On Operads

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Contents

Introduction

Even though J.P. May was the first to use the name *operad* ([**May72**]), the concept was introduced earlier by J.M. Boardman and R.M.Vogt in [**BV68**]. In their work operads appeared as the data, determining certain kinds of theories, called *categories of operators in standard form.* This definition was inspired by earlier notions like the PACTs and PROPs of Adams and MacLane or the formal group laws of Lazard.

In a few short words an operad encodes algebraic structures on objects of symmetric monoidal categories, like the categories of topological spaces, of differentially graded modules over a ring k, or of categories (topological and discreet). More precisely it consists of a family $\{A(n)\}_{n\in\mathbb{N}}$ of objects, encoding families of composable n-ary operations $X^{\otimes n} \to X$, together with actions of the symmetric groups Σ_n, permuting the arguments. After their successful premiere in algebraic topology, operads were and are applied in homological algebra, category theory, algebraic geometry and mathematical physics.

This thesis consists of four independent parts, containing results about different topics in the field of operads. Each of them is based on ideas or discoveries which arose during my attempt to prove the following

CONJECTURE. *Let $f : A \to A'$ be a topological equivalence of topological operads (i.e. f is a map of operads and each map $f(n) : A(n) \to A'(n)$ is a homotopy equivalence) and B a "cofibrant" operad. Then the map $f \otimes id_B : A \otimes B \to A' \otimes B$ is a topological equivalence of operads.*

Unfortunately I did not succeed in this particular task. But the results given here may justify the effort.

The first part, *Strongly Homotopy Commutative Monoids Revisited*, is already published ([**Bri00**]). It has two connections with operads, even though there is no explicit mention of them. The described structure on monoids is an example of two interchanging operadic structures (cmp. section 7), namely those of the associative monoid multiplication and an additional non-associative multiplication. Another connection is the usage of the W-construction, which was originally introduced by Boardman and Vogt to provide a functorial, cofibrant resolution for PRO(P)s, and therefore for operads. In this part a result of Sugawara ([**Sug60**]) is extended by proving that the classifying space of an associative, well-pointed and grouplike monoid has a non-associative multiplication, if and only if the multiplication of the monoid is a homomorphism up to coherent homotopies. Furthermore we define a homotopy category of monoids, such that the classifying space and the

Moore loop space functor induce an adjoint pair of functors to the homotopy category of based spaces.

The second part, *The Tensor Product of Little Cubes*, is an extension of a result of G. Dunn ([**Dun88**]) regarding the little n-cube operads C_n. These were some of the first, and perhaps still the most important, examples of operads. Boardman and Vogt introduced and used them in [**BV68**] and [**BV73**] to recognize and approximate n-fold loop spaces, as did May in [**May72**]. In short, I prove there that the tensor product $C_{n_1} \otimes \cdots \otimes C_{n_k}$ is topologically equivalent to $C_{n_1+\cdots+n_k}$, i.e. the underlying spaces are homotopy equivalent but the maps do not preserve the structures.

The third and longest part, *Homotopy Algebras and Lax Operads*, generalizes the concept of operads in two ways. First I introduce, based on an idea of Boardman and Vogt, *colored operads*, which allow a "unification" of the notions of classical, cyclic and modular operads by describing them as algebras over certain colored operads. Furthermore a reformulation of topological categories in operadic terms is given. Secondly the structure of operads themselves is weakened up to coherent homotopies, introducing lax operads and algebras over them. Again this is done via the W-construction of Boardman and Vogt and their notions of *homotopy algebras* and *homomorphisms*. In addition this leads to a homotopy theory of operads and homotopy algebras, and to a model for the localization of the category of algebras over operads along the topological equivalences. Finally the operadic description of topological categories is used to define a topological analogue of A_∞-categories.

In the fourth part, *The Milgram Non-Operad*, which is already published (cmp. [**Bri99**]), it is proved that the operad structure claimed by C. Berger in [**Ber96**] on the Milgram models for free loop spaces (cmp. [**Mil66**]), is not well-defined.

Strongly Homotopy-Commutative Monoids Revisited

In [**Sug60**] Sugawara examined structures on topological monoids, which induce H-space multiplications on the classifying spaces. He introduced a form of coherently homotopy commutative monoids, which he called *strongly homotopy commutative*. His main result is that a countable CW-group G is strongly homotopy-commutative if and only if its classifying space BG is an H-space. The proof proceeds as follows. One first shows that the multiplication $G \times G \to G$ of a strongly homotopy commutative group is a homotopy homomorphism (Sugawara called such maps strongly homotopy multiplicative), i.e. a homomorphism up to coherent homotopies. Then one shows that this map induces an H-space structure on BG. The proof of the converse is very sketchy and far from convincing.

We start with an easy to handle reformulation of the notion of homotopy homomorphisms. The well-pointed and grouplike monoids (cmp. Def. 2.4) and homotopy classes of these homotopy homomorphisms form a category $\mathcal{H}\mathbf{Gr}_H$. If $\mathfrak{Top}^*_H$ is the category of well-pointed spaces and based homotopy classes of maps, then the classifying space and the Moore loop space functors induces functors $B_H : \mathcal{H}\mathbf{Gr}_H \to \mathfrak{Top}^*_H$ and $\Omega_H : \mathfrak{Top}^*_H \to \mathcal{H}\mathbf{Gr}_H$. We first prove the following strengthening of a result of Fuchs ([**Fuc65**]).

THEOREM (3.7). *The functor B_H is left adjoint to Ω_H.*

The adjunction induces an equivalence of the full subcategories of monoids in $\mathcal{H}\mathbf{Gr}_H$ of the homotopy type of CW-complexes and of the full subcategory of $\mathfrak{Top}^*_H$ of connected spaces of the homotopy type of CW-complexes.

We then reexamine Sugawara's result starting with grouplike monoids whose multiplications are homotopy homomorphisms. They give rise to H-objects (i.e. Hopf objects) in the category $\mathcal{H}\mathbf{Gr}_H$. We obtain the following extension of Sugawara's theorem.

THEOREM (3.8 and 4.2). *The classifying space of a grouplike and well-pointed monoid M is an H-space if and only if M is an H-object in $\mathcal{H}\mathbf{Gr}_H$.*

As mentioned above, the multiplication of a strongly homotopy commutative monoid is a homotopy homomorphism. We were not able to prove the converse and consider it an open question.

This part of my thesis is already published ([**Bri00**]).

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1. The W-construction

Let **Mon** be the category of well-pointed, topological monoids and continuous homomorphisms between them. Here well-pointed means that the inclusion of the unit is a closed cofibration.

REMARK 1.1. One can functorially replace any monoid M by well-pointed one by adding a whisker (cmp. [**BV68**], pg 1130f.). This does not change the (unbased) homotopy type of M.

DEFINITION 1.2. Let M and N be topological monoids. A homotopy $H_t : M \to N$ is called a *homotopy through homomorphisms* if for each $t \in I$ the map $H_t : M \to N$ is a homomorphism.

DEFINITION 1.3. (cmp. [**BV73**],[**Vog73**],[**SV86**]) We define a functor $W : \mathbf{Mon} \to \mathbf{Mon}$. For $M \in \mathfrak{ob}\,\mathbf{Mon}$ the monoid WM is the space

$$WM = \coprod_{n\in\mathbb{N}} M^{n+1} \times I^n / \sim$$

with the relation

$$(x_0, t_1, x_1, \ldots, t_n, x_n) =$$
$$\begin{cases} (x_0, \ldots, t_{i-1}, x_{i-1}x_i, t_{i+1}, \ldots, x_n) & \text{for } t_i = 0 \\ (x_1, t_2, \ldots, x_n) & \text{for } x_0 = e \\ (x_0, \ldots, x_{i-1}, \max(t_i, t_{i+1}), x_{i+1}, \ldots, x_n) & \text{for } x_i = e \\ (x_0, \ldots, t_{n-1}, x_{n-1}) & \text{for } x_n = e. \end{cases}$$

The multiplication is given by

$$(x_0, \ldots, t_n, x_n) \cdot (y_0, s_1, \ldots, y_k) = (x_0, \ldots, t_n, x_n, 1, y_0, s_1, \ldots, y_k).$$

A continuous homomorphism $F : M \to N$ is mapped to $WF : WM \to WN$ with

$$WF(x_0, t_1, x_1 \ldots, x_n) = \big(F(x_0), t_1, F(x_1), \ldots, F(x_n))\big).$$

The *augmentation* $\varepsilon_M : WM \to M$ with $\varepsilon_M(x_0, \ldots, x_n) = x_0 \cdot \cdots \cdot x_n$ defines a natural transformation $\varepsilon : W \to \mathrm{id}$. If $i_M : M \to WM$ is the inclusion, which maps every element x of M to the chain (x), we get $\varepsilon_M \circ i_M = \mathrm{id}_M$ and a non-homomorphic homotopy $h_t : WM \to WM$ from $i_M \circ \varepsilon_M$ to id_M, given by

$$h_t(x_0, t_1, x_1, \ldots, t_n, x_n) = (x_0, tt_1, x_1, \ldots, tt_n, x_n).$$

Therefore ε_M is a homotopy equivalence and M a strong deformation retract of WM at *space level*, i.e. its homotopy inverse is no homomorphism.

One of the most important properties of the W-construction is the following lifting theorem, which is a slight variation of [**SV86**, 4.2] and is proven in the same way.

THEOREM 1.4. *Given the following diagram in* **Mon** *with* $0 \leq n \leq \infty$ *such that*

$$\begin{tikzcd} WM \arrow[rr, dotted, "H"] \arrow[dr, "F"'] & & B \arrow[dl, "L"] \\ & N & \end{tikzcd}$$

1. *M is well-pointed and*
2. *L is a homotopy equivalence.*

Then there exists a homomorphism $H : WM \to B$ and a homotopy $K_t : WM \to N$ through homomorphisms from $L \circ H$ to F. Furthermore H is unique up to homotopy through homomorphisms.

2. Homotopy homomorphisms

Definition 2.1. Let M and N be two well-pointed monoids. A *homotopy homomorphism* F from M to N is a homomorphism $F : WM \to WN$. The map $f := \varepsilon_N \circ F \circ i_M : M \to N$ is the *underlying map* of F.

Let $\mathcal{H}\mathbf{Mon}$ be the category whose objects are well-pointed, topological monoids, and whose morphisms are homotopy homomorphisms.

Remark 2.2. Our homotopy homomorphisms are closely related to Sugawara's approach. If we compose a homotopy homomorphism with the augmentation, we obtain a map $WM \to N$ which is, up to the conditions for the unit, a strong homotopy multiplicative map in Sugawara's sense. Since ε_N is a homotopy equivalence, the resulting structures are equivalent, after passage to the homotopy category.

The Moore loop-space construction $\Omega_M X$ and the classifying space functor B define functors $\Omega_W : \mathfrak{Top}^* \to \mathcal{H}\mathbf{Mon}$ and $B_W : \mathcal{H}\mathbf{Mon} \to \mathfrak{Top}^*$ by $\Omega_W(X) = \Omega_M X$ and $B_W(M) = B(WM)$ on objects and $\Omega_W(f) = W\Omega_M f$ and $B_W(F) = BF$ on morphisms.

For a based map $f : X \to Y$ let $[f]_*$ denote its based homotopy class. For a homomorphism F of monoids, let $[F]$ denote its homotopy class with respect to homotopies through homomorphisms.

Let $\mathfrak{Top}^*_H$ be the category of based, well-pointed spaces and based homotopy classes of based spaces and $\mathcal{H}\mathbf{Mon}_H$ the category of well-pointed monoids and homotopy classes of homotopy homomorphisms.

Remark 2.3. One can prove that the homotopy homomorphisms, which are homotopy equivalences on space level, represent isomorphisms in $\mathcal{H}\mathbf{Mon}_H$.

Since Ω_W and B_W preserve homotopies, they induce a pair of functors.

$$B_H : \mathfrak{Top}^*_H \rightleftarrows \mathcal{H}\mathbf{Mon}_H : \Omega_H$$

Definition 2.4. A monoid M with multiplication μ and unit e is called *grouplike*, if there a continuous map $i : M \to M$ such that the maps $x \mapsto \mu(x, i(x))$ and $x \mapsto \mu(i(x), x)$ are homotopic to the constant map on e.

Since the Moore loop-spaces are grouplike and since this notion is homotopy invariant, an additional restriction is necessary for Theorem 3.7 to be true. Let $\mathcal{H}\mathbf{Gr}$ be the full subcategory of $\mathcal{H}\mathbf{Mon}$, whose objects are grouplike, and let $\mathcal{H}\mathbf{Gr}_H$ be the corresponding homotopy category. Then B_H and Ω_H give rise to a pair of functors

$$B_H : \mathfrak{Top}^*_H \rightleftarrows \mathcal{H}\mathbf{Gr}_H : \Omega_H.$$

We make use of a construction from [**SV86**]. For an arbitrary monoid M let EM be the contractible space with right M-action such that $EM/M \simeq BM$. We define a monoid structure on the Moore path space

$$P(EM; e, M) := \\ \left\{(\omega, l) \in EM^{\mathbb{R}_+} \times \mathbb{R}_+ : \omega(0) = e, \omega(l) \in M, \omega(t) = \omega(l) \text{ for } t \geq l\right\}.$$

The product of two paths (ω, l) and (ν, k) is given by $(\rho, l + k)$, with

$$\rho(t) = \begin{cases} \omega(t) & \text{if } 0 \leq t \leq l \\ \omega(l) \cdot \nu(t - l) & \text{if } l \leq t \leq l + k. \end{cases}$$

The end-point projection $\pi_M : P(EM; e, M) \to M, (\omega, l) \mapsto \omega(l)$ is a continuous homomorphism. Since $P(EM; e, M)$ is the homotopy fiber of the inclusion $i : M \hookrightarrow EM$ and since EM is contractible, π_M is a homotopy equivalence.

By Theorem 1.4 there exists a homomorphism $\bar{T}_M : WM \to P(EWM; e, WM)$ such that the following diagram commutes up to homotopy through homomorphisms.

$$\begin{tikzcd} WM \arrow[rr, "\bar{T}_M"] \arrow[dr, equal] & & P(EWM; e, WM) \arrow[dl, "\pi_{WM}"] \\ & WM & \end{tikzcd}$$

Because π_{WM} is strictly natural in WM, $\bar{T}_M$ is natural up to homotopy through homomorphism.

Obviously we have $P(BWM, *, *) = \Omega_M BWM$. Hence the projection $p_{WM} : EWM \to BWM$ induces a natural homomorphism $P(p_{WM}) : P(EWM; e, WM) \to \Omega_M BWM$. Because WM is grouplike, $P(p_{WM})$ is a homotopy equivalence. Therefore we obtain a homomorphism $T_M : WM \to W\Omega_M BWM$, which is induced by Theorem 1.4 and the following diagram.

$$\begin{tikzcd} WM \arrow[r, "T_M"] \arrow[d, "\bar{T}_M"'] & W\Omega_M BWM \arrow[d, "\varepsilon_{\Omega_M BWM}"] \\ P(EWM; e, WM) \arrow[r, "P(p_M)"'] & \Omega_M BWM \end{tikzcd}$$

Since all morphisms are natural up to homotopy through homomorphisms, the T_M form a natural transformation $[T]$ from $\mathrm{id}_{\mathcal{H}\mathbf{Gr}_H}$ to $\Omega_H B_H$ and each T_M is a homotopy equivalence and hence an isomorphism in $\mathcal{H}\mathbf{Gr}_H$. Its inverse $[K_M]$ can be constructed by Theorem 1.4 and the following diagram.

$$\begin{tikzcd} W\Omega_M BWM \arrow[rr, dotted, "K_M"] \arrow[dr, equal] & & WM \arrow[dl, "T_M"] \\ & W\Omega_M BWM & \end{tikzcd}$$

For each well-pointed space X, we chose E_X to be the dotted arrow in the following diagram.

$$\begin{tikzcd}
BW\Omega_M BW\Omega_M X \arrow[rr, "BK_{\Omega_M X}"] \arrow[d, "B\varepsilon_{\Omega_M BW\Omega_M X}"'] & & BW\Omega_M X \arrow[d, "B\varepsilon_{\Omega_M X}"] \\
B\Omega_M BW\Omega_M X \arrow[d, "e_{BW\Omega_M X}"'] & & B\Omega_M X \arrow[d, "e_X"] \\
BW\Omega_M X \arrow[rr, dotted, "E_X"'] & & X
\end{tikzcd}$$

Here the $e_\bullet$ are the maps described in Proposition 5.1. Since all solid arrows, except for e_X, are based homotopy equivalences the morphism E_X exists and is uniquely determined up to based homotopy. The naturality of E_X follows from the naturality up to homotopy of all other maps. Hence we have a natural transformation $[E]_*$ from $B_H\Omega_H$ to the identity on $\mathfrak{Top}^*_H$.

THEOREM 2.5. *The functor $B_H : \mathcal{H}\mathbf{Gr}_H \to \mathfrak{Top}^*_H$ is left adjoint to Ω_H. The natural isomorphism $[T]$ is the unit, and the natural transformation $[E]_*$ the counit of this adjunction.*

PROOF. The definition of E_{BWM} and the naturality of several morphisms imply

$$[E_{BWM} \circ BT_M \circ e_{BWM}]_* = [e_{BWM}]_*$$

and since e_{BWM} is a based homotopy equivalence by Proposition 5.1 this results in

$$[E_{B_H(M)}]_* \circ B_H[T_M] = [E_{BWM}]_* \circ [BT_M]_* = [\mathrm{id}_{BM}]_*.$$

The definition of E_X implies

$$[W\Omega_M E_X \circ W\Omega_M e_{BW\Omega_M X} \circ W\Omega_M B\varepsilon_{\Omega_M BW\Omega_M X} \circ W\Omega_M BT_{\Omega_M X}] = \\ [W\Omega_M e_X \circ W\Omega_M B\varepsilon_{\Omega_M X}]$$

and the naturality of several maps leads to

$$[W\Omega_M E_X \circ W\Omega_M e_{BW\Omega_M X} \circ W\Omega_M B\varepsilon_{\Omega_M BW\Omega_M X} \circ W\Omega_M BT_{\Omega_M X}] = \\ [W\Omega_M e_X \circ W\Omega_M B\varepsilon_{\Omega_M X} \circ W\Omega_M BW\Omega_M E_X \circ W\Omega_M BT_{\Omega_M X}]\,.$$

Since $\varepsilon_{\Omega_M X}$ and $\Omega_M e_X$ are homotopy equivalences the homomorphisms $W\Omega_M e_X$ and $W\Omega_B\varepsilon_{\Omega_M X}$ represent isomorphisms in $\mathcal{H}\mathbf{Gr}_H$. Therefore we have

$$[W\Omega_M BW\Omega_M E_X \circ W\Omega_M BT_{\Omega_M X}] = [\mathrm{id}_{W\Omega_M BW\Omega_M X}]\,.$$

The facts that $T_{\Omega_M X}$ is an isomorphism in $\mathcal{H}\mathbf{Gr}_H$ and that

$$[T_{\Omega_M X} \circ W\Omega_M E_X \circ T_{\Omega_M X}] = \\ [W\Omega_M BW\Omega_M E_X \circ W\Omega_M BT_{\Omega_M X} \circ T_{\Omega_M X}]$$

imply

$$\Omega_H[E_X]_* \circ [T_{\Omega_H(X)}] = [W\Omega_M E_X \circ T_{\Omega_M X}] = [\mathrm{id}_{W\Omega_M X}]\,.$$

□

3. Hopf-objects

Definition 3.1. An H- or *Hopf-object* (X,μ,ρ) in a monoidal category[1] $(\mathcal{C},\otimes,e)$ is a non-associative monoid, i.e. an object X of $\mathcal{C}$ together with morphisms $\mu: X\otimes X\to X$ and $\rho: e\to X$ such that the following diagram commutes.

$$\begin{tikzcd}
e\otimes X \arrow[r, "\rho\otimes \mathrm{id}_X"] \arrow[dr, "\simeq"'] & X\otimes X \arrow[d, "\mu"] & X\otimes e \arrow[l, "\mathrm{id}_X\otimes\rho"'] \arrow[dl, "\simeq"] \\
 & X. &
\end{tikzcd}$$

A morphism of H-objects (or H-morphism) $f: X\to Y$ is a morphism such that $\mu_Y\circ(f\otimes f) = f\circ\mu_X$. The H-objects of $\mathcal{C}$ and the H-morphisms form a category $Hopf\mathcal{C}$.

Proposition 3.2. *Let $(\mathcal{C},\odot,e_{\mathcal{C}})$ and $(\mathcal{D},\otimes,e_{\mathcal{D}})$ be monoidal categories and*

$$(F,G,\eta,\varepsilon):\mathcal{C}\to\mathcal{D}$$

an adjunction of monoidal functors[2] such that the diagrams

$$\begin{tikzcd}
Y\odot Y \arrow[r, "\eta_Y\odot\eta_Y"'] \arrow[d, "\eta_{Y\odot Y}"] & GFY\odot GFY \arrow[d] & FGX\otimes FGX \arrow[r] \arrow[d, "\varepsilon_X\otimes\varepsilon_X"'] & F(GX\odot GX) \arrow[d] \\
GF(Y\odot Y) & G(FY\otimes FY) \arrow[l] & X\otimes X & FG(X\otimes X) \arrow[l]
\end{tikzcd}$$

commute for each $X\in\mathcal{C}$ and $Y\in\mathcal{D}$, then there exists an adjoint pair of functors

$$HopfF: HopfH\mathcal{C}\leftrightarrows Hopf\mathcal{D}: HopfG.$$

Proof. $HopfF$ is given by

$$HopfF(X,\mu,\rho) = (FX, F\mu\circ\varphi, F\rho) \text{ and } HopfF(f) = Ff,$$

with $\varphi: FX\otimes FX\to F(X\odot X)$ the natural transformation. Its adjoint $HopfG$ is given analogously. The two commutative diagrams imply that the units η_X and the counits ε_Y of the adjunction are H-morphisms. Therefore they form the unit and counit of an adjunction. $\square$

Example 3.3. $\mathfrak{Top}^*_H$ with its product is a monoidal category. The H-objects in $\mathfrak{Top}^*_H$ are precisely the H-spaces with the base point as unit. The homotopy class $[\mu]_*$ of the multiplication is called *H-space structure* of X. H-morphisms are the homotopy classes of H-space morphisms up to homotopy.

Example 3.4. $\mathcal{H}\mathbf{Gr}_H$ has a monoidal structure $\otimes$ given on objects by $M\otimes N = M\times N$. For morphisms $F: WM\to WM'$ and $G: WN\to WN'$ we define $F\otimes G: W(M\times N)\to W(M'\times N')$ as follows: Let $S_{M,N} =$

[1]For a definition of monoidal categories see [**McL71**].
[2]For a definition of monoidal functors see [**BFSV98b**]

$(W\mathrm{pr}_M, W\mathrm{pr}_N) : W(M \times N) \to WM \times WN$ be induced by the two projections. Then the diagram

$$\begin{tikzcd} W(M \times N) \arrow[rr, "S_{M,N}"] \arrow[dr, "\varepsilon_{M\times N}"'] & & WM \times WN \arrow[dl, "\varepsilon_M \times \varepsilon_N"] \\ & M \times N. & \end{tikzcd}$$

commutes. Obviously $S_{M,N}$ is a homotopy equivalence. By Theorem 1.4 the homotopy class of $S_{M,N}$ in $\mathcal{H}\mathbf{Mon}$ is uniquely determined.

For two homotopy homomorphisms $F : WM \to WM'$ and $G : WN \to WN'$, we define $F \otimes G : W(M \times N) \to W(M' \times N')$ to be the lifting in the following diagram.

$$\begin{tikzcd} W(M \times N) \arrow[r, "F\otimes G"] \arrow[d, "S_{M,N}"'] & W(M' \times N') \arrow[d, "S_{M',N'}"] \\ WM \times WN \arrow[r, "F\times G"'] & WM' \times WN'. \end{tikzcd}$$

This construction is compatible with the composition and we can define a functor $\otimes : \mathcal{H}\mathbf{Gr}_H \times \mathcal{H}\mathbf{Gr}_H \to \mathcal{H}\mathbf{Gr}_H$ with $M \otimes N = M \times N$ and $[F] \otimes [G] = [F \otimes G]$.

The projections $[P_M]$ and $[P_N]$ on $M \otimes N$ are given by $[p_i \circ S_{M,N}]$, where p_i is the according projection from $WM \times WN$. It is easy to check that $\otimes$ and these projections form a product in $\mathcal{H}\mathbf{Gr}_H$ and that the trivial monoid $*$ is a terminal and initial object of $\mathcal{H}\mathbf{Gr}_H$. Therefore $\mathcal{H}\mathbf{Gr}_H$ is monoidal and we have a notion of H-objects in $\mathcal{H}\mathbf{Gr}_H$.

The unit of an H-object in $\mathcal{H}\mathbf{Gr}_H$ is always the unit of the underlying monoid.

LEMMA 3.5. *If $(M, [F])$ is a H-object in $\mathcal{H}\mathbf{Gr}_H$, then the underlying map f of F is homotopic to the multiplication μ of M.*

PROOF. The homomorphism $\bar{F} = \varepsilon_M \circ F$ has the property $[\bar{F} \circ Wi_k] = [\varepsilon_M]$ for $k = 1, 2$. The homotopy $h_t : M \times M \to M$ with $h_t(x, y) = \bar{F}\big((x, e), t, (e, y)\big)$ runs from $f(x, y)$ to $f(x, e)f(e, y)$, and hence f and μ are based homotopic. □

Thus the multiplication μ of an H-object $(M, [F])$ in $\mathcal{H}\mathbf{Gr}_H$ is homotopic to the underlying map of F, and therefore homotopy-commutative with the commuting homotopy from xy to yx derived from $F\big((e, y), t, (x, e)\big)$. The relations in $W(M \times M)$ define higher homotopies so that the underlying monoid is homotopy commutative in a strong sense.

We now want to examine the structure on a monoid M, that leads to the existence of an H-space multiplication on its classifying space.

PROPOSITION 3.6. *B_H and Ω_H are monoidal functors.*

PROOF. For $M, N \in \mathcal{H}\mathbf{Gr}_H$ the morphism

$$s_{M,N} : BW(M \times N) \to BWM \times BWN$$

is given by the based homotopy equivalence (BWp_1, BWp_2), where $p_1, p_2 : M \times M \to M$ are the projections.

For $X, Y \in \mathfrak{Top}^*_H$ the morphism $\Omega_H(X \times Y) \simeq \Omega_H X \otimes \Omega_H Y$ is given by $W(\Omega_M p_1, \Omega_M p_2) : W\Omega_M(X \times Y) \to W(\Omega_M X \times \Omega_M Y)$. □

Theorem 3.2 now implies

THEOREM 3.7. *B_H and Ω_H induce an adjunction*

$$HopfB_H : Hopf\mathcal{H}\mathbf{Gr}_H \leftrightarrows Hopf\mathfrak{Top}^*_H : Hopf\Omega_H$$

with

$$HopfB_H(M, [F]) = (BWM, [BF \circ s_{M,M}]_*)$$

and

$$Hopf\Omega_H(X, [\mu]_*) = (\Omega_M X, [W\Omega_M \mu \circ R_{X,X}]).$$

THEOREM 3.8. *The classifying space BM of a grouplike and well-pointed monoid M is an H-space if and only if M is an H-object in $\mathcal{H}\mathbf{Gr}_H$.*

PROOF. If M is an H-object, then BWM and thus BM are H-spaces.

Now let BM be an H-space. Then $\Omega_M BWM$ is an H-object in $Hopf\mathcal{H}\mathbf{Gr}_H$. Since $T_M : WM \to W\Omega_M BWM$ is a homotopy equivalence, M is an H-object, too. □

4. Extensions

A monoid in $Hopf\mathfrak{Top}^*_H$ is a homotopy-associative H-space (X, μ). A monoid in $Hopf\mathcal{H}\mathbf{Gr}_H$ consists of a well-pointed and grouplike monoid together with homotopy homomorphisms $F_2 : W(M \times M) \to WM$ and $F_3 : W(M \times M \times M) \to WM$ such that $(M, [F_2])$ is an H-object and

$$[F_2 \circ (F_2 \otimes \mathrm{id})] = [F_3] = [F_2 \circ (\mathrm{id} \otimes F_2)].$$

We call the H-object $(M, [F_2])$ *associative.*

Since these structures are invariant under isomorphisms we obtain, similar to the non-associative case, the following

THEOREM 4.1. *The classifying space BM of a well-pointed, grouplike monoid M is an homotopy associative H-space, if M is an associative H-object in $\mathcal{H}\mathbf{Gr}_H$.*

As we realized earlier, the morphism $e_X : B\Omega_M X \to X$ need not be a homotopy equivalence. But by Proposition 5.1 $\Omega_M e_X$ is a based homotopy equivalence. Hence, if we restrict to connected, based spaces of the homotopy type of CW-complexes, e_X is a homotopy equivalence.

This implies that the adjunction

$$B_H : \mathcal{H}\mathbf{Gr}_H \leftrightarrows \mathfrak{Top}^*_H : \Omega_H$$

induces an equivalence of categories, if we restrict to the full subcategories of based spaces of the homotopy type of connected CW-complexes and grouplike monoids of the homotopy type of CW-complexes.

THEOREM 4.2. *The full subcategories $Hopf\mathcal{H}\mathbf{Gr}_H^{CW} \subset Hopf\mathcal{H}\mathbf{Gr}_H$ of H-objects of the homotopy type of CW-complexes, and $Hopf\mathfrak{Top}_H^{*,CW} \subset Hopf\mathfrak{Top}_H^*$ of connected H-spaces of the homotopy type of CW-complexes, are equivalent.*

5. Appendix: The evaluation map

This section is dedicated to the proof of the following theorem.

PROPOSITION 5.1. *For each based space X there exists a natural map $e_X : B\Omega_M X \to X$ such that*

1. *$\Omega_M e_X$ is a homotopy equivalence for each based space X and*
2. *if M is a grouplike wellpointed monoid then e_{BM} is a homotopy equivalence.*

To prove this we will use based simplicial spaces. A *based simplicial space* is a functor from the dual of the category Δ of finite, ordered sets $[n] = \{0, 1, \ldots, n\}$ to $\mathfrak{Top}_*$. The *based standard simplices* $\nabla_*(n)$ are given by the quotient space $\nabla(n)/V_n$ with $\nabla(n)$ the n-th standard simplex and V_n its subspace of vertices. They induce a based cosimplicial space $\nabla_* : \Delta \to \mathfrak{Top}_*$.

We define the *based geometric realization* of a based simplicial space $\mathfrak{X}$ as

$$|\cdot|_* = \coprod_n \mathfrak{X}(n) \wedge \nabla_*(n)/\sim$$

with the relation $\sim$ generated by the same equalities as in the unbased case. This induces a functor $|\cdot|_*$ from the category of based simplicial spaces to $\mathfrak{Top}_*$.

Analogous to the unbased singular complex we can define the *based singular complex $S_*X : \Delta^{op} \to \mathfrak{Top}_*$* of a based space X by

$$[n] \mapsto \mathfrak{Top}_*(\nabla_*(n), X).$$

S_* induces a functor from $\mathfrak{Top}_*$ to the category of based simplicial sets. As in the unbased case this right adjoint to the based realization $|\cdot|_*$. The unit $\tau_* : \mathrm{id} \to S_*|\cdot|_*$ is given by

$$\tau_{*,\mathfrak{X}}(x) = (t \mapsto (x,t)), \quad x \in \mathfrak{X}_n, t \in \nabla_*(n)$$

and the counit $\eta_* : |S_* \cdot|_* \to \mathrm{id}$ by

$$\eta_{*,X}(\omega, t) = \omega(t), \quad \omega \in S_*Y(n), t \in \nabla_*(n).$$

DEFINITION 5.2. (cmp. [**Seg74**, A.4.]) A based simplicial space $\mathfrak{X}$ is *good* if for each n and $0 \leq i \leq n$ the inclusion $s_i(\mathfrak{X}_{n-1}) \hookrightarrow \mathfrak{X}_n$ is a closed cofibration.

Now observe that the based realization $|\mathfrak{X}|_*$ coincides with the unbased realization $|\mathfrak{X}|$ if the simplicial space $\mathfrak{X}$ has only one 0-simplex. Therefore we obtain the following lemma from well-known facts.

LEMMA 5.3. *(cmp. [**Seg74**, A.1]) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be good, based simplicial spaces with $\mathfrak{X}_0 = * = \mathfrak{Y}_0$ and let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a based simplicial map. If each map $\mathfrak{f}_n$ is a based homotopy equivalence, then the map*

$$|\mathfrak{f}|_* : |\mathfrak{X}|_* \to |\mathfrak{Y}|_*$$

is a based homotopy equivalence.

In the following we will show that the nerve $\Omega_M^\bullet X$ of the Moore loop space of an arbitrary wellpointed space X is homotopy equivalent to its based simplicial complex. There exists a based simplicial map $a : \Omega_M^\bullet X \to S_* X$, given by

$$a_n(\omega_1, \ldots, \omega_n)(t_0, \ldots, t_n) = (\omega_1 + \cdots + \omega_n)\left(\sum_{i=1}^{n}\sum_{j=1}^{i} t_i l_j\right)$$

(l_j is the length of the loop ω_j and $+$ the loop addition). Let $\mathfrak{e}_j = (t_0, \ldots, t_n)$ be the vertex of $\nabla(n)$ given by $t_j = 1, t_k = 0, k \neq j$. Then a maps the loop ω_j to the edge running from $\mathfrak{e}_{j-1}$ to $\mathfrak{e}_j$.

$E_n := \{(t_0, \ldots, t_n) \in \nabla(n) : t_i + t_{i+1} = 1 \text{ for some } i\}$ is a strong deformation retract of $\nabla(n)$ and there exists a sequence of homotopy equivalences

$$\mathfrak{Top}_*(\nabla_*(n), X) \simeq \mathfrak{Top}_*(E_n, X) \simeq (\Omega X)^n \simeq (\Omega_M X)^n$$

such that the composition of a with these maps is the endomorphism of $(\Omega_M X)^n$ which changes the length of the loops to length 1. This map is homotopic to the identity, and hence a is a homotopy equivalence. Furthermore a is natural in X and defines a natural transformation from $\Omega_M^\bullet$ to S_*. If X and hence $\Omega_M X$ and $\mathfrak{Top}_*(\nabla_*(n), X)$ are wellpointed, then a_X is a based homotopy equivalence.

The map $e_X := \eta_{*,X} \circ |a_X|_* : |\Omega_M^\bullet X|_* \to X$ is natural in X and therefore induces a natural transformation from $|\Omega_M^\bullet \cdot|_*$ to id. Since $\Omega_M^\bullet$ is the nerve of a topological monoid, e is in fact a natural transformation from $B\Omega_M$ to $\mathrm{id}_{\mathfrak{Top}_*}$.

By [**Seg74**, 1.5] the canonical map $\tau_{\Omega_M X} : \Omega_M X \to \Omega B \Omega_M X$ with $\tau_{\Omega_M X}(\omega)(t) = (\omega; 1-t, t)$ is a homotopy equivalence because $\Omega_M X$ is grouplike. The composition $\Omega e_X \circ \tau_{\Omega_M X} : \Omega_M X \to \Omega X$ is the map normalizing the loops to length 1 and hence a homotopy equivalence. Therefore Ωe_X is a homotopy equivalence. Since the maps $\Omega_M X \to \Omega X$ are natural in X, this implies the first statement of Proposition 5.1.

Let M be a wellpointed grouplike monoid. Using the adjunction of the based realization and the based singular complex functors, we obtain a sequence

$$BM = |M^\bullet|_* \xrightarrow[|\tau_{*,M}|_*]{} |S_* BM|_* \xrightarrow[\eta_{*,BM}]{} |M^\bullet|_* = BM$$

The map $\eta_{*,BM} \circ |\tau_{*,M^\bullet}|_*$ is the identity. $S_*BM(1)$ is precisely the non-associative loop space ΩBM and, by [**Seg74**, 1.5], the map $\tau_{*,M^\bullet}$ is a homotopy equivalence on the 1-simplices. Furthermore $S_*BM(n)$ is based homotopy equivalent to $(\Omega_M BM)^n$ and $S_*BM(n)$ is special, i.e. it satisfies the conditions of [**Seg74**, 1.5]. Therefore $\tau_{*,M^\bullet}$ is a based homotopy equivalence in each dimension and thus $|\tau_{*,M^\bullet}|_*$ and $\eta_{*,BM}$. Since $|a_{BM}|_*$ is a based homotopy equivalence this implies the second statement of Proposition 5.1.

The Tensor Product of Little Cubes

Operads were introduced by Boardman and Vogt in 1968 to study the algebraic structure of iterated loop spaces (they called them *categories of operators in standard form*) [**BV68**]. Their results were refined in [**BV73**] and independently by May in [**May72**]. They proved that any n-fold loop space is homotopy equivalent to a grouplike C_n-space and vice versa, where C_n is the operad of little n-cubes.

For $n \geq 2$ the iterated loop space $\Omega^n X$ has a homotopy-commutative multiplication, satisfying an increasing number of coherence conditions, which are codified by actions of the operad C_n. This lead to the definition of E_n-spaces, as spaces on which an operad D, homotopy equivalent to C_n, operates.

Since an $(n+k)$-fold loop space can be regarded as a k-fold loop space in the category of n-fold loop spaces, one might think that an E_{n+k}-space is an E_k-space in the category of E_n-spaces. This type of structure, i.e. a D-space in the category of C-spaces, where C and D are operads, is codified by the *tensor product* $C \otimes D$ of operads (see section 7 below). Therefore the naive assumption arises, that the tensor product of an E_n-operad with an E_k-operad is homotopy equivalent to C_{n+k}, and hence an E_{n+k}-operad.

In general this is not true. The operad $\mathcal{M}$ of associative monoids is an E_1-operad, i.e. its grouplike algebras are precisely the one-fold loop spaces. But the tensor product with itself is the operad of commutative monoids, which is an E_∞-operad.

A better version of the naive approach is the following

CONJECTURE. *The tensor product of a* cofibrant *E_n-operad with an E_k-operad is an E_{n+k}-operad.*

Here the notion *cofibrant* has to be made precise. One possible choice is given in [**Vog99**].

A step in this direction was made by Dunn in [**Dun88**]. He proved that the n-fold tensor product of C_1 with itself, i.e. $C_1^{\otimes n}$ is homotopy equivalent to C_n. But unfortunately this result does not imply the equivalence of $C_n \otimes C_m$ and C_{n+m}, since the tensor product of operads does not respect homotopy equivalences.

