} \dim(F_y^{\Gamma_i}) \geq \dim(F_y^\Delta).$$

Proof. Without loss of generality we can assume that $F_x^\Delta = \operatorname{conv}(v_0, \ldots, v_t)$ with $t \in \{0, \ldots, d\}$. Then let

$$\Gamma_i := \operatorname{conv}(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_d), \quad i = 0, \ldots, t,$$

be the simplices resulting from the stellar subdivision by x to Δ. Furthermore, we have

$$F_y^{\Gamma_i} = F_y^{\Gamma_j}$$

if $y \in \Gamma_i$ and $y \in \Gamma_j$. Otherwise, $F_y^{\Gamma_i} \cap F_y^{\Gamma_j}$ would be a face of Γ_i and Γ_j which is strictly contained in $F_y^{\Gamma_i}$ and in $F_y^{\Gamma_j}$ and contains y. This is a contradiction.

Let now $l \in \{0, \ldots, t\}$ such that $y \in \Gamma_l$. Now we distinguish between two cases.

Case 1. $x \notin F_y^{\Gamma_l}$. It follows that $F_y^\Delta = F_y^{\Gamma_l}$. Therefore,

$$\sum_{i=0}^{t} \dim(F_y^{\Gamma_i}) \geq \dim(F_y^{\Gamma_l}) = F_y^\Delta.$$

Case 2. $x \in F_y^{\Gamma_l}$. Let $F_y^{\Gamma_l} = \operatorname{conv}(x, w_1 \ldots, w_s)$ with $w_i \in \{v_0, \ldots, v_d\}$ for all i. Then we define $A := \{v_0, \ldots, v_t\}$ and $B := \{w_1, \ldots, w_s\}$. Due to the definition of Γ_i $(i = 0, \ldots, t)$ it follows that $y \in \Gamma_i$ if and only if $v_i \notin B$. Therefore,

$$|\{\Gamma_i : y \in \Gamma_i\}| = |A \setminus B|.$$

This implies that

$$\sum_{i=0}^{t} \dim(F_y^{\Gamma_i}) = |\{\Gamma_i : y \in \Gamma_i\}| \cdot s = |A \setminus B| \cdot |B|.$$

On the other hand, we can deduce that $y \in \operatorname{conv}(v_0, \ldots, v_t, w_1, \ldots, w_s)$, hence

$$\dim(F_y^{\Delta}) \leq |A \cup B| - 1.$$

Since $|A \cup B| - |B| = |A \setminus B|$, it follows that

$$\sum_{i=0}^{t} \dim(F_y^{\Gamma_i}) \geq \dim(F_y^{\Delta}).$$

□

Lemma 4.2.9. *Let $\Delta = \operatorname{conv}(v_0, v_1, \ldots, v_d) \subset \mathbb{R}^d$. be lattice simplex. Then a triangulation of Δ into empty lattice simplices resulting from successive stellar subdivisions consists of at least*

$$\tau_{\Delta} := \sum_{x \in \Delta \cap \mathbb{Z}^d} \dim(F_x^{\Delta}) + 1$$

simplices.

Proof. Let $\Delta \subset \mathbb{R}^d$ be a lattice d-simplex. We will prove the lemma by induction on the standardized volume $\operatorname{vol}(\Delta)$ of Δ. If $\operatorname{vol}(\Delta) = 1$, then Δ is unimodular. Hence Δ is empty and $\tau(\Delta) = 1$. Therefore, the statement is correct in this case.

So let $\operatorname{vol}(\Delta) > 1$. If Δ is empty, we are done. So let us assume that there exists $x \in \Delta \cap \mathbb{Z}^d$ such that $x \notin \{v_0, \ldots, v_d\}$. Then we apply stellar subdivision by x to the simplex Δ, which results into a triangulation $\Delta = \Gamma_1 \cup \ldots \cup \Gamma_r$. Because $\operatorname{vol}(\Gamma_i) < \operatorname{vol}(\Delta)$ for all i, it follows by induction that a triangulation of Δ into empty simplices – starting with the stellar subdivision by x – consists of a least

$$\sum_{i=1}^{r} \tau_{\Gamma_i} = \sum_{y \in (\Delta \cap \mathbb{Z}^d) \setminus \{x\}} \left(\sum_{i=1}^{r} \dim(F_y^{\Gamma_i}) \right) + r,$$

simplices, where $\dim(F_y^{\Gamma_i}) = 0$ if $y \notin \Gamma_i$.