REMARK 5.4. The little cube operads C_n are not cofibrant in the sense of [**Vog99**].

In this paper we extend Dunn's result to our

MAIN THEOREM. *For all $l \geq 2, n_1, \dots, n_l \in \mathbb{N}$ and $n = n_1 + \cdots + n_l$ there exists a map $C_{n_1} \otimes \cdots \otimes C_{n_l} \to C_n$ of operads, which is a local Σ-equivalence.*

In the first three sections we recall the definition of operads in the topological setting, give a short overview over the interchange and the tensor product of operads and repeat the definition of the little cubes operads, which is extended to an operad of compact spaces in section 9. In addition we introduce a model $C_n|C_m$ for the tensor product $C_n \otimes C_m$ as a suboperad of C_{n+m}, which is based on Dunn's ideas.

The last three sections contain an analysis of this model, which leads together with some tools of Dunn to our main theorem.

Throughout this paper we work in the category $\mathfrak{Top}$ of compactly generated Hausdorff spaces in the sense of [**Vog71**].

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6. Topological operads and trees

DEFINITION 6.1. A *collection* is a family $\{A(j)\}_{j\in\mathbb{N}}$ of spaces in $\mathfrak{Top}$ such that Σ_j acts on $A(j)$ from the right. For $\alpha \in A(j)$ we call j the *number of inputs* of α.

A *map of collections* $f : A \to B$ is a family $\{f_j : A(j) \to B(j)\}_{j\in\mathbb{N}}$ of equivariant maps.

The category of collections and maps between them is called $\Sigma\mathfrak{Top}$.

DEFINITION 6.2. A *local* Σ*-equivalence* between two collections A and B is a map $f : A \to B$ of collections such that each $f_j : A(j) \to B(j)$ is an Σ_j-equivariant homotopy equivalence.

DEFINITION 6.3. An *operad* A is a collection, together with a unit $\mathrm{id} \in A(1)$ and a series of *compositions* $- \circ - : A(k) \times A(j_1) \times \cdots \times A(j_k) \to A(j_1 + \cdots + j_k)$ such that

- $\alpha\sigma \circ (\beta_1, \ldots, \beta_k) = \alpha \circ (\beta_{\sigma^{-1}(1)}, \ldots, \beta_{\sigma^{-1}(k)}) \circ \bar{\sigma}$ for each $\alpha \in A(k), \beta_i \in A(j_i)$ and $\sigma \in \Sigma_k$, where $\bar{\sigma}$ permutes the blocks given by $j_1, \ldots, j_k)$ according to σ,
- $\alpha \circ (\mathrm{id}, \ldots, \mathrm{id}) = \alpha$ and $\mathrm{id} \circ \alpha = \alpha$ and
- $\alpha \circ \big(\beta_1 \circ (\gamma^1_1, \ldots, \gamma^1_{i_1}), \ldots, \beta_j \circ (\gamma^j_1, \ldots, \gamma^j_{i_j})\big) =$
$$\big(\alpha \circ (\beta_1, \ldots, \beta_j)\big) \circ (\gamma^1_1, \ldots, \gamma^1_{i_1}, \ldots, \gamma^j_{i_j})$$

A map $f : A \to B$ of operads is a map of the underlying collections such that $f(\mathrm{id}) = \mathrm{id}$ and

$$f\big(\alpha \circ_A (\beta_1, \ldots, \beta_k)\big) = f(\alpha) \circ_B \big(f(\beta_1), \ldots, f(\beta_k)\big),$$

where $\circ_A$ is the composition of A, and $\circ_B$ the one of B.

The category of operads and maps between them is called *oper*$\mathfrak{Top}$.

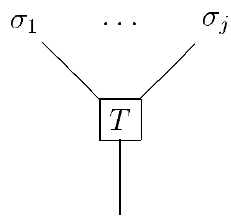
REMARK 6.4. Since we require an operad to have a unit id, our notion is equivalent to the $\circ_i$-approach of Markl in [**Mar96**].

A very good notion for the work with operads - if not the best (free operads are constructed this way) - are trees. Since all results in the following are well-known, we just give a short description of all the terms, ideas and constructions needed. For details the reader is referred to the literature.

An edge of a graph is called *internal* if it is bounded by two vertices and *external* otherwise. External edges of directed graphs, with no vertex at their starting point are called *inputs*, and edges with no vertex at their end point are called *output*. A *tree* T is a connected, directed graph without loops, with exactly one output such that each vertex has precisely one output. The *valence* of a vertex v in a tree is the number $in(v)$ of its incoming edges. A vertex of valence 0 is called a *stump*.

REMARK 6.5. The graph with no vertex and only one external edge, is a tree.

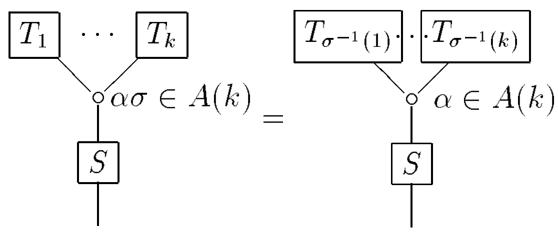
A *labeled* planar tree is a tree T together with a bijection $\sigma : in(T) \to \{1, \ldots, |in(T)|\}$ from the set of inputs of T. We represent it graphically by



where T is a tree with j inputs.

It is well-known that the labeled trees form a topological operad $\mathfrak{Tree}$ such that $\mathfrak{Tree}(j)$ is the set of trees with j inputs. The composition is given by grafting the trees along their roots and inputs.

DEFINITION 6.6. The j-the space of the *free operad* FA of a collection A is the quotient of the space of all labeled trees with vertex labels, i.e. each vertex v of a tree is assigned a label $\alpha_v \in A(in(v))$, under the relation



The topology on $FA(j)$ is the topology of the according quotient space of

$$\coprod_{T \in \mathfrak{Tree}(j)} \left(\prod_{v \in T} A\big(in(v)\big) \right).$$

The unit of FA is the trivial tree with no vertex.

The free operads imply a functor $F : \Sigma\mathfrak{Top} \to oper\mathfrak{Top}$, which is left-adjoint to the forgetful functor $U : oper\mathfrak{Top} \to \Sigma\mathfrak{Top}$.

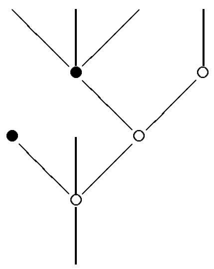
REMARK 6.7. Since we need an order to define the product, we use the natural order on the vertices of a tree, given by left-traversion.

For our purposes we need a slight extension of this notion of trees.

DEFINITION 6.8. A *bi-colored tree* (T, c) consists of a tree T and a map $c : ver(T) \to \{0, 1\}$ from the set of vertices of T. The number $c(v)$ is called the *color* of the vertex v. An internal edge is called *monochrome*, if its vertices have the same color.

Graphically we represent bi-colored trees by trees whose vertices are white ($c(v) = 0$) or black ($c(v) = 1$).

EXAMPLE 6.9.



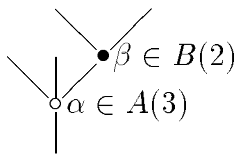
The sets $\mathfrak{BiTree}(j)$ of labeled bi-colored trees form an operad $\mathfrak{BiTree}$. As in the monochrome case the composition is given by the grafting of trees.

Bi-colored trees are very useful in the description of the direct sum $A \sqcup B$ of operads. Let T be a bi-colored tree with j inputs and $(A, B)_T$ the space

$$(A, B)_T = \prod_{\substack{v \in ver(T) \\ c(v) = 0}} A(in(v)) \times \prod_{\substack{v \in ver(T) \\ c(v) = 1}} B(in(v)).$$

Then the free operad $F(A \sqcup_\Sigma B)$, generated by the coproduct $A \sqcup_\Sigma B$ of the underlying collections, is given by the spaces $\coprod_{T \in \mathfrak{BiTree}(j)} (A, B)_T$ modulo the relations of Definition 6.6. The composition is induced by the grafting of trees. The identity (or unit) is the trivial tree with no vertex.

EXAMPLE 6.10.



LEMMA 6.11. *$A \sqcup B(j)$ is the quotient of $F(A \sqcup_\Sigma B)(j)$, by the relations*

1. *Monochrome edges may be shrunk and their vertices composed,*
2. *The identities of A and B are identified with the trivial tree and*
3. *The relation of Definition 6.6*

7. Interchange

The concept of interchange of operad structures and the tensor product of operads is well-known. Boardman and Vogt used it in [**BV73**] to describe homomorphisms between theories and algebras over theories, and May's notion of a pairing of two operads is closely related to the interchange of the two structures.

DEFINITION 7.1. Let A, B and C be operads and $f : A \to C$ and $g : B \to C$ two maps of operads. We say f and g *interchange*, if the diagram

$$\begin{tikzcd}
A(j) \times B(k) \arrow[r, "\mathrm{id}\times\Delta"] \arrow[d, "\simeq"'] & A(j) \times B(k)^j \arrow[r, "f_j \times g_k^j"] & C(j) \times C(k)^j \arrow[dd, "-\circ-"] \\
B(k) \times A(j) \arrow[d, "\mathrm{id}\times\Delta"'] & & \\
B(k) \times A(j)^k \arrow[r, "g_k \times f_j^k"'] & C(k) \times C(j)^k \arrow[r, "-\circ-"'] & C(jk)
\end{tikzcd}$$

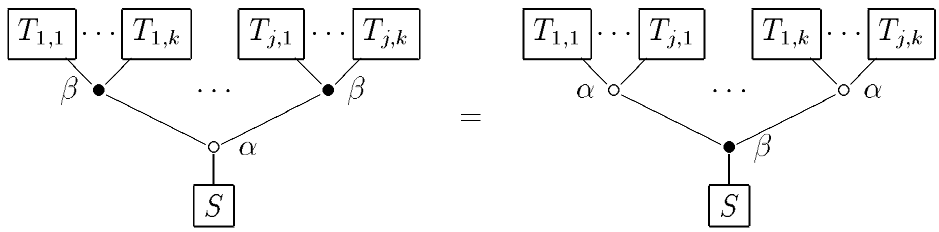
commutes for all $j, k \in \mathbb{N}$. Here Δ always means the appropriate diagonal.

If we apply this definition to algebras over A and B, i.e. if we choose $C = End_X$, then the structures of A and B on X interchange if and only if the diagrams

$$\begin{tikzcd}
(X^k)^j \arrow[rr, "\beta^j"] \arrow[d, "\simeq"'] & & X^j \arrow[d, "\alpha"] \\
(X^j)^k \arrow[r, "\alpha^k"'] & X^k \arrow[r, "\beta"'] & X
\end{tikzcd}$$

commute for all $\alpha \in A(j)$ and $\beta \in B(k)$.

The tensor product $A \otimes B$ of two operads A and B is an operad, which codifies the interchange of operad maps (cmp. [**BV73**]). This means that there exist two maps $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$ such that the operad maps $f : A \to C$ and $g : B \to C$ interchange if and only if there exists a map $h : A \otimes B \to C$ such that $f = h \circ i_A$ and $g = h \circ i_B$. Its j-th space $A \otimes B(j)$ is the quotient of $A \sqcup B(j)$ under the additional *shuffle-relation*



As Dunn noted in [**Dun88**] the tensor product $A \otimes B$ is universal for pairings of operads in the sense of [**May80**].

8. The little cubes

For convenience we will use the following notations. The n-dimensional interval $[a_1, b_1] \times \cdots \times [a_n, b_n]$ of $\mathbb{R}^n$ will be denoted with $[a, b]$. For $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ we will write $a < b$ if $a_j < b_j$ for each j. In the same fashion we will write $a \leq b$. We denote the vector $(a_1 b_1, \ldots, a_n b_n)$ with ab.

DEFINITION 8.1. Let $C_n(j)$, $j \geq 1$, be given as the set of ordered j-tuples of n-dimensional intervals $[a^i, b^i]$ in $I^n = [0, 1]^n$ with disjoint and non-empty

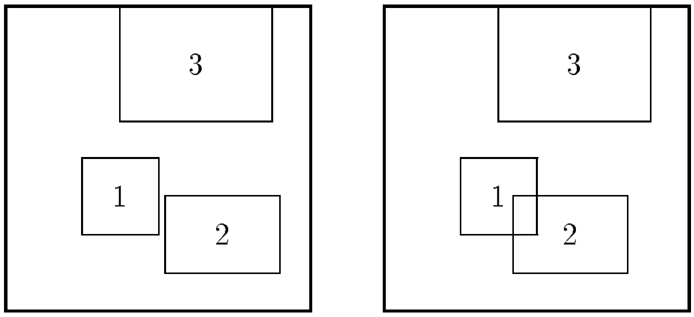


FIGURE 1. The left example is an element of $C_2(3)$ the right is not

interiors, i.e. with $a^i < b^i$. The space $C_n(0)$ consists only of the empty tupel ().

The composition $\alpha \circ (\beta_1, \ldots, \beta_k)$ of $\alpha = \big([a^1, b^1], \ldots, [a^k, b^k]\big) \in C_n(k)$ with $\beta_i = ([c^{1,i}, d^{1,i}], \ldots, [c^{j_i,i}, d^{j_i,i}]) \in C_n(j_i)$ for $1 \leq i \leq k$ is given by replacing the i-th interval $[a_i, b_i]$ of α with the following j-tupel

$$\big([a^i + (b^i - a^i)c^{1,i}, a^i + (b^i - a^i)d^{1,i}], \ldots \\ \ldots, [a^i + (b^i - a^i)c^{j_i,i}, a^i + (b^i - a^i)d^{j_i,i}]\big)$$

(recall that the $a^i, b^i, c^{i,j}$ and $d^{i,j}$ are vectors). This operation corresponds to the replacement of the i-th interval of α with a scaled-down copy of β_i.

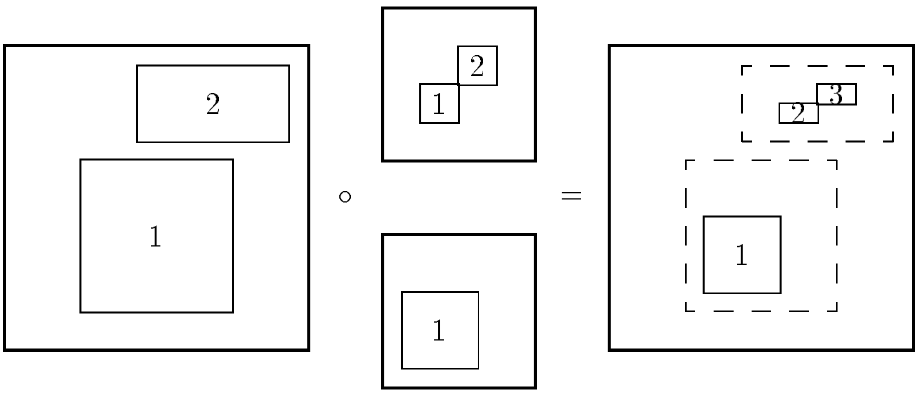


FIGURE 2. Example of a composition in C_2

DEFINITION 8.2. A little cube $c \in C_n(j)$ is called *decomposable*, if ...

1. ... $j \in \{0, 1, 2\}$ or
2. ... there exist a $d \in C_n(2)$ and decomposable c_1, c_2 with $c_k \in C_n(j_k)$ for $j_k > 0, k = 1, 2$ such that $c = \mu(d; c_1, c_2)$.

It is easy to see that the decomposable cubes of C_n form a suboperad D_n. Furthermore $D_1 = C_1$ and $D_n(j) = C_n(j)$ for $j \leq 3$.

A more geometrical description of decomposability is given by the insertion of a hyper plane. $c \in C_n(j)$ is decomposable, if and only if there exists an $1 \leq i \leq n$ and a hyper plane L of codimension 1, parallel to the i-axis, which hits no interior of the component cubes of C, such that each of the two parts is decomposable and contains at least one component cube (cmp. [**Dun88**]). We call such a hyper plane *separating*.

PROPOSITION 8.3. *(cmp.* [**Dun88**, Prop. 2.3.]*) The inclusion $D_n \to C_n$ is a local Σ-equivalence.*

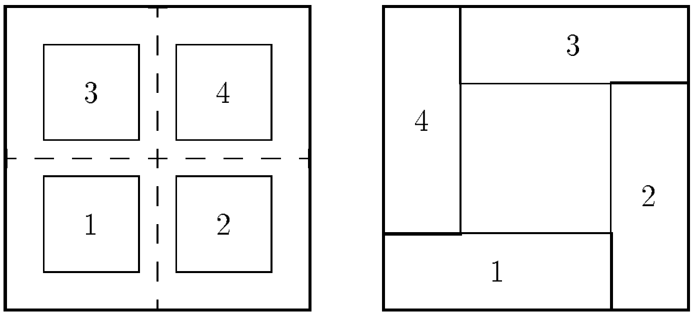


FIGURE 3. The left cube is decomposable (the dashed lines are separating hyper planes), the right is not.

Now let $H \subset C_n$ and $V \subset C_m$ be two suboperads. Each of them can be embedded into C_{n+m} as a suboperad. For H we use the inclusion

$$\big([a_1, b_1], \dots, [a_k, b_k]\big) \mapsto \big([(a_1, 0), (b_1, 1), \dots, (a_k, 0), (b_k, 1)]\big),$$

where $(a_i, 0)$ is the $(n+m)$-tupel $(a_i^1, \dots, a_i^n, 0, \dots, 0)$ and $(b_i, 1)$ is the tupel $(b_i^1, \dots, b_i^n, 1, \dots, 1)$. Similar we have an inclusion of V into C_{n+m} with

$$\big([c_1, d_1], \dots, [c_l, b_l]\big) \mapsto \big([(0, c_1), (1, d_1)], \dots, [(0, c_l), (1, d_l)]\big).$$

Graphically the two inclusions i_H and i_V are described by Figure 4.

These two operad morphisms induce two maps $H(j) \times V(k) \to C_{n+m}(jk)$ of collections for each pair j, k of natural numbers, given by

$$(h, v) \mapsto i_H(h) \circ \underbrace{(i_V(v), \dots, i_V(v))}_{k-\text{times}}$$

and

$$(h, v) \mapsto i_V(v) \circ \underbrace{(i_H(h), \dots, i_H(h))}_{l-\text{times}}.$$

The image of the first map is called $h|v$.

It is easy to check that the first morphism is given by

$$\Big(\big([a_1, b_1], \dots, [a_k, b_k]\big), \big([c_1, d_1], \dots, [c_l, k_l]\big)\Big) \mapsto$$
$$\big([(a_1, c_1), (b_1, d_1)], \dots, [(a_1, c_l), (b_1, d_l)], [(a_2, c_1), (b_2, d_1)], \dots\big)$$

and the second by

$$\Big(\big([a_1, b_1], \dots, [a_k, b_k]\big), \big([c_1, d_1], \dots, [c_l, k_l]\big)\Big) \mapsto$$
$$\big([(a_1, c_1), (b_1, d_1)], \dots, [(a_k, c_1), (b_k, d_1)], [(a_1, c_2), (b_1, d_2)], \dots\big).$$

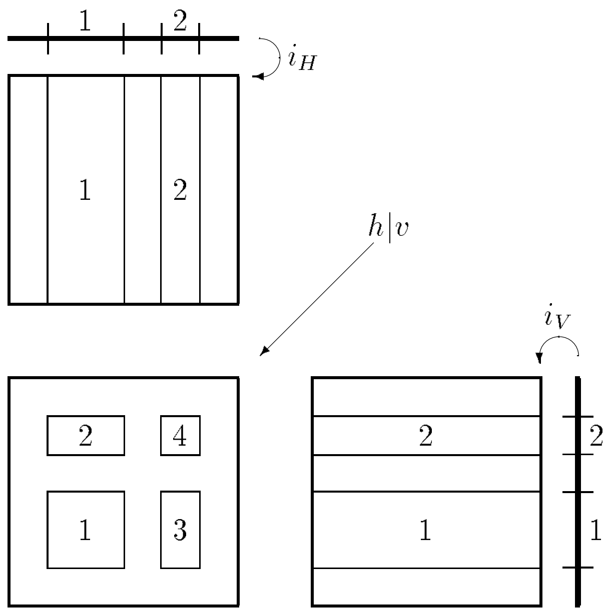


FIGURE 4. The inclusions i_H and i_V and the cube $h|v = i_H(h) \circ (i_V(v), i_V(v))$.

If we order the tuples (a_i, c_j) and (b_i, d_j) lexicographically by their indeces, the we see that the two images coincide up to a transposition. Comparing this with the interchange condition shows that i_H and i_V interchange. This leads to the existence of a morphism $H \otimes V \to C_{n+m}$. Let $H|V \subset C_{n+m}$ be the image of this morphism and $\varphi : H \otimes V \to H|V$ the induced map of morphisms.

Since this construction is based on the addition of ¨trivial¨ coordinates, it is easy to see that the suboperads $(H|M)|V$ and $H|(M|V)$ of C_{n+l+m} with $H \subset C_n$, $M \subset C_l$ and $V \subset C_m$ are equal.

9. The closed cubes

For the proofs of the main theorem we need an extension $\bar{C}_n$ of the little n-cubes such that each $\bar{C}_n(j)$ is a compact subset of $\mathbb{R}^{2nj}$. We start with an alternative description of $C_n(j)$. Let $\alpha = ([a^1, b^1], \ldots, [a^j, b^j])$ be an element of $C_n(j)$. The property that all intervals $[a^i, b^i]$ have non-empty interiors can be described by the inequalities $a^i < b^i$. The disjointness of the interiors of different cubes is more difficult.

Let $C_n^{(i,k)}(j)$ be the space of tupels $([a^1, b^1], \ldots, [a^j, b^j]) \in I^{2nj}$ with non-empty interior for $1 \leq i < k \leq j$ such that $[a^i, b^i]$ and $[a^k, b^k]$ have disjoint interiors. Obviously we have

$$C_n(j) = \bigcap_{1 \leq i < k \leq j} C_n^{(i,k)}(j).$$

The cube $[a^i, b^i]$ defines $2n$ parts of I^n, which are of the form

$$\Big(\big[(0, \ldots, 0), (1, \ldots, a^i_l, \ldots, 1)\big]\Big) \text{ or } \Big(\big[(0, \ldots, b^i_l, 0), (1, \ldots, 1)\big]\Big),$$

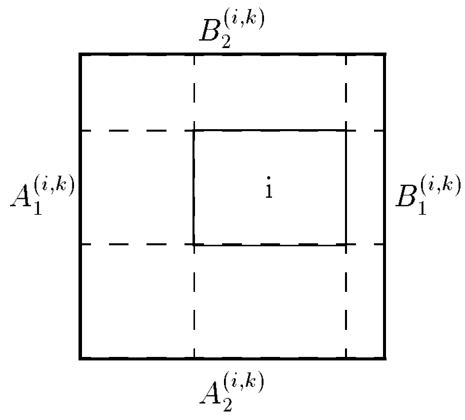


FIGURE 5. The $A_l^{(i,k)}$s and $B_l^{(i,k)}$s

whose union is the complement of the interior of $[a^i, b^i]$ (recall that $a^i = (a^i_1, \dots, a^i_n)$ and $b^i = (b^i_1, \dots b^i_n)$). Let $A_l^{(i,k)}(j)$ be the subspace of $C_n^{(i,k)}(j)$ such that $[a^k, b^k]$ lies in the l-th part of the first form and let $B_l^{(i,k)}(j)$ be the subspace of $C_n^{(i,k)}(j)$ such that $[a^k, b^k]$ lies in the l-th part of the second form.

$[a^k, b^k]$ and $[a^i, b^i]$ have disjoint interiors, if and only if $[a^k, b^k]$ lies in one of these parts. Hence $C_n^{(i,k)}(j)$ is the union of the $2n$ subspaces $A_l^{(i,k)}(j)$ and $B_l^{(i,k)}(j)$ of $\mathbb{R}^{2nj}$,

$$C_n(j) = \bigcap_{1 \le i < k \le j} \left(\bigcup_{1 \le l \le n} A_l^{(i,k)}(j) \cup B_l^{(i,k)}(j) \right).$$

Now we use this (quite complicated) description to obtain a closed (and hence compact) subset $\bar{C}_n(j)$ of I^{2nj}, which contains $C_n(j)$. We define

$$\bar{C}_n(j) = \bigcap_{1 \le i < k \le j} \left(\bigcup_{\le l \le n} \bar{A}_l^{(i,k)}(j) \cup \bar{B}_l^{(i,k)}(j) \right),$$

where $\bar{A}_l^{(i,k)}(j)$ is the set of all tupels $([a^1, b^1], \dots, [a^j, b^j])$ in I^{2nj} such that $a^i \le b^i$, i.e. the interiors are allowed to be empty, and $[a^k, b^k]$ lies in the l-th part of I^{2nj}, generated by $[a^i, b^i]$. $\bar{B}_l^{(i,k)}(j)$ is defined accordingly. Since these properties can be described by the inequalities $a^i_m \le b^i_m$ for $1 \le i \le j$ and $1 \le m \le n$, and either $b^k \le (1, \dots, a^i_l, \dots, 1)$ or $(0, \dots, b^i_l, \dots, 0) \le a^k$, these two spaces are closed in I^{2nj}.

REMARK 9.1. $\bar{C}_n(2)$ does not consist of all little n-cubes with arbitrary interior. For example the configuration in Figure 6 is not an element in $\bar{C}_2(2)$, since each of the intervals does not lie in one of the four parts defined by the other.

In $C_2(3)$ and $C_2(4)$ the same configuration can appear, since then we can split one or two of the intervals at their intersection.

In fact a tupel $\alpha = ([a^1, b^1], \dots, [a^j, b^j])$ of j intervals in I^n is an element of $A_l^{(i,k)}(j)$ if and only if the inequality $b^k_l \le a^i_l$ holds. And it is an element

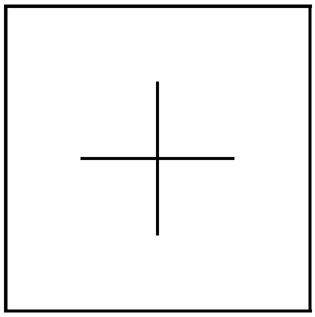


FIGURE 6. A non-example of a closed cube

of $B_l^{(i,k)}$ if and only if $b_l^i \leq a_l^k$. Hence α is an element of $C_n(j)$ if and only if there exists an $1 \leq l \leq n$ for each pair $1 \leq i < k \leq j$ such that either $b_l^k \leq a_l^i$ or $b_l^i \leq a_l^k$.

Now Let $\alpha = ([a^1, b^1], \dots, [a^j, b^j])$ be an element of $\bar{C}_n(j)$ and $\gamma_i = ([c^{1,i}, d^{1,i}], \dots, [c^{k,i}, d^{k,i}]), 1 \leq i \leq j$, elements of $\bar{C}_n(k_i)$. As in $\mathcal{C}_n$, we can define $\alpha \circ (\gamma_1, \dots, \gamma_k)$. It is not very hard to see that this is an element in $\bar{C}_n(k_1 + \dots + k_j)$. Therefore the $\bar{C}_n(j)$ form an operad $\bar{C}_n$, which contains C_n as a suboperad. We call $\bar{C}_n$ the *operad of closed n-cubes.*

As for C_n and C_m, we obtain a suboperad $\bar{C}_n|\bar{C}_m$ of $\bar{C}_{n+m}$, which contains $C_n|C_m$ as a suboperad. Again $\bar{C}_n|\bar{C}_m$ is given as the image of a morphism $\bar{C}_n \otimes \bar{C}_m \to \bar{C}_{n+m}$.

DEFINITION 9.2. Let

$$\alpha = \big([a^1, b^1], \dots, [a^j, b^j]\big) \text{ and } \beta = \big([c^1, d^1], \dots, [c^k, d^k]\big)$$

be two elements of $\bar{C}_n$. α is called a *frame* of β if there exists a surjective map $\varphi : \mathbf{k} \to \mathbf{j}$ such that

$$[c^i, d^i] \subset [a^{\varphi(i)}, b^{\varphi(i)}]$$

for all $i \in \mathbf{k}$. The map φ is called a *framing* of β into α.

DEFINITION 9.3. Let $\alpha \in \bar{C}_n(j)$ and $\alpha' \in \bar{C}_n(l)$ be two frames of $\beta \in \bar{C}_n(k)$. If α' if a frame of α, then α is called *tighter* than α'.

LEMMA 9.4. *Let $\alpha \in \bar{C}_n(j)$ and $\alpha' \in \bar{C}_n(j')$ be two frames of $\beta \in \bar{C}_n(k)$. Then there exists a frame $\alpha \cap \alpha'$ of β, which is tighter than α and α'.*

PROOF. Let α be of the form $(\dots[a^i, b^i]\dots)$ and α' of the form $(\dots[\bar{a}^i, \bar{b}^i]\dots)$ and β of the form $(\dots[c^i, d^i]\dots)$. Furthermore let φ_α and $\varphi_{\alpha'}$ be two framings of β into α and α'.

The intervals of $\alpha \cap \alpha'$ are all intervals of the form

$$[a^{\varphi_\alpha(i)}, b^{\varphi_\alpha(i)}] \cap [\bar{a}^{\varphi_{\alpha'}(i)}, \bar{b}^{\varphi_{\alpha'}(i)}]$$

for each $1 \leq i \leq k$. Then each interval $[c^i, b^i]$ of β is contained in the i-th intersection. The intervals of $\alpha \cap \alpha'$ can be ordered arbitrarily. In addition the map $(\varphi_\alpha, \varphi_{\alpha'}) : \mathbf{k} \to \mathbf{j} \times \mathbf{j}'$ implies a surjective map from $\mathbf{k}$ into its image. This map is a framing of β into $\alpha \cap \alpha'$ (the latter one has as many inputs as the image has elements). The maps $\mathbf{k} \to \mathbf{j} \times \mathbf{j}' \to \mathbf{j}$ and $\mathbf{k} \to \mathbf{j} \times \mathbf{j}' \to \mathbf{j}'$ induce framings of $\alpha \cap \alpha'$ into α and α'. $\square$

LEMMA 9.5. *Let $\alpha \in \bar{C}_n(j)$ and $\alpha' \in \bar{C}_n(j')$ be two frames of β such that α is tighter than α' and vice versa. Then α and α' coincide up to a permutation, i.e. there exists a permutation τ such that $\alpha\tau = \alpha'$.*

PROOF. Let $\varphi : \mathbf{j} \to \mathbf{j}'$ be a framing of α into α' and $\psi : \mathbf{j}' \to \mathbf{j}$ a framing of α' into α. Since both maps are surjective, their compositions are. This again implies that $\varphi\psi$ and $\psi\varphi$ are bijective and that $j = j'$.

Let $\sigma \in \Sigma_j$ be the map $\psi\varphi$. We know $\sigma^{j!} = \mathrm{id}$ and furthermore

$$[c^i, d^i] \subset [a^{\varphi(i)}, b^{\varphi(i)}] \subset [c^{\sigma(i)}, d^{\sigma(i)}] \subset \cdots \subset [c^{\sigma^{j!}(i)}, d^{\sigma^{j!}(i)}] = [c^i, d^i],$$

where $[c^i, d^i]$ is the i-th interval of α and $[a^i, b^i]$ the one of α'. The i-th interval of α is precisely the $\varphi(i)$-th interval of α'. Since φ is bijective the statement follows. $\square$

Obviously we have

PROPOSITION 9.6. *If $\beta \in \bar{C}_n(j)$ is of the form $\alpha \circ (\gamma_1, \ldots, \gamma_k)$ with $\alpha \in \bar{C}_n(k)$ such that each γ_i has at least one input, then α is a frame of β.*

LEMMA 9.7. *Let $\alpha \in \bar{C}_n(1)$ be a frame of $\beta \in \bar{C}_n(j)$. Then there exists a $\beta' \in \bar{C}_n(j)$ such that $\beta = \alpha \circ \beta'$.*

PROOF. Let $[a, b]$ be the only interval of α and let $[c^i, d^i], 1 \le i \le j$, be the i-th interval of β. We define

$$\bar{c}^i_l = \begin{cases} \frac{c^i_l - a_l}{b_l - a_l} & \text{if } b_l \neq a_l \\ \frac{i-1}{j} & \text{if } b_l = a_l \end{cases} \text{ and } \bar{d}^i_l = \begin{cases} \frac{d^i_l - a_l}{b_l - a_l} & \text{if } b_l \neq a_l \\ \frac{i}{j} & \text{if } b_l = a_l, \end{cases}$$

for $1 \le i \le j$ and $1 \le l \le n$. Now let β' be given by $([\bar{c}^1, \bar{d}^1], \ldots, [\bar{c}^j, \bar{d}^j])$. We have to check, that this sequence of intervals is a complete cube.

Choose $\le i < k \le j$. Following Remark 9.1 we have to find $1 \le l \le n$ such that either $\bar{d}^k_l \le \bar{c}^i_l$ or $\bar{d}^i_l \le \bar{c}^k_l$. We know that there exists an l such that either $d^k_l \le c^i_l$ or $d^i_l \le c^k_l$ holds. If $a_l \neq b_l$ we are done. Otherwise we have two cases. In the first, $k \le i - 1$, we have

$$\bar{d}^k_l = \frac{k}{j} \le \frac{i-1}{j} = \bar{c}^i_l.$$

For $i + 1 \le k$ we have

$$\bar{d}^i_l = \frac{i}{j} \le \frac{k-1}{j} = \bar{c}^k_l.$$

$\square$

COROLLARY 9.8. *Let $\alpha \in \bar{C}_n(k)$ be a frame of $\beta \in \bar{C}_n(j)$. Then there exist $\beta'_i \in \bar{C}_n, 1 \le i \le k$ such that $\beta = \alpha \circ (\beta'_1, \ldots, \beta'_k)$.*

PROOF. Let φ be a framing of β into α. Let $I_i \subset \{1, \ldots, j\}$ be the preimage of $i \in \{1, \ldots, k\}$ of under φ. Then we can kill all inputs of β, except for the inputs whose label is in I_i, by composition with stumps. We obtain $\beta_i \in \bar{C}_n$. Furthermore the i-th interval $[a^i, b^i]$ of α is a frame of β_i. By Lemma 9.7 exists a β'_i such that $\beta_i = ([a^i, b^i]) \circ \beta'_i$. This implies

$$\beta = \alpha \circ (\beta'_1 \ldots, \beta'_k).$$

$\square$

LEMMA 9.9. *Let $\alpha \in \bar{C}_n(k)$ and $\beta \in \bar{C}_n(j)$ and $([a,b]) \in \bar{C}_n(1)$ such that $([a,b]) \circ \alpha$ is a frame of $([a,b]) \circ \beta$. Then there exists a β' such that α is a frame of β' and such that*

$$([a,b]) \circ \beta = ([a,b]) \circ \beta'.$$

PROOF. Let $[a^i, b^i]$ be the i-th interval of α and $[c^i, d^i]$ the i-th interval of β and φ a framing of $([a,b]) \circ \beta$ into $([a,b]) \circ \alpha$. We chose β' to be the tupel $([\bar{c}^1, \bar{d}^1], \dots, [\bar{c}^j, \bar{d}^j])$ with

$$\bar{c}^i_l = \begin{cases} c^i_l & \text{if } a_l \neq b_l \\ \frac{a^{\varphi(i)}_l + b^{\varphi(i)}_l}{2} & \text{if } a_l = b_l \end{cases} \text{ and } \bar{d}^i_l = \begin{cases} d^i_l & \text{if } a_l \neq b_l \\ \frac{a^{\varphi(i)}_l + b^{\varphi(i)}_l}{2} & \text{if } a_l = b_l, \end{cases}$$

for $1 \leq i \leq j$ and $1 \leq l \leq n$.

First we prove, that β is in $\bar{C}_n(j)$. Let $1 \leq i < k \leq j$. Since $\beta \in \bar{C}_n(j)$, we know that there exists a $1 \leq l \leq n$ such that either $d^i_l \leq c^k_l$ or $d^k_l \leq c^i_l$. If $a_l \neq b_l$, we are done. If $a_l = b_l$ we have $\bar{c}^i_l = \bar{d}^i_l$ and $\bar{c}^k_l = \bar{d}^k_l$. Hence one of the necessary inequalities holds.

Now let φ be the framing of $([a,b]) \circ \beta$ into $([a,b]) \circ \alpha$. Then the inequality

$$a_l + (b_l - a_l)a^{\varphi(i)}_l \leq a_l + (b_l - a_l)c^i_l \leq a_l + (b_l - a_l)d^i_l \leq a_l + (b_l - a_l)b^{\varphi(i)}_l$$

holds for each $1 \leq l \leq n$ and $1 \leq i \leq j$. If $a_l \neq b_l$ this immediately leads to

$$a^{\varphi(i)}_l \leq c^i_l \leq d^i_l \leq b^{\varphi(i)}_l.$$

If $b_l = a_l$ we have

$$a^{\varphi(i)}_l \leq \bar{c}^i_l = \bar{d}^i_l \leq b^{\varphi(i)}_l.$$

Thus α is a frame of β' and φ is a framing of β' into α.

The fact that $([a,b]) \circ \beta$ is equal to $([a,b]) \circ \beta'$ is easy to see. □

Together with Corollary 9.8 this leads to

COROLLARY 9.10. *Let α and β be two elements of $\bar{C}_n$ such that $([a,b]) \circ \alpha$ is a frame of β. Then there exists a β' such that α is a frame of β' and $\beta = ([a,b]) \circ \beta$.*

10. Reduced representations

Obviously the map $\bar{C}_n \sqcup \bar{C}_m(j) \to \bar{C}_n \otimes \bar{C}_m(j)$ is a surjection for each $j \in \mathbb{N}$ and the map $\bar{C}_n \otimes \bar{C}_m(j) \to \bar{C}_n|\bar{C}_m(j)$ is surjective by definition. Therefore every element of $\bar{C}_n|\bar{C}_m(j)$ and every element of $\bar{C}_n \otimes \bar{C}_m(j)$ can be represented by an element of $(\bar{C}_n, \bar{C}_m)_T$ with T a labeled, bi-colored tree with j inputs.

REMARK 10.1. In the following we denote an element of $F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ and the trees underlying its representations with the same name. It should be clear from the context whether the vertex labels are of importance.

DEFINITION 10.2. A labeled, bi-colored tree with $j > 1$ inputs is *reduced*, if

- it contains no monochrome edge,
- it contains no vertex of valence 0 and

- it contains no sequence of valence 1, i.e. there is no subtree with more than two vertices which all have valence 1.

A tree with 0 inputs is reduced if and only if it is a stump, and a tree with one input is reduced if it contains at most two vertices of valence 1 and different colors.

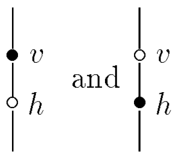
There are only finitely many reduced trees with j inputs. The maximal number of vertices a reduced tree with j inputs can have, is given by the number of vertices of a binary tree with j inputs, plus the number of all edges (split an edge by one vertex of valence 1), i.e. $(j-1)+(2j-1)$.

LEMMA 10.3. *For each $c \in \bar{C}_n|\bar{C}_m(j)$ there exists a reduced tree T with j inputs and an representation $T_c \in (\bar{C}_n, \bar{C}_m)_T$ of c.*

PROOF. For $j = 0$ the statement is trivial, since $\bar{C}_{n+m}(0)$ consists only of one point. For a given representation $S_c \in (\bar{C}_n, \bar{C}_m)_S$ of $c \in \bar{C}_n|\bar{C}_m(j)$ for $j \geq 1$, we construct a reduced representation T_c. If S contains monochrome edges, we can shrink them them by composing the labels at their vertices. Hence we can replace S_c with a representation which contains no monochrome edge.

Now assume that S_c contains no monochrome edge. Since the images of vertices of valence 0 of both colors coincide in $\bar{C}_{n+m}(0)$, their colors can be changed without affecting the image of the tree. Hence all outgoing edges of a vertex of valence 0 can be assumed to be monochrome. Therefore we can shrink them by composing their vertices. This kills one input of the root of the corresponding edge and the stump.

Now we assume that S_c contains no monochrome edge and no stump. It is easy to see that the two trees



represent the same element in $\bar{C}_{n+m}(1)$. Therefore we can change the order in a sequence of valence 1 arbitrarily. Thus we can sort them by color and then shrink the obtained monochrome edges. Hence we can assume that each sequence of valence 1 consists only of two vertices of different colors. For $j = 1$ we are done now. For $j > 1$, this sequence is connected to another vertex of arbitrary color (either at the input or at the output). If the connecting edge is not monochrome, we exchange the two vertices of valence 1 and obtain at least one monochrome edge, which again can be shrunk. This last step kills (at least) one of the two vertices of valence 1. $\square$

COROLLARY 10.4. *$\bar{C}_n|\bar{C}_m(j)$ is the union of finitely many compact subspaces and hence compact.*

PROOF. For each reduced tree T with j inputs, the space $(\bar{C}_n, \bar{C}_m)_T$ is compact, because it is a product of compact spaces. Therefore its image K_T in $\bar{C}_n|\bar{C}_m(j)$ is compact. Since each element is represented by a reduced tree, $\bar{C}_n|\bar{C}_m(j)$ is the union of the finitely many K_T. $\square$

In the same way we get

COROLLARY 10.5. *$\bar{C}_n \otimes \bar{C}_m(j)$ is compact.*

Since there exists a continuous morphism $\bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$, two elements of $F(\bar{C}_n \sqcup_\Sigma \bar{C}_M)$, which represent the same element in $\bar{C}_n \otimes \bar{C}_m$, represent the same element in $\bar{C}_n|\bar{C}_m$. For the proof of the converse situation, we construct ¨minimal¨ representations.

11. Minimal representations

DEFINITION 11.1. A *$\circ$-representation* of $\alpha \in \bar{C}_n|\bar{C}_m(j)$, $j > 0$, is a representation of α in $(\bar{C}_n, \bar{C}_m)_T$ such that the root vertex of T has the color $\circ$ (or 0). Similarly a *$\bullet$-representation* is a representation of α, whose root has the color $\bullet$ (or 1). If there exists a $\circ$-representation of α, with $h \in \bar{C}_n$ as root, then h is called a *$\circ$-root* of α.

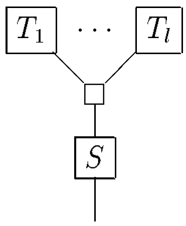
DEFINITION 11.2. A *$\circ$-frame* of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ is an element $h \in \bar{C}_n$ such that $h|\mathrm{id}$ is a frame of α. Similarly a *$\bullet$-frame* of α is an element v of $\bar{C}_m$ such that $\mathrm{id}|v$ is a frame of α.

The following lemma is a consequence of the proof of Lemma 9.4.

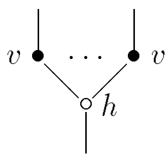
LEMMA 11.3. *If h and h' are $\circ$-frames of α, then the intersection $h|id \cap h'|id$ is given by a $\circ$-frame $h \cap h'$.*

LEMMA 11.4. *For $\alpha \in \bar{C}_n|\bar{C}_m(j)$ with $j > 1$, exists a $\circ$-root $h \in \bar{C}_n(k)$ or a $\bullet$-root $v \in \bar{C}_m(k)$ with $k > 1$.*

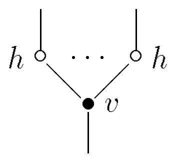
PROOF. If α has more than one input, then there exists a reduced representation (either $\circ$ or $\bullet$), which has at least one vertex of a valence higher than 1. In general it is of the form



where $\square$ is either $\circ$ or $\bullet$. Since we know that the trees



and



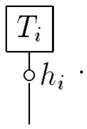
represent the same element in $\bar{C}_n|\bar{C}_m$, we can push the lowest vertex of valence > 1 down to the root. $\square$

LEMMA 11.5. *$h \in \bar{C}_n(k)$ is a $\circ$-frame of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ if and only if it is a $\circ$-root.*

PROOF. By Proposition 9.6 each $\circ$-root of α is a $\circ$-frame.

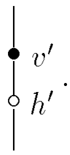
Now let $h \in \bar{C}_n(k)$ be a $\circ$-frame of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ and φ a framing of α into $h|\mathrm{id}$. Assume that the statement is true for $k = 1$. Then for $k > 1$ we can kill all inputs of α, which do not lie in the i-th input of $h|id$, i.e. all $l \in \mathbf{j}$ with $\varphi(j) \neq i$. We obtain $\alpha_i \in \bar{C}_n|\bar{C}_m$, which is framed by $h_i := ([a^i, b^i])$, where

$[a^i, b^i]$ is the i-th interval of h. Therefore we can find a $\circ$-representation of α_i with the root h_i, i.e. it is of the form



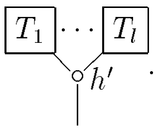
If β_i is the cube represented by T_i, we obtain $(h|\mathrm{id}) \circ (\beta_1, \ldots, \beta_k)$. Therefore h is a $\circ$-root of α.

We still have to prove the theorem for $k = 1$. For $j = 1$ the statement is quite obvious, since each reduced $\circ$-representation is of the form



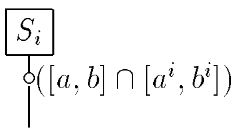
If h is a $\circ$-frame of α, it is obviously a frame of h', and by Lemma 9.7 there exists a $h'' \in \bar{C}_n$ with $h' = h \circ h''$.

For $j > 1$ and $k = 1$ we have to use the fact that there exists at least one representation ($\circ$ or $\bullet$), whose root has a valence greater than 1 (cmp. Lemma 11.4). Let it be of the form



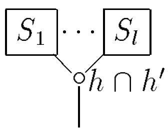
If h is a frame of h', then we are done, since by Corollary 9.8 there exists an $h'' \in \bar{C}_n(l)$ such that $h' = h \circ h''$.

If h is not a frame of h', we consider $\alpha_i := ([a^i, b^i])|\mathrm{id} \circ \beta_i$ for each $1 \le i \le l$, where $[a^i, b^i]$ is the i-th interval of h' and β_i is the cube represented by T_i. Since α_i can be obtained from α by the composition with stumps at all inputs, which do not belong to T_i, it is an element of $\bar{C}_n|\bar{C}_m(j_i)$, where $1 \le j_i < j$. By induction there exists a $\circ$-representation of α_i, of the form



because $([a, b] \cap [a^i, b^i])$ is a $\circ$-frame of α_i.

The $\circ$-frame $h \cap h'$ of α consist of the intervals $[a, b] \cap [a^i, b^i]$, and hence α is represented by



Since h is a frame of $h \cap h'$, we can find a $h'' \in \bar{C}_n(l)$ with $h \cap h' = h \circ h''$.

If the root of valence greater than 1 has the color $\bullet$, we can proof the statement by similar means. $\square$

LEMMA 11.6. *Let* $\alpha \in \bar{C}_n|\bar{C}_m(j), h \in \bar{C}_n(k), H' \in \bar{C}_n(k')$ *and* $[a, b] \subset I^n$ *an interval such that*

- h *is a* $\circ$*-frame of* α*,*

- *$([a,b])\circ h$ is a frame of H' and*
- *H' is a $\circ$-frame of $([a,b])\circ\alpha$.*

Then there exists $h'\in\bar{C}_n(k')$ such that

- *$H'=([a,b])\circ h'$,*
- *h is a frame of h'.*
- *h' is a $\circ$-frame of α.*

Proof. Let $\varphi:\mathbf{j}\to\mathbf{k}$ be a framing of α into $h|\mathrm{id}$ and hence from $([a,b])|\mathrm{id}\circ\alpha$ into $(([a,b])\circ h)|\mathrm{id}$ and $\Phi':\mathbf{j}\to\mathbf{k}'$ a framing from $([a,b])|\mathrm{id}\circ\alpha$ into $H'|\mathrm{id}$ and $\psi:\mathbf{k}'\to\mathbf{k}$ a framing of H' into $([a,b])\circ h$. We can assume that $\psi\circ\Phi'=\varphi$.

We define $h'\in\bar{C}_n(k')$ to be the tupel $(\ldots[\bar{c}^i,\bar{d}^i]\ldots)$ with

$$\bar{c}^i_l=\begin{cases}\frac{c^i_l-a_l}{b_l-a_l} & \text{if } a_l\neq b_l\\ min\big(a^s_l : s\in\Phi'^{-1}(i)\big) & \text{if } a_l=b_l\end{cases}$$

and

$$\bar{d}^i_l=\begin{cases}\frac{d^i_l-a_l}{b_l-a_l} & \text{if } a_l\neq b_l\\ max\big(b^s_l : s\in\Phi'^{-1}(i)\big) & \text{if } a_l=b_l,\end{cases}$$

for $1\le l\le n$, where $[a^i,b^i]$ is the i-th interval of α and $[c^i,d^i]$ the i-th interval of H'.

First we prove that h' is an element of $\bar{C}_n(k')$. Let $1\le i<h\le k'$. If $[\bar{a}^i,\bar{b}^i]$ is the i-th interval of h, then there exists an l with $1\le l\le n$ such that either $\bar{a}^{\psi(i)}_l\ge\bar{b}^{\psi(h)}_l$ or $\bar{a}^{\psi(h)}_l\ge\bar{b}^{\psi(i)}_l$. Since ψ is a framing of H' into $([a,b])\circ h$, we know

$$a_l+(b_l-a_l)\bar{a}^{\psi(i)}_l\le c^i_l\le d^i_l\le a_l+(b_l-a_l)\bar{b}^{\psi(i)}_l.$$

If $b_l\neq a_l$, this immediately leads to

$$\bar{a}^{\psi(i)}_l\le\bar{c}^i_l\le\bar{d}^i_l\le\bar{b}^{\psi(i)}_l.$$

The same inequality follows for h instead of i. Together they imply that either $\bar{c}^i_l\ge\bar{d}^h_l$ or $\bar{c}^h_l\ge\bar{d}^i_l$ holds.

If $a_l=b_l$, we use the fact that $[a^s,b^s]\subset[\bar{a}^i,\bar{b}^i]$ and hence

$$\bar{a}^i_l\le a^s_l\le b^s_l\le\bar{b}^i_l$$

for all $s\in\Phi^{-1}(i)$. Since $\Phi=\psi\Phi'$ we also have $\Phi'^{-1}(i)\subset\Phi^{-1}(\psi(i))$, and therefore

$$\bar{c}^i_l=min\big(a^s_l : s\in\Phi'^{-1}(i)\big)\ge min\big(a^s_l : s\in\Phi^{-1}(\psi(i))\big)\ge\bar{a}^{\psi(i)}_l$$

and

$$\bar{d}^i_l=max\big(b^s_l : s\in\Phi'^{-1}(i)\big)\le max\big(b^s_l : s\in\Phi^{-1}(\psi(i))\big)\le\bar{b}^{\psi(i)}_l.$$

This implies that either $\bar{c}^i_l\ge\bar{a}^{\psi(i)}_l\ge\bar{b}^{\psi(h)}_l\ge\bar{b}^h_l$ or $\bar{c}^h_l\ge\bar{a}^{\psi(h)}_l\ge\bar{b}^{\psi(i)}_l\ge\bar{d}^i_l$. From these observations also follows, that ψ is a framing of h' into h.

To prove that h' is a $\circ$-frame of α with framing Φ', we have to check that for each $1\le l\le n$ and $1\le i\le j$, the inequality

$$\bar{c}^{\Phi'(i)}_l\le a^i_l\le b^i_l\le\bar{d}^{\Phi'(i)}_l$$

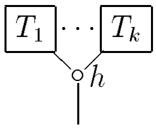
is true. If $a_l \neq b_l$ this follows from

$$c_l^{\Phi'(i)} \leq a_l + (b_l - a_l)a_l^i \leq a_l + (b_l - a_l)b_l^i \leq d_l^{\Phi'(i)},$$

which again holds, because Φ' is a framing of $H'|\mathrm{id}$ to $([a,b]) \circ \alpha$. For $a_l = b_l$ the inequality is fulfilled since $i \in \Phi'^{-1}(\Phi(i))$.

It remains to check, that $H' = ([a,b]) \circ h'$. But this is an immediate consequence of the definition of h'. $\square$

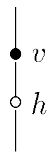
THEOREM 11.7. *For each $\alpha \in \bar{C}_n|\bar{C}_m(j)$ there exists a (up to permutations) uniquely determined $\circ$-frame $h \in \bar{C}_n(k)$ of α and a $\circ$-representation of the form*



such that each $\circ$-frame h' of α is also a frame of h.

PROOF. We prove the theorem via induction over the number j of inputs of α. For $j = 0$ the only reduced $\circ$-representation is the stump of color $\circ$. Hence the theorem holds trivially.

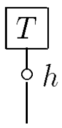
For $j = 1$ the reduced $\circ$-representation of α is uniquely determined and of the form



.

Obviously h is tighter than any other $\circ$-frame of α.

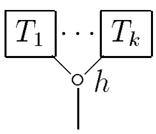
If α has more than one input we have to differentiate between two basic cases. First let all $\circ$-representations of α have a root of valence 0, i.e. every $\circ$-representation is of the form



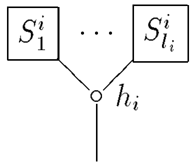
For each $1 \leq i \leq k$ we can chose an interval $h^i = ([c^i, d^i]) \in \bar{C}_n(1)$ such that $d_i^i - c_i^i$ is minimal under all $\circ$-roots. (Intervals of this kind exist, since $\bar{C}_n|\bar{C}_m(j)$ is a Hausdorff-space and since $(\bar{C}_n, \bar{C}_m)_T$ is compact for each bicolored, labelled tree T.) Each h^i is a $\circ$-frame of α. Hence their intersection h is a $\circ$-frame and therefore a $\circ$-root of α.

If h' is another $\circ$-frame of α, then the intersection $h'' := h \cap h'$ would be a frame, which is tighter than h and h'. Hence there exists a $\circ$-representation with h'' as root. Since h is the intersection of ¨minimal¨ roots h^i, this implies $h'' = h$, because otherwise there has to exists a coordinate such that h'' is ¨smaller¨ in the i-th direction than the according h^i.

In the second case we assume that there exists a reduced $\circ$-representation of α of the form

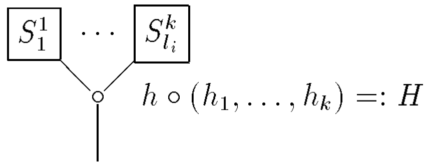


for $k > 1$. Then each T_i has k_i inputs with $1 \leq k_i < j$. By induction we can find $h_i \in \bar{C}_n(l_i)$ and $\circ$-representations of $\beta_i \in \bar{C}_n|\bar{C}_m(k_i)$, represented by T_i, of the form



such that every $\circ$-frame of β_i is wider than h_i.

Together these form a $\circ$-representation



of α.

Now let H' be another frame of α. Without restriction we can assume that H' is tighter than H (replace it with $H \cap H'$). By composition with stumps, we can kill all inputs of α, H and H', which are represented by inputs on another subtree than T_i. We obtain $\alpha_i \in \bar{C}_n|\bar{C}_m(k_i)$ and two $\circ$-frames H_i and H_i' such that H_i' is tighter than H_i. The cube α_i is the composition $([a^i, b^i])|\mathrm{id} \circ \beta_i$ and H_i the composition $([a^i, b^i]) \circ h_i$, where $[a^i, b^i]$ is the i-th interval of h. By Lemma 11.6 there exists an $h_i' \in \bar{C}_n$ such that

- $H_i' = ([a^i, b^i]) \circ h_i'$,
- h_i' is a $\circ$-frame of β_i and
- h_i is a frame of h_i'.

The second property implies that h_i' is a frame of h_i. Together with the third property and Lemma 9.5 this implies that h_i and h_i' coincide up to permutation. Hence we can assume that they are equal. Therefore we have $H_i = H_i'$. This again implies that H and H' are equal up to a permutation. Hence H is a minimal $\circ$-root.

The uniqueness of H is an immediate consequence of Lemma 9.5. $\square$

DEFINITION 11.8. We call the (up to permutation) unique root of Theorem 11.7 the *minimal $\circ$-root*. We define a *minimal $\bullet$-root* analogously.

DEFINITION 11.9. A reduced representation T of $\alpha \in \bar{C}_n|\bar{C}_m$ is called *minimal*, if every vertex is a minimal root of the element represented by the subtree with the vertex as root.

The algorithm for the construction of a minimal representation is quite clear. We choose the color of the root, construct the minimal root of this color, and then recursively construct the minimal representations of the subtrees whose root has the other color. Since the minimal roots are uniquely determined (up to permutation) we obtain the following

PROPOSITION 11.10. *There exists an (up to permutations) uniquely determined minimal $\circ$-representation for each $\alpha \in \bar{C}_n|\bar{C}_m(j)$-*

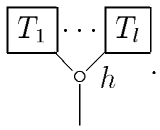
REMARK 11.11. With ¨up to permutations¨ we mean one permutation for each vertex of the tree.

THEOREM 11.12. *The images of an arbitrary reduced* $\circ$*-representation* $T \in F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ *of* $\alpha \in \bar{C}_n|\bar{C}_m(j), j > 0$, *and the minimal* $\circ$*-representation under the projection* $p : F(\bar{C}_n \sqcup_\Sigma \bar{C}_m) \to \bar{C}_n \otimes \bar{C}_m$ *coincide.*

PROOF. First recall, that the application of a permutation to a vertex of a tree in $F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ does not change the image under p. Hence we can ignore ambiguities which occur when choosing permutations.

For $j = 1$ the reduced $\circ$-representation is minimal. Hence the statement is trivial.

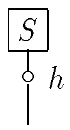
If α has more than one input and its minimal $\circ$-root h_{min} has more than one input, then, as seen in the proof of Theorem 11.7, h_{min} is the composition of the root h of T and the minimal $\circ$-roots h_i of the $\beta_i := pr(T_i)$, where T is of the form



Since each T_i has at least one input and less than j, the statement follows by induction.

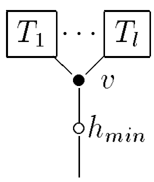
If α has more than one input and its $\circ$-root only has one input, then its $\bullet$-root has more than one input (follows from Lemma 11.4). As above we can prove that each reduced $\bullet$-representation has the same image as the minimal $\bullet$-representation, if the minimal $\bullet$-root has more than one input.

T has to be of the form

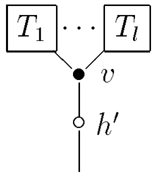


where S is a $\bullet$-representation of an element $\beta \in \bar{C}_n|\bar{C}_m(j)$. The minimal $\bullet$-root of T has more than one input (otherwise the minimal $\circ$-root of T has to have more than one). Thus $p(S) = p(S_{min})$, where S_{min} is the minimal $\circ$-representation of β.

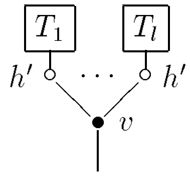
Let the minimal $\circ$-representation T_{min} of α be of the form



with $l > 1$. We know that h is a frame of the minimal $\circ$-root h_{min} of α. By Corollary 9.8 we can find an $h' \in \bar{C}_n(1)$ such that $h_{min} = h \circ h'$. Thus the tree



is a representation of β. Since the interchange condition holds, the image of this tree coincides with the image of the tree T', which is of the form



which is a $\bullet$-representation of β. Hence we now that it image under p coincides with the one of S_{min} and hence with the one of S. Since p is a map of operads, it follows that the images of T and T_{min} coincide. $\square$

The construction of a reduced from an arbitrary representation in Lemma 10.3 shows that their image under p coincide. Hence we obtain the following

COROLLARY 11.13. *Let T and S be two $\circ$-representations of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ for $j < 0$. Then $p(T) = p(S) \in \bar{C}_n \otimes \bar{C}_m(j)$.*

12. The tensor product of little cubes

Now we use the minimal representations, to construct a homeomorphism between the two operads $\bar{C}_n|\bar{C}_m$ and $\bar{C}_n \otimes \bar{C}_m$. We then show, that $C_n \otimes C_m$ is locally Σ-equivalent to $\mathcal{C}_{n+m}$. One direction of the homeomorphism, namely $\bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$, is already known. The minimal representations make it possible to construct an inverse map.

THEOREM 12.1. *The morphism $\varphi : \bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$ is a homeomorphism of operads.*

PROOF. First we construct an inverse map $\psi_j : \bar{C}_n|\bar{C}_m(j) \to \bar{C}_n \otimes \bar{C}_m(j)$ for each $j \geq 0$. For $j = 0$ the map is trivial, since both spaces are. For $j > 0$ we choose ψ to be given by $\psi(x) := p(T)$ where $T \in F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ is a $\circ$-representation of x, and $p : F(\bar{C}_n \sqcup_\Sigma \bar{C}_m) \to \bar{C}_n \otimes \bar{C}_m$ and $q : F(\bar{C}_n \sqcup_\Sigma \bar{C}_m) \to \bar{C}_n|\bar{C}_m$ are the projections. Let T and T' be two $\circ$-representations of $x \in \bar{C}_n|\bar{C}_m(j)$. By Corollary 11.13 this implies $p(T) = p(T')$ and hence ψ is well-defined for each j.

Since $\varphi \circ p = q$ holds, we have

$$\psi \circ \varphi \circ p(T) = \psi \circ q(T) = p(T)$$

for each $\circ$-representation T. Every element of $\bar{C}_n \otimes \bar{C}_M$ has a $\circ$-representation and this implies $\psi \circ \varphi = \mathrm{id}$. On the other hand we have

$$\varphi \circ \psi \circ q(T) = \varphi \circ p(T) = q(T).$$

which leads to $\varphi \circ \psi = \mathrm{id}$. Hence φ and ψ are bijective maps of set operads.

It remains to prove that ψ is continuous. By Corollary 10.5 $\bar{C}_n \otimes \bar{C}_m(j)$ is compact. Since φ is continuous and bijective and $\bar{C}_n|\bar{C}_m(j)$ is a Hausdorff-space, φ is a homeomorphism. $\square$

Since $\varphi : \bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$ maps non-degenerated cubes, i.e. elements of $C_n \otimes C_m$, surjectively to non-degenerate cubes, i.e. elements of $C_n|C_m$, we obtain

COROLLARY 12.2. *$\varphi : C_n \otimes C_m \to C_n|C_m$ is a homeomorphism of operads.*

Up to this point we just examined tensor products of two little cubes operads. But a look at the results of the preceding sections reveals, that it is possible to adapt the proof to the tensor product of three or more little cubes. In the following we just give a short overview over the necessary changes.

In the combinatorial part, i.e. the construction of the minimal representations, we just have to use multi-colored trees instead of trees with only two colors, i.e. we need one color i for each factor C_{n_i} in the tensor product $C_{n_1} \otimes \cdots \otimes C_{n_l}$. In addition we have to modify the notion of reduced representations. They still are not allowed to have monochrome edges and vertices of valence 0. But they are allowed to have sequences of valence 1 of a length less than l such that each vertex of this sequence has a different color. As in the bicolored case we can find a reduced representation for each element of $\bar{C}_{n_1}|\ldots|\bar{C}_{n_l} \subset \bar{C}_n$ with $n = n_1 + \cdots + n_l$.

With this modification, the results of sections 10 and 11 remain valid. We just have to take the increased number of colors into account. Basically this results in more bookkeeping. But we still obtain

COROLLARY 12.3. *The map $\varphi : C_{n_1} \otimes \cdots \otimes C_{n_l} \to C_{n_1}|\ldots|C_{n_l}$ is a homeomorphism of operads for each $l \geq 2$ and each choice $n_1, \ldots, n_l \in \mathbb{N}$*

In the following we use the suboperad of decomposable cubes $D_n(j) \subset C_n(j)$, to obtain our final result.

LEMMA 12.4. *$D_n|D_m(j)$ is precisely the space $D_{n+m}(j)$.*

PROOF. For $j = 0, 1$ the spaces $D_n(j)$ and $C_n(j)$ coincide. Hence the equality of $D_n|D_m(j)$ and $D_{n+m}(j)$ follows directly from the fact that $C_n|C_m(j) = C_{n+m}(j)$ for $j = 0, 1$.

An element α of $C_{n+m}(j), j \geq 2$, is decomposable, if and only if there exists an $i \in \{1, \ldots, n+m\}$ and a hyper plane orthogonal to the i-th axis, which separates α into two non-trivial parts (i.e. parts with at least on input each). This is equivalent to the existence of $\beta \in C_{n+m}(2)$ of the form

$$\beta = \Big(\big[(0, \ldots, 0), (1, \ldots, \underset{i}{r}, \ldots, 1)\big], \big[(0, \ldots, \underset{i}{r}, \ldots, 0), (1, \ldots, 1)\big] \Big)$$

and $\beta_k \in D_{n+m}(j_k)$ for $k = 1, 2$ such that $1 \leq j_k < j$ and $\alpha = \beta \circ (\beta_1, \beta_2)$. Obviously β is an element of $C_n|C_m(2)$. It even is of one of the forms $h|\mathrm{id}$ or $\mathrm{id}|v$ with $h \in D_n(2) = C_n(2)$ or $v \in D_m(2) = C_m(2)$, depending whether i is less or equal to n or not. Hence we see, by induction over the number of inputs of α, that $D_{n+m}(j)$ is a subspace of $D_n|D_m(j)$ for each j.

On the other hand each element of the form $h|\mathrm{id}$ with $h \in D_n(j)$ is decomposable in $C_{n+m}(j)$ and the same holds for $\mathrm{id}|v$ for $v \in D_m(j)$. This implies that each element of $D_n|D_m(j)$ is a composition of decomposable elements in $C_{n+m}(j)$ and hence itself decomposable, what again leads to $D_n|D_m(j) \subset D_{n+m}(j)$ for each j. □

COROLLARY 12.5. *$D_{n_1}|\ldots|D_{n_l}(j) \subset C_{n_1}|\ldots|C_{n_l}(j)$ is precisely the space $D_n(j) \subset C_n(j)$ for all $j, n_1, \ldots, n_l \in \mathbb{N}, l \geq 2$ and $n = n_1 + \cdots + n_l$.*

LEMMA 12.6. *For each $j \in \mathbb{N}, l \geq 2$ and all $n_1, \dots, n_l \in \mathbb{N}$ the space $D_{n_1}|\dots|D_{n_l}(j)$ is a Σ_j-equivariant deformation retract of $C_{n_1}|\dots|C_{n_l}(j)$.*

PROOF. Since $D_n(j) = C_n(j)$ for all n and $j = 0, 1, 2$, the statement is trivial in this cases.

Now let $n = n_1 + \dots + n_l$. Following [**Dun88**, Lem. 2.2.] there exists an equivariant deformation retraction $h : I \times C_n(j) \to C_n(j)$ of $C_n(j)$ to $D_n(j)$ and a map $u : C_n \to I$ such that

$$h(s, \alpha) = \alpha \circ \big(\beta_{su(\alpha)}, \dots, \beta_{su(\alpha)}\big),$$

where $\beta_t \in C_n(1)$ is of the form

$$\beta_t = \left(\left[\frac{1}{2}t, 1 - \frac{1}{2}t\right]^n\right).$$

h maps $C_{n_1}|\dots|C_{n_l}(j)$ into itself, because $C_{n_1}|\dots|C_{n_l}(1) = C_n(1)$. Together with the equality of $D_{n_1}|\dots|D_{n_l}(j)$ and $D_n(j)$, this implies the statement. $\square$

Putting together all the collected pieces, we obtain the diagram

$$\begin{tikzcd}
C_{n_1} \otimes \dots \otimes C_{n_l}(j) \arrow[r] \arrow[dd, "\varphi"'] & C_n(j) \\
 & D_n(j) \arrow[u, hook, "\simeq"'] \arrow[d, equal] \\
C_{n_1}|\dots|C_{n_l}(j) \arrow[uur] & D_{n_1}|\dots|D_{n_l}(j) \arrow[l, "\simeq"]
\end{tikzcd}$$

for each $j \geq 0$. Since all maps, except for the diagonal and the map at the top, are known to be either homeomorphisms or local Σ-equivalences, we obtain

MAIN THEOREM. *The operad-map $C_{n_1} \otimes \dots \otimes C_{n_l} \to C_n$ is a local Σ-equivalence for all $l \geq 2, n_1, \dots, n_l \in \mathbb{N}$ and $n = n_1 + \dots + n_l$.*

Homotopy Algebras and Lax Operads

In topology it is often useful to weaken algebraic structures up to coherent homotopies. The main goal is the description of homotopy invariant structures, i.e. structures which are preserved if the underlying spaces are changed up to homotopy equivalence or the underlying maps up to homotopy. If the structure can be described by an operad, there are two possible approaches.

On one hand, one can construct an operad which encodes the homotopy structure and the homotopies. One example for this are Stasheff's associahedra (cmp. [**Sta63**]) which form a non-symmetric operad, whose algebras are spaces with a homotopy associative multiplication, such that there exist coherent homotopies of finite products, the so-called *A_∞-spaces*. In some sense the little cube operads of Boardman and Vogt (cmp. [**BV68**]) are other examples. They encode coherently homotopy commutative and homotopy associative multiplications.

On the other hand one can weaken the axioms of operads to obtain *lax operads*, whose structures are only given up to coherent homotopies. In this paper we will use the same language to describe both approaches.

Our main tool, *Colored operads* originate from [**BV73**] where Boardman and Vogt introduced *colored PROs* and *PROPs*, or categories of operators, the predecessors of operads. The main idea behind colored operads is the restriction of the compositions. We apply colors, i.e. elements of an arbitrary set, to the inputs and output of an operation and only allow them to be composed, if the color of the output and the corresponding input coincide. It is immediately clear, that a monochrome operad in this sense is precisely a classical operad.

As we will see, operads can be described as algebras over a certain operad, colored by the natural numbers. Furthermore we obtain descriptions of *cyclic* and *modular* operads, as introduced by Getzler, Kapranov and Markl (cmp. [**GK98**], [**Mar96**]), as algebras over certain colored operads. Another notion, which can be described using colored operads, are topological categories with discret object sets.

We apply the homotopy theory of PRO(P)s and algebras over them, as described by Boardman and Vogt in [**BV73**], to operads and develop a theory of homotopy algebras over colored operads. Using the W-construction of Boardman and Vogt, we relax the conditions on algebras over operads. Instead of strictly commutative diagrams, we just require that the composition of the operad and the evaluation on the algebra are compatible up to coherent homotopies. As a consequence we also have to relax the axioms of morphisms between algebras over operads. This leads to the notions of *homotopy algebras* over an operad and *homotopy homomorphisms* between them.

A homotopy algebra over a given colored operad A is an algebra over a certain operad WA, which is a kind of cofibrant resolution of A. A homotopy homomorphism between two such homotopy algebras is an algebra over the cofibrant resolution $W\mathbf{Mor}_A$ of the operad $\mathbf{Mor}_A$, describing morphisms between A-algebras. We will describe a category $\mathfrak{Map}_A$ whose objects are the homotopy algebras over A and whose morphismsm are homotopy classes of homotopy homomorphisms.

By identifying homotopic A-algebra morphisms, we obtain another category $\mathfrak{HomTop}^A$. The homotopy category of A-algebras is the localization $\mathfrak{HomTop}^A[\Sigma^{-1}]$ of this category along the class Σ of *topological equivalences*, i.e. the morphisms of A-algebras whose underlying maps are homotopy equivalences. As Boardman and Vogt did in the case of PROPs, we will prove that the category $\mathfrak{Map}_A$ is equivalent to the homotopy category $\mathfrak{HomTop}^A[\Sigma^{-1}]$ of A-algebras.

Using these notions of homotopy algebras and the description of operads as algebras over a colored operad $\mathbf{Op}$, we can define *lax operads* as algebras and *lax operad morphisms* as homotopy homomorphisms between them. The universal properties of the W-construction imply several homotopy invariance properties of this notion. For example the structure of a lax operad (and hence a strict one) can be transferred to any homotopy equivalent family of spaces.

In the last section we use the description of categories as algebras over operads, to define *topological A_∞-categories* and *-functors*. Furthermore we prove that the homotopy category of topological categories is equivalent to our category of A_∞-categories.

13. Colored operads

In the following, let $(\mathcal{V}, \otimes, k)$ be a closed, symmetric monoidal category with product $\otimes$ and unit object k. We assume that $\mathcal{V}$ is complete and cocomplete. Furthermore let C be a non-empty set, the set of colors.

13.1. C-colored collections. Let Σ be the category of all finite (ordered) sets $\mathbf{n} = \{1, \ldots, n\}$ including the empty set $\mathbf{0}$, and bijective maps. Then $C^\Sigma := \Sigma \downarrow C$ denotes the category, whose objects are maps $\alpha : \mathbf{n} \to C$ and whose morphisms $\sigma : (\alpha : \mathbf{n} \to C) \to (\beta : \mathbf{n} \to C)$ are permutations $\sigma : \mathbf{n} \to \mathbf{n}$ such that $\beta \circ \sigma = \alpha$.

Definition 13.1. A *C-colored collection* (or shorter *C-collection*) A in $\mathcal{V}$ is a functor $A : C \times (C^\Sigma)^{op} \to \mathcal{V}$. A *$C$-morphism $f : A \to B$ of C-collections* is a natural transformation of functors $C \times (C^\Sigma)^{op}$. The category of C-collections and C-morphisms in $\mathcal{V}$ will be denoted with $coll_C\mathcal{V}$.

If C consists of only one element, a C-collection A will be called *monochrome*.

Each map $\alpha : \mathbf{n} \to C$ can be interpreted as an n-tupel $(\alpha_1, \ldots, \alpha_n)$ in C such that $\alpha_i = \alpha(i)$ for each i. Hence A consists of a family

$$\{A(o; \alpha_1, \ldots, \alpha_n)\}_{o, \alpha_j \in C}$$

of objects in $\mathcal{V}$, and (iso)morphisms

$$\sigma^* : A(o; \alpha_1, \ldots, \alpha_n) \to A(o; \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$$

for each $\sigma \in \Sigma_n$ such that

$$(\sigma\tau)^* = \tau^*\sigma^*.$$

The α_i are the colors of the inputs, and o is the output color. A morphism $f : A \to B$ of C-collections in $\mathcal{V}$ consists of a family of morphisms $f(o; \alpha_1, \ldots, \alpha_n) : A(o; \alpha_1, \ldots, \alpha_n) \to B(o; \alpha_1, \ldots, \alpha_n)$ such that

$$f(o; \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})\sigma^* = \sigma^* f(o; \alpha_1, \ldots, \alpha_n).$$

Since $coll_C\mathcal{V}$ is a category of functors, the following lemma is obvious.

LEMMA 13.2. *$coll_C\mathcal{V}$ is (co)complete, if $\mathcal{V}$ is.*

If there exists a map $\varphi : D \to C$ of sets, we obtain a functor $coll\varphi : coll_C\mathcal{V} \to coll_D\mathcal{V}$, with

$$coll\varphi A(o; a_1, \ldots, a_n) = A(\varphi(o); \varphi(a_1), \ldots, \varphi(a_n)).$$

And for each monoidal functor $F : \mathcal{V} \to \mathcal{W}$, there exists a functor $collF : coll_C\mathcal{V} \to coll_C\mathcal{W}$, given by

$$collFA(o; a_1, \ldots, a_n) = FA(o; a_1, \ldots, a_n).$$

13.2. C-colored operads. Basically a C-colored operad is an operad with restricted compositions. Similar to a category, we are not allowed to compose any two operations. Instead we have to take the colors of outputs and inputs into account.

For easy notation, we introduce the following compositions on the symmetric groups. For $\sigma \in \Sigma_n, \tau \in \Sigma_m$ and $1 \le l \le n$ the permutation $\sigma \circ_l \tau \in \Sigma_{n+m-1}$ is given by

$$\sigma \circ_l \tau(i) = \begin{cases} \sigma(i) & 1 \le i < l, \sigma(i) < \sigma(l) \\ \sigma(i) + m - 1 & 1 \le i < l, \sigma(i) > \sigma(l) \\ \sigma(l) + \tau(i - l + 1) - 1 & l \le i < l + m \\ \sigma(i - m + 1) & l + m \le i < n + m, \sigma(i) < \sigma(l) \\ \sigma(i - m + 1) + m - 1 & l + m \le i < n + m, \sigma(i) > \sigma(l). \end{cases}$$

In fact this composition is exactly the composition of the (pseudo) operad of monoids. Hence the $\circ_i$ compositions, defined above, are associative. This means

$$(\sigma \circ_i \tau) \circ_j \rho = \begin{cases} (\sigma \circ_j \rho) \circ_{i+l-1} \tau & 1 \le j \le i - 1 \\ \sigma \circ_i (\tau \circ_{j-i+1} \rho) & i \le j \le i + m - 1 \\ (\sigma \circ_{j-m+1} \rho) \circ_i \tau & i + m \le j \le n + m - 1 \end{cases}$$

for $\sigma \in \Sigma_n, \tau \in \Sigma_m, \rho \in \Sigma_l$ and $1 \le i \le n, 1 \le j \le n + m - 1$. Furthermore we have

$$((\sigma\tau) \circ_i (\rho\pi)) = (\sigma \circ_i \rho)(\tau \circ_{\sigma(i)} \pi).$$

For $\alpha \in C^n$ and $\beta \in C^m$, we define $\alpha \circ_i \beta \in C^{n+m-1}$ to be the tupel

$$\alpha \circ_i \beta = (\alpha_1, \ldots, \alpha_{i-1}, \beta_1, \ldots, \beta_m, \alpha_{i+1}, \ldots, \alpha_n),$$

where α_i is the i-th coordinate of α and β_j the j-th of β. For $\sigma \in \Sigma_n$ the tupel $\alpha\sigma$ will be given by

$$\alpha\sigma = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}).$$

DEFINITION 13.3. A C-*operad* A in $\mathcal{V}$ consists of a C-collection A in $\mathcal{V}$, and *compositions*

$$- \circ_i - : A(o;\alpha) \otimes A(\alpha_i;\beta) \to A(o;\alpha \circ_i \beta),$$

for all $n, m \in \mathbb{N}, 1 \leq i \leq n$ and $o \in C, \alpha \in C^n, \beta \in C^m$ such that the following axioms hold.