But due to Lemma 4.2.8 it follows that

$$\sum_{y \in (\Delta \cap \mathbb{Z}^d) \setminus \{x\}} \left(\sum_{i=1}^{r} \dim(F_y^{\Gamma_i}) \right) \geq \sum_{y \in (\Delta \cap \mathbb{Z}^d) \setminus \{x\}} \dim(F_y^{\Delta}).$$

Because $r - 1 = \dim(F_x^{\Delta})$, a triangulation of Δ into empty simplices resulting from successive stellar subdivision (starting with the stellar subdivision by x) has at least

$$\tau_{\Delta} := \sum_{z \in \Delta \cap \mathbb{Z}^d} \dim(F_z^{\Delta}) + 1$$

simplices. $\square$

Theorem 4.2.10. *Let* $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ *(*$d > 2$*) be a lattice simplex with empty faces. If*

$$|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)| + d - 1}{d},$$

then Δ *admits an* ISUT.

Proof. Let $\Delta = \operatorname{conv}(0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ $(d > 2)$ be a lattice d-simplex with empty faces such that

$$|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)| + d - 1}{d}.$$

We will prove the theorem by induction on the number $m := |\operatorname{par}(\Delta) \cap \Delta|$. If $m = 1$, then it follows, due to the identity $|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)|+d-1}{d}$, that $|\operatorname{par}(\Delta)| = 1$. Therefore, Δ is unimodular and admits an ISUT.

Let now $m > 1$. Due to Lemma 4.2.9 Δ admits a triangulation into at least $|\operatorname{par}(\Delta)|$ simplices, which means that Δ admits a unimodular triangulation. And because the faces of Δ are empty and $|\operatorname{par}(\Delta)| > 1$, there must exist a vector $x \in \operatorname{par}(\Delta) \cap \Delta$ such that

$$x = \frac{a_1}{V} v_1 + \cdots + \frac{a_d}{V} v_d$$

with $V := |\operatorname{par}(\Delta)|$, $a_1 = 1$ and $\gcd(a_i, V) = 1$ for all i. Then let us apply stellar subdivision by x to the simplex Δ. If $\Gamma_0, \ldots, \Gamma_d$ with $\Gamma_i = \operatorname{conv}(0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_d)$ are the resulting simplices, then we have

$$\sum_{i=0}^{d} |\operatorname{par}(\Gamma_i)| = |\operatorname{par}(\Delta)|. \tag{$*$}$$

(In a strict sense $|\operatorname{par}(\Gamma_0)|$ is not defined, because $0 \notin \Gamma_0$. So let $\Phi = \operatorname{conv}(u_0, \ldots, u_d)$ be an arbitrary lattice d-simplex with $0 \notin \Phi$. Then we set $|\operatorname{par}(\Phi)| := |\operatorname{par}(0, u_1 - u_0, \ldots, u_d - u_0)|$. Analogously, we set $|\operatorname{par}(\Phi) \cap \Phi| := |\operatorname{par}(0, u_1 - u_0, \ldots, u_d - u_0) \cap (-u_0 + \Phi)|$. Obviously, these definitions are independent from the choice of u_0.) Now we will distinguish between two cases. Either the faces of all resulting simplices $\Gamma_0, \ldots, \Gamma_d$ are empty or they are not.

Case 1. The faces of all resulting simplices $\Gamma_0, \ldots, \Gamma_d$ are empty. Then we have

$$|\operatorname{par}(\Gamma_i) \cap \Gamma_i| \leq \frac{|\operatorname{par}(\Gamma_i)| + d - 1}{d}, \tag{$**$}$$

for all i. Otherwise there would exist a t such that, due to Lemma 4.2.9, Γ_t admits a triangulation into

$$\tau_{\Gamma_t} = d \cdot (|\operatorname{par}(\Gamma_i) \cap \Gamma_i| - 1) + 1 > |\operatorname{par}(\Gamma_i)|$$

empty simplices. But this is a contradiction. Moreover, because all faces of the simplices Γ_i are empty and because $x \in \Gamma_i$ for all i, it follows that $\sum_{i=0}^{d} |\operatorname{par}(\Gamma_i) \cap \Gamma_i| = |\operatorname{par}(\Delta) \cap \Delta| + d - 1$. Due to $(*)$ and $(**)$, this implies

$$|\operatorname{par}(\Gamma_i) \cap \Gamma_i| = \frac{|\operatorname{par}(\Gamma_i)| + d - 1}{d}$$

for all i. Therefore, we conclude by induction that all simplices Γ_i admit an ISUT. Hence Δ also admits an ISUT.