1. **Associativity.** For $\alpha \in C^n, \beta \in C^m$ and $\gamma \in C^l$, the following holds on $A(o;\alpha) \otimes A(\alpha_j;\beta) \otimes A((\alpha \circ_j \beta)_i;\gamma)$.
$$\circ_i(\circ_j \otimes \mathrm{id}) = \begin{cases} \circ_{j+l-1}(\circ_i \otimes \mathrm{id})(\mathrm{id} \otimes \tau) & 1 \leq i < j \leq n \\ \circ_j(\mathrm{id} \otimes \circ_{i-j+1}) & j \leq i \leq j+m-1 \\ \circ_j(\circ_{i-n+1} \otimes \mathrm{id})(\mathrm{id} \otimes \tau) & j+m \leq i, \end{cases}$$
where τ is the commuting isomorphism of $\mathcal{V}$.
2. **Equivariance.** On $A(o;\alpha) \otimes A(\alpha_i;\beta)$ the following holds for any $\sigma \in \Sigma_n, \rho \in \Sigma_m$ and $1 \leq i \leq n$
$$\circ_i(\sigma^* \otimes \rho^*) = (\sigma \circ_i \rho)^* \circ_{\sigma(i)} .$$

By comparison with the axioms in [**Mar96**] we see that a monochrome Operad in our sense is precisely a pseudo-operad in the sense of Markl. In this case we often write $A(n)$ for the object $A(*;\alpha), \alpha \in \{*\}^n$, of the underlying collection.

A morphism $f : A \to B$ of C-operads is a morphism of C-collections, which respects the compositions. Therefore it is a collection of maps $f(o;\alpha) : A(o;\alpha) \to B(o;\alpha)$ for $o \in C$ and $\alpha \in C^n$ such that for any $1 \leq i \leq n$ and $\beta \in C^m$

$$\begin{tikzcd} A(o;\alpha) \otimes A(\alpha_i;\beta) \arrow[r, "\circ_i"] \arrow[d, "f\otimes f"'] & A(o;\alpha \circ_i \beta) \arrow[d, "f"] \\ B(o;\alpha) \otimes B(\alpha_i;\beta) \arrow[r, "\circ_i"'] & B(o;\alpha \circ_i \beta) \end{tikzcd}$$

and for each $\sigma \in \Sigma_n$

$$\begin{tikzcd} A(o;\alpha) \arrow[r, "\sigma^*"] \arrow[d, "f"'] & A(o;\alpha\sigma) \arrow[d, "f"] \\ B(o;\alpha) \arrow[r, "\sigma^*"'] & B(o;\alpha\sigma). \end{tikzcd}$$

Since these conditions are compatible with the composition of morphisms, the C-operads in $\mathcal{V}$ and the morphisms between them, form a category $oper_C\mathcal{V}$. As for the collections, we obtain functors $oper\varphi : oper_C\mathcal{V} \to oper_D\mathcal{V}$ for each map $\varphi : D \to C$. But only a monoidal functor $F : \mathcal{V} \to \mathcal{W}$ induces a functor $operF : oper_C\mathcal{V} \to oper_C\mathcal{W}$.

NOTATION 13.4. Let A be a C-operad. The morphism

$$- \circ - : A(o;\alpha) \otimes \bigotimes_{i=1}^{n} A(\alpha_i;\beta^i) \to A(o;\beta),$$

where $\beta \in C^m, m = m_1 + \cdots + m_n$, is the tupel obtained by combining the $\beta^i \in C^{m_i}$, is given by the iterative composition

$$(- \circ_{m_1+\cdots+m_{n-1}+1} -) \circ \cdots \circ (- \circ_1 - \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}), i.e.$$
$$x \circ (y_1, \ldots, y_n) = ((x \circ_1 y_1) \circ_{m_1+1} y_2) \circ_{m_1+m_2+1} y_3 \cdots .$$

EXAMPLE 13.5. **The Endomorphism C-operad.** Since $\mathcal{V}$ is a closed symmetric monoidal category, there exists an internal hom-functor $\underline{Hom}$: $\mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$ together with a natural adjunction isomorphism

$$\mathcal{V}[X \otimes Y, Z] \simeq \mathcal{V}[X, \underline{Hom}(Y,Z)]$$

for all objects X, Y, Z in $\mathcal{V}$, and a composition

$$\underline{Hom}(X,Y) \otimes \underline{Hom}(Y,Z) \to \underline{Hom}(X,Z).$$

Since C can be interpreted as a discrete category, there exists a functor category $\mathcal{V}^C$. For a functor $X : C \to \mathcal{V}$ and $\alpha \in C^n$ let

$$X(\alpha) := \bigotimes_{j=1}^{n} X(\alpha_j)$$

if $\alpha \in C^n$ with $n > 0$ and $X(\alpha) = k$ if $\alpha \in C^0$. Each permutation $\sigma \in \Sigma_n$ induces an isomorphism $X(\sigma) : X(\alpha) \to X(\alpha\sigma)$, by permuting the objects in the product. This defines a functor $X : (C^\Sigma)^{op} \to \mathcal{V}$.

Applying the internal hom-functor of $\mathcal{V}$, we obtain a C-collection $\mathbf{End}_C X$ with

$$\mathbf{End}_C X(o;\alpha) := \underline{Hom}\left(X(\alpha), X(o)\right).$$

With the natural adjunction isomorphisms and the composition, we can define morphisms

$$- \circ_i - : \mathbf{End}_C X(o;\alpha) \otimes \mathbf{End}_C X(\alpha_i;\beta) \to \mathbf{End}_C X(o;\alpha \circ_i \beta)$$

such that $\mathbf{End}_C X$ is a C-operad, the *Endomorphism operad* of X. The associativity and equivariance of the compositions follow directly from the associativity of the composition in $\mathcal{V}$ and the symmetry of $\otimes$.

NOTATION 13.6. If X is a $C \times \{0,\ldots,n\}$-family in $\mathcal{V}$, then the C-family X_i with $0 \leq i \leq n$ is given by $X_i(o) = X(o,i)$. The endomorphism operad $\mathbf{End}_{C\times\{0,\ldots,n\}}(X)$ will also be denoted with $\mathbf{End}_C(X_0,\ldots,X_n)$.

13.3. The non-symmetric case. As in the monochrome case, we can drop the equivariance axioms from the definition of C-collections and C-operads. This leads us to the non-symmetric C-collections and -operads.

DEFINITION 13.7. A *non-symmetric C-collection* A in $\mathcal{V}$ is a functor $A : C \times (C^{\mathbb{N}})^{op} \to \mathcal{V}$, where $C^{\mathbb{N}}$ is the discrete category of maps from finite sets $\mathbf{n} = \{1,\ldots,n\}$, including the empty set $\mathbf{0}$, to C (i.e. its objects are those of C^Σ).

A *non-symmetric C-operad* in $\mathcal{V}$ is a non-symmetric C-collection, together with compositions

$$- \circ_l - : A(o;\alpha) \otimes A(\alpha(l);\beta) \to A(o;\alpha \circ_l \beta),$$

for each $\alpha : \mathbf{n} \to C, \beta : \mathbf{m} \to C$ and $l \in \mathbf{n}$ such that the associativity axiom of Definition 13.3 holds.

Exactly as in the symmetric case, we have the categories $n - coll_C\mathcal{V}$ and $n - oper_C\mathcal{V}$ of non-symmetric C-collections and -operads. Obviously there exists a forgetful functor $U_C : n - oper_C\mathcal{V} \to n - coll_C\mathcal{V}$.

Furthermore we get functors $U_C^{coll} : coll_C\mathcal{V} \to n - coll_C\mathcal{V}$ and $U_C^{oper} : oper_C\mathcal{V} \to n - oper_C\mathcal{V}$ by forgetting the equivariance. There exist functors $S_C^{coll} : n - coll_C\mathcal{V} \to coll_C\mathcal{V}$ and $S_C^{oper} : n - oper_C\mathcal{V} \to oper_C\mathcal{V}$, given by

$$S_C(A)(o;\alpha) = \bigoplus_{\sigma \in \Sigma_n} A(o;\alpha\sigma),$$

for $o \in C, \alpha \in C^n$ and $\sigma \in \Sigma_n$. For $\sigma \in \Sigma_n$ the morphism $\sigma^* : S_CA(o;\alpha) \to S_CA(o;\alpha\sigma)$ is given on the Summand $A(o;\alpha\tau)$ by

$$(\tau^{-1}\sigma\tau) : A(o;\alpha\tau) \to A(o;\alpha\sigma\tau) \to S_CA(o;\alpha\sigma).$$

If A is a non-symmetric operad, the composition

$$- \circ_i - : S_CA(o;\alpha) \otimes S_C(\alpha_i;\beta) \to S_CA(o;\alpha \circ_i \beta)$$

of the *symmetrization* S_CA is induced by the morphisms

$$A(o;\alpha\sigma) \otimes A(\alpha_i;\beta\tau) \xrightarrow{\circ_{\sigma^{-1}(i)}} A(o;\alpha\sigma \circ_{\sigma^{-1}} \beta\tau).$$

S_C^{coll} is left adjoint to U_C^{coll} and S_C^{oper} to U_C^{oper}.

13.4. The unitary case. Until now, we have not used units. In this section we will give a description, of how to include them into the concept of colored-operads. The unit k of $\mathcal{V}$ plays an important role in this setting. Furthermore $\mathcal{V}$ contains an initial object $\emptyset$, because it is cocomplete. The *unit C-collection I_C* is given by

$$I_C(o;\alpha) = \begin{cases} k & \text{if } \alpha = (o) \in C^1 \\ \emptyset & \text{otherwise.} \end{cases}$$

I_C is a C-operad with trivial compositions.

DEFINITION 13.8. A C-operad A is called *unitary*, if there exists a morphism $u : I_C \to A$ of C-operads such that for each $\alpha \in C^n, o \in C$ and $1 \le i \le n$, the following diagrams commute.

$$\begin{tikzcd} k \otimes A(o;\alpha) \arrow[d, "u(o;o)\otimes \mathrm{id}"'] \arrow[dr, "\simeq"] & \\ A(o;o) \otimes A(o;\alpha) \arrow[r, "\circ_1"'] & A(o;\alpha) \end{tikzcd} \qquad \begin{tikzcd} A(o;\alpha) \otimes k \arrow[d, "\mathrm{id}\otimes u(\alpha_i;\alpha_i)"'] \arrow[dr, "\simeq"] & \\ A(o;\alpha) \otimes A(\alpha_i;\alpha_i) \arrow[r, "\circ_i"'] & A(o;\alpha) \end{tikzcd}$$

The main difference to the monochrome case lies in the fact, that a unitary C-operad has an identity for each color $c \in C$.

EXAMPLE 13.9. Let $\mathcal{D}$ be a $\mathcal{V}$-enriched category. Then we can define a unitary $ob\mathcal{D}$-colored operad in $\mathcal{V}$, by

$$\mathcal{D}(o;\alpha) = \begin{cases} \mathcal{D}(o,\alpha) & \text{if } o,\alpha \in ob\mathcal{D} \\ \emptyset & \text{otherwise.} \end{cases}$$

for each object o in $\mathcal{D}$ and each $\alpha \in ob\mathcal{D}^n$. The operad-compositions are given by the compositions in the category $\mathcal{D}$. The morphism $I_{ob\mathcal{D}} \to \mathcal{D}$ is given by the identities.

EXAMPLE 13.10. If X is an object of $\mathcal{V}^C$, then the endomorphism C-operad $\mathbf{End}_C X$ is unitary. The morphism $u : I_C \to \mathbf{End}_C X$ is given by the morphisms $k \to \mathcal{V}(X(o), X(o))$, which is adjoint to the identity of $X(o)$.

REMARK 13.11. Our definition of colored operads does not correspond to the "usual" definition of operads as monoids in the monoidal category of collections. Instead we use Markl's pseudo-operads. But the usual operads are unitary pseudo-operads.

13.5. Algebras over a C-operad.

DEFINITION 13.12. Let A be a C-operad. An *A-algebra* (or an algebra over A) is a family $X = \{X(c)\}_{c\in C}$ of objects in $\mathcal{V}$ together with morphisms $ev : A(o;\alpha) \otimes X(\alpha) \to X(o)$ such that the diagram

$$\begin{tikzcd} A(o;\alpha) \otimes A(\alpha_i;\beta) \otimes X(\alpha \circ_i \beta) \arrow[r] \arrow[d, "\circ_i \otimes \mathrm{id}"'] & A(o;\alpha) \otimes X(\alpha) \arrow[d, "ev"] \\ A(o;,\alpha \circ_i \beta) \otimes X(\alpha \circ_i \beta) \arrow[r, "ev"'] & X(o), \end{tikzcd}$$

where the top row is basically the evaluation

$$A(\alpha_i;\beta) \otimes X(\beta) \to X(\alpha_i),$$

and the diagram

$$\begin{tikzcd} A(o;\alpha) \otimes X(\alpha) \arrow[r, "ev"] \arrow[d, "\sigma^* \otimes \sigma^*"'] & X(o) \\ A(o;\alpha\sigma) \otimes X(\alpha\sigma) \arrow[ur, "ev"'] & \end{tikzcd}$$

commute for all α and β in C^Σ, and $1 \le i \le n$.

A *morphism* $f : X \to Y$ *of A-algebras* is a family $f = \{f(c) : X(c) \to Y(c)\}_{c\in C}$ such that the following diagram commutes for all $o \in C$ and $\alpha \in C^n$.

$$\begin{tikzcd} A(o;\alpha) \otimes X(\alpha) \arrow[r, "ev"] \arrow[d, "\mathrm{id}\otimes f(\alpha)"'] & X(o) \arrow[d, "f(o)"] \\ A(o;\alpha) \otimes Y(\alpha) \arrow[r, "ev"] & Y(o) \end{tikzcd}$$

The category of algebras over A and morphisms between them is called $\mathcal{V}^A$.

By checking the axioms and definitions and using the adjunction in the closed monoidal category $\mathcal{V}$, we get the following

PROPOSITION 13.13. *If A is a C-operad, then $X \in \mathcal{V}^C$ is an A-algebra if and only if there exists a morphism $A \to \mathbf{End}_C X$ of C-operads.*

DEFINITION 13.14. A *unitary algebra* X over a unitary C-operad A, is an A-algebra such that the following diagram commutes for each $o \in C$.

$$\begin{tikzcd} k \otimes X(o) \arrow[d, "u(o;o)\otimes \mathrm{id}"'] \arrow[dr, "\simeq"] & \\ A(o;o) \otimes X(o) \arrow[r, "ev"] & X(o) \end{tikzcd}$$

REMARK 13.15. We do not distinguish between non-symmetric, symmetric, unitary and non-unitary algebras, since we always assume that an algebra X over a C-operad A has the same attributes as A. Thus if A is unitary, then X is unitary.

EXAMPLE 13.16. A unitary algebra X of a unitary C-operad A, is an A-algebra such that the C-operad-morphism $A \to \mathbf{End}_C X$ is unitary.

Since C can be regarded as a discrete category, we have a functor category $\mathcal{V}^C$, whose objects are families $X = \{X(c)\}_{c\in C}$ of objects in $\mathcal{V}$ and whose morphisms from X to Y are families $\{f(c) : X(c) \to Y(c)\}_{c\in C}$. If A is a C-operad, we have a functor $U : \mathcal{V}^A \to \mathcal{V}^C$, which maps an A-algebra to its underlying C-family.

EXAMPLE 13.17. Let C be an arbitrary set of colors. Then there exists a C-operad O_C in $\mathfrak{Sets}$ with

$$O_C(o;\alpha) = \begin{cases} * & \text{if } \alpha \in C \text{ and } \alpha_1 = o \\ \emptyset & \text{otherwise} \end{cases}$$

The compositions are trivial.

An algebra over this operad is a C-family and morphisms between algebras over O_C are maps of C-families.

We often identify C and the operad O_C.

We are going to describe the free algebra of a C-operad A generated by a C-family $X \in \mathcal{V}^C$. We only describe the symmetric construction.

There exists a functor $X : C^{\Sigma,op} \to \mathcal{V}$, with $X(\alpha) = \bigotimes_{i=1}^n X(\alpha_i)$ and $\sigma^* : X(\alpha) \to X(\alpha\sigma)$, which is given by the permutation of the coordinates. Together with the functor $A : C \times C^{\Sigma,op} \to \mathcal{V}$, we get a functor $\bar{A}X : C \times C^{\Sigma,op} \to \mathcal{V}$, given by

$$\bar{A}X(o;\alpha) = A(o;\alpha) \otimes X(\alpha).$$

Now let AX be the colimit of this functor over $C^{\Sigma,op}$,

$$AX = colim_{\mathcal{V}^{\Sigma,op}} \bar{A}X.$$

Therefore AX is a C-family. We can define morphisms $ev : A(o;\alpha) \otimes AX(\alpha) \to AX(o)$, which are induced by the compositions

$$A(o;\alpha) \otimes \bigotimes_{i=1}^{n} A(\alpha_i;\beta^i) \otimes X(\beta^i) \to A(o;\alpha \circ (\beta^1,\dots,\beta^n)) \otimes \bigotimes_{i=1}^{n} X(\beta^i),$$

where $\beta^j \in C^{n_j}$ and

$$\alpha \circ (\beta^1, \dots, \beta^n) = (\dots(\alpha \circ_n \beta^n) \circ_{n-1} \dots) \circ_1 \beta^1.$$

The associativity of the compositions ensures, that these evaluations are compatible with the compositions of A. The equivariance of the compositions of A shows the equivariance of the evaluations. Hence AX is an A-algebra.

Now let $f : X \to Y$ be a morphism of families. $Af : AX \to AY$, induced by $id \otimes f(\alpha)$ for each $\alpha \in C^n$, is a morphism of A-algebras and compatible with the composition of C-family morphisms. Hence we have a functor $F : \mathcal{V}^C \to \mathcal{V}^A$ with $F(X) = AX$ and $F(f) = Af$.

It is obvious that AX is a unitary algebra, if A is a unitary C-operad. In this case we even get the following

LEMMA 13.18. *Let A be a* **unitary** *C-operad. Then $F : \mathcal{V}^C \to \mathcal{V}^A$ is left adjoint to $U : \mathcal{V}^A \to \mathcal{V}^C$.*

PROOF. Let X be an arbitrary C-family. $\eta_X : X \to AX$ is the natural C-family morphism given by

$$X(o) \simeq k \otimes X(o) \to A(o;o) \otimes X(o) \to AX(o).$$

For an A-algebra Y the morphism $\varepsilon_Y(o) : AY(o) \to Y(o)$ is induced by the evaluation

$$ev : A(o;\alpha) \otimes Y(\alpha) \to Y(o).$$

These form a natural morphism $\varepsilon_Y : AY \to Y$ of A-algebras.

The morphism $\varepsilon_Y \eta_Y : Y \to AY \to Y$ is induced by the top row in the diagram

$$\begin{tikzcd} Y(o) \arrow[r, "\simeq"] & k \otimes Y(o) \arrow[r, "u(o)\otimes \mathrm{id}"] \arrow[rr, bend right, "\simeq"'] & A(o;O) \otimes Y(o) \arrow[r, "ev"] & Y(o), \end{tikzcd}$$

and therefore the identity.

$\varepsilon_{AX} A\eta_X$ is induced by the top row and the right column of the diagram

$$\begin{tikzcd} A(o;\alpha) \otimes X(\alpha) \arrow[r, "\simeq"] \arrow[rdd, equal, bend right] & A(o;\alpha) \otimes \bigotimes_{i=1}^n (k \otimes X(\alpha_i)) \arrow[d] \\ & A(o;\alpha) \otimes \bigotimes_{i=1}^n (A(\alpha_i;\alpha_i) \otimes X(\alpha_i)) \arrow[d] \\ & A(o;\alpha) \otimes \bigotimes_{i=1}^n X(\alpha_i) \end{tikzcd}$$

and hence the identity. ◻

If $\varphi : D \to C$ is a map of sets, A is a C-operad and X is an A-algebra, then we obtain a φA-algebra φX with

$$\varphi X(o;\alpha) = X(\varphi(o);\varphi\alpha).$$

For each morphism $f : X \to Y$ of A-algebras, we obtain a morphism $\varphi f : \varphi X \to \varphi Y$ of φA-algebras, which is given by

$$\varphi f(o;\alpha) = f(\varphi(o);\varphi\alpha).$$

Since this definition is compatible with the composition, we obtain a functor $\varphi : \mathcal{V}^A \to \mathcal{V}^{\varphi A}$.

EXAMPLE 13.19. Let C be a set of colors and A a C-operad. For each subset D of C, we have an inclusion $i : D \hookrightarrow C$ and a D-operad $B = iA$ given by $B(o;\alpha) = A(o;\alpha)$ for $o \in D$ and $\alpha \in D^n$. The compositions are precisely the compositions of A. We call B the *suboperad of A given by the colors in D*. In the same way we can restrict morphisms of C-operads to morphisms of suboperads given by D.

If X is an A-algebra, we also have a B-algebra $Y = iX$, which is given by $Y(d) = X(d)$. The evaluations are the evaluations of X. We call Y the *subalgebra of X given by the colors in D*. And again we can restrict morphisms of A-algebras to morphisms of subalgebras given by D.

13.6. Homogeneous families and multiplicative maps. In the following, we will often use operads whose colors consist of two coordinates, i.e. operads whose sets of colors are of the form $C \times L$. we call the C-component of a color the *primary* and the L component the *secondary* color. Most times we will write the color $(o,l) \in C \times L$ as o^l. If α is a tupel in C^n the symbol α^l for $l \in L$ will denote the tupel $(\alpha_1^l, \ldots, \alpha_n^l) \in (C \times L)^n$.

As we will see, we often can restrict our attention to operations whose inputs all have the same secondary color. Operations of this type are called *homogeneous.* More precisely we define

DEFINITION 13.20. (cmp. section II.7 of [**BV73**]) Let A be a $C \times L$-collection. The *homogeneous collection $H_L A$ of A* consists of all spaces $A(o^l;\alpha^{l'})$ with $l, l' \in L, o \in C$ and $\alpha \in C^n$ for some $n \in \mathbb{N}$. If A is an operad, then $H_L A$ is called the *homogeneous operad of A.*

Obviously the homogeneous operad of an operad is not a suboperad, since the composition of two homogeneous operations, does not need to be homogeneous. But nonetheless it makes sense to consider the composition.

PROPOSITION 13.21. *Let A be a $C \times L$-operad. Then the C-families A_l with $A_l(o;\alpha) = A(o^l;\alpha^l)$ are C-operads.*

PROOF. Obviously each A_l is a C-family. The compositions $- \circ_i -$ are induced by the compositions

$$A(o^l;\alpha^l) \otimes A(\alpha_i^l;\beta^l) \to A(o^l;(\alpha \circ_i \beta)^l).$$

□

DEFINITION 13.22. Let A and B be $C \times L$-operads. A morphism $f : H_L A \to H_L B$ of families is called *multiplicative*, if it respects the compositions, i.e. the following diagram commutes for all choices of colors.

$$\begin{tikzcd}
A(o^{l_1};\alpha^{l_2}) \otimes \bigotimes_{i=1}^n A(\alpha_i^{l_2};\beta^{i,l_3}) \arrow[r, "f \otimes \otimes_{i=1}^n f"] \arrow[d, "\circ"'] & B(o^{l_1};\alpha^{l_2}) \otimes \bigotimes_{i=1}^n B(\alpha_i^{l_2};\beta^{i,l_3}) \arrow[d, "\circ"] \\
A(o^{l_1};(\beta^1,\ldots,\beta^n)^{l_3}) \arrow[r, "f"'] & B(o^{l_1};(\beta^1,\ldots,\beta^n)^{l_3})
\end{tikzcd}$$

In fact in the unitary setting the homogeneous families are a generalization of operads. Each C-colored operad can be viewed as the homogeneous family of an $C \times *$-colored operad (namely itself). A multiplicative morphism between two such (unitary) homogeneous families is precisely a morphism of (unitary) operads.

Remark 13.23. Boardman and Vogt did not require that homogeneous PROPs are parts of complete PROPs. But since our examples all will be of this type, we restrict to our notion.

14. Colored trees

14.1. The unlabelled colored trees. An *unlabelled C-tree* is a finite, oriented, planar tree T, drawn with the root at the bottom, whose edges (including the inputs and the outputs) have colors in C.

For planar trees, we have a natural order on the inputs, vertices and edges of a tree, given by left traversion of the tree. Thus the inputs are ordered from left to right. For the edges, the order is described by the following facts:

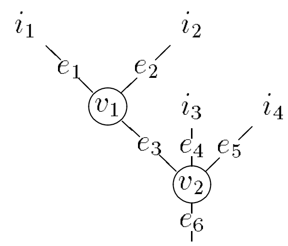
1. The output of a vertex is bigger than all inputs of this vertex.
2. The inputs of a vertex are ordered clockwise, starting with the input next to the output.

A similar description describes the order of the vertices.

1. The children of a vertex are smaller than the vertex itself.
2. The children of a vertex are ordered clockwise, starting with the child next to the output.

We call a vertex w of a tree a *child* of a vertex v, if the output of w is an input of v. v is called the *parent* of w.

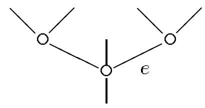
Example 14.1.

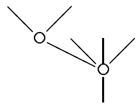


For this tree we have the order $i_1 < i_2 < i_3 < i_4$ on its inputs, $e_1 < e_2 < e_3 < e_4 < e_5 < e_6$ on its edges and $v_1 < v_2$ on its vertices.

For $o \in C$ and $\alpha \in C^n$ let $n - \mathfrak{Tree}_C(o; \alpha)$ be the set of C-trees with n inputs, output color o and color α_i on the i-th input. Let $T \in n - \mathfrak{Tree}_C(o; \alpha)$ and $S \in n - \mathfrak{Tree}_C(\alpha_i; \beta)$ be C-trees. The $\circ_i$-composition in $n - \mathfrak{Tree}_C$ is given by the C-tree $T \circ_i S$, whose underlying uncolored tree is obtained by grafting the output of S along the i-th input of T, and the edges are colored according to T and S. The coloring is well-defined, since the only edge, which appears in both trees is the output of S, resp. the i-th input of T, and both are assumed to have the same color. Since the grafting of trees is associative, the $\circ_i$-compositions in $n - \mathfrak{Tree}_C$ are associative. Therefore $n - \mathfrak{Tree}_C$ is a C-operad in the category $\mathfrak{FSets}$ of finite sets.

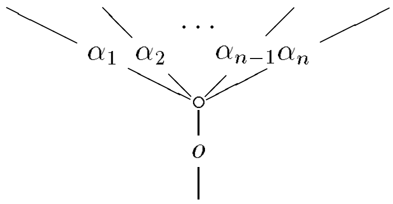
For each set $E \subset edges(T)$ of internal edges of a planar, finite tree T, we can define a tree T/E, obtained from T, by shrinking the internal edges in E. For uncolored trees and $E = \{e\}$ this operation is described by the following picture.

$T =$ 

$T/e =$ 

We have $edges(T/E) = edges(T) \setminus E$. In colored trees the colors of the shrunken edges are simply dropped.

We write $T > S$ if there exists a non-empty set $E \subset edges(T)$ such that $S = T/E$. This induces a partial order on $n - \mathfrak{Tree}_C(o;\alpha)$ for each $o \in C$ and $\alpha \in C^n$. Obviously the minimal element of $n - \mathfrak{Tree}_C(0;\alpha)$ is



The partial order allows us to regard $n - \mathfrak{Tree}_C$ as a non-symmetric operad in the category of categories.

14.2. Labelled colored trees. To obtain a symmetric C-operad, we introduce a labelling of the inputs. A *labelled* C-tree, is a pair (T,τ), with T a C-tree with n inputs and a bijective map $\tau : \mathbf{n} \to inputs(T)$, called the *labelling* of the tree. For $n > 0$ the map τ assigns to each label in $\mathbf{n} = \{1,\ldots,n\}$ a unique input of T. Since we have a natural order on the inputs, we can assume τ to be a permutation.

For $o \in C$ and $\alpha \in C^n$ let $\mathfrak{Tree}_C(o;\alpha)$ be the set of all labelled C-trees (T,τ) such that α_i is the color of the input with label i. Therefore $\mathfrak{Tree}_C(o;\alpha)$ is the set of all pairs (T,τ) with $\tau \in \Sigma_n$ and $T \in n - \mathfrak{Tree}_C(o;\alpha\tau^{-1})$.

For each permutation $\sigma \in \Sigma_n$, we have a map $\sigma^* : \mathfrak{Tree}_C(o;\alpha) \to \mathfrak{Tree}_C(o;\alpha\sigma)$, given by $(T,\tau) \mapsto (T,\tau\sigma)$. Obviously we have $\rho^*\sigma^* = (\sigma\rho)^*$ and hence a C-collection $\mathfrak{Tree}_C$ of labelled C-trees.

The grafting of trees along the input with **label** i induces compositions

$$\circ_i : \mathfrak{Tree}_C(o;\alpha) \times \mathfrak{Tree}_C(\alpha_i;\beta) \to \mathfrak{Tree}_C(o;\alpha \circ_i \beta),$$

given by $(T,\tau) \circ_i (S,\sigma) = (T \circ_{\tau(i)} S, \tau \circ_i \sigma)$. These are associative and equivariant and hence define a C-operad structure on $\mathfrak{Tree}_C$ in $\mathfrak{FSets}$.

As in the unlabelled case, we can regard $\mathfrak{Tree}_C(o;\alpha)$ as a partial ordered set. We write $(T,\tau) > (S,\tau)$, if $T > S$.

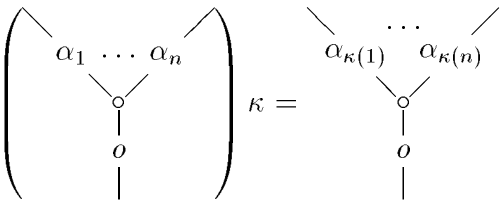
14.3. Shape orbits. For each unlabelled C-tree $T \in n - \mathfrak{Tree}(o;\alpha)$ with n inputs, we define the set $\Sigma(T)$ by

$$\Sigma(T) = \bigotimes_{v \in ver(T)} \Sigma_{|v|},$$

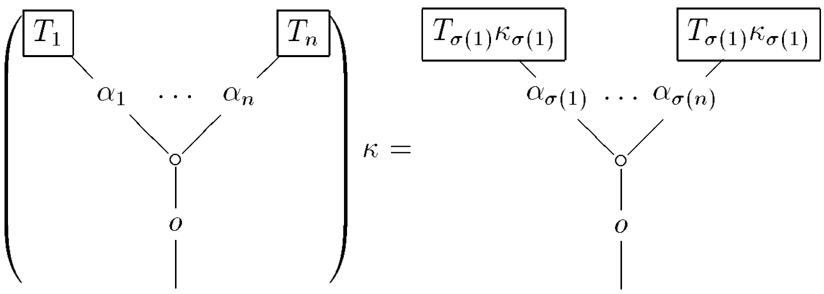
where $|v|$ is the number of inputs of the vertex v. Each element of $\Sigma(T)$ can be interpreted as a vertex-labelling of the tree T, where the label of a vertex is a permutation of its inputs. Using the non-symmetric pseudo-operad structure on the permutations, we can evaluate this labelled tree and get a permutation in Σ_n. Hence we get an evaluation $ev : \Sigma(T) \to \Sigma_n$ for each tree $T \in n - \mathfrak{Tree}_C(o; \alpha)$ with $o \in C$ and $\alpha \in C^n$.

For $\kappa \in \Sigma(T)$ we define the unlabelled C-tree $T\kappa$ recursively by the following equations.

- For $\kappa \in \Sigma_n$ we have



- For $\kappa = (\sigma; \kappa_1, \ldots, \kappa_n)$ with $\sigma \in \Sigma_n, \kappa_j \in \Sigma(T_j)$ we have



Hence the coordinate of $\Sigma(T)$ which corresponds to the vertex v, permutes the subtrees on its inputs. It is clear that $\Sigma(T)$ and $\Sigma(T\kappa)$ are isomorphic (we don't change the valence of vertices, we just change the order). Therefore it is possible to compose an element κ of $\Sigma(T)$ with an element ρ of $\Sigma(T\kappa)$. The composition $\rho\kappa \in \Sigma(T)$ is given by the coordinate wise composition, after reordering ρ accordingly.

If T is an arbitrary unlabelled C-tree, then its *shape orbit* Λ_T is the set of all trees, which are obtained from T, by an iterated application of elements of $\Sigma(T)$. Therefore Λ_T is a set with a right $\Sigma(T)$-action. Often we will write $\Sigma(\Lambda_T)$ for $\Sigma(T)$.

The isotropy group $Sym(T) \subset \Sigma(T)$ of T, i.e. the subgroup which consists of all elements κ such that $T\kappa = T$, is called *symmetry group of the shape* T.

The $\Sigma(T)$-action on the unlabelled C-trees can be extended in a natural way to the labelled trees. We just permute the labels together with the subtrees. Formally this can be expressed by the equation

$$(T, \tau)\kappa = (T\kappa, \kappa^{-1}\tau) := (T\kappa, ev(\kappa)^{-1}\tau).$$

By taking a close look on the action of $\Sigma(T)$, we see that for each tree $(T, \tau) \in \mathfrak{Tree}_C(o; \alpha)$ and each $\kappa \in \Sigma(T)$, the tree $(T, \tau)\kappa = (T\kappa, \kappa^{-1}\tau)$ is an element of $\mathfrak{Tree}_C(o; \alpha)$, too. This fact can be used to define a category $\mathfrak{Tree}_C(o; \alpha)$ for each $o \in C$ and $\alpha \in C^n$. The objects are the labelled C-trees

(T,τ). The morphisms $\kappa_* : (T,\tau) \to (T,\tau)\kappa$ are the elements of $\Sigma(T)$, and the composition is given by $\rho_*\kappa_* = (\rho\kappa)_*$.

15. Topological C-operads

We are mainly interested in topological C-operads. Therefore we restrict our attention to the case $\mathcal{V} = \mathfrak{Top}$, where $\mathfrak{Top}$ is the category of compact-generated topological spaces in the sense of [**Vog71**].

15.1. The free topological C-operad. Let A be a topological C-collection for an arbitrary non-empty set C of colors. For $o \in C$ and $\alpha \in C^n$ let $\bar{F}_C A(o;\alpha)$ be the space

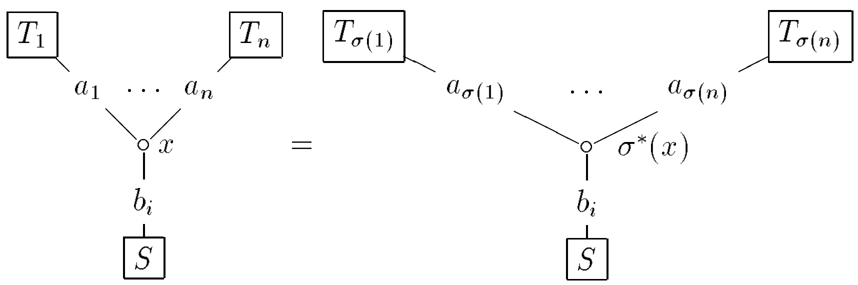
$$\bar{F}_C A(o;\alpha) = \bigoplus_{(T,\tau)\in \mathfrak{Tree}_C(o;\alpha)} A(T,\tau)$$

where

$$A(T,\tau) = A(T) = \bigotimes_{v\in ver(T)} A(out(v); in(v)).$$

Here $out(v)$ is the output color of the vertex v and $in(v)$ is the tuple $\alpha \in C^{|v|}$, whose i-th coordinate is the color of the i-th input of v. Its elements can be interpreted as C-colored trees, whose inputs are numbered and whose vertices are labeled by elements in the acorresponding space $A(out(v); in(v))$ of A.

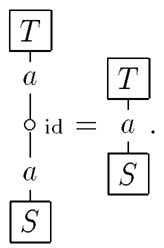
The space $F_C A(o;\alpha)$ of the free C-operad generated by A is the quotient of $\bar{F}_C A(o;\alpha)$ with the following relation



The composition $T \circ_i S$ of two such trees is given by grafting the output of S along the input with label i of T. The map $\sigma^* : F_C A(o;\alpha) \to F_C A(o;\alpha\sigma)$ for $\sigma \in \Sigma_n$ is given by relabelling the inputs of the tree.

REMARK 15.1. If A is a non-symmetric C-collection, we have no morphisms $\sigma^* : A(o;\alpha) \to A(o;\alpha\sigma)$. Therefore the free non-symmetric C-operad of A consists of the spaces $\bar{F}_C A(o;\alpha)$.

REMARK 15.2. If A is a unitary C-collection, the free unitary C-operad $F_C A$ is obtained from the non-unitary version by the application of an additional relation, namely by



A morphism $f : A \to B$ of C-collections induces a map

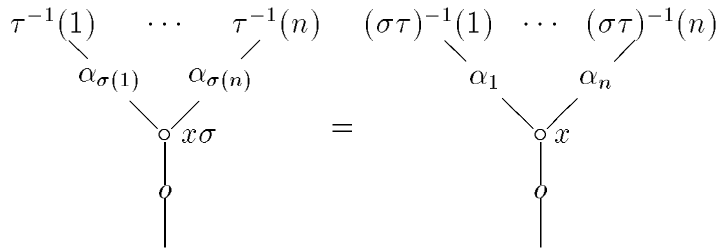
$$f(T,\tau) = \bigotimes_{v \in ver(T)} f(out(v); in(v)) : A(T,\tau) \to B(T,\tau)$$

for each labelled C-tree (T,τ) and thus a map

$$\bar{F}_C f(o;\alpha) : \bar{F}_C A(o;\alpha) \to \bar{F}_C B(o;\alpha).$$

Since f is equivariant, $\bar{F}_C f$ respects the relation on $F_C A$ and $F_C B$ and hence induces a map $F_C f : F_C A \to F_C B$ of C-operads. The definition of this map is compatible with the composition of morphisms of C-collections and enables us to define a functor $F_C : coll_C\mathfrak{Top} \to oper_C\mathfrak{Top}$.

We still have to show, that F_C is left adjoint to the forgetful functor $U_C : oper_C\mathfrak{Top} \to coll_C\mathfrak{Top}$. Since the tree $T_{o;\alpha}$ with one vertex, output color $o \in C$ and input colors $\alpha \in C^n$ is an element of $\mathfrak{Tree}_C(o;\alpha)$, and since $A(T_{o;\alpha},\tau) = A(o;\alpha)$, we have a map $\eta(o;\alpha) : A(o;\alpha) \to F_C A(o;\alpha)$. The equality



in $\bar{F}_C A$ implies the commutativity of the diagram

$$\begin{tikzcd}
A(o;\alpha) \arrow[r, "\eta(o;\alpha)"] \arrow[d, "\sigma^*"'] & F_C A(o;\alpha) \arrow[d, "\sigma^*"] \\
A(o;\alpha\sigma) \arrow[r, "\eta(o;\alpha\sigma)"'] & F_C A(o;\alpha\sigma)
\end{tikzcd}$$

for each permutation $\sigma \in \Sigma_n$. In addition the diagram

$$\begin{tikzcd}
A(o;\alpha) \arrow[r, "\eta_A"] \arrow[d, "f"'] & F_C A(o;\alpha) \arrow[d, "F_C f"] \\
B(o;\alpha) \arrow[r, "\eta_B"'] & F_C B(o;\alpha)
\end{tikzcd}$$

commutes. Therefore the η_A induce a natural transformation $\eta : \mathrm{id} \to U_C F_C$ of endofunctors of $coll_C\mathfrak{Top}$.

Now let A be a topological C-operad. We can define a continuous map $\bar{\varepsilon}_A(o;\alpha) : \bar{F}_C A(o;\alpha) \to A(o;\alpha)$, by composing the vertex labels and applying the labelling of the tree to the result. The equivariance of the composition in A ensures, that $\bar{\varepsilon}_A$ respects the relations on $F_C A$. Therefore we obtain a continuous map $\varepsilon_A : F_C U_C A \to A$ of C-operads. If $f : A \to B$ is a morphism of C-operads the following diagram commutes.

$$\begin{tikzcd} F_CU_CA(o;\alpha) \arrow[r, "F_CU_Cf"] \arrow[d, "\varepsilon_A"'] & F_CU_CB(o;\alpha) \arrow[d, "\varepsilon_B"] \\ A(o;\alpha) \arrow[r, "f"'] & B(o;\alpha) \end{tikzcd}$$

Thus they form a natural transformation $\varepsilon : F_CU_C \to \mathrm{id}$ of endofunctors of $oper_C\mathfrak{Top}$.

By definition the composition $\varepsilon_A\eta_A$ is the identity for any C-operad A. If B is a C-collection, the map $\varepsilon_{F_CB}F\eta_B : F_CB \to F_CU_CF_CB \to F_CB$ is the identity, too. $F_C\eta_B$ maps the vertex label in B of a tree in F_CB to the corresponding vertex labels in F_CB. Then ε_{F_CB} composes these label-trees to form a tree in F_CB, which is exactly the original tree. Thus F_C is left-adjoint to U_C.

Remark 15.3. In all settings (non-symmetric, symmetric, unitary, non-unitary) we will denote the free C-operad of a C-collection by F_CA. If the context is not clear, we will use the appropriate adjectives.

Remark 15.4. The following diagram of forgetful functors commutes. Hence the two left adjoints F_CS_C and S_CF_C of the two paths in the diagram are isomorphic. Here the S_C are the symmetrization functors, which were introduced in section 13.3.

$$\begin{tikzcd} oper_C\mathcal{V} \arrow[r] \arrow[d] & n-oper_C\mathcal{V} \arrow[d] \\ coll_C\mathcal{V} \arrow[r] & n-coll_C\mathcal{V} \end{tikzcd}$$

Let $\bar{G}$ be a non-symmetric C-collection and $G = S_C\bar{G}$. Since F_CG is the free symmetric C-operad generated by $G = S_C\bar{G}$, we can assume that it is the symmetrization $S_CF_C\bar{G}$ of the free non-symmetric C-operad $F_C\bar{G}$. Translated into the language of labelled trees, this implies that it does not matter if we apply the labelling to the inputs of the vertices of the trees, or to the inputs of the tree itself. Obviously the second choice is easier to handle.

15.2. Generators and relations. All important examples of C-operads we are going to use will be given in terms of generators and relations between them. In this section we will give a precise definition of this terminology in the topological case. In algebraic settings a similar thing can be done in a more natural way.

Definition 15.5. Let A be a (topological) C-operad. An *ideal* I of A is a C-collection such that for each $o \in C, \alpha \in C^n$ and $\beta \in C^m$ the following holds.

1. $I(o;\alpha) \subset A(o;\alpha) \times A(o;\alpha)$
2. $(x,x) \in I(o;\alpha)$ for each $x \in A(o;\alpha)$
3. $x \circ_i y := (x_1 \circ_i y, x_2 \circ_i y) \in I(o;\alpha\circ_i\beta)$ for each $x = (x_1,x_2) \in I(o;\alpha), y \in A(\alpha_i;\beta)$ and $1 \leq i \leq n$

4. $z \circ_i x := (z \circ_i x_1, z \circ_i x_2) \in I(o; \alpha \circ_i \beta)$ for each $z \in A(o;\alpha), x = (x_1, x_2) \in I(\alpha_i; \beta)$ and $1 \leq i \leq n$
5. $\sigma^*(x) := (\sigma^*(x_1), \sigma^*(x_2)) \in I(o; \alpha\sigma)$ for each $x = (x_1, x_2) \in I(o;\alpha)$ and $\sigma \in \Sigma_n$

REMARK 15.6. For $x = (x_1, x_2) \in I(o; \alpha)$ and $y = (y_1, y_2) \in I(\alpha_i; \beta)$, the pair $x \circ_i y := (x_1 \circ_i y_1, x_2 \circ_i y_2)$ obviously is an element of $I(o; \alpha \circ_i \beta)$. Thus we have compositions

$$- \circ_i - : I(o;\alpha) \times I(\alpha_i;\beta) \to I(o; \alpha \circ_i \beta).$$

LEMMA 15.7. *Let A be a C-operad and I an ideal of A. Then the spaces $A/I(o;\alpha)$, which are the coequalizers of the diagram*

$$\begin{tikzcd} I(o;\alpha) \arrow[r, shift left] \arrow[r, shift right] & A(o;\alpha) \arrow[r] & A/I(o;\alpha), \end{tikzcd}$$

where the maps on the left are the projections onto the first and second coordinate, form a C-operad A/I. The maps $A(o;\alpha) \to A/I(o;\alpha)$ form a map $p : A \to A/I$ of C-operads.

PROOF. For each permutation $\sigma \in \Sigma_n$ and $o \in C, \alpha \in C^n$, the following diagram commutes.

$$\begin{tikzcd} I(o;\alpha) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\sigma^*"'] & A(o;\alpha) \arrow[r] \arrow[d, "\sigma^*"] & A/I(o;\alpha) \arrow[d, dotted, "\sigma^*"] \\ I(o;\alpha\sigma) \arrow[r, shift left] \arrow[r, shift right] & A(o;\alpha\sigma) \arrow[r] & A/I(o;\alpha\sigma) \end{tikzcd}$$

The induced maps $\sigma^* : A/I(o;\alpha) \to A/I(o;\alpha\sigma)$ are the maps of the underlying C-collection.

Now let $\beta \in C^m$ and $1 \leq i \leq n$, Then the diagram

$$\begin{tikzcd} I(o;\alpha) \times I(\alpha_i;\beta) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\circ_i"'] & A(o;\alpha) \times A(\alpha_i;\beta) \arrow[r] \arrow[d, "\circ_i"'] & A/I(o;\alpha) \times A/I(\alpha_i;\beta) \arrow[d, dotted, "\circ_i"] \\ I(o;\alpha \circ_i \beta) \arrow[r, shift left] \arrow[r, shift right] & A(o;\alpha \circ_i \beta) \arrow[r] & A/I(o;\alpha \circ_i \beta) \end{tikzcd}$$

commutes and induces a map $\circ_i : A/I(o;\alpha) \times A(\alpha_i;\beta) \to A/I(o;\alpha \circ_i \beta)$. These maps are the compositions of the C-operad A/I. The associativity and equivariance follow directly from the properties of the compositions on A. The definition of the compositions and the maps σ^* imply that the maps $A(o;\alpha) \to A/I(o;\alpha)$ form a map $p : A \to A/I$ of C-operads.

If A is unitary then A/I is it, too. The unit morphism $u : I_C \to A/I$ is given by the composition $I_C \to A \to A/I$. $\square$

COROLLARY 15.8. *Let $f : A \to B$ be a map of C-operads and I an ideal of A such that*

$$\begin{tikzcd} I \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. Then there exists a uniquely determined map $\bar{f} : A/I \to B$ of C-operads such that

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "f"'] & A/I \arrow[d, "\bar{f}"] \\ & B. \end{tikzcd}$$

PROOF. Obviously there exist uniquely determined maps

$$\bar{f}(o;\alpha) : A/I(o;\alpha) \to B(o;\alpha)$$

for each $o \in C$ and $\alpha \in C^n$. The equivariance of these maps is a result of the fact that the composition $\sigma^*\bar{f} : A/I(o;\alpha) \to B(o;\alpha\sigma)$ is induced by $\sigma^* f = f\sigma^*$. The second term in this equation induces $\bar{f}\sigma^*$. The compatibility of $\bar{f}$ with the compositions follows in a similar fashion. □

LEMMA 15.9. *Let A be a C-operad and J a family of spaces*

$$J = \{J(o;\alpha) \subset A(o;\alpha) \times A(o;\alpha)\}_{o\in C,\alpha\in C^n}.$$

Let $(J)(o;\alpha)$ be the space, which contains all elements of $A(o;\alpha) \times A(o;\alpha)$, which are of one of the following forms.

1. (x,x) *for all* $x \in A(o;\alpha)$,
2. $\sigma^*(x) = (\sigma^*(x_1), \sigma^*(x_2))$ *for* $x \in J(o;\alpha)$ *and* $\sigma \in \Sigma_n$,
3. $\sigma^*(y \circ_i x) \in A(o;\alpha) \times A(o;\alpha)$ *for* $y \in A, x \in J$ *and* $\sigma \in \Sigma_n$,
4. $\sigma^*(x \circ_j y) \in A(o;\alpha) \times A(o;\alpha)$ *for* $y \in A, x \in J$ *and* $\sigma \in \Sigma_n$.
5. $\sigma^*(z \circ_i (x \circ_j y)) \in A(o;\alpha) \times A(o;\alpha)$ *for* $x \in J, z, y \in A$ *and* $\sigma \in \Sigma_n$.

These spaces form an ideal (J) of A. This ideal is called the ideal generated by J in A.

If I is an ideal of A such that $J(o;\alpha) \subset I(o;\alpha)$ for each $o \in C$ and $\alpha \in C^n$, then we have $(J)(o;\alpha) \subset I(o;\alpha)$.

PROOF. $\sigma^* : (J)(o;\alpha) \to (J)(o;\alpha\sigma)$ is given by

$$x = (x_1, x_2) \mapsto (\sigma^*(x_1), \sigma^*(x_2)).$$

Therefore (J) is a C-collection. The fact that (J) is an ideal of A follows directly from the definition.

Since I is an ideal, each element $\sigma^*(z \circ_i (x \circ_j y))$ of (J) is also an element of I. This proves the second statement. □

REMARK 15.10. In the non-symmetric case we can drop the application of σ^* in the definition of (J).

Ideals of C-operads and the notion of an ideal generated by a family of subspaces of $A \times A$ allow us to define C-operads using the notions of generators and relations. The generators are the elements of a C-collection G and the relations are pairs (x_1, x_2) of objects in $F_C G(o;\alpha)$, which form a family $R = \{R(o;\alpha)\}_{o\in C,\alpha\in C^n}$ of spaces. The C-operad *generated by G with relations R* is the C-operad $F_C G/(R)$.

LEMMA 15.11. *Let A and B be C-operads such that A is generated by the collection G with relations R, that is $A = F_C G/(R)$. If $f : G \to B$ is a* *morphism of C-collections such that for all $o \in C$ and $\alpha \in C^n$ and each pair $x = (x_1, x_2) \in R(o;\alpha)$ the equation*

$$\bar{f}(x_1) = \bar{f}(x_2), \text{ where } \bar{f} : F_C G \to B \text{ is induced by } f,$$

holds, then there exists a uniquely determined map $F : A \to B$ of C-operads such that

$$\begin{tikzcd} F_C G \arrow[r] \arrow[dr, "\bar{f}"'] & A \arrow[d, "F"] \\ & B \end{tikzcd}$$

commutes.

Proof. It suffices to show, that the diagram

$$\begin{tikzcd} (R) \arrow[r, "p_1"] \arrow[d, "p_2"'] & F_C G \arrow[d, "\bar{f}"] \\ F_C G \arrow[r, "\bar{f}"'] & B \end{tikzcd}$$

commutes. Hence we have to show that (in the worst case) the equation

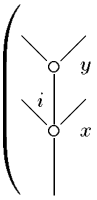
$$\bar{f}\sigma^*(z \circ_i (x_1 \circ_j y)) = \bar{f}\sigma^*(z \circ_i (x_2 \circ_j y))$$

holds for each pair $x = (x_1, x_2) \in R, x, y \in F_C G$ and each permutation σ. But this is true, since $\bar{f}$ is a morphism of C-operads. $\square$

Before we proceed to the examples, we observe, that in fact any C-operad can be described by generators and relations. But unfortunately this description is trivial.

Lemma 15.12. *Let A be a C-operad. Then A is homeomorphic to $F_C A/(R)$, where R is the family of spaces given by*

$$R(o;\alpha) = \left\{ \left(\quad , x \circ_i y \right) : x \in A(o;\beta), y \in A(\beta_i;\gamma), \beta \circ_i \gamma = \alpha \right\}.$$



Proof. Since the square

$$\begin{tikzcd} (R) \arrow[r, "p_1"] \arrow[d, "p_2"'] & F_C A \arrow[d, "\varepsilon_A"] \\ F_C A \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

commutes, we have a uniquely determined map $\varphi : F_C A/(R) \to A$ of C-operads such that

$$\begin{tikzcd} F_C A \arrow[r, "p"] \arrow[dr, "\varepsilon_A"'] & F_C A/(R) \arrow[d, "\varphi"] \\ & A. \end{tikzcd}$$

In the opposite direction we have the composition $\psi = \eta_A p : A \to F_C A \to F_C A/(R)$.

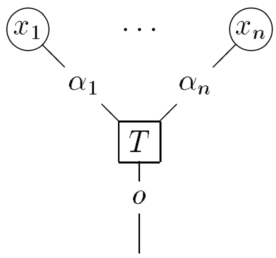
Since $\varepsilon_A \eta_A = \mathrm{id}$ the composition $\varphi\psi$ is the identity. The other composition $\psi\varphi$ is the top row in the following commuting diagram.

$$\begin{tikzcd}
F_C A/(R) \arrow[r, "\varphi"] & A \arrow[r, "\eta_A"] & F_C A \arrow[r, "p"] & F_C A/(R) \\
F_C A \arrow[u, "p"] & & & \\
A \arrow[u, "\eta_A"] \arrow[uur, equal] \arrow[rrr, "\eta_A"'] & & & F_C A \arrow[uu, "p"']
\end{tikzcd}$$

By the universal property of $F_C A$ the map $\psi\varphi p : F_C A \to F_C A/(R)$ is induced by the map $p\eta_A : A \to F_C A/(R)$ and hence we have $\psi\varphi p = p$. Because of the universal property of the coequalizer this induces $\psi\varphi = \mathrm{id}$. $\square$

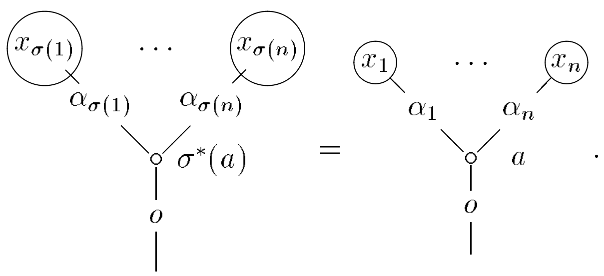
15.3. The free algebra. The free A-algebra generated by a C-family X, can be described with *cherry-trees*. A cherry tree (T, τ, χ) consists of a (labelled) C-tree (T, τ) with vertex labels in A and a map $\chi : \mathbf{n} \to \bigoplus_{c \in C} X(c)$, which assigns to each label i an element in $X(\alpha_i)$ (in the non-symmetric case, we assume the labelling to be the identity). x_i is called the *i-th cherry* of (T, τ, χ).

Cherry trees will be represented graphically in the form



where T is a tree in $\mathfrak{Tree}_C(o; \alpha)$ and $x_i \in X(\alpha_i)$.

The space $AX(o)$ of the free A-algebra is a quotient of the space of all cherry trees with output color o and only one vertex. The relations are given by



The evaluation $ev : A(o; \alpha) \times AX(\alpha) \to AX(o)$ is given by the composition of the vertices of the cherry trees with the operation in $A(o; \alpha)$.

If the C-operad A is given in terms of generators G and relations R, the vertex label of an element in AX is represented by a tree in $F_C G$. If we replace the vertex by this tree, we obtain a cherry tree in $F_C G(o; \alpha) \times X(\alpha)$, which represents the same element. The relations on

$$\bigoplus_{\alpha \in C^n} F_C G(o;\alpha) \times X(\alpha)$$

given by the relation above and the ones given by R. In addition the cherries must be permuted if the labelling of the inputs is changed.

16. Examples of colored operads and algebras

As already mentioned, the classical notion of operads is the special case of a monochrome operad in the colored setting. But we have more than that. A classical operad is also an algebra over a certain $\mathbb{N}$-colored operad. By choosing the correct set of colors and the right colored operad, we can describe cyclic and modular operads in this way. Furthermore we will give a describtion of topological categories in the terms of colored operads.

16.1. Monochrome and colored operads. We want to describe an $\mathbb{N}$-colored operad **Op** in $\mathfrak{Top}$ such that its algebras are precisely the unitary topological (pseudo-)operads. If X is an operad, we have spaces $X(n)$ for each $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$ and each $1 \leq i \leq n$ and $\sigma \in \Sigma_n$ we also have a composition $\circ_i : X(n) \otimes X(m) \to X(n+m-1)$ and a map $\sigma^* : X(n) \to X(n)$. In addition we have an element $\mathrm{id} \in X(1)$, the *identity* such that $x \circ_i \mathrm{id} = x$ and $\mathrm{id} \circ_1 x = x$ for each $x \in X(n), n \in \mathbb{N}$.

To encode this, we define the $\mathbb{N}$-collection O to be the symmetrization of the non-symmetric $\mathbb{N}$-collection $\bar{O}$ with

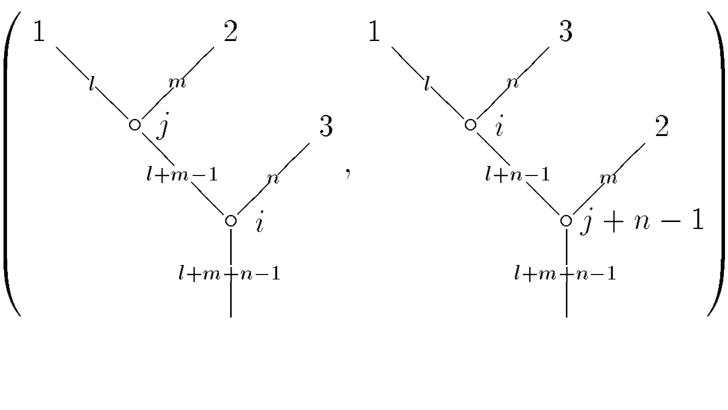
$$\bar{O}(o;\alpha) = \begin{cases} * & \text{if } o = 1 \text{ and } \alpha \in \mathbb{N}^0 \\ \Sigma_o & \text{if } \alpha \in \mathbb{N} \text{ and } \alpha_1 = o \\ \{1, \ldots, \alpha_1\} & \text{if } \alpha \in \mathbb{N}^2 \text{ and } o = \alpha_1 + \alpha_2 - 1 \\ \emptyset & \text{otherwise .} \end{cases}$$

Each $O(o;\alpha)$ has the discrete topology. This is a unitary non-symmetric $\mathbb{N}$-collection. The identity in $O(n;n)$ is the identity of Σ_n.

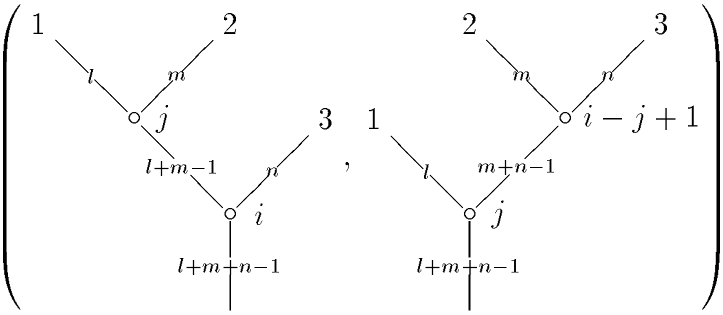
It is clear that each algebra X over $FO = F_{\mathbb{N}}O$ consists of spaces $X(n)$ for each $n \in \mathbb{N}$, with compositions $\circ_i$ and morphisms σ^* as described above. But this does not suffice for an operad-structure on X. In addition we have to introduce relations, which encode the axioms of operads.

The associativity of compositions can be described by the equality of the elements of the following pairs.

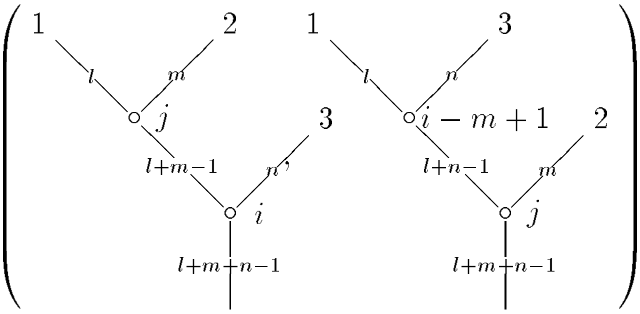
1. For $i \leq j - 1$



2. For $j \le i \le j+l-1$

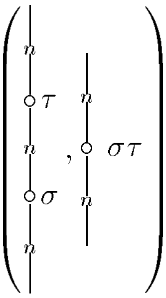


3. For $j+m \le i \le n+m-1$

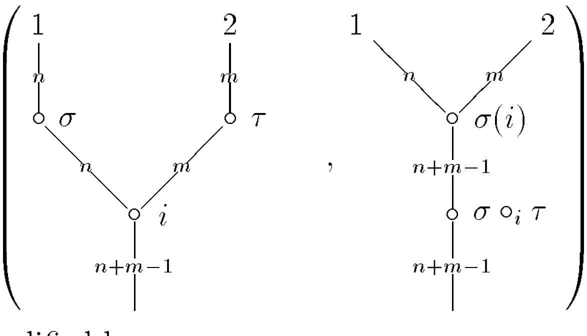


The equivariance conditions correspond to the pairs

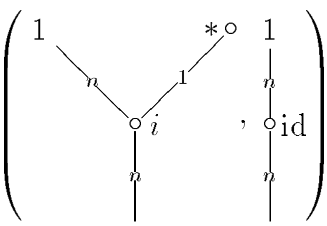
1. For $\sigma, \tau \in \Sigma_n$



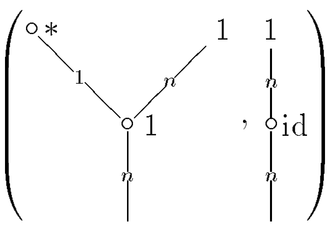
2. For $\sigma \in \Sigma_n, \tau \in \Sigma_m$ and $1 \le i \le n$



The identity is codified by



and



.

Now let R be the family $\{R(o;\alpha)\}$ which consists of these pairs. Then **Op** is the $\mathbb{N}$-operad generated by O with the relations R. That means

$$\mathbf{Op} = F_{\mathbb{N}}O/(R).$$

THEOREM 16.1. *The topological operads are precisely the algebras over* **Op**, *and the morphisms between operads are precisely the morphisms between* **Op**-*algebras.*

PROOF. Let X be a topological operad. As mentioned above X is an algebra over FO. Hence we have a map $f : FO \to \mathbf{End}_{\mathbb{N}}X$ of $\mathbb{N}$-operads. The operad-axioms ensure, that we have $f(x_1) = f(x_2)$ for each pair $x = (x_1, x_2)$ given above. This implies the existence of a map $\mathbf{Op} \to \mathbf{End}_{\mathbb{N}}X$ of $\mathbb{N}$-operads, which in turn is equivalent to the fact that X is an **Op**-algebra.

Now let X be an **Op**-algebra. That means it is a family $\{X(n) : n \in \mathbb{N}\}$ of spaces, together with maps $\circ_i : X(n) \times X(m) \to X(n+m-1)$ for $1 \leq i \leq n$ and $\sigma^* : X(n) \to X(n)$ for $\sigma \in \Sigma_n$. The relations imply that these maps fulfill the axioms of an operad.

Now let $f : X \to Y$ be a morphism of operads. Since X and Y are **Op**-algebras, there exist morphisms $e_X : \mathbf{Op} \to \mathbf{End}_{\mathbb{N}}X$ and $e_Y : \mathbf{Op} \to \mathbf{End}_{\mathbb{N}}Y$ of $\mathbb{N}$-operads. The axioms for morphisms of monochrome operads imply that the diagram

$$\begin{tikzcd} & \mathfrak{Top}(X(\alpha), X(o)) \arrow[dr, "f(o)_*"] & \\ O(o;\alpha) \arrow[ur, "e_X(o;\alpha)"] \arrow[dr, "e_Y(o;\alpha)"'] & & \mathfrak{Top}(X(\alpha), Y(o)) \\ & \mathfrak{Top}(Y(\alpha), Y(o)) \arrow[ur, "f(\alpha)^*"'] & \end{tikzcd}$$

commutes for all $o \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Since the compositions in $\mathbf{End}_{\mathbb{N}}X$ and $\mathbf{End}_{\mathbb{N}}Y$ are given by the composition of maps, this implies the commutativity of

$$\begin{tikzcd} & \mathfrak{Top}(X(\alpha), X(o)) \arrow[dr, "f(o)_*"] & \\ FO(o;\alpha) \arrow[ur, "e_X(o;\alpha)"] \arrow[dr, "e_Y(o;\alpha)"'] & & \mathfrak{Top}(X(\alpha), Y(o)). \\ & \mathfrak{Top}(Y(\alpha), Y(o)) \arrow[ur, "f(\alpha)^*"'] & \end{tikzcd}$$

We obtain an adjoint diagram

$$\begin{tikzcd} FO(o;\alpha) \times X(\alpha) \arrow[r, "ev_X"] \arrow[d, "id \times f(\alpha)"'] & X(o) \arrow[d, "f(o)"] \\ FO(o;\alpha) \times Y(\alpha) \arrow[r, "ev_Y"'] & Y(o) \end{tikzcd}$$

which commutes. The evaluations ev_X and ev_Y are given by the **Op**-algebra structures on X and Y (they are adjoint to e_X and e_Y). Therefore both paths in this diagram induce the same morphism $\mathbf{Op}(o;\alpha) \times X(\alpha) \to Y(o)$ and hence f is a morphism of $\mathbf{Op}$-algebras.

Now let $f : X \to Y$ be a morphism of $\mathbf{Op}$-algebras. X and Y are operads. The compositions in X are given by the map

$$X(n) \times X(m) \to O(n+m-1; n, m) \times X(n) \times X(m) \to X(n+m-1)$$

with $(x, y) \mapsto (i, x, y) \mapsto x \circ_i y$ and the action of Σ_n on $X(n)$ is given by $X\sigma = \sigma^*(x)$. The same holds for Y. Since f is a morphism of $\mathbf{Op}$-algebras, the diagram

$$\begin{tikzcd} O(o;\alpha) \times X(\alpha) \arrow[r, "ev"] \arrow[d, "\mathrm{id}\times f"'] & X(o) \arrow[d, "f"] \\ O(o;\alpha) \times Y(\alpha) \arrow[r, "ev"'] & Y(o) \end{tikzcd}$$

commutes. Hence we have $f(x \circ_i y) = f(x) \circ_i f(y)$ and $f(x\sigma) = f(x)\sigma$, which again proves that f is a morphism of operads. $\square$

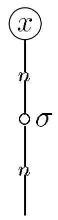
Corollary 16.2. *The category $oper_*\mathfrak{Top}$ of (monochrome) topological operads is isomorphic to $\mathfrak{Top}^{\mathbf{Op}}$, the category of algebras over* $\mathbf{Op}$.

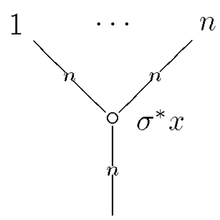
This isomorphism of categories has an interesting interpretation. We have two different ways to describe the free monochrome operad generated by an $\mathbb{N}$-collection X. The first is the description in $oper_*\mathfrak{Top}$. In this case we interpret an operation in $FX(n)$ as a labelled tree with n inputs, and vertex-labels in X. The composition is described by the grafting of trees. Since the vertex labels correspond to operations in X, we call this trees *operation-trees.*

The second approach is the free $\mathbf{Op}$-algebra in $\mathfrak{Top}^{\mathbf{Op}}$. Here we think of an operation in $\mathbf{Op}X(n)$ as a tree with output color n, cherries in X and natural numbers and permutations as vertex-labels. These trees will be called *composition-trees*, because the vertex labels correspond to the compositions of operations (which are the cherries).

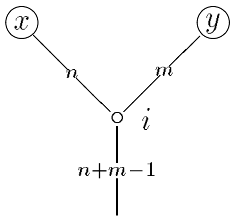
The two approaches are isomorphic, since the universal properties coincide. Therefore there has to be a one-to-one correspondence between composition- and operation-trees. In fact we can describe a recursive algorithm, which translates a composition tree into the corresponding operation tree.

We start with the most trivial composition trees, i.e. with those which have only one vertex.

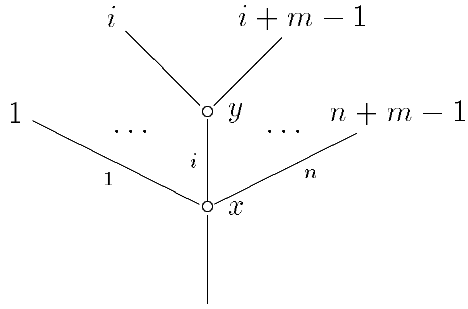


corresponds to


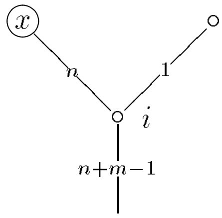
and



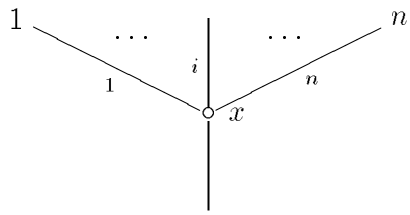
corresponds to



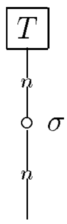
A stump in a composition-tree is ignored, i.e.



corresponds to

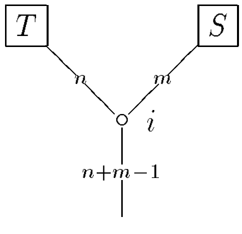


To describe the correspondence in the general case, we use the observation that each composition-tree is semi-binary, that means its vertices have at most two inputs. To construct the operation-tree, we start with the root vertex of the composition-tree. If it is unary, we take the operation-tree which corresponds to the subtree above the root vertex and apply the permutation, labelling the root vertex, to it (i.e. we relabel the inputs). If it is binary and has the label i, we take the operation-tree, corresponding to the left subtree and compose it along its input with label i with the operation-tree, which corresponds to the right subtree. In our graphical language



corresponds to $T\sigma$

and

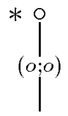


corresponds to $T \circ_i S$.

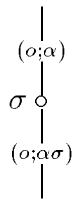
In addition to the standard composition, there exists an additional composition. A cherry of a composition-tree can be replaced with a whole new subtree, which has the same output color as the cherry. In the terms of operation-trees this corresponds to the replacement of a vertex with a subtree.

16.1.1. *Colored Operads.* Now Let C be an arbitrary set of colors and let T_C be the set of all tupels $(o;\alpha)$ with $o \in C, \alpha \in C^n$ and $n \in \mathbb{N}$. Then we can define, similar to the monochrome case, a T_C-colored operad $\mathbf{Op}_C$, whose algebras are precisely the C-colored operads.

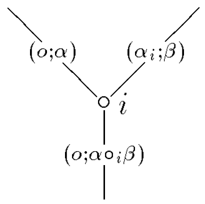
The generating vertices are of the following forms.



for each $o \in C$,



for each $o \in C, \alpha \in C^n$ and $\sigma \in \Sigma_n$



for each $o \in C, \alpha \in C^n, \beta \in C^m$ and $1 \leq i \leq n$.

The relations are given in analogy to the monochrome case.

EXAMPLE 16.3. Let $C = \{0,\ldots,n\}$ and $T_n := T_C$. The T_n operad $\mathbf{Op}_{n+1} := \mathbf{Op}_C$ is the operad, whose algebras are operads with $n+1$ colors.

16.2. Cyclic Operads. Cyclic operads were introduced by Getzler and Kapranov in [**GK95**]. Basically they are operads, which not only allow to permute the inputs (via the action of the symmetric group), but also the output. This exchange of inputs and outputs is realized by an additional action of the cyclic group $\mathbb{Z}_{n+1}$ on $X(n)$, the n-th space of the underlying operad X. The elements of the cyclic group correspond to the powers of the cycle $(12\ldots n+1)$ in Σ_{n+1} and the permutations in Σ_n to the permutations in Σ_{n+1} with fix point $n+1$. Hence every element of Σ_{n+1} is generated by a uniquely determined pair of elements in $\mathbb{Z}_{n+1}$ and Σ_n. Therefore we have an action of Σ_{n+1} on $X(n)$. Together with this extension we get an additional relation given by

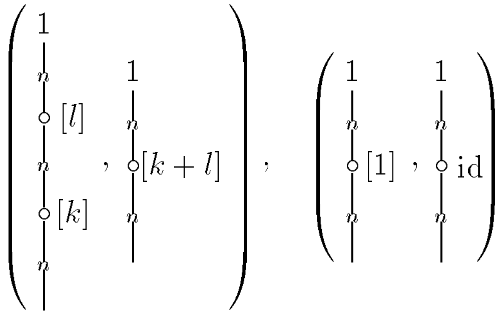
$$(a \circ_n b)^* = b^* \circ_1 a^*$$

for any $a \in X(n)$ and $b \in X(m)$, where a^* is the element $a[1]$ and $[1] \in \mathbb{Z}_{n+1}$ the generator.

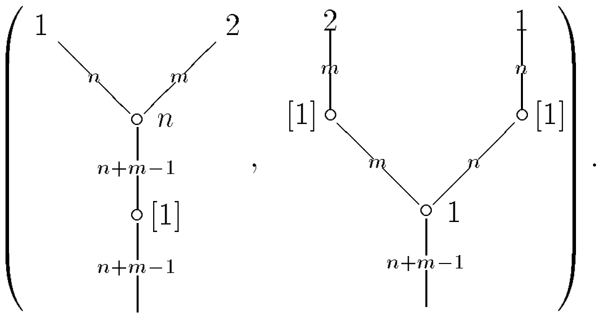
To construct **Cyc**, the $\mathbb{N}$-colored operad, whose algebras are the cyclic operads, we have to extend our description of **Op**. First we have to introduce additional generators representing the action of $\mathbb{Z}_{n+}$. Let $\bar{Z}$ be the non-symmetric $\mathbb{N}$-collection, which is given by

$$\bar{Z}(o;\alpha) = \begin{cases} * & \text{if } o = 1 \text{ and } \alpha \in \mathbb{N}^0 \\ \Sigma_o \cup \mathbb{Z}_{o+1} & \text{if } \alpha \in \mathbb{N} \text{ and } \alpha_1 = o \\ \{1,\ldots,\alpha_1\} & \text{if } \alpha \in \mathbb{N}^2 \text{ and } o = \alpha_1 + \alpha_2 - 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Let Z be the symmetrization $S_{\mathbb{N}}\bar{Z}$. Let $R = \{R(o;\alpha)\}_{o\in\mathbb{N},\alpha\in\mathbb{N}^n}$ be the family, which consists of the pairs for the monochrome operads and of the pairs



and



By checking the axioms in [**GK98**] we get the following

THEOREM 16.4. *The topological cyclic operads are precisely the algebras over* **Cyc**, *and the morphisms between cyclic operads are precisely the morphisms between* **Cyc**-*algebras.*

COROLLARY 16.5. *The category* $cyc\mathfrak{Top}$ *of topological cyclic operads is isomorphic to* $\mathfrak{Top}^{\mathbf{Cyc}}$, *the category of algebras over* **Cyc**.

Let $\mathfrak{Graph}^0(n)$ be the set of all graphs without loops and $n+1$ legs, i.e. edges which only have vertex such that the legs are labelled by the natural numbers from 0 to n. The leg with label 0 is the output of the graph and the remaining legs are inputs. These sets form a generic cyclic operad, a generalization of trees. The i-th composition $x \circ_i y$ of two such graphs is given by grafting the output of y along the i-th input of x. The action of the symmetric group Σ^n on $\mathfrak{Graph}^0(n)$ is given by relabelling of the inputs. The action of the cyclic group $\mathbb{Z}_{n+1}$ is generated by [1], which relabels inputs and outputs. The output becomes the input with label n and the first input the new output. The remaining labels are shifted accordingly.