Case 2. There exist $z \in \mathbb{Z}^d \setminus \{0, v_1, \ldots, v_d, x\}$ and $t \in \{0, \ldots, d\}$ such that z lies on one of the faces of Γ_t. Let us assume that $F_z^{\Gamma_t} = \operatorname{conv}(w_1, \ldots, w_r)$ with $w_i \in \{0, v_1, \ldots, v_d, x\}$ for all i. Then $x \in \{w_1, \ldots, w_r\}$.

Now we will show that

$$\dim\left(F_z^{\Gamma_i}\right) = r - 1 = 1.$$

for all i with $z \in \Gamma_i$. Since $|\{\Gamma_i : z \in \Gamma_i\}| = (d + 1 - r)$, we have

$$\sum_{i=0}^{d} \dim\left(F_z^{\Gamma_i}\right) = (d + 1 - r) \cdot (r - 1).$$

So, if $1 < \dim\left(F_z^{\Gamma_i}\right) < d$, then $\sum_{i=0}^{d} \dim\left(F_z^{\Gamma_i}\right) > d$. But due to Lemma 4.2.9 Δ admits a triangulation into at least

$$\sum_{i=0}^{d} \tau_{\Gamma_i} = \sum_{y \in (\Delta \cap \mathbb{Z}^d) \setminus \{x\}} \left(\sum_{i=0}^{d} \dim\left(F_y^{\Gamma_i}\right) \right) + d + 1$$

empty simplices. Because $\sum_{i=0}^{d} \dim\left(F_z^{\Gamma_i}\right) > d$ and, due to Lemma 4.2.8,

$$\sum_{i=0}^{d} \dim\left(F_y^{\Gamma_i}\right) \geq \dim\left(F_y^{\Delta}\right) = d$$

for all $y \in (\Delta \cap \mathbb{Z}^d) \setminus \{0, \ldots, v_d, x\}$, it follows that

$$\sum_{i=0}^{d} \tau_{\Gamma_i} > |\operatorname{par}(\Delta)|,$$

since $|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)|+d-1}{d}$. This is a contradiction.

Finally, we know that z lies on a line segment $\overline{u_1 u_2}$ with $u_1, u_2 \in \{0, v_1, \ldots, v_d, x\}$. Because the line segments $\overline{v_i v_j}$ are empty, since all faces of Δ are empty and the line segments $\overline{v_i x}$ $(i \neq 1)$ are empty, because the simplex $\Gamma_1 = \operatorname{conv}(0, x, v_2, \ldots, v_d)$ is unimodular, it follows that

$$z \in \operatorname{conv}(v_1, x).$$

Now we will show that there exist no other lattice points on the simplices Γ_i than than the points on the line segment $\operatorname{conv}(v_1, x)$. We have already excluded that there are any other lattice points on any faces F of the simplices Γ_i with $\dim(F) < d$. Consequently, if there is any lattice point in one of the simplices Γ_i, it would have to be a point in the interior of one of these simplices. Hence let us assume that there exist $s \in \{0, \ldots, d\}$ and $w \in \Gamma_s \cap \mathbb{Z}^d$ such that $w \in \operatorname{int}(\Gamma_s)$. This means that $\dim(F_w^{\Gamma_s}) = d$.