Similar as in the classical case, the free cyclic operad of a monochrome cyclic collection X, i.e. a monochrome collection with an additional action of $\mathbb{Z}_{n+1}$ on $X(n)$, is given by adding labels to the vertices of such graphs.

16.3. Modular Operads. Another generalization of operads are modular operads, which were introduced by Getzler and Kapranov in [**GK98**]. Basically a (topological) modular operad X is a family $\{X(g,n)\}_{g,n\in\mathbb{N}}$ of spaces such that the family $X(n) = \cup_{g\in\mathbb{N}} X(g,n)$ is a cyclic operad such that the compositions preserve the additional grading given by g, i.e. we have compositions

$$\circ_i : X(g,n) \times X(h,m) \to X(g+h, n+m-1)$$

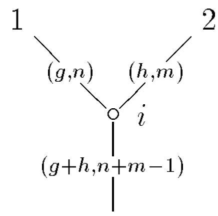
for each $1 \leq i \leq n$, which satisfy relations analogous to the ones of monochrome and cyclic operads.

In addition there are *contractions* $\zeta_{ij} : X(g,n) \to X(g+1, n-2)$ for $1 \leq i,j \leq n$ with $i \neq j$ such that

1. $\zeta_{ij}\sigma^* = \hat{\sigma}^*\zeta_{\sigma(i)\sigma(j)}$ for each $\sigma \in \Sigma_n, a \in X(g,n)$, where $\hat{\sigma} \in \Sigma_{n-2}$ is obtained from σ by removing i and j from the set $\{1,\dots,n\}$ and mapping it bijectively and order preserving to $\{1,\dots,n-2\}$.
2. $\zeta_{ij}\zeta_{kl} = \zeta_{kl}\zeta_{il}$ for pairwise different i,j,k and l.
3. (a) $\zeta_{12}(\alpha \circ_n \beta) = \zeta_{12}(\alpha) \circ_{n-2} \beta$
   (b) $\zeta_{n,n+1}(\alpha \circ_n \beta) = \alpha \circ_n \zeta_{12}(\beta)$
   (c) $\zeta_{n-1,n}(\alpha \circ_n \beta) = \zeta_{n+m-2,n+m-1}(\alpha \circ_{n-1} \beta^*)$
   for $\alpha \in X(n,g)$ and $\beta \in X(m,g)$.

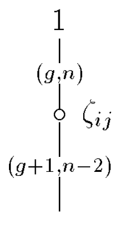
For $X(g,n)$ the number g is called *genus* and n *valence*.

Again it is possible to construct a colored operad **Mod**, whose algebras are exactly the modular operads. This time the set of colors is $\mathbb{N} \times \mathbb{N}$. The generators for the compositions are given by trees



.

The generators for the permutations and the elements of the cyclic groups are given as for **Cyc**, but for each genus g. The generators for **Cyc** for each genus g, plus the generators for the contractions



.

The relations are given by the relations for **Cyc**, preserving the genus, plus the pairs which codify the axioms given above. The precise formulation of this relations is left to the reader.

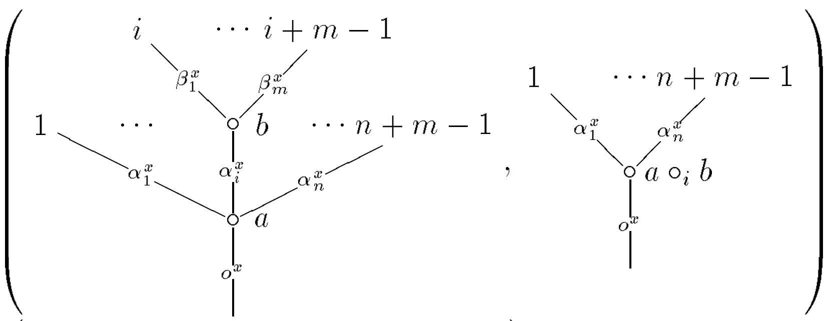
Again we have a generic modular operad, which "generates" the free construction. It is given by the sets $\mathfrak{Graph}(g,n)$ of graphs with genus g and $n+1$ legs. Again the legs are labelled by the natural numbers from 0 to n, where 0 is the label of the output. The action of Σ_n and $\mathbb{Z}_{n+1}$ and the compositions are given as in the cyclic case. The contractions ζ_{ij} are given by joining the i-th and the j-th input of the tree, which increases the genus by one, but decreases the number of inputs by two.

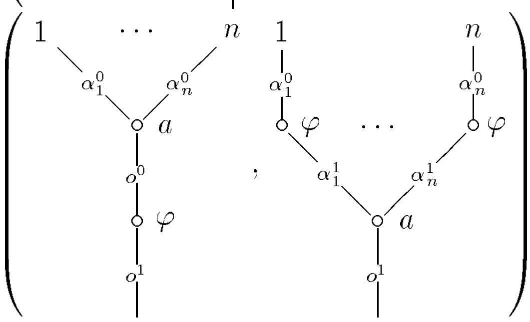
16.4. Morphisms of algebras. The next example will be a colored operad, whose algebras are morphisms between algebras over a C-operad A. This operad $\mathbf{Mor}_A$ is a $C \times \{0,1\}$-Operad. For easy reference, we will write the second color as superscript, i.e. o^x is the pair $(o,x) \in C \times \{0,1\}$. For $\alpha \in C^n$ and $x \in \{0,1\}$ the symbol α^x will denote the tupel $(\alpha_1^x, \ldots, \alpha_n^x)$. Furthermore we will call the base *primary* and the exponent *secondary* color.

The generators of $\mathbf{Mor}_A$ are given by the family A_φ with

1. $A_\varphi(o^x; \alpha^x) = A(o;\alpha)$ for $x \in \{0,1\}$ and $o \in C, \alpha \in C^n$ and
2. $A_\varphi(o^1; o^0) = \{\varphi\}$ for $o \in C$.

In all other cases the set of generators will be empty. The following relations are applied for $x \in \{0,1\}$.

1. 

2. 

There exist two canonical inclusions $d^i : C \to C \times \{0,1\}, i = 0,1$, given by $d^i(c) = (c,i)$. Hence an algebra over $\mathbf{Mor}_A$ consists of two C-families $X_i, i = 0,1$ with $X_i(c) = X(c,i)$. The first type of generators and the first relation ensures that X_0 and X_1 are A-algebras. The second set of generators corresponds to the existence of maps $\varphi(c) : X_0(c) \to X_1(c)$. The second relation guarantees that these form a morphism of A-algebras. Hence every $\mathbf{Mor}_A$-algebra is a morphism of A-algebras and every morphism of A-algebras induces a $\mathbf{Mor}_A$-algebra. In fact this correspondence is a bijection.

REMARK 16.6. In [**BV73**] this operad is called $A \otimes \mathcal{L}_1$, where $\otimes$ is the tensor product of operads, and $\mathcal{L}_1$ is the $\{0,1\}$-colored unitary operad with only one non-trivial operation in $\mathcal{L}_1(1;0)$.

Now let H, G be $\mathbf{Mor}_A$-algebras, which correspond to morphisms $h : X \to Z$ and $g : Z \to Y$. Then we can define a $\mathbf{Mor}_A$-algebra $G \circ H$, which corresponds to the composition $g \circ h$. We set $(G \circ H)_0 = X$ and $(G \circ_H)_1 = Y$. For the generator φ of $\mathbf{Mor}_A$, the evaluation $(G \circ H)(o,0) \to (G \circ H)(o,1)$ is given by

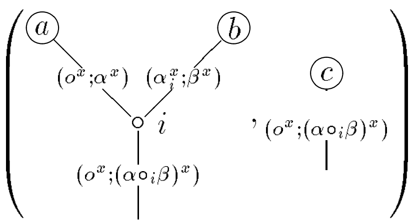
$$X(o) \xrightarrow{h} Z(o) \xrightarrow{g} Y(o) .$$

The evaluations for the generators (A, i) are given by the A-algebra structures on X_0 and Y_1. The relations are respected, since both, g and h are A-algebra morphisms.

Remark 16.7. In the following, we often identify a homomorphism $f : X \to Y$ of A-algebras with the induced homomorphism $\mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$. We denote both with f.

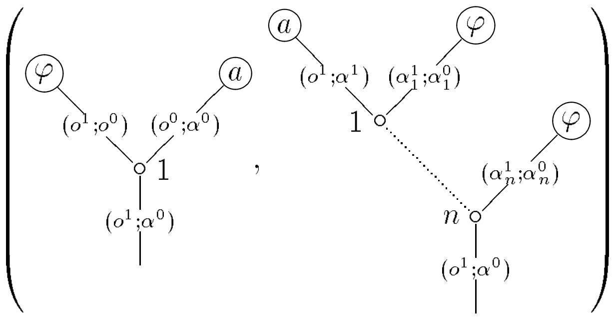
This description of $\mathbf{Mor}_A$ uses operation-trees, i.e. we use the description of the free operad in *oper*$\mathfrak{Top}$ and then apply additional relations. Alternatively we can use the description of the free operad as a free algebra over $\mathbf{Op}_{T_{C \times \{0,1\}}}$. In this case $\mathbf{Mor}_A$ is generated by all cherry trees in $\mathbf{Op}_{T_{C \times \{0,1\}}}$ whose cherries are the given generators of $\mathbf{Mor}_A$.

The relations are given by the pairs



with $c = a \circ_i b$

and

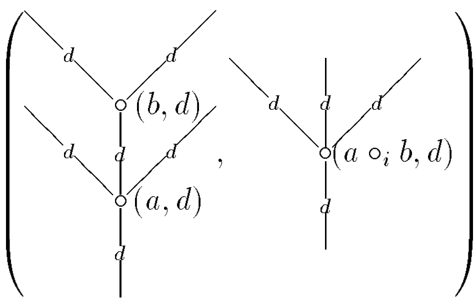


16.5. Diagrams of algebras. The previous example can be generalized to diagrams of algebras. Let $\mathcal{D}$ be a small category and A a C-operad. The $C \times \mathfrak{ob}\,\mathcal{D}$-operad $A_{\mathcal{D}}$ is given by the following generators

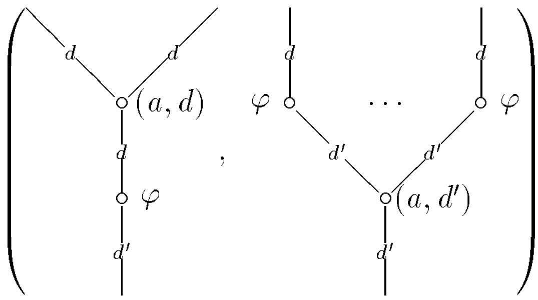
1. the *A-generators* $(a, d) \in A_{\mathcal{D}}((o, d); \alpha^d)$ for each $d \in \mathfrak{ob}\,\mathcal{D}, o \in C, \alpha \in C^n$ and $a \in A(o; \alpha)$,
2. and the *$\mathcal{D}$-generators* $\varphi \in A_{\mathcal{D}}((o, d') : (o, d))$ for each morphism $\varphi : d \to d'$ in $\mathcal{D}$

and the following relations.

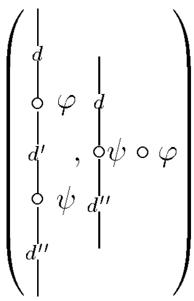
1. Each internal edge whose vertices are A-generators is deleted and the labels are composed.



2. An φ-generator can be *pushed up*.



3. Two adjacent $\mathcal{D}$-generators can be composed.



If X is an $A_{\mathcal{D}}$-algebra, then there exists an A-algebra X_d for each object d of $\mathcal{D}$. It is given by $X_d(c) = X(c, d)$ for each $c \in C$. The A-structure is given by the restriction to the suboperad of $A_{\mathcal{D}}$ of the colors (c, d). For each morphism $\varphi : d \to d'$ exists a $\mathbf{Mor}_A$-Algebra X_φ given by $X_\varphi(c, 0) = X(c, d)$ and $X_\varphi(c, 1) = X(c, d')$. Furthermore we have $X_\varphi \circ X_\psi = X_{\varphi\circ\psi}$ for each composable pair of morphisms in $\mathcal{D}$. Hence an $A_{\mathcal{D}}$-algebra is precisely a functor from $\mathcal{D}$ to $\mathfrak{Top}^A$, the category of A-algebras. On the other hand each functor $F : \mathcal{D} \to \mathfrak{Top}^A$ gives rise to an $A_{\mathcal{D}}$-algebra X. It is easy to see that these two correspondences are inverse to each other.

THEOREM 16.8. *Let $\mathcal{D}$ be a small category. Then the categories $\mathfrak{Top}^{A_{\mathcal{D}}}$ and $Func(\mathcal{D}, \mathfrak{Top}^A)$ are isomorphic.*

EXAMPLE 16.9. Let $\mathbf{Iso}_A$ be the operad, which belongs to the diagram consisting of two objects and two morphisms between them, which are inverse to each other, i.e.

$$\begin{tikzcd} 0 \arrow[r, bend left, "\varphi"] & 1 \arrow[l, bend left, "\psi"] \end{tikzcd}$$

with $\psi \circ \varphi = \mathrm{id}_0$ and $\varphi \circ \psi = \mathrm{id}_1$.

Let $H : \mathbf{Iso}_A \to \mathbf{End}_C(X, Y)$ be a homomorphism. By restriction to trees of constant secondary color, we obtain two homomorphisms $H_X : A \to \mathbf{End}_C(X)$ and $H_X : A \to \mathbf{End}_C(Y)$. Similar we obtain a homomorphism $H_{X,Y} : \mathbf{End}_A \to \mathbf{End}_C(X, Y)$ by interpreting every tree in $\mathbf{Mor}_A$ as an element in $\mathbf{Iso}_A$. If we also exchange the secondary colors, we obtain a homomorphism $H_{Y,X} : \mathbf{Mor}_A \to \mathbf{End}_C(Y, X)$.

If $H : \mathbf{Iso}_A \to \mathbf{End}_C(X, Y)$ is a morphism of operads, then the homomorphisms $H_{X,Y} : \mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$ and $H_{Y,X} : \mathbf{Mor}_A \to \mathbf{End}_C(Y, X)$ of A-algebras are inverse to each other.

EXAMPLE 16.10. Let $\mathbf{Mor}_A^n$ be the operad, which belongs to the diagram $\mathcal{L}_n$, consisting of the objects $0, \ldots, n$ and exactly one morphism $i \to j$ if $i \leq j$. Thus a $\mathbf{Mor}_A^n$ algebra consists of $n+1$ algebras over A and a sequence of morphisms between them.

The generators of $\mathbf{Mor}_A^n$ are the elements of the family A_φ^n with

1. $A_\varphi^n(o^x; \alpha^x) = A(o; \alpha)$ for $0 \leq x \leq n$ and $o \in C, \alpha \in C^n$ and
2. $A_\varphi^n(o^j; o^i) = \{\varphi\}$ for $o \in C$ and $0 \leq i < j \leq n$.

For $0 \leq i \leq n$ and $\omega \in T_{C \times \{0,\ldots,n-1\}}$ let $\delta_i(\omega) : \mathbf{Mor}_A^{n-1}(\omega) \to \mathbf{Mor}_A^n(\delta_i\omega)$ be the map, which changes the secondary colors according the injective, order preserving map $\delta_i : \{0, \ldots, n-1\} \to \{0, \ldots, n\}$, given by

$$\delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

Similar we can define $\sigma_i(\omega) : \mathbf{Mor}_A^{n+1}(\omega) \to \mathbf{Mor}_A^n(\sigma_i\omega)$ for each surjective, order preserving map $\sigma_i : \{0, \ldots, n+1\} \to \{0, \ldots, n\}$ with

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

The images $\delta_i(\mathbf{Mor}_A^{n-1}(\omega)) \subset W\mathbf{Mor}_A^n(\delta_i\omega)$ form a $T_{C \times \{0,\ldots,n-1\}}$-operad, which we will call $D_i\mathbf{Mor}_A^n$. Similar the images $\sigma_i(\mathbf{Mor}_A^{n+1}(\omega))$ form a $T_{C \times \{0,\ldots,n+1\}}$-operad $S_i\mathbf{Mor}_A^n$.

If $X_0, \ldots, X_n$ are spaces, then a homomorphism

$$F : \mathbf{Mor}_A^n \to \mathbf{End}_C(X_0, \ldots, X_n)$$

induces homomorphisms

$$d_iF : \mathbf{Mor}_A^{n-1} \to \mathbf{End}_C(\ldots, X_{i-1}, X_{i+1}, \ldots)$$

for $0 \leq i \leq n$, which are given by $d_iF(T) = F(\delta_iT)$. Similar we have homomorphisms

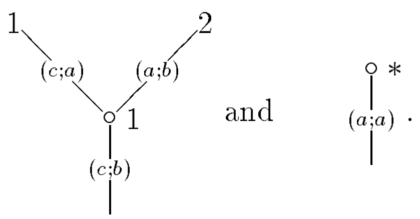
$$s_iF : \mathbf{Mor}_A^{n+1} \to \mathbf{End}_C(\ldots, X_i, X_i, \ldots)$$

given by $s_iF(T) = F(\sigma_iT)$.

The homomorphisms $F : \mathbf{Mor}_A^n \to \mathbf{End}(X_0, \ldots, X_n)$ are the n-simplices of a simplicial class. The simplicial operations are given by the two constructions above.

REMARK 16.11. $\mathbf{Mor}_A^n$ is the operad $A \otimes \mathcal{L}_n$ of [**BV73**].

16.6. Categories. Let S be an arbitrary set of colors. The $(S \times S)$-operad $\mathbf{Cat}_S \subset \mathbf{Op}_S$ is the suboperad, generated by all vertices of the form



Hence the $\mathbf{Cat}_S$-algebras are all S-operads, which contain only unary operations. But there is another interpretation. Each $\mathbf{Cat}_S$-algebra A consists of a family $\{A(a,b)\}_{a,b\in S}$ of spaces, together with maps $\mathrm{id}_a : * \to A(a,a)$, which correspond to an element id_a, the *identity* of a, and *compositions*

$$\mu_{a,b,c} : A(a,b) \times A(b,c) \to A(a,c) \text{ for all } a,b,c \in S.$$

These compositions are associative, i.e. the diagram

$$\begin{tikzcd}
A(a,b) \times A(b,c) \times A(c,d) \arrow[r, "\mu_{a,b,c}\times A(c,d)"] \arrow[d, "A(a,b)\times\mu_{b,c,d}"'] & A(a,c) \times A(c,d) \arrow[d, "\mu_{a,c,d}"] \\
A(a,b) \times A(b,d) \arrow[r, "\mu_{a,b,d}"'] & A(a,d)
\end{tikzcd}$$

commutes for all $a,b,c,d \in S$. Furthermore the following diagrams commute.

$$\begin{tikzcd}
* \times A(a,b) \arrow[r, "\mathrm{id}_a \times A(a,a)"] \arrow[dr, "\simeq"'] & A(a,a) \times A(a,b) \arrow[d, "\mu_{a,a,b}"] \\
& A(a,b)
\end{tikzcd}$$

$$\begin{tikzcd}
A(a,b) \times * \arrow[r, "A(a,a)\times \mathrm{id}_b"] \arrow[dr, "\simeq"'] & A(a,b) \times A(b,b) \arrow[d, "\mu_{a,b,b}"] \\
& A(a,b)
\end{tikzcd}$$

Therefore the (topological) $\mathbf{Cat}_S$-algebras are precisely the small topological categories with object set S.

REMARK 16.12. The operad $\mathbf{Cat}_S$ is not only a reduction of $\mathbf{Op}_S$, but also an extension of the operad of associative topological monoids, which is precisely $\mathbf{Cat}_*$.

Since the colors are not changed, the morphisms of $\mathbf{Cat}_S$-algebras are precisely the functors between topological operads with object set S, which are the identity on the object sets. We denote the category $\mathfrak{Top}^{\mathbf{Cat}_S}$ by $\mathfrak{TCat}_S$. For the description of arbitrary functors, we have to expand our construction.

We will denote a small topological category by a pair (A,S), where S is the set of objects of the category, and A the $\mathbf{Cat}_S$-algebra consisting of the morphism spaces. An arbitrary functor $(F,\varphi) : (A,S) \to (B,T)$, consists of a map $\varphi : S \to T$ and a family of continuous maps

$$F(a,b) : A(a,b) \to B\left(\mathfrak{ob}\, F(a), \mathfrak{ob}\, F(b)\right),$$

which respect the compositions and which map the identites to identites, i.e. the following diagram commutes.

$$\begin{tikzcd}
& * \arrow[dl, "\mathrm{id}_a"'] \arrow[dr, "\mathrm{id}_{\varphi(a)}"] & \\
A(a,a) \arrow[rr, "F(a,a)"'] & & B\left(\varphi(a), \varphi(a)\right)
\end{tikzcd}$$

Via the map φ, we can define a $(S\times S)$-family φB with

$$\varphi B(a,b) = B\left(\varphi(a), \varphi(b)\right) \text{ for all } a,b \in S.$$

The compositions in B induce associative compositions on φB. Since the maps $F(a,a)$ map identities to identities, this makes φB into a $\mathbf{Cat}_S$-algebra. The family $\{F(a,b)\}_{a,b\in S}$ induces a morphism $F : A \to \varphi B$ of $\mathbf{Cat}_S$-algebras.

On the other hand each morphism $F : A \to \varphi B$ of $\mathbf{Cat}_S$-algebras induces a functor $F : (A,S) \to (B,T)$ of the underlying categories. We will use this correspondence to formulate a operadic interpretation of topological categories.

For the sake of readability, we denote the category $\mathfrak{Top}^{\mathbf{Cat}_S}$ with $\mathfrak{TCat}_S$. For each map $\varphi : S \to T$ of sets, we obtain a functor $\mathfrak{TCat}_\varphi : \mathfrak{TCat}_T \to \mathfrak{TCat}_S$, given by

$$\mathfrak{TCat}_\varphi(A)(a,b) = \varphi A(a,b) = A(\varphi(a),\varphi(b))$$

and

$$\mathfrak{TCat}_\varphi(F)(a,b) = \varphi F(a,b) = F(\varphi(a),\varphi(b)) \text{ for } F : A \to B.$$

Together the form a functor

$$\mathfrak{TCat}_\bullet : \mathfrak{Sets}^{op} \to \mathfrak{Cat}.$$

Now we can give an alternative description of the category $\mathfrak{TCat}$ of small topological categories in this operadic setting. The objects of $\mathfrak{TCat}$ are pairs (A,S) with S a set and A a topological category with object set S. The morphisms are pairs $(F,\varphi) : (A,S) \to (B,T)$ with $\varphi : S \to T$ a map of sets and $F : A \to \varphi B$ a morphism of $\mathbf{Cat}_S$-algebras. If $(F,\varphi) : (A,S) \to (B,T)$ and $(G,\psi) : (B,S) \to (C,U)$ are two such morphisms, then their composition is given by

$$(G,\psi)\circ(F,\varphi) := (\varphi G \circ F, \psi \circ \varphi).$$

The identity of (A,S) is given by $(\mathrm{id}_A, \mathrm{id}_S)$. The functor induced by the composition $(G,\psi)\circ(F,\varphi)$ is given by the family

$$(\varphi G \circ F)(a,b) = \varphi G(a,b) \circ F(a,b) = G(\varphi(a),\varphi(b)) \circ F(a,b).$$

Hence it corresponds to the composition of the induced functors.

17. The bar-construction

Let A be a C-operad and X an algebra over A. It is easy to see that we lose the structure if we replace X by a homotopy equivalent family Y. The same holds if we replace a morphism $f : X \to X'$ of A-algebras by a homotopic map of C-families. In [**BV73**] Boardman and Vogt described a way to obtain homotopy invariant algebras and morphisms over PROs and PROPs. We use the close relation of colored operads to their notions to obtain a homotopy invariant notion of *homotopy algebras* and *(strong) homotopy homomorphisms* over a C-operad A.

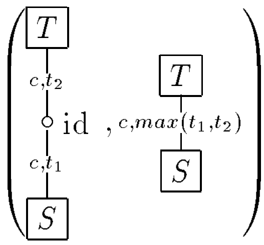
As in the monochrome case, which is described in [**Vog99**], we can construct a "cofibrant resolution" of a colored operad. This means for each topological C-operad A we can construct a C-operad WA, whose underlying spaces are homotopy equivalent to the spaces of A, and which satisfies certain universal homotopy-invariant properties.

Remark 17.1. From now on we assume all operads to be unitary.

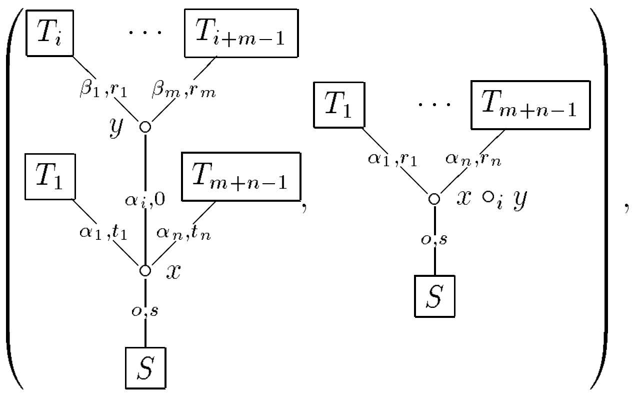
Let A be a unitary C-operad and $TA(o;\alpha)$ for $o \in C$ and $\alpha \in C^n$ the set of all labelled C-trees with vertex labels in A and lengths $l_e \in [0,1]$ for each internal edge e. Hence an element of $TA(o;\alpha)$ is a tupel (T,τ,v,l) with (T,τ) a labelled C-tree, v a map which assigns to each vertex of T an element in $A(out(v);in(v))$ and l a map which assigns to each internal edge a length in $[0,1]$. Each external edge (that is each input or output) is assumed to have the length 1. The topology on $TA(o;\alpha)$ is induced by the vertex labels in A and the lengths of the edges. The grafting of trees defines compositions $\circ_i$ of these trees. The newly formed internal edge is assigned the length 1.

Now let $R(o;\alpha) \subset TA(o;\alpha) \times TA(o;\alpha)$ be the spaces which are given by pairs of the following forms.

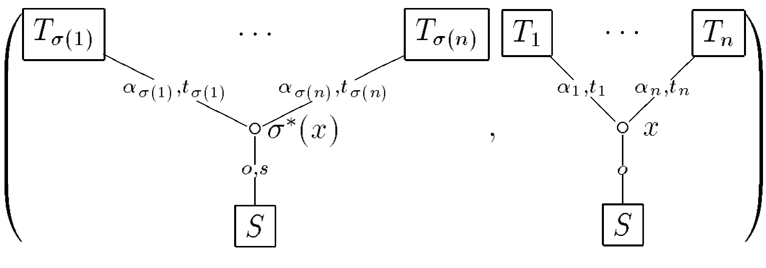
1. For all trees S and T in TA such that at least one contains at least one vertex, the pair



   is contained in R, i.e. vertices labelled by an identity can be deleted. The length of the new edge is the maximum of the lengths of the deleted edges.
2. For each $y \in A(\alpha_i;\beta)$ and $x \in A(o;\alpha)$ we have



   i.e. an edge of length 0 can be shrunk. The labels of its vertices are composed.
3. For each $\sigma \in \Sigma_n$ and $x \in A(o;\alpha)$ with $o \in C$ and $\alpha \in C^n$ we have



Remark 17.2. We write the lengths of the edges as a second color. Thus the pair α_i, t on an edge corresponds to the color α_i and the length t.

Definition 17.3. The *bar- or W-construction* WA of a C-operad A is the C-operad

$$WA = TA/(R).$$

The homomorphism $\varepsilon_A(o;\alpha) : WA(o;\alpha) \to A(o;\alpha)$ is given by shrinking the edges of a tree in WA to length 0. These form a morphism $\varepsilon_A : WA \to A$ of C-operads, the *augmentation* of A.

Definition 17.4. A homomorphism $f : A \to B$ of topological C-operads is called a *topological equivalence* if each map $f(o;\alpha)$ is a based and equivariant homotopy equivalence.

Here equivariant and based means, that there exist homotopy inverses $g(o;\alpha)$ of $f(o;\alpha)$ for a topological equivalence $f : A \to B$ such that

$$\begin{tikzcd} B(o;\alpha) \arrow[r, "\sigma^*"] \arrow[d, "g(o;\alpha)"'] & B(o;\alpha\sigma) \arrow[d, "g(o;\alpha\sigma)"] \\ A(o;\alpha) \arrow[r, "\sigma^*"'] & A(o;\alpha\sigma) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B(o;o) \arrow[dd, "g(o;o)"] \\ * \arrow[ur] \arrow[dr] & \\ & A(o;o) \end{tikzcd}$$

commute. Obviously this notion can be adapted to the non-symmetric and/or non-unitary cases.

Theorem 17.5. *(cmp. Prop. 3.6 of* [**BV73**]*) The augmentation $\varepsilon_A : WA \to A$ is a topological equivalence. Its inverse is given by the map $i : A \to WA$ of families, which maps an operation a in A to the tree in WA with only one vertex and label a.*

17.1. Lifting results. We will now recollect some lifting results from [**BV73**]. Most of them are stated in the homogeneous setting. But since each unitary C-operad can be viewed as the homogeneous part of a $C \times *$-operad these results can be applied to the non-homogeneous case also.

Definition 17.6. Two homomorphisms $f_0, f_1 : A \to B$ of C-operads are called *homotopic* in $oper_C\mathfrak{Top}$, if there exists a homotopy $f_t(o;\alpha) : A(o;\alpha) \to B(o;\alpha)$ such that for each $t \in [0,1]$, the maps f_t form a homomorphism $f_t : A \to B$ of C-operads.

Definition 17.7. A topological equivalence $f : A \to B$ is called a *homotopy equivalence of operads*, if there exists a morphism $g : B \to A$ of operads such that $f \circ g$ is homotopic in $oper_C\mathfrak{Top}$ to id_B and $g \circ f$ to id_A.

Since we require homogeneous operads to be parts of regular operads, these two definitions directly imply the corresponding notion for the homogeneous case. We just have to replace "morphism of operads" by "multiplicative morphism".

Definition 17.8. A $C \times L$-tree T, labelled or unlabelled, with or without vertex labels and lengths, is called *homogeneous* if all inputs have the same secondary color.

Let A be a $C \times L$-operad and T a homogeneous, unlabelled $C \times L$-tree with output color o^l and input colors $\alpha^{l'}$. The group $\Sigma(T)$ (see section 14.3) acts from the right on the space of all trees in TA, whose underlying tree lies in the shape orbit Λ_T. The action is described by the second relation on WA.

For $q \in \mathbb{N}, q \geq 1$, let $P_{T,q}$ be the subspace of $TA(o^l; \alpha^{l'})$ of all trees of shape T, which have at most q edges of a length less than 1 and whose inputs are labelled by the identity, i.e. numbered from left to right. The symmetry group $Sym(T) \subset \Sigma(T)$ of the shape T acts on $P_{T,q}$ from the right, by first applying the action of $\Sigma(T)$ and then relabelling the tree.

The subspace $Q_{T,q} \subset P_{T,q}$ is given by all elements such that either

1. at least one vertex is labelled by the identity,
2. at least one internal edge has the length 0,
3. there exists a collection of edges of length 1, which separates the tree into homogeneous trees or
4. there are less than q edges of a length less than 1.

In the unitary, non-homogeneous case the third condition is automatically satisfied, since there exists only one secondary color.

DEFINITION 17.9. Let A be a $C \times L$-operad and B a suboperad of WA. For each unlabelled, homogeneous $C \times L$-tree T let $B_{T,q}$ be the subspace of $P_{T,q}$, which consist of all elements representing an element in B. The suboperad B is called *admissible in $H_L WA$* if the following statements hold.

1. Each $B_{T,q}$ is closed in $P_{T,q}$.
2. Each inclusion $Q_{T,q} \cup B_{T,q} \hookrightarrow P_{T,q}$ is an $Sym(T)$-equivariant, closed cofibration.
3. If $x \circ_i y$ is an element of $H_L B$, then x and y are elements of $H_L B$.

THEOREM 17.10. *(cmp. Prop 3.14 of* [**BV73**]*) Let A and D be $C \times L$-operads and B an admissible suboperad of WA. Suppose there exists a multiplicative map $f : H_L WA \to H_L D$ and a homotopy $h_t : H_L B \to H_L D$ through multiplicative maps such that h_0 is the restriction of f to B. Then there exists a homotopy $f_t : H_L WA \to H_L D$ of multiplicative maps such that f_t is an extension of h_t and $f_0 = f$.*

THEOREM 17.11. **Lifting-Theorem** *(cmp. Thm. 3.17 of* [**BV73**]*)*
Given a diagram

$$\begin{tikzcd} H_L B \arrow[r, "k"] \arrow[d, hook, "i"'] \arrow[dr, dotted, "h_t"] & H_L E \arrow[d, "f"] \\ H_L WA \arrow[r, "g"'] & H_L D \end{tikzcd}$$

of the homogeneous parts of $C \times L$-operads and multiplicative maps between them such that

1. *B is an admissible suboperad of WA,*
2. *f is a topological equivalence, and*
3. *h_t is a homotopy of multiplicative maps from $f \circ k$ to $g \circ i$.*

Then there exists a lift $\bar{g} : H_L WA \to H_L E$ *such that* $\bar{g} \circ i = k$, *and a homotopy* H_t *of multiplicative maps from* $f\bar{g}$ *to* g, *which extends* h_t. *Any two such lifts are homotopic through multiplicative maps.*

REMARK 17.12. This theorem is a slight variation of the original lifting theorem in [**BV73**]. But the proof there can be adapted to show this more general result.

In the unitary, non-homogeneous case we assume $L = *$ and hence we can drop the restriction to the homogeneous parts and replace multiplicative by morphisms of C-operads.

17.2. Examples of admissible suboperads.

DEFINITION 17.13. A unitary C-operad X is called *wellpointed* if all maps $u_C(o) : * \to A(o; o)$ are closed cofibrations.

PROPOSITION 17.14. *Let* A *be a wellpointed* C*-operad. Then the suboperad* B *consisting of the identities is admissible.*

PROOF. We regard A as a $C \times *$-operad. Since the inclusion of $\{0, 1\}$ into the unit interval and the inclusion of the identity into $A(o; o)$ are closed cofibrations, the inclusion $Q_{T,q} \to P_{T,q}$ is a $Sym(T)$-equivariant closed cofibration for each unlabelled tree T and each natural number q. Furthermore $B_{T,q}$ is contained in $Q_{T,q}$, because a vertex of a tree in $B_{T,q}$ is either labelled by an identity, or it is part of a subtree with at least two vertices, whose edges all have length 0. Hence $B_{T,q}$ is contained in $Q_{T,q}$.

If a tree in $B_{T,q}$ contains an edge of length 1, then both parts represent an identity, since otherwise the tree can not be equivalent to an identity. □

LEMMA 17.15. *If* A *is wellpointed, then the trees in the* $T_C \times \{0, \ldots, n\}$*-operad* $W\mathbf{Mor}_A^n$, *whose edges all have the same secondary color, form an admissible suboperad of* $W\mathbf{Mor}_A^n$.

PROOF. First we have to prove that the trees with constant secondary color form a suboperad. But this is clear, because it is only possible to compose two of these trees if their secondary colors coincide.

Now let B be this suboperad. Then the spaces $B_{T,q}$ are either $P_{T,q}$ if the tree T contains only one secondary color, or empty otherwise. Hence the statement follows, if each $Q_{T,q} \hookrightarrow P_{T,q}$ is a closed, $Sym(T)$-equivariant cofibration. But this is true, since A is wellpointed.

If a tree in $B_{T,q}$ contains an edge of length 1, then the two parts have constant secondary colors, and hence are elements of $B_{T,q}$. □

As for $\mathbf{Mor}_A^n$, we have maps $\delta_i : W\mathbf{Mor}_A^{n-1}(\omega) \to W\mathbf{Mor}_A^n + (\delta_i \omega)$ for each $\omega \in T_{C \times \{0,\ldots,n-1\}}$, induced by the injective order preserving maps $\delta_i : \{0, \ldots, n-1\} \to \{0, \ldots, n\}$ for $0 \leq i \leq n$. Again only the secondary colors are changed. The images $\delta_i(W\mathbf{Mor}_A^{n-1}(\omega))$ form a $T_{C \times \{0,\ldots,n-1\}}$-operad, which we will denote by $D_i W\mathbf{Mor}_A^n$.

LEMMA 17.16. *Let* A *be a wellpointed* C*-operad and* $B \subset W\mathbf{Mor}_A^{n+1}$ *a suboperad generated by all or some faces* $D_i W\mathbf{Mor}_A^{n+1}$ *for* $i \in I \subset \{0, \ldots, n+1\}$. *Then* B *is admissible.*

PROOF. We are in the non-homogeneous case, i.e. we assume that $W\mathbf{Mor}_A^{n+1}$ is a $(T_C \times \{0, \ldots, n+1\}) \times *$-operad.

Let T be an unlabelled $T_C \times \{0, \ldots, n+1\}$-tree. If there exists an $i \in I$ such that no edge of T has the secondary color i, then each element of the space $P_{T,q}$ of Definition 17.9 represents an element in the i-th face of $W\mathbf{Mor}_A^{n+1}$. Hence in this case the space $B_{T,q}$ is precisely $P_{T,q}$.

If T contains edges of all secondary colors $i \in I$, then an element in $P_{T,q}$ represents an element in D if and only if there exist edges of length 1, which separate the tree into several subtrees which represent elements in one of the faces. Hence $B_{T,q}$ is a closed subspace of $Q_{T,q}$. Since A is wellpointed, $\mathbf{Mor}_A^{n+1}$ is also. Therefore the inclusion $Q_{T,q} \hookrightarrow P_{T,q}$ is a closed, $Sym(T)$-equivariant cofibration.

Since a tree in $W\mathbf{Mor}_A^{n+1}$ is decomposable if and only if it contains an edge of length 1, the two parts of a composition $x \circ_i y \in B$ are again elements of B. $\square$

18. Homotopy algebras and homotopy homomorphisms

18.1. Homotopy algebras. Now we can use the "cofibrant resolution" WA to define homotopy invariant notions of algebras over a C-operad and morphisms between them.

DEFINITION 18.1. A *homotopy algebra* (X, φ) of a C-operad A is a WA-algebra X with a structure homomorphism $\varphi : WA \to \mathbf{End}_C(X)$.

For each homomorphism $\Phi : W\mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$ of $C \times \{0, 1\}$-operads, we can define a homomorphism $d_1\Phi : WA \to \mathbf{End}_C(X)$, which is given by

$$d_1\Phi(T)(x_1, \ldots, x_n) = \Phi(T')(x_1, \ldots, x_n)$$

where $T' \in W\mathbf{Mor}_A(o^0; \alpha^0)$ is obtained from $T \in WA(o; \alpha)$ simply by adding the secondary color 0 to all edge colors. Or, in other words, $d_1\Phi$ is completely described by the images of monochrome trees of secondary color 0. Similar we can define $d_0\Phi : WA \to \mathbf{End}(Y)$.

DEFINITION 18.2. A *homotopy homomorphism* $(h, H) : (X, \varphi) \to (Y, \psi)$ of homotopy A-algebras consists of a $C \times \{0, 1\}$-operad homomorphism $H : W\mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$ such that $\varphi = d_1 H : WA \to \mathbf{End}_C(X)$ and $\psi = d_0 H : WA \to \mathbf{End}_C(Y)$, and a map $h : X \to Y$ of C-families, given by

$$h(o)(x) = H\left(\begin{array}{c} x \\ | \\ o^0 \\ \circ \ \ \varphi \\ | \\ o^1 \\ | \end{array}\right).$$

h is called the *underlying map* of H. (h, H).

In fact there are strong parallels to the algebraic setting, as described by M. Markl in [**Mar99**]. A homotopy algebra is an algebra over a cofibrant resolution of an C-operad, and a homotopy morphism between such algebras is an algebra over a cofibrant resolution of $\mathbf{Mor}_A$.

Unfortunately there is no obvious or natural way to define a composition of two homotopy homomorphisms. But Boardman and Vogt defined a simplicial class, which satisfies the restricted Kan-condition (see definition 4.8. of [**BV73**]). This implies the existence of a *fundamental category*, whose objects are the 0-simplices of the simplicial class, and whose morphisms are the simplicial homotopy classes of 1-simplices.

Let $F : W\mathbf{Mor}_A^n \to \mathbf{End}_C(X_0, \dots, X_n)$ be a homomorphism. Then we can define a homomorphism

$$d_i F : W\mathbf{Mor}_A^{n-1} \to \mathbf{End}_C(\dots, X_{i-1}, X_{i+1}, \dots)$$

for $0 \leq i \leq n$, given by $d_i F(T) = F(\delta_i T)$, where $\delta_i T$ is obtained from T by replacing the secondary colors according to the usual injective and order preserving map $\delta_i : \{0, \dots, n-1\} \to \{0, \dots, n\}$ (cmp. Example 16.10). Similarly the homomorphism

$$s_i F : W\mathbf{Mor}_A^{n+1} \to \mathbf{End}_C(\dots, X_i, X_i, \dots)$$

is given by $s_i F(T) = F(\sigma_i T)$ for the surjective and order preserving map $\sigma_i : \{0, \dots, n+1\} \to \{0, \dots, n\}$.

DEFINITION 18.3. **(and Lemma)** $\mathcal{S}\mathfrak{Map}_A$ is the simplicial class, whose n-simplices are the homomorphisms $W\mathbf{Mor}_A^n \to \mathbf{End}_C(X_0, \dots, X_n)$ of T_n-colored operads, with $X_0, \dots, X_n$ arbitrary spaces. The simplicial operations are given as above.

REMARK 18.4. The homomorphisms $d_1\Phi : WA \to \mathbf{End}_C(X)$ and $d_0\Phi : WA \to \mathbf{End}_C(Y)$, induced by $\Phi : WA \to \mathbf{End}_C(X, Y)$, are precisely the 0- and 1-face of the 1-simplex Φ in $\mathcal{S}\mathfrak{Map}_A$.

DEFINITION 18.5. (def. 4.10 of [**BV73**]) Two homotopy homomorphisms $(f, F), (g, G) : (X, \varphi) \to (Y, \psi)$ are called *simplicially homotopic*, if there exists a 2-simplex $H : W\mathbf{Mor}_A^2 \to \mathbf{End}_C(X, Y, Y)$ in $\mathcal{S}\mathfrak{Map}_A$ such that $d_2 H = F, d_1 H = G$ and $d_0 H = s_0 \psi$.

PROPOSITION 18.6. *(Thm. 4.9 of* [**BV73**]*) The simplicial class $\mathcal{S}\mathfrak{Map}_A$ satisfies the restricted Kan-condition.*

DEFINITION 18.7. Two homotopy homomorphisms $(f_0, F_0), (f_1, F_1) : (X, \varphi) \to (Y, \psi)$ are *homotopic*, if there exist a homotopy $F_t : W\mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$ through homomorphisms such that each F_t is a homotopy homomorphism from (X, φ) to (Y, ψ). The induced homotopy homomorphisms form a *homotopy* $(f_t, F_t) : (X, \varphi) \to (Y, \psi)$.

REMARK 18.8. Note that for the second version of homotopy the structures on the underlying spaces, induced by each F_t, do not change. This means $d_1 F_t = \varphi$ and $d_0 F_t = \psi$ for all $t \in [0, 1]$.

PROPOSITION 18.9. *(cmp. Lem. 4.9 of* [**BV73**]*) Two homotopy homomorphisms are simplicially homotopic if and only if they are homotopic.*

This allows us to define the fundamental category of $\mathcal{S}\mathfrak{Map}_A$ in the following way.

DEFINITION 18.10. ([**BV73**], section IV.2.) Let A be a topological C-operad. $\mathfrak{Map}_A$ is the category, whose objects are homotopy A-algebras (X, φ) and whose morphisms are homotopy classes of homotopy homomorphisms $(f, F) : (X, \varphi) \to (Y, \psi)$. The *composition* $(g, G)\square(f, F) : (X, \varphi) \to (Z, \zeta)$ of two such homotopy classes, represented by $(f, F) : (X, \varphi) \to (Y, \psi)$ and $(g, G) : (Y, \psi) \to (Z, \zeta)$, is given by the face $d_1 H$, of a 2-simplex H with $d_0 H = G$ and $d_2 H = F$.

Several homotopy invariance properties of these notions were proved in [**BV73**]. We just state the results and give a reference. For the formulation of these results, we use the same notation as in Example 16.9. This means that for a homomorphism $H : W\mathbf{Iso}_A \to \mathbf{End}_C(X, Y)$ we have homomorphisms $H_X : WA \to \mathbf{End}_C(X)$ and $H_Y : WA \to \mathbf{End}_C(Y)$, obtained by restriction to trees of constant secondary color. In addition we have homomorphisms $H_{X,Y} : W\mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$ and $H_{Y,X} : W\mathbf{Mor}_A \to \mathbf{End}_C(Y, X)$, given by interpreting every tree in $W\mathbf{Mor}_A$ as a tree in $W\mathbf{Iso}_A$ and, in the second case, exchanging the secondary colors.

PROPOSITION 18.11. *([**BV73**], 4.14) Let $(f, F) : (X, \varphi) \to (Y, \psi)$ be a homotopy homomorphism of homotopy A-algebras and $f = \{f(o)\}$ the family of underlying maps. If $g = \{g(o) : X(o) \to Y(o)\}$ is a map of C-families and h_t a family of homotopies from f to g, then there exists a homotopy homomorphism $(g, G) : (X, \varphi) \to (Y, \psi)$, with g as underlying map and a homotopy (h_t, H_T) from (f, F) to (g, G).*

PROPOSITION 18.12. *([**BV73**], 4.16) Let X and Y be two C-families and $\{p(c) : X(c) \to Y(c)\}_{c\in C}$ a family of topological equivalences. Then there exists a morphism $H : W\mathbf{Iso}_C \to \mathbf{End}_C(X, Y)$ such that the p are the underlying maps of the homomorphism $H_{X,Y}$.*

PROPOSITION 18.13. *([**BV73**], 4.17) Let $H : W\mathbf{Iso}_A \to \mathbf{End}_C(X, Y)$ be a homomorphism of operads. Then the two homotopy homomorphisms $(h_{X,Y}, H_{X,Y}) : (X, H_X) \to (Y, H_Y)$ and $(h_{Y,X}, H_{Y,X}) : (Y, H_Y) \to (X, H_X)$ are inverse to each other in $\mathfrak{Map}_A$.*

PROPOSITION 18.14. *([**BV73**], 4.18) Let B be a sub-C-operad of A such that each inclusion $B(o; \alpha) \to A(o; \alpha)$ is an equivariant, closed cofibration.*

If (X, φ) is a homotopy B-algebra, (Y, ψ) a homotopy A-algebra and if there exist a homomorphism $H : W\mathbf{Iso}_B \to \mathbf{End}_C(X, Y)$, with $H_Y = \psi|_{WB}$ and $H_X = \varphi$, then there exists a homomorphism $\bar{H} : W\mathbf{Iso}_A \to \mathbf{End}_C(X, Y)$ such that $H = \bar{H}|_{W\mathbf{Iso}_B}$ and $\bar{H}_Y = \psi$. This implies that there exists a homotopy B-algebra structure $\bar{H}_X$, extending φ.

PROPOSITION 18.15. *([**BV73**], 4.19) Let B be a sub-C-operad of A such that each inclusion $i(o; \alpha) : B(o; \alpha) \to A(o; \alpha)$ is an equivariant, closed cofibration. Furthermore let $(p, P) : (X, \varphi) \to (Y, \psi)$ be a homotopy homomorphism of homotopy A-algebras such that $p : X \to Y$ is a topological equivalence.*

If there exists a homomorphism $\bar{Q} : W\mathbf{Iso}_B \to \mathbf{End}_C(X, Y)$ such that $\bar{Q}_{X,Y} = P|_{W\mathbf{Mor}_B}$, then there exists an extension $Q : W\mathbf{Iso}_A \to \mathbf{End}_C(X, Y)$ of $\bar{Q}$ such that $Q_{X,Y} = P$.

18.2. Reduced Homotopy Homomorphisms. Before we describe a left adjoint of the functor $J : \mathfrak{HomTop}^A \to \mathfrak{Map}^A$, we restrict ourself to a special kind of homotopy homomorphism.

Let $W_r\mathbf{Mor}_A(o;\alpha)$ be the quotient of $W\mathbf{Mor}_A(o;\alpha)$ under the relation, that a tree T is equivalent to the tree T', obtained by shrinking all edges of secondary color 1 to length 0. These spaces form a C-operad $W_r\mathbf{Mor}_A$, which is topologically equivalent to $\mathbf{Mor}_A A$. The projections form a topological equivalence $\pi : W\mathbf{Mor}_A \to W_r\mathbf{Mor}_A$ of $C \times \{0,1\}$-operads.

Since each homomorphism $F : W_r\mathbf{Mor}_A \to \mathbf{End}_C(X,Y)$ can be interpreted as a homomorphism $W\mathbf{Mor}_A \to \mathbf{End}_C(X,Y)$, there exist homomorphisms $d_1(F \circ \pi) : WA \to \mathbf{End}(X)$ and $d_0(F\circ\pi) : WA \to \mathbf{End}(Y)$. But the latter one factors through A. Therefore we interpret d_1F as a homomorphism from A to $\mathbf{End}(Y)$ if $F : W_r\mathbf{Mor}_A \to \mathbf{End}(X,Y)$ is a homomorphism.

Definition 18.16. Let A be a C-operad, (X,φ) a homotopy A-algebra and (Y,ψ) a strict one. A *reduced homotopy homomorphism* $(f,F) : (X,\varphi) \to (Y,\psi)$ consists of a morphism $F : W_r\mathbf{Mor}_A \to \mathbf{End}_C(X,Y)$ such that $d_0F = \varphi$ and $d_1F = \psi$, and the underlying map $f : X \to Y$ of $F \circ \pi$.

On the first view, our definition of reduced homotopy homomorphisms seems to differ from the one introduced by Boardman and Vogt. They only require the homogeneous part to be reduced. But, as they noted in chapter 4, section 5 of [**BV73**], their reduced version of $W\mathbf{Mor}_A$ is precisely the homogeneous part of our version. As we will see in Lemma 18.21, this implies that the two notions are equivalent.

Notation 18.17. We will write HA for the homogeneous part $H_{\{0,1\}}A$ of a $C \times \{0,1\}$-operad A. We will use this notation especially for morphism operads.

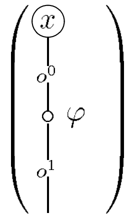
Definition 18.18. Let A be a C-operad and (X,φ) and (Y,ψ) be two homotopy algebras over A. A *homogeneous homotopy homomorphism* $(f,F) : (X,\varphi) \to (Y,\psi)$ consists of a multiplicative map $F : HW\mathbf{Mor}_A \to H\mathbf{End}_C(X,Y)$ with $d_0F = \varphi$ and $d_1F = \psi$, and an *underlying map* $f : X \to Y$ of C-families, given by

$$f(o)(x) = F\left(\begin{array}{c} (x) \\ | \\ o^0 \\ \circ \;\; \varphi \\ | \\ o^1 \\ | \end{array}\right)$$

Definition 18.19. Let A be a C-operad, (X,φ) a homotopy A-algebra and (Y,ψ) a strict one. A *reduced homogeneous homotopy homomorphism* $(f,F) : (X,\varphi) \to (Y,\psi)$ consists of a multiplicative map $F : HW_r\mathbf{Mor}_A \to H\mathbf{End}_C(X,Y)$ with $d_0F = \varphi$ and $d_1F = \psi$, and an *underlying map* $f : X \to$

Y of C-families, given by

$$f(o)(x) = F$$



LEMMA 18.20. *Let A be a C-operad and (X,φ) and (Y,ψ) homotopy A-algebras. Suppose there exists a family*

$$F(o^1;\alpha^0) : W\mathbf{Mor}_A(o^1;\alpha^0) \to \mathbf{End}_C(X,Y)(o^1;\alpha^0)$$

of maps such that

1. $F(R \circ (S_1,\dots,S_n)) = F(R) \circ (\varphi(S_1),\dots,\varphi(S_n))$ *if each S_i contains only edges of secondary color* 0, *and*
2. $F(R \circ (S_1,\dots,S_n)) = \psi(R) \circ (F(S_1),\dots,F(S_n))$ *if R contains only edges of secondary color* 1.

Then there exists a uniquely determined homogeneous homotopy homomorphism $(f,F) : (X,\varphi) \to (Y,\psi)$, which extends the given family.

The same result holds for reduced homogeneous homotopy homomorphisms if (Y,ψ) is a strict A-algebra and the given family is reduced.

PROOF. Obviously we have to use the structures φ and ψ to extend the given family to a map $F : HW_{(r)}\mathbf{Mor}_A \to H\mathbf{End}_C(X,Y)$ of families. The two conditions imply that this family is multiplicative. □

LEMMA 18.21. *Let A be a C-colored operad, (X,φ) a homotopy A-algebra and (Y,ψ) a strict one. If $(f,F) : (X,\varphi) \to (Y,\psi)$ is a reduced, homogeneous homotopy homomorphism, then there exists a unique reduced homotopy homomorphism $(f,\bar{F}) : (X,\varphi) \to (Y,\psi)$ such that $\bar{F}$ is an extension of the morphisms F on the homogeneous part of $W_r\mathbf{Mor}_A$.*

PROOF. Let T be a non-homogeneous tree of $W_r\mathbf{Mor}_A$, i.e. its output has the secondary color 1 and at least one of its inputs has the secondary color 0. Since we are in the reduced setting, the tree T is a composition of a tree of constant secondary color 1 and several reduced, homogeneous trees. Hence the image of T is uniquely determined by the given data.

Since $F : HW_r\mathbf{Mor}_A \to H\mathbf{End}_C(X,Y)$ is multiplicative, this construction is compatible with the relations on $W_r\mathbf{Mor}_A$. Hence we obtain an extension $\bar{F} : W_r\mathbf{Mor}_A \to \mathbf{End}_C(X,Y)$ of $\bar{F}$. □

For the composition of homotopy homomorphisms with reduced ones, Boardman and Vogt used a reduced version of $W\mathbf{Mor}^n_A$. Their "suboperad" consisted of all operations of $W\mathbf{Mor}^n_A$ whose inputs either all have the secondary color n or all have a secondary color less than n. In addition they introduced the additional relation, that all edges of secondary color n in these trees may be shrunk and their vertices composed. We will call this family $\tilde{W}_r\mathbf{Mor}^n_A$. Similar to homogeneous families, we have a restricted composition on this family. Therefore we can consider multiplicative maps.

We do not restrict to special operations, as Boardman and Vogt did. Instead we use the operad $W_r\mathbf{Mor}_A$, obtained from $W\mathbf{Mor}_A$, by applying the additional relation, that each tree T is equivalent to the tree T' obtained from T by shrinking each edge of secondary color n to length 0. The family $\tilde{W}_r\mathbf{Mor}^n_A$ is a subfamily of this operad. Using the same technique as in Lemma 18.21, we can prove

LEMMA 18.22. *Let $H : \tilde{W}_r\mathbf{Mor}^n_A \to \mathbf{End}(X_0, \ldots, X_n)$ be a multiplicative map of families. The there exists a homomorphism $\bar{H} : W_r\mathbf{Mor}^n_A \to \mathbf{End}(X_0, \ldots, X_n)$ of operads such that the restriction of $\bar{H}$ to $\tilde{W}_r\mathbf{Mor}^n_A$ is H.*

PROOF. If the output of a tree in $W_r\mathbf{Mor}^n_A$ has a secondary color less than n, then it is an element of $\tilde{W}_r\mathbf{Mor}^n_A$, and its image is given by H. If The output has the secondary color n and at least one input has a different secondary color, we can decompose the tree into one tree whose edges all have secondary color n and several trees in $\tilde{W}_r\mathbf{Mor}^n_A$. In this case the image is uniquely determined by the composition of the images of the smaller trees. If the tree has only inputs of secondary color n, then it is again contained in $\tilde{W}_r\mathbf{Mor}^n_A$, and its image is given by H. $\square$

There exist homotopy lifting results for the version of Boardman and Vogt, similar to Theorem 17.10 and Theorem 17.11, for reduced homotopy homomorphisms. For details the reader is referred to section 5 of chapter 4 of [**BV73**]. We only need a corollary of these results.

PROPOSITION 18.23. *Given a homotopy homomorphism $(f, F) : (X, \varphi_X) \to (Y, \varphi_Y)$ and a reduced homotopy homomorphism $(g, G) : (Y, \varphi_Y) \to (Z, \varphi_Z)$. Then there exists a reduced homotopy homomorphism $(h, H) : (X, \varphi_X) \to (Z, \varphi_Z)$, which is homotopic to the composition $(g, G)\Box(f, F)$ in $\mathfrak{Map}_A$.*

PROOF. Following prop. 4.46 of [**BV73**], there exists a multiplicative map $H : \tilde{W}_r\mathbf{Mor}^2_A \to \mathbf{End}(X, Y, Z)$ such that the homotopy homomorphism $d_2H : W\mathbf{Mor}_A \to \mathbf{End}(X, Y)$, given by the secondary colors 0 and 1, is precisely F, and such that the reduced homotopy homomorphism $d_0H : \tilde{W}_r\mathbf{Mor}_A \to \mathbf{End}(Y, Z)$ is G. By Lemma 18.22 this induces a homomorphism $\bar{H} : W\mathbf{Mor}^2_A \to W_r\mathbf{Mor}^2_A \to \mathbf{End}(X, Y, Z)$, with $d_2\bar{H} = F, d_0\bar{H} = G$. The composition $d_1\bar{H}$ of (f, F) and (g, G) in $\mathfrak{Map}_A$, is a reduced homotopy homomorphism. $\square$

COROLLARY 18.24. *Each homotopy homomorphism $(f, F) : (X, \varphi) \to (Y, \psi \circ \varepsilon)$ into a strict A-algebra is homotopic to a reduced one.*

PROOF. We use the fact that the identity of Y induces a reduced homotopy homomorphism from $(Y, \psi \circ \varepsilon)$ to (Y, ψ). Therefore there exists a reduced homotopy homomorphism, which is homotopic to the composition of (f, F) with the identity, and thus to (f, F) itself. $\square$

DEFINITION 18.25. (cmp. Remark 4.5 of [**BV73**]) Let (X, φ_X) be a homotopy A-algebra and (Y, φ_Y) and (Z, φ_Z) be two strict ones. Furthermore let $(f,F):(X,\varphi_X)\to(Y,\varphi_Y\circ\varepsilon)$ and $(g,G):(X,\varphi_X)\to(Z,\varphi_Z\circ\varepsilon)$ be two (reduced) homotopy homomorphisms and $h:(Y,\varphi_Y)\to(Z,\varphi_Z)$ a strict homomorphism. (g,G) is called a *(reduced) canonical composition* of (f,F) and h, if for all $o\in C$, $\bar\alpha\in(C\times\{0,1\})^n$, $T\in W\mathbf{Mor}_A(o^1;\bar\alpha)$ and $u_1,\dots,u_n$, with $u_i\in X(\alpha_i)$ for $\bar\alpha_i=\alpha_i^0$ and $u_i\in Y(\alpha_i)$, for $\bar\alpha_i=\alpha_i^1$, the following equation holds:

$$h\left(F(T)(u_1,\dots,u_n)\right)=G(T)(v_1,\dots,v_n),$$

where $v_i=u_i$ if $\bar\alpha_i=\alpha_i^0$ and $v_i=h(u_i)$ if $\bar\alpha_i=\alpha_i^1$.

LEMMA 18.26. *If $(f,F):(X,\varphi_X)\to(Y,\varphi_Y)$ is a reduced homotopy homomorphism and $h:(Y,\varphi_Y)\to(Z,\varphi_Z)$ a strict homomorphism of A-algebras, then there exists a uniquely determined reduced canonical composition $h(f,F)$ of h and (f,F). The underlying homomorphism $G:W_r\mathbf{Mor}_A\to\mathbf{End}(X,Z)$ of $h(f,F)$ is given by*

$$G(T)(u_1,\dots,u_n)=h\left(F(T)(u_1,\dots,u_n)\right)$$

for all homogeneous trees $T\in W_r\mathbf{Mor}_A(o^1;\alpha^0)$. Furthermore $h(f,F)$ is the only reduced canonical composition.

PROOF. The equation defines a family $G(o^1;\alpha^0):W_r\mathbf{Mor}_A(o^1;\alpha^0)\to\mathbf{End}_C(X,Z)(o^1;\alpha^0)$ of maps, which satisfy the conditions of Lemma 18.20. Hence there exists a reduced homogeneous homotopy homomorphism $(g,G):(X,\varphi_X)\to(Z,\varphi_Z\circ\varepsilon)$, which, following Lemma 18.21, can be extended to a reduced homotopy homomorphism $h(f,F)$.

Since each tree in $W_r\mathbf{Mor}_A$ whose output has the secondary color 1 either is homogeneous or can be decomposed into a tree whose edges all have secondary color 1 and several homogeneous trees, (g,G) is a reduced canonical composition.

As we have seen, the equation in the statement already uniquely describes a reduced canonical composition. Since the equation is required by the properties of a canonical composition, this implies that each reduced canonical composition is of the given form. □

A simple check of the construction of the reduced canonical composition proves the following

COROLLARY 18.27. *Let $f:(X,\varphi_X)\to(Y,\varphi_Y)$ and $g:(Y,\varphi_Y)\to(Z,\varphi_Z)$ be homomorphisms of A-algebras and $(e,E):(W,\varphi_W)\to(X,\varphi_X)$ a reduced homotopy homomorphism. Then we have*

$$g\left(f(e,E)\right)=(g\circ f)(e,E).$$

THEOREM 18.28. *Let (X,φ_X) be a homotopy A-algebra and (Y,φ_Y) and (Z,φ_Z) two strict ones. If $(g,G):(X,\varphi_X)\to(Y,\varphi_Y)$ is a reduced homotopy homomorphism and $f:(Y,\varphi_Y)\to(Z,\varphi_Z)$ a strict homomorphism, then the reduced canonical composition $f(g,G)$ is a composition of Jf and (g,G) in $\mathfrak{Map}_A$.*

PROOF. Let B be the operad, which consists of the quotients of $W\mathbf{Mor}_A^2$ under the additional relation, that each tree T is equivalent to the tree T', obtained by shrinking the lengths of all edges of secondary colors 1 and 2 to 0. The compositions on B are induced by the compositions in $W\mathbf{Mor}_a^2$, i.e. by the grafting of trees. Therefore the projection $W\mathbf{Mor}_A^2 \to B$ is a homomorphism of operads.

For $i \in \{0,1\}$ let

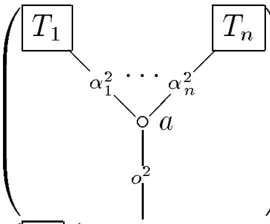
$$\bar{H} : B(o^i;\bar{\alpha}) \to \mathbf{End}(X,Y,Z)(o^i;\bar{\alpha})$$

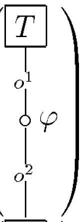
be given by G. Since the relations on $B(o^i;\bar{\alpha})$ are precisely the relations on $W_r\mathbf{Mor}_A$, this maps are well defined.

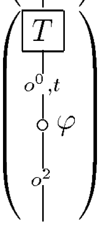
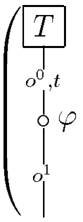
For $B(o^2;\bar{\alpha})$, we use a recursive definition. Let T be a tree representing an element in $B(o^2;\bar{\alpha})$. If T has no vertex, $\bar{H}$ is given by

$$\bar{H}\left(\Big|_{o^2} \right) = \mathrm{id}_Z.$$

If T has at least one vertex, we have three cases.

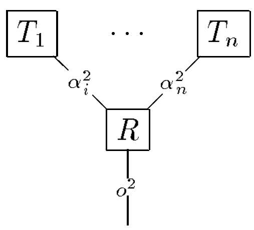
1. $\bar{H}\Bigg($  $\Bigg) = \varphi_Z(a) \circ \big(\bar{H}(T_1),\dots,\bar{H}(T_n)\big)$

2. $\bar{H}\Bigg($  $\Bigg) = f\,(G(T))$

3. $\bar{H}\Bigg($  $\Bigg) = f\Bigg(G\Bigg($  $\Bigg)\Bigg)$.

In all three cases we neglect the lengths of all edges of secondary color 1 or 2 and assume that they have length 1. Therefore $\bar{H}$ respects the additional relations on B. It remains to prove that it respects the relations on $W\mathbf{Mor}_A^2$.

Since G respects all relations on all trees of B, whose output has the secondary color 0 or 1, it suffices to check all relations involving edges of secondary color 2. But first observe that for a tree of the form

$T =$ 

the equation

$$\bar{H}(T) = \varphi_Z(R) \circ \big(\bar{H}(T_1),\dots\bar{H}(T_n)\big)$$

holds. Therefore we only need to check the relations occurring at the root vertex. Otherwise, we can decompose the given tree as above such that one of the root vertices of the T_i is involved.

The equivariance relation is respected, since we have

$$\varphi_Z(\sigma^*(a))\left(\bar{H}(T_{\sigma(1)}),\dots \bar{H}(T_{\sigma(n)})\right) = \varphi_Z(a)\circ\left(\bar{H}(T_1),\dots,\bar{H}(T_n)\right).$$

If the root vertex is labelled by an identity, then we have

$$\varphi_Z(\mathrm{id})\circ(\bar{H}(T_1)) = \bar{H}(T_1),$$

which implies that we can delete it.

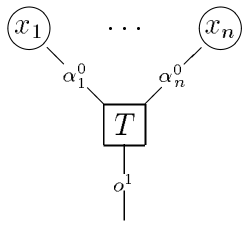
In the case that an input of the root vertex has the length 0, we use the fact, that f is a strict and G a reduced homotopy homomorphism. This implies that the map respects the composition of two vertices as well, as the interchange with φ-vertices. $\square$

18.3. The rectification. In this section we describe a rectification of a homotopy A-algebra X, i.e. we construct a strict A-algebra M_AX, which is topologically equivalent to X. Furthermore M_AX satisfies several universal properties, regarding reduced homotopy homomorphisms.

For $o\in C$ let $\bar{M}_AX(o)$ be the space

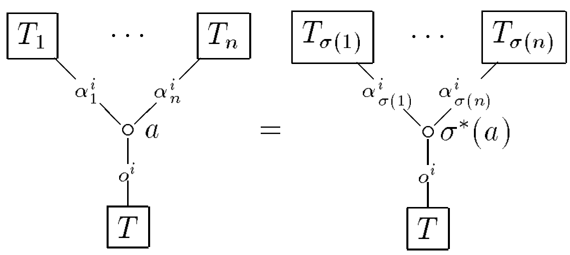
$$\bar{M}_AX(o) = \bigoplus_{\alpha\in C^n} W_r\mathbf{Mor}_A(o^1;\alpha^0)\times X(\alpha).$$

Each element $(T;x_1,\dots,x_n)\in\bar{M}_AX(o)$ can be interpreted as a cherry tree of the following form.



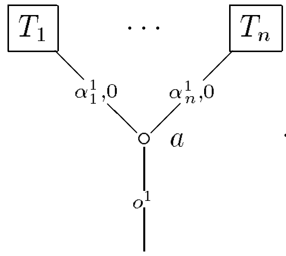
$M_AX(o)$ is the quotient of $\bar{M}_AX(o)$ under the following relations:

1. $(T\circ_i S;x_1,\dots,x_{m+n-1}) = (T;x_1,\dots,x_{i-1},y,x_{i+n},\dots,x_{m+n-1})$ with $y=\varphi(S)(x_i,\dots,x_{i+n-1})$ for $T\in W_r\mathbf{Mor}_A(o^1;\alpha^0)$ and $S\in W_r\mathbf{Mor}_A(\alpha_i^0;\beta^0), \alpha\in C^m,\beta\in C^n$ and $1\le i\le m$. In other words a cherry tree of secondary color 0 above an edge of length 1 can be replaced by its value in X.
2. For all cherry trees $T_1,\dots,T_n, a\in A(o;\alpha)$, each tree T and $\sigma\in\Sigma_n$ we have



This also holds, if the trees T_i consist only of cherries.

The evaluations $ev : A(o;\alpha) \times M_A X(\alpha) \to M_A X(o)$ are given by

$(a; (T_1, \mathbf{x}^1), \dots, (T_n, \mathbf{x}^n)) \mapsto$ 

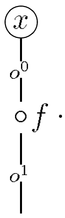
.

This means we graft the cherry trees $T_1, \dots, T_n$ along their roots to the inputs of a tree with one vertex, output color o^1, input colors α_i^1 and vertex label a. Since the new internal edges have the color 1, their lengths are not important and can be set to 0. The reduction of edges of secondary color 1 implies that this defines an A-algebra structure $\bar{\varphi}$ on $M_A X$.

THEOREM 18.29. *(Thm. 4.49 of* [**BV73**]*) Let A be a C-operad and (X,φ) a homotopy A-algebra and (Y,ψ) a strict one.*

1. *There exists a reduced homotopy homomorphism $(i_X, I_X) : (X,\varphi) \to (M_A X, \bar{\varphi})$.*
2. *For each reduced homotopy homomorphism $(f,F) : (X,\varphi) \to (Y, \psi \circ \varepsilon)$ there exists a unique homomorphism $\bar{f} : (M_A X, \bar{\varphi}) \to (Y,\psi)$ of A-algebras such that $(f,F) = \bar{f}(i_X, I_X)$.*
3. *If $(g,G) : (X,\varphi) \to (Y,\psi\circ\varepsilon)$ is homotopic to (f,F), then the induced homomorphisms $\bar{f}$ and $\bar{g}$ are homotopic through homomorphisms of A-algebras.*

PROOF. Let $i_X(o) : X(o) \to M_A X(o)$ be given by

$x \mapsto$ 

.

For $T \in W_r\mathbf{Mor}_A(o^1;\bar{\alpha})$ with $o \in C$ and $\bar{\alpha} \in (C \times \{0,1\})^n$, the map $I_X : W_r\mathbf{Mor}_A \to \mathbf{End}_C(X, M_A X)$ is given by

$$I_X(T)(u_1, \dots, u_n) = (T \circ (S_1, \dots, S_n); \mathbf{v}_1, \dots, \mathbf{v}_n)$$

where

$$(S_i, \mathbf{v}_i) = \begin{cases} (\mathrm{id}, u_i) & \text{if } \bar{\alpha}_i = \alpha_i^0 \text{ and } u_i \in X \\ (S, \mathbf{z}) & \text{if } \bar{\alpha}_i = \alpha_1^1 \text{ and } u_i = (S, \mathbf{v}) \in M_A X. \end{cases}$$

Graphically we reinterpret the cherries of $(T; u_1, \dots, u_n)$ again as cherry trees. For $\bar{\alpha}_i = \alpha_i^1$ the cherry y_i is a cherry tree $(S, \mathbf{v})$ in $W_r\mathbf{Mor}_A$, with output color 1 and input colors 0. For $\bar{\alpha}_i = \alpha_i^0$ the cherry $y_i \in X(\alpha_i)$ can be interpreted as the cherry tree consisting of one vertex with the identity as label, the cherry y_i and output and input color 0. Since the output colors of these small trees are the same as the color of the corresponding inputs of T, we can compose them and obtain a bigger cherry tree with output color 1 and input colors 0, which represents an element in $M_A X$. For

$T \in W_r\mathbf{Mor}_A(o^0; \alpha^0)$ the homomorphism $I_X : W_r\mathbf{Mor}_A \to \mathbf{End}(X, M_AX)$ is given by the structure homomorphisms φ of X.

The first relation on M_AX ensures that the restriction of I_X to $W_r\mathbf{Mor}_A(o^1; \alpha^1)$ is exactly the structure $\bar{\varphi} \circ \varepsilon$ of M_AX. Therefore we have a reduced homotopy homomorphism $(i_X, I_X) : W_r\mathbf{Mor}_A \to \mathbf{End}_C(X, M_AX)$.

Now let $(f, F) : (X, \varphi) \to (Y, \psi \circ \varepsilon)$ be a reduced homotopy homomorphism of homotopy A-algebras. Since for each homogeneous tree $T \in W_r\mathbf{Mor}_A(o^1; \alpha^0)$ and $(x_1, \dots, x_n) \in X(\alpha)$ the equation

$$I(T)(x_1, \dots, x_n) = (T; x_1, \dots, x_n) \in M_AX(o)$$

holds, the map $\bar{f}(o) : M_AX(o) \to Y(o)$ is uniquely determined by the equation of Definition 18.25.

For $T \in W_r\mathbf{Mor}_A(o^1; \bar{\alpha})$ and $u_i \in X(\alpha_i)$ if $\bar{\alpha}_i = \alpha_i^0$ and $u_i \in M_AX(\alpha_i)$ if $\bar{\alpha}_i = \alpha_i^1$, we have

$$\begin{aligned}
\bar{f}(I(T)(u_1, \dots, u_n)) &= \bar{f}(T \circ (S_1, \dots, S_n); \mathbf{v}_1, \dots, \mathbf{v}_n) \\
&= F(T \circ (S_1, \dots, S_n))(\mathbf{v}_1, \dots, \mathbf{v}_n) \\
&= F(T)(F(S_1)(\mathbf{v}_1), \dots, F(S_n)(\mathbf{v}_n)),
\end{aligned}$$

where the $(S_i, \mathbf{v}_i)$ are given as above. Since this equation holds, and since (f, F) is reduced, the map $\bar{f}$ of C-families is an A-homomorphism. This proves part (b) of the theorem.

(c) is a direct consequence of (b). If $H_t : (X, \varphi) \to (Y, \psi \circ \varepsilon)$ is a homotopy through reduced homotopy homomorphisms from F to G, then there exists a unique homomorphism $\bar{h}_t : (M_AX, \bar{\varphi}) \to (Y, \psi)$ for each $t \in [0, 1]$ with

$$\bar{h}_t(o)(T; x_1, \dots, x_n) = H_t(T)(x_1, \dots, x_n)$$

for each $(T; x_1, \dots, x_n) \in M_AX(o)$. Therefore the $\bar{h}_t$ form a homotopy through homomorphisms of A-algebras from $\bar{f}$ to $\bar{g}$. $\square$

NOTATION 18.30. The image of a reduced homotopy homomorphism $(f, F) : (X, \varphi) \to (Y, \psi \circ \varepsilon)$ will be called $\bar{f}$.

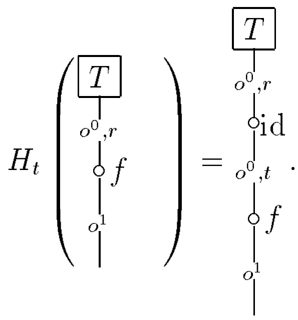
COROLLARY 18.31. *Let (X, φ) be a homotopy A-algebra and (Y, ψ) a strict one. Then the set of reduced homotopy homomorphisms $(f, F) : (X, \varphi) \to (Y, \psi \circ \varepsilon)$ is bijective with the set of homomorphisms $\bar{f} : (M_AX, \bar{\varphi}) \to (Y, \psi)$. The bijection is given by $(f, F) \mapsto \bar{f}$ and its inverse by $\bar{f} \mapsto \bar{f}\square(i_X, I_X)$.*

THEOREM 18.32. *(cmp. Thm 4.49 of* [**BV73**]*) The maps $i_X(o) : X(o) \to M_AX(o)$, underlying the reduced homotopy homomorphism $(i_X, I_X) : (X, \varphi) \to (M_AX, \bar{\varphi} \circ \varepsilon)$, are homotopy equivalences.*

PROOF. The inclusion $X(o) \to M_AX(o)$ is given as above.

Obviously each element $(S, \mathbf{x})$ of $M_AX(o)$ can be represented by an element $(S', \mathbf{x}')$ such that S' has no internal edge of color 1, i.e. its only edge of color 1 is the root. The deformation retraction $H_t : M_AX(o) \to M_AX(o)$

is given on these representations by



For $t = 0$ this is the identity and for $t = 1$, we can reduce the tree above to its image in $X(o)$, and therefore obtain an element in the image of the inclusion i_X. $\square$

18.4. The Homotopy categories. In addition to the simplicially defined category $\mathfrak{Map}_A$, we can define a homotopy category of strict A-algebras in a more classical way.

Definition 18.33. Let (X, φ) and (Y, ψ) be two strict A-algebras. Two homomorphisms $f, g : (X, \varphi) \to (Y, \psi)$ of A-algebras are called *homotopic* if there exists a homotopy $h_t : (X, \varphi) \to (Y, \psi)$ from f to g such that each $h_t : (X, \varphi) \to (Y, \psi)$ is a homomorphism.