Then we apply stellar subdivision by the vector w to the simplex Γ_s (see Figure 4.4 for the situation in dimension $d = 2$). This gives us the

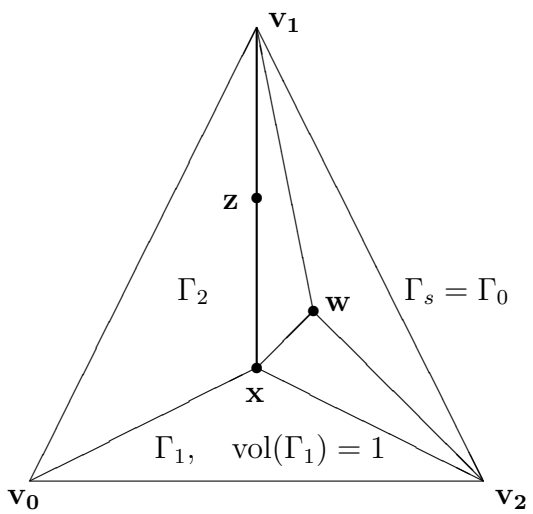

FIGURE 4.4

triangulation $\Gamma_s = \Lambda_0 \cup \ldots \cup \Lambda_d$. Due to Lemma 4.2.9 Δ admits a triangulation into at least

$$n := \sum_{i \neq s} \tau_{\Gamma_i} + \sum_{i=0}^{d} \tau_{\Lambda_i}$$

empty simplices. But from Lemma 4.2.8 we can conclude $\sum_{i=0}^{d} \dim\left(F_y^{\Gamma_i}\right) \geq \dim\left(F_y^{\Delta}\right)$ and $\sum_{i=0}^{d} \dim\left(F_y^{\Lambda_i}\right) \geq \dim\left(F_y^{\Gamma_s}\right)$. Hence it follows, due to the

definition of τ_Δ, that

$$n \geq \sum_{y \in (\Delta \cap \mathbb{Z}^d) \setminus \{x,w,z\}} \dim(F_y^\Delta) + \sum_{i \neq s} \dim(F_z^{\Gamma_i}) + \sum_{i=0}^{d} \dim(F_z^{\Lambda_i}) + 2d + 1.$$

Furthermore, $\dim(F_z^{\Gamma_i}) = 1$ for all $i \neq 1$, $\dim(F_z^{\Lambda_i}) = 1$ for all i with $z \in \Lambda_i$. But $|\{\Lambda_i : z \in \Lambda_i\}| = d - 1$. Out of this reason, we have (with $\epsilon := |\operatorname{par}(\Delta) \cap \Delta|$)

$$n \geq (\epsilon - 4)d + (d-1) + (d-1) + 2d + 1 = \epsilon \cdot d - 1.$$

This implies that Δ admits a triangulation into empty simplices with at least $n \geq |\operatorname{par}(\Delta)| + d - 2$ simplices. This is a contradiction if $d > 2$.

Therefore, the only points $y \in (\Delta \cap \mathbb{Z}^d) \setminus \{0, \ldots, v_d\}$ are the ones on the line segment $\operatorname{conv}(v_1, x)$. On the other hand, we already mentioned that, due to Lemma 4.2.9, each triangulation of Δ into empty simplices (resulting from successive stellar subdivision) is unimodular. It follows that $a_2 = \cdots = a_d$. (Recall that $x = \frac{a_1}{V} v_1 + \cdots + \frac{a_d}{V} v_d$ with $V := |\operatorname{par}(\Delta)|$, $a_1 = 1$ and $\gcd(a_i, V) = 1$ for all i.) As a result, there also exists a lattice point

$$x' = \frac{b}{V} v_1 + \frac{1}{V} v_2 + \cdots + \frac{1}{V} v_d.$$

Of course, $x' \in \Delta$. Otherwise Δ would not admit a unimodular triangulation due to Theorem 4.2.2.

Now we can forget about the point x. Instead we will apply stellar subdivision to Δ by x'. Let $\Delta = \Gamma'_0 \cup \ldots \cup \Gamma'_d$ be the resulting triangulation such that $\Gamma'_i = \operatorname{conv}(0, \ldots, v_{i-1}, x', v_{i+1}, \ldots, v_d)$. Then the faces of all these simplices must be empty. This is because Γ'_i is unimodular if $i \neq 1$.

Hence we are in Case 1. Finally, Theorem 4.2.10 is proven. □

Remark 4.2.11. At first, one might suppose that, because, roughly speaking, much points are needed in the set $M := \operatorname{par}(\Delta) \cap \Delta$ (due to Theorem 4.2.10) to come up with an ISUT, only polytopes of the form given in Example 4.2.6 (2) admit an ISUT. But in fact, Example 4.2.6 (1) shows that this is not the case.

Remark 4.2.12. Lemma 4.2.9 also shows that $r := d^{-1}(|\operatorname{par}(\Delta)| + d - 1)$ is an upper bound for the number of points in the set $M := \operatorname{par}(\Delta) \cap \Delta$ if Δ is a lattice simplex with empty faces. Otherwise we could provide a triangulation of this lattice simplex Δ with more than $n := |\operatorname{par}(\Delta)|$ lattice simplices. But this is a contradiction. Moreover, Example 4.2.6 (2) tells us

that this upper bound is optimal in a sense that in every dimension $d \geq 3$ there exists an empty lattice simplex Δ for which $|M| = r$. Furthermore, the proof of Theorem 4.2.10 is also constructive. It provides us with an algorithm to construct an ISUT for a lattice simplex with empty faces which fulfills the property of Theorem 4.2.10.

Remark 4.2.13. In dimension $d = 2$ things are different. There the condition $|\operatorname{par}(\Delta) \cap \Delta| = d^{-1}(|\operatorname{par}(\Delta)| + d - 1)$ is not sufficient.

On the one hand, we have that all lattice simplices $\Delta \subset \mathbb{R}^2$ which have a nonempty face F do not admit an ISUT, because by definition all vectors x used for stellar subdivision have to be elements of the interior of Δ. On the other hand, if Δ is a lattice simplex for which all faces F with $\dim(F) = 1$ are empty, then we have

$$|\operatorname{par}(\Delta) \cap \Delta| = \frac{|\operatorname{par}(\Delta)| + 1}{2}.$$

Why is that true? Let $\Delta = \operatorname{conv}(0, v_1, v_2)$. Then for every $x \in \operatorname{par}(\Delta) \setminus \{0\}$ with $x = \alpha_1 v_1 + \alpha_2 v_2$ we have $y := \beta_1 v_1 + \beta_2 v_2 \in \operatorname{par}(\Delta)$, where $\beta_1 := (1 - \alpha_1)$ and $\beta_2 := (1 - \alpha_2)$, because $\alpha_1, \alpha_2 \neq 0$ (since there are no points on the faces along the vectors v_1 and v_2). Therefore, either $\alpha_1 + \alpha_2 < 1$ or $\beta_1 + \beta_2 < 1$ (the case $\alpha_1 + \alpha_2 = 1$ can be excluded, since there is no point on the line segment $\overline{v_1 v_2}$). This implies that exactly $\frac{|\operatorname{par}(\Delta)|+1}{2}$ of the vectors $x \in \operatorname{par}(\Delta)$ are also elements of Δ, since $0 \in \Delta$.

But not all lattice simplices for which all faces are empty admit an ISUT as Example 4.2.5 shows. Therefore, the condition is not sufficient in dimension $d = 2$.

Do the theorems above also shed any light on the problem with SUTs? First of all, it gives us a criterion by which we could check whether a simplex has a SUT in dimension $d > 2$, simply because an ISUT is also a SUT. But, of course, the condition is just sufficient in the case of SUTs, and not necessary. For example, the lattice simplex $\Delta \subset \mathbb{R}^3$ with

$$\operatorname{par}(\Delta) = \left\langle \frac{1}{8} v_1 + \frac{3}{8} v_2 + \frac{3}{8} v_3 \right\rangle$$

admits a SUT (simply apply successively stellar subdivision to Δ by $x_1 := \frac{1}{8} v_1 + \frac{3}{8} v_2 + \frac{3}{8} v_3$ and $x_2 := \frac{3}{8} v_1 + \frac{1}{8} v_2 + \frac{1}{8} v_3$), but

$$2 = |\operatorname{par}(\Delta) \cap \Delta| \neq \frac{|\operatorname{par}(\Delta)| + d - 1}{d} = \frac{10}{3}.$$

Honestly, we must say that apart from the above statement, the theorems do (on the first view) not shed any light on SUTs. In fact, we just hope that they might be helpful to gain similar results concerning SUTs or even unimodular triangulations. We hope that we could provide a similar relation between the number of points in the set $M := \mathrm{par}(\Delta) \cap \Delta$ and the existence of a SUT, or even an arbitrary unimodular triangulation, of Δ.

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