Remark 18.34. If we interpret h_t as a homotopy $h_t : \mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$ of homomorphisms, then the homomorphisms $d_0 h_t : A \to \mathbf{End}_C(X)$ and $d_1 h_t : A \to \mathbf{End}_C(Y)$, induced by the trees of secondary color 0 and 1, are precisely φ and ψ.

Definition 18.35. Let A be a topological C-operad. $\mathfrak{HomTop}^A$ is the category, whose objects are A-algebras (X, φ) and whose morphisms are homotopy classes of strict A-homomorphisms $f : (X, \varphi) \to (Y, \psi)$. The composition is given by the composition of the representing morphisms.

Since the augmentation $\varepsilon_A : WA \to A$ is a morphism of C-operads, we can interpret every A-algebra (X, φ) as a homotopy A-algebra $(X, \varphi \circ \varepsilon)$. A strict A-morphism $f : (X, \varphi) \to (Y, \psi)$ induces a homotopy morphism $(f, F) : (X, \varphi \circ \varepsilon_{\mathbf{Mor}_A}) \to (Y, \psi \circ \varepsilon_{\mathbf{Mor}_A})$, with $F = f \circ \varepsilon_{\mathbf{Mor}_A} : W\mathbf{Mor}_A \to \mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$. Therefore we obtain a functor $J : \mathfrak{HomTop}^A \to \mathfrak{Map}^A$ with $J(X, \varphi) = (X, \varphi \circ \varepsilon)$.

The rectification of homotopy algebras induces a functor $M_A : \mathfrak{Map}_A \to \mathfrak{HomTop}^A$. The image $M_A(f, F) : (M_A X, \bar{\varphi}) \to (M_A Y, \bar{\psi})$ of a homotopy homomorphism $(f, F) : (X, \varphi) \to (Y, \varphi)$ is given by the homomorphism induced by the composition $(i_Y, I_Y) \square (f, F)$.

$$\begin{tikzcd}
(X, \varphi) \arrow[r, "{(i_X, I_X)}"] \arrow[d, "{(f,F)}"'] & (M_A X, \bar{\varphi} \circ \varepsilon) \arrow[d, dashed, "{M_A(f,F)}"] \\
(Y, \psi) \arrow[r, "{(i_Y, I_Y)}"'] & (M_A Y, \bar{\psi} \circ \varepsilon).
\end{tikzcd}$$

Here we use that, according Proposition 18.23, the composition $(i_Y, I_y)\Box(f, F)$ can be chosen to be a reduced homotopy homomorphism. This defines a functor, because the diagram

$$\begin{tikzcd}
(X,\varphi_X) \arrow[r, "(i_X,I_X)"] \arrow[d, "(f,F)"'] & (M_AX, \bar\varphi_X\circ\varepsilon) \arrow[d, "M_A(f,F)"] \\
(Y,\varphi_Y) \arrow[r, "(i_Y,I_Y)"] \arrow[d, "(g,G)"'] & (M_AY, \bar\varphi_Y\circ\varepsilon) \arrow[d, "M_A(g,G)"] \\
(Z,\varphi_Z) \arrow[r, "(i_Z,I_Z)"] & (M_AZ, \bar\varphi_Z\circ\varepsilon)
\end{tikzcd}$$

commutes in $\mathfrak{Map}_A$ for any two homotopy homomorphisms (g, G) : $(Y, \varphi_Y) \to (Z, \varphi_Z)$ and $(f, F) : (X, \varphi_X) \to (Y, \varphi_Y)$, and because

$$\begin{aligned}(M_A(g,G)\circ M_A(f,F))\,(i_x, I_X) &= J\,(M_A(g,G)\circ M_A(f,F))\Box(i_x, I_X)\\ &= JM_A(g,G)\Box JM_A(f,F)\Box(i_x,I_X).\end{aligned}$$

Therefore the uniqueness of the construction implies that $M_A(g,G)\circ M_A(f,F)$ is homotopic to $M_A\left((g,G)\Box(f,F)\right)$. According to Corollary 18.31, we even have

Corollary 18.36. *(cmp. 4.51 of* **[BV73]***) The functor $M_A : \mathfrak{Map}_A \to \mathfrak{HomTop}^A$ is left adjoint to the functor $J : \mathfrak{HomTop}^A \to \mathfrak{Map}_A$. The unit of this adjunction is given by the reduced canonical homotopy homomorphisms $(i_X, I_X) : (X,\varphi) \to (M_AX, \bar\varphi)$. The counit $\eta_Y : (M_AY, \overline{\psi\circ\varepsilon}) \to (Y,\psi)$ is induced by the identity of an strict A-algebra (Y,ψ), i.e.*

$$\begin{tikzcd}
(Y,\psi\circ\varepsilon) \arrow[r, "(i_Y,I_Y)"] \arrow[dr, "id"'] & (M_AY, \overline{\psi\circ\varepsilon}) \arrow[d, "J\eta_Y"] \\
 & (Y,\psi\circ\varepsilon)
\end{tikzcd}$$

Proof. Corollary 18.31 implies that the set of homotopy classes of homomorphisms $\bar f : (M_AX,\bar\varphi) \to (Y,\psi)$ is bijective with the set of homotopy classes of reduced homotopy homomorphisms $(f,F) : (X,\varphi) \to (Y,\psi)$. According to Corollary 18.24 each homotopy homomorphism is homotopic to a reduced one. Hence we obtain a bijection

$$\mathfrak{HomTop}^A\left((M_AX,\bar\varphi),(Y,\psi)\right) \simeq \mathfrak{Map}_A\left((X,\varphi),(YX,\psi\circ\varepsilon)\right).$$

The unit of the resulting adjunction is given by the images of the identities of $(M_AX,\bar\varphi)$, and therefore by (i_X, I_X). The counit is given by the images of the identities of $(Y,\psi\circ\varepsilon)$, and therefore by η_Y. $\Box$

Now we will recollect the results of Boardman and Vogt which show that $\mathfrak{Map}_A$ is a model of the homotopy category of topological A-algebras.

Definition 18.37. Let $\mathcal{V}$ be a category and $\Sigma \subset \mathcal{V}$ a family of morphisms in $\mathcal{V}$. The *localization* $\mathcal{V}[\Sigma^{-1}]$ of $\mathcal{V}$ along Σ is a category together with a functor $P : \mathcal{V} \to \mathcal{V}[\Sigma^{-1}]$ such that for each functor $F : \mathcal{V} \to \mathcal{D}$ which maps all morphisms of Σ to isomorphisms, there exists a unique functor $\bar{F} : \mathcal{V}[\Sigma^{-1}] \to \mathcal{D}$ with $F = \bar{F} \circ P$.

It is well known that the localization $\mathcal{V}[\Sigma^{-1}]$ can be constructed similarly to the free group of a monoid (see for example [**GZ67**]). Furthermore $\mathcal{V}[\Sigma^{-1}]$ has the same objects as $\mathcal{V}$.

Now let Σ be the class of topological equivalences in $\hom \mathfrak{Top}^A$. Since J preserves these equivalences, it induces a functor $\hat{J} : \mathfrak{HomTop}^A[\Sigma^{-1}] \to \mathfrak{Map}_A$.

THEOREM 18.38. *(cmp. 4.53 of* [**BV73**]*) The functor $\hat{J}$ is an equivalence of categories, whose adjoint is $P \circ M$. The unit $id \to \hat{J} \circ (P \circ M) = J \circ M$ of this adjunction is given by the natural, reduced homotopy homomorphisms $(i_X, I_X) : (X, \varphi) \to (M_A X, \bar{\varphi})$.*

PROOF. The natural transformation $(i_X, I_X) : \mathrm{id} \to J \circ M = \hat{J} \circ (P \circ M)$ is a natural isomorphism. In addition the counit $\eta : M \circ J \to \mathrm{id}$ induces a natural transformation from $(P \circ M) \circ J$ to P. According to lemma 1.2 in chapter 1.1 of [**GZ67**], this implies that the morphisms $P\eta$ form a natural transformation from $(P \circ M) \circ \hat{J}$ to the identity on $\mathfrak{HomTop}^A[\Sigma^{-1}]$. Since each $\eta_{(Y,\psi_Y)} : (M_A Y, \overline{\psi \circ \varepsilon}) \to (Y, \psi)$ is a topological equivalence, $P\eta$ is a natural isomorphism. $\square$

Since the projection $\mathfrak{Top}^A \to \mathfrak{HomTop}^A$ maps topological equivalences to topological equivalences, we also get a functor $\mathfrak{Top}^A[\Sigma^{-1}] \to \mathfrak{HomTop}^A[\Sigma^{-1}]$. Here we abusively use Σ for both, the class of topological equivalences in $\mathfrak{Top}^A$ and in $\mathfrak{HomTop}^A$, but this should not cause problems.

PROPOSITION 18.39. *(cmp. 4.54 of* [**BV73**]*) The functor $\mathfrak{Top}^A[\Sigma^{-1}] \to \mathfrak{HomTop}^A[\Sigma^{-1}]$ is an equivalence of categories.*

18.5. Homotopies between Homotopy Homomorphisms. Up to this point we have considered only homotopies $H_t : W\mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$ through (reduced) homotopy homomorphisms, which are constant on the monochrome parts, i.e. $d_i H_t = d_i H_0$. This was necessary, since each stage of the homotopy should represent the same (reduced) homotopy homomorphism between the homotopy (or strict) A-algebras $(X, d_1 H_0)$ and $(Y, d_0 H_0)$. In this section we are going to examine homotopies $H_t : W\mathbf{Mor}_A \to \mathbf{End}_C(X, Y)$, which do not fulfill this condition.

LEMMA 18.40. *Let A be a wellpointed C-operad and X a space. If there exists a homotopy $\varphi_t : WA \to \mathbf{End}_C(X)$ through homomorphisms, then the homotopy A-algebras (X, φ_0) and (X, φ_1) are isomorphic.*

PROOF. Following Lemma 17.16 the suboperad $B \subset W\mathbf{Mor}_A$ generated by the faces $D_0 W\mathbf{Mor}_A$ and $D_1 W\mathbf{Mor}_A$, i.e. the two copies of WA, is admissible. Let $\bar{\Phi}_t : B \to \mathbf{End}_C(X, X)$ be given on $D_1 W\mathbf{Mor}_A$ by the constant homotopy on φ_0 and on $D_0 W\mathbf{Mor}_A$ by the homotopy φ_t. The homomorphism $\bar{\Phi}_0$ is a restriction of the 1-simplex $s_1\varphi_0 : W\mathbf{Mor}_A \to \mathbf{End}_C(X, X)$ in $S\mathfrak{Map}_A$. By Theorem 17.10 exists an extension $\Phi_t : W\mathbf{Mor}_A \to \mathbf{End}_C(X, X)$

of $\bar{\Phi}_t$ such that $\Phi_0 = s_1\varphi_0$. Graphically we represent this extension by the diagram

$$\begin{tikzcd}
\varphi_0 \arrow[r, dotted, "\Phi_1"] \arrow[d, equal] \arrow[dr, phantom, "\Phi_t"] & \varphi_1 \arrow[d, dash, "\varphi_t"] \\
\varphi_0 \arrow[r, equal, "s_1\varphi_0"'] & \varphi_0
\end{tikzcd}$$

Similar we obtain an extension $\Psi_t : W\mathbf{Mor}_A \to \mathbf{End}_C(X,X)$ of the form

$$\begin{tikzcd}
\varphi_1 \arrow[r, dotted, "\Psi_1"] \arrow[d, dash, "\varphi_t"'] \arrow[dr, phantom, "\Psi_t"] & \varphi_0 \arrow[d, equal, "s_1\varphi_0"] \\
\varphi_0 \arrow[r, equal] & \varphi_0
\end{tikzcd}$$

Now let $D \subset W\mathbf{Mor}_A^2$ be the admissible suboperad generated by all faces of $W\mathbf{Mor}_A^2$. The homotopy $\bar{H}_t : D \to \mathbf{End}_C(X,X,X)$ is given on $D_2W\mathbf{Mor}_A^2$ by Φ_t, on $D_0W\mathbf{Mor}_A^2$ by Ψ_T and on $D_1W\mathbf{Mor}_A^2$ by $s_1\varphi_0$.

$$\begin{tikzcd}
\varphi_0 \arrow[rr, equal] \arrow[dr] \arrow[dd, equal] & & \varphi_0 \arrow[dd, equal] \\
 & \varphi_1 \arrow[ur] \arrow[dd, dash] & \\
\varphi_0 \arrow[dr, equal] \arrow[ur, phantom, "\Phi_t"] & & \varphi_0 \arrow[dl, equal] \arrow[ul, phantom, "\Psi_t"] \\
 & \varphi_0 &
\end{tikzcd}$$

Since the homotopies Φ_t and Ψ_t and the constant homotopy on $s_1\varphi_0$ coincide on the monochrome parts, i.e. on the intersections $D_iD_jW\mathbf{Mor}_A^2$ of two faces, this homotopy is well defined.

The homomorphism $\bar{H}_0$ is the restriction of $s_1s_1\varphi_0$. By Theorem 17.10 exists a homotopy $H_t : W\mathbf{Mor}_A^1 \to \mathbf{End}_C(X,X,X)$ through homomorphisms, which extends $\bar{H}_t$ and such that $H_0 = s_1(s_1\varphi_0)$. The homomorphism $D_1H_1 : W\mathbf{Mor}_A \to \mathbf{End}_C(X,X,X)$ represents a composition $(\psi_1, \Psi_1)\Box(\varphi_1, \Phi_1)$ in $\mathfrak{Map}_A$. Hence we obtain

$$(\psi_1, \Psi_1)\Box(\varphi_1, \Phi_1) = s_1\varphi_0 = \mathrm{id}_{(X,\varphi_0)}.$$

Very similar we can define a homotopy $\bar{G}_t : D \to \mathbf{End}_C(X,X,X)$ through homomorphisms such that $\bar{G}_t$ is given on $D_2W\mathbf{Mor}_A^2$ by Ψ_t, on

$D_0W\mathbf{Mor}^2_A$ by Φ_t and on $D_1W\mathbf{Mor}^2_A$ by the homotopy $s_1\varphi_t$.

$$\begin{tikzcd}
\varphi_1 \arrow[rr, equal] \arrow[dr] \arrow[dd, dash] & & \varphi_1 \arrow[dd, dash] \\
& \varphi_0 \arrow[ur] \arrow[dd, equal] \arrow[dl, phantom, "\Psi_t"] \arrow[dr, phantom, "\Phi_t"] & \\
\varphi_0 \arrow[rr, equal] \arrow[dr, equal] & & \varphi_0 \arrow[dl, equal] \\
& \varphi_0 &
\end{tikzcd}$$

Then $\bar{G}_0$ is an extension of $s_1s_1\varphi_0$ and there exists a homotopy G_t : $W\mathbf{Mor}^2_a \to \mathbf{End}_C(X,X,X)$ through homomorphisms, which extends $\bar{G}_t$ such that $G_0 = s_1(s_1\varphi_0)$. Therefore we have

$$(\varphi_1,\Phi_1)\Box(\psi_1,\Psi_1) = s_1\varphi_1 = \mathrm{id}_{(X,\varphi_1)}.$$

$\Box$

LEMMA 18.41. *Let $F_t : W\mathbf{Mor}_A \to \mathbf{End}_C(X,Y)$ be a homotopy through homomorphisms. If $d_1(f_t,F_t) : (X,d_1F_0) \to (X,d_1F_1)$ is the isomorphism induced by the homotopy d_1F_t, and $d_0(f_t,F_t) : (Y,d_0F_0) \to (Y,d_0F_1)$ the one induced by d_0F_t, then the following diagram commutes.*

$$\begin{tikzcd}
(X,d_1F_0) \arrow[r, "(f_0,F_0)"] \arrow[d, "d_1(f_t,F_t)"'] & (Y,d_0F_0) \arrow[d, "d_0(f_t,F_t)"] \\
(X,d_1F_1) \arrow[r, "(f_1,F_1)"'] & (Y,d_0F_1)
\end{tikzcd}$$

PROOF. Let $B \subset W\mathbf{Mor}^2_A$ be the admissible suboperad, which is generated by the faces $D_0W\mathbf{Mor}^2_A$ and $D_2W\mathbf{Mor}^2_A$. The homotopy $\bar{H}_t$: $B \to \mathbf{End}_C(X,X,X)$ is given on $D_0W\mathbf{Mor}^2_A$ by the homotopy F_t, and on $D_2W\mathbf{Mor}^2_A$ by the homotopy induced by the construction of $d_1(f_t,F_t)$. Graphically we have

$$\begin{tikzcd}
(X,d_1F_0) \arrow[r, dotted] & (X,d_1F_1) \arrow[r, "(f_1,F_1)"] & (Y,d_0F_1) \\
(X,d_1F_0) \arrow[u, equal] \arrow[r, equal] & (X,d_1F_0) \arrow[u, dash, "d_1F_t"] \arrow[r, "(f_0,F_0)"'] \arrow[ur, phantom, "F_t"] & (Y,d_0F_0) \arrow[u, dash]
\end{tikzcd}$$

This is well defined, since the two homotopies coincide on $D_1D_0W\mathbf{Mor}^2_A$.

The homomorphism $\bar{H}_0$ is the restriction of the two-simplex s_0F : $W\mathbf{Mor}^2_A \to \mathbf{End}_C(X,X,Y)$ in $S\mathfrak{Map}_A$. By Theorem 17.10 exists an extension $H_t : W\mathbf{Mor}^2_A \to \mathbf{End}_C(X,X,Y)$ of $\bar{H}_t$ such that $H_0 = s_0F$. Furthermore we have

$$d_1H_1 = (f_1,F_1)\Box d_1(f_t,F_t).$$

Let $D \subset W\mathbf{Mor}^2_A$ be the suboperad generated by all faces. The homotopy d_1H_t can be combined with the homotopy, which induces the isomorphism $d_0(f_t,F_t)$, and the constant homotopy F_0, to a homotopy $\bar{G}_t : D \to$

$\mathbf{End}_C(X, Y, Y)$.

$$
\begin{tikzcd}
(X, d_1F_0) \arrow[rr, "d_1H_1"] \arrow[dr, "(f_0,F_0)"] \arrow[dd, equal] \arrow[ddr, phantom, "F_0"] & & (Y, d_0F_1) \arrow[dd, dash, "d_0F_t"] \\
& (Y, d_0F_0) \arrow[ur, dotted] \arrow[dd, equal] & \\
(X, d_1F_0) \arrow[rr, "(f_0,F_0)"] \arrow[dr, "(f_0,F_0)"'] & & (Y, d_0F_0) \arrow[dl, equal] \\
& (Y, d_0F_0) &
\end{tikzcd}
$$

Since $\bar{G}_0$ is the restriction of s_2F_0, Theorem 17.10 implies the existence of an extension $G_t : W\mathbf{Mor}^2_A \to \mathbf{End}_C(X, Y, Y)$ of $\bar{G}$ such that $G_0 = s_2F$. Therefore we have

$$d_0(f_t, F_t)\square(f_0, F_0) = d_1H_1 = (f_1, F_1)\square d_1(f_t, F_t).$$

$\square$

Let $f : A \to B$ be a homomorphism of C-operads. Then there exists a functor $\mathfrak{Map}_f : \mathfrak{Map}_B \to \mathfrak{Map}_A$, given by

$$\mathfrak{Map}_f(X, \varphi) = (X, \varphi \circ Wf) \text{ and}$$
$$\mathfrak{Map}_f(g, G) = (g \circ \mathbf{Mor}_f, G \circ W\mathbf{Mor}_f),$$

for each homotopy homomorphism $(g, G) : (X; \varphi) \to (Y, \psi)$ of homotopy B-algebras. The homomorphism $\mathbf{Mor}_f : \mathbf{Mor}_A \to \mathbf{Mor}_B$ is given by the application of f to all A-labels of the representing trees.

For each composition $(g_1, G_1)\square(g_0, G_0)$ of homotopy homomorphisms of B-algebras, exists a 2-simplex H in $S\mathfrak{Map}_B$ such that $d_2H = G_0$ and $d_0H = G_1$. The remaining face d_1H represents the composition. By composition with the homomorphism $W\mathbf{Mor}^2_f : W\mathbf{Mor}^2_A \to W\mathbf{Mor}^2_B$, which is given in analogy to $W\mathbf{Mor}_f$, we obtain a 2-simplex in $S\mathfrak{Map}_A$ such that $d_i(H \circ W\mathbf{Mor}^2_f) = d_iH \circ W\mathbf{Mor}_f$. Therefore the image of the composition $(g_1, G_1)\square(g_0, G_0)$ is a composition of the images. Similar we have $s_1(\varphi \circ Wf) = s_1\varphi \circ W\mathbf{Mor}_f$, for each B-algebra (X, φ). This proves that $\mathfrak{Map}_f$ is a functor.

If g is an homomorphism $B \to D$ of C-operads, then the composition $\mathfrak{Map}_f \circ \mathfrak{Map}_g : \mathfrak{Map}_D \to \mathfrak{Map}_A$ is the functor $\mathfrak{Map}_{g\circ f}$. Since $\mathfrak{Map}_{\mathrm{id}_A}$ is the identity on $\mathfrak{Map}_A$, we have a contravariant functor $\mathfrak{Map}$ from the category $oper_C\mathfrak{Top}$ of topological C-operads to the category $\mathfrak{Cat}$ of categories.

LEMMA 18.42. *Let $f_0, f_1 : A \to B$ be two homomorphisms of C-operads and $f_t : A \to B$ a homotopy through homomorphisms between them. Then the functors $\mathfrak{Map}_{f_0}$ and $\mathfrak{Map}_{f_1}$ are naturally isomorphic.*

PROOF. For each homotopy B-algebra (X, φ), the structures of the images $\mathfrak{Map}_{f_i}(X, \varphi)$ are given by $\varphi \circ Wf_i$. Therefore there exists a homotopy through homomorphisms between them, given by $\varphi \circ W f_t$. Following Lemma 18.40 we have an isomorphism $\kappa_{(X,\varphi)} : \mathfrak{Map}_{f_0}(X,\varphi) \to \mathfrak{Map}_{f_1}(X,\varphi)$.

If $(g,G) : (X,\varphi) \to (Y,\psi)$ is a homotopy homomorphism between homotopy B-algebras, then $\mathfrak{Map}_{f_0}(g,G)$ and $\mathfrak{Map}_{f_1}(g,G)$ are homotopic through homomorphisms. The homotopy is given by $G \circ W\mathbf{Mor}_{f_t}$. The faces $d_i(G \circ W\mathbf{Mor}_{f_t})$ are precisely $d_i G \circ W f_t$ for $i = 0,1$, where $d_1 G = \varphi$ and $d_0 G = \psi$. By Lemma 18.41 the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{Map}_{f_0}(X,\varphi) \arrow[r, "\mathfrak{Map}_{f_0}(g,G)"] \arrow[d, "\kappa_{(X,\varphi)}"'] & \mathfrak{Map}_{f_0}(Y,\psi) \arrow[d, "\kappa_{(Y,\psi)}"] \\
\mathfrak{Map}_{f_1}(X,\varphi) \arrow[r, "\mathfrak{Map}_{f_1}(g,G)"] & \mathfrak{Map}_{f_1}(Y,\psi)
\end{tikzcd}$$

Therefore the isomorphisms $\kappa_{(X,\varphi)}$ form a natural isomorphism between $\mathfrak{Map}_{f_0}$ and $\mathfrak{Map}_{f_1}$. □

THEOREM 18.43. *Let A and B be two C-operads and $f : A \to B$ a homotopy equivalence between them. Then the functor $\mathfrak{Map}_f : \mathfrak{Map}_B \to \mathfrak{Map}_A$ is an equivalence of categories.*

PROOF. Let $g : B \to A$ be a homotopy inverse of f. Then there exist a homotopy $h_t : A \to A$ through homomorphisms from $g \circ f$ to the identity of A. Since $\mathfrak{Map}_f \circ \mathfrak{Map}_g = \mathfrak{Map}_{g \circ f}$ the natural isomorphism to $\mathfrak{Map}_{\mathrm{id}_A} = \mathrm{id}_{\mathfrak{Map}_A}$ is induced by Lemma 18.42. □

19. Lax operads

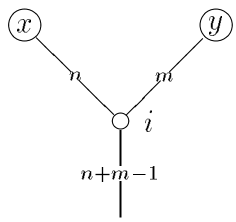
A *lax C-operad* (or homotopy C-operad) should be a C-operad, whose compositions are only associative up to coherent homotopies. In addition this notion should be homotopy invariant. Since a C-operad is an algebra over the T_C-operad $\mathbf{Op}_C$ a natural choice is the notion of homotopy algebras over $\mathbf{Op}_C$.

DEFINITION 19.1. A *lax C-operad* (A,η) is a homotopy algebra A over $\mathbf{Op}_C$, i.e. an algebra over $W\mathbf{Op}_C$, with structure homomorphism $\eta : W\mathbf{Op}_C \to \mathbf{End}_{T_C}(A)$. A morphism $(f,F) : (A,\eta) \to (B,\mu)$ of lax C-operads is a homotopy homomorphism. The category $\mathfrak{LaxOp}_C$ of lax C-operads and morphisms between them is the category $\mathfrak{Map}_{\mathbf{Op}_C}$.

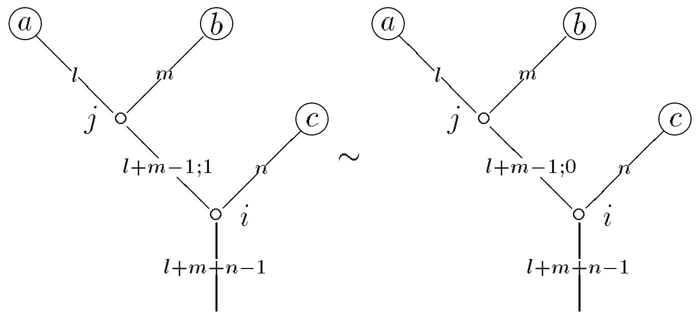
NOTATION 19.2. In the following we will drop the index $\mathbf{Op}_C$ from the notation if we are working with the rectification $M_{\mathbf{Op}_C} A$ of a lax C-operad.

In the following description of the structure of a lax operad, we restrict to the monochrome case. To obtain the general case, one just has to replace the $\mathbb{N}$-colors of the edges by appropriate tupels in T_C.

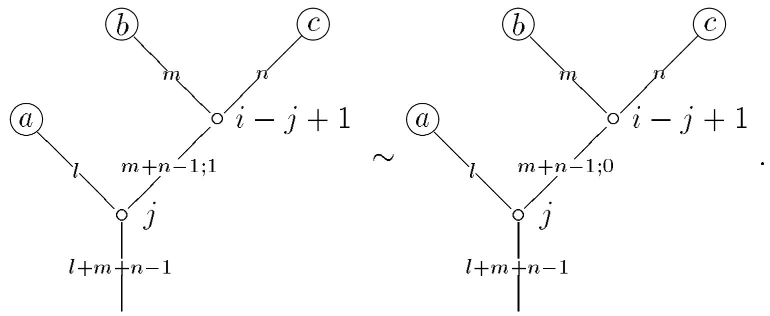
If A is a lax operad, we have compositions $\circ_i : A(n) \times A(m) \to A(n+m-1)$ for $1 \le i \le n$, given by the evaluations of the following cherry trees.



These compositions are not associative. Instead we have homotopies, which connect the appropriate compositions. For $j \le i \le j+l-1$ for example, we have the homotopies



and



If we denote the two equivalent trees on the right by $(a, b, c)_{j,i}$, we obtain a homotopy

$$(a \circ_j b) \circ_i c \sim (a, b, c)_{j,i} \sim a \circ_j (b \circ_{i-j+1} c).$$

Similarly we obtain homotopies

$$(a \circ_j b) \circ_i c \sim (a, b, c)_{j,i} \sim (a \circ_i c) \circ_{j+n-1} b$$

for $i \le j-1$ and

$$(a \circ_j b) \circ_i c \sim (a, b, c)_{j,i} \sim (a \circ_{i-l+1} c) \circ_j b.$$

For $a \in A(n), b \in A(m), c \in A(l), d \in A(h)$ and $1 \le i \le n, 1 \le j \le l$ and $i \le k \le i+m-1$, we get the model of Figure 7.

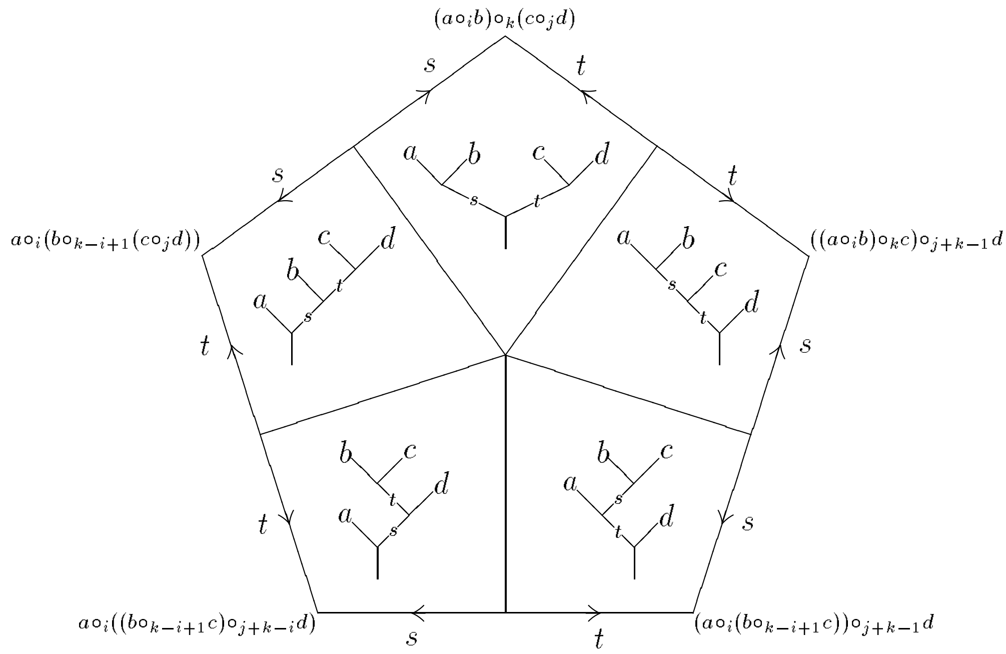
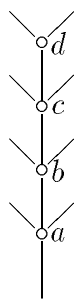


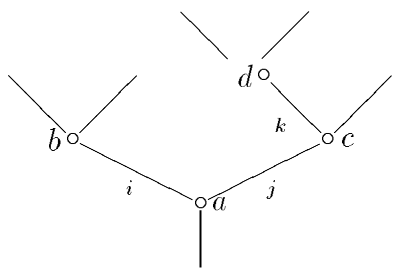
FIGURE 7. The "standard" associahedron

This model corresponds to an operation-tree of the form


,

i.e. each corner of the pentagon corresponds to a certain bracketing of the word $abcd$ and we are in the case of a two-dimensional associahedron, with the usual trees describing its cellular decomposition.

For the operation-tree



we also get a subdivision of the pentagon (see Fig. 8). But the trees used for the parts are different. We have not only a choice of how to analyze the associativity of a, c and d, i.e. how to choose the bracketing, but also a choice of the order in which we evaluate the tree. We can first compose a with b and then with the composition of c and d or, we can compose a with c, then with b and then compose the resulting operation with d. This results in the occurrence of several permutations in the trees, given by a reordering of the inputs.

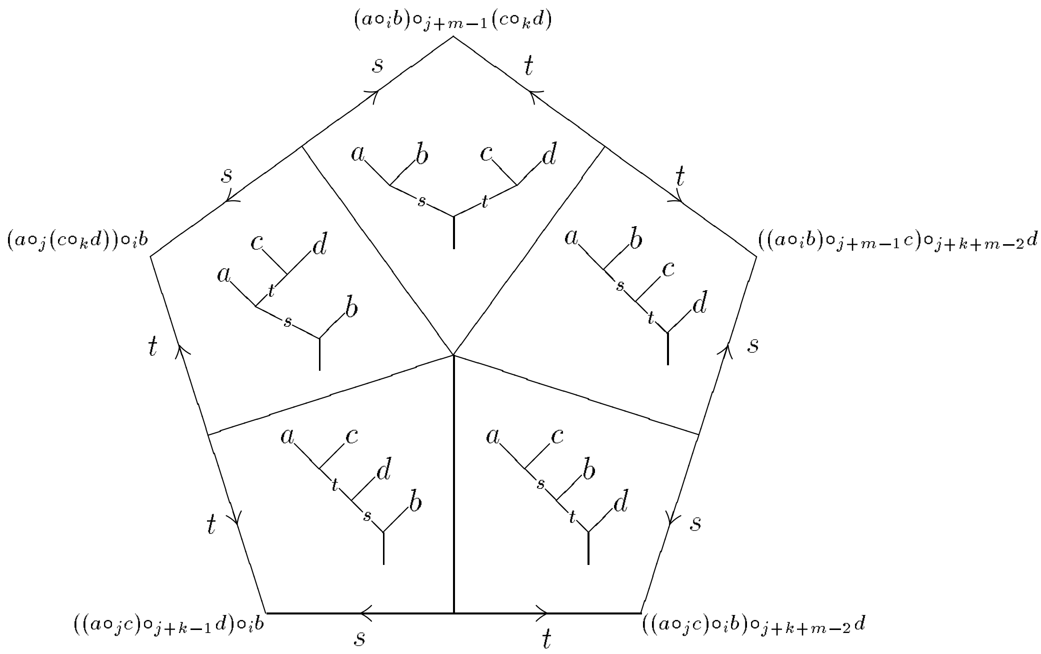
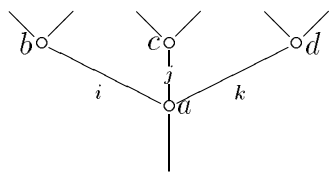


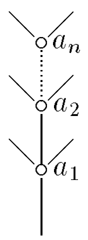
FIGURE 8. The "non-standard" associahedron

But for the operation tree



with $1 \leq i < j < k \leq n$, we do not get a pentagon. Since the associativity relations on **Op** correspond to the transposition of two of the operations b, c and d, we obtain a hexagon, whose vertices correspond to the possible permutations of three elements. This means that for this tree we have the 2-dimensional permutohedron (cmp. Fig. 9).

For an increasing number of vertices the situation becomes more complicated. For example the operation tree the a "linear" tree with n vertices, i.e. a tree of the form

 ,

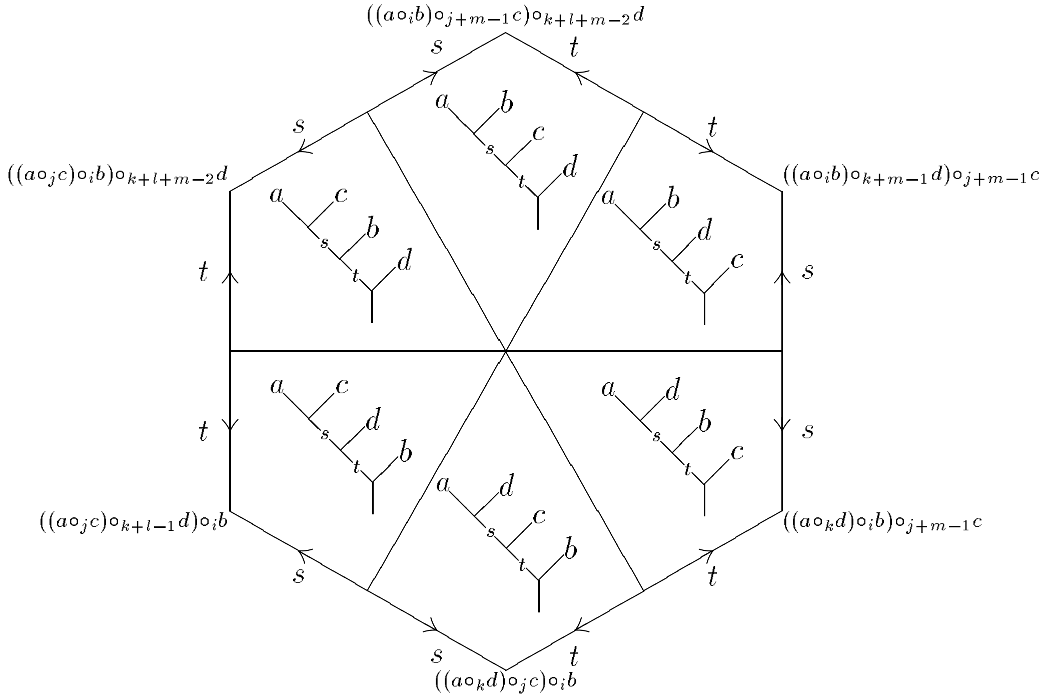
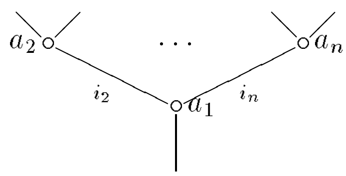


Figure 9. The permutohedron

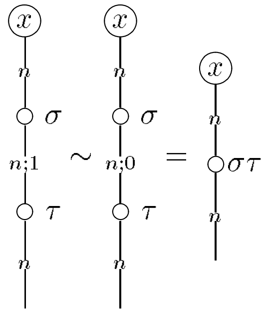
leads to the n-th associahedron, since there is only a choice for the brackets. But a tree of the form



is associated with the $(n-1)$st permutohedron, whose vertices correspond to the permutations of the vertices a_2 to a_n. Hence the resulting model depends on the form of the corresponding operation tree. In the two extreme cases we either have the associahedron or the permutohedron. The boundary of the models of intermediate steps is given by combinations of lower dimensional models of these two basic polyhedra.

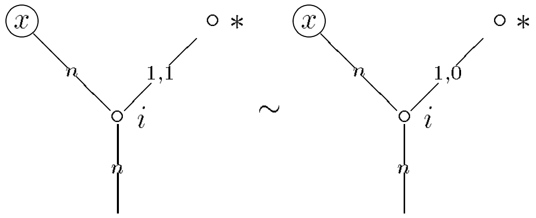
The action $A(n) \times \Sigma_n \to A(n)$ of the symmetric group on a lax operad is also weakened up to coherent homotopies. We only have a homotopy of the maps $A(n) \times \Sigma_n \times \Sigma_n \to A(n)$, given by $(x, \sigma, \tau) \mapsto x(\sigma\tau)$ and $(x, \sigma, \tau) \mapsto$

$(x\sigma)\tau$ of the form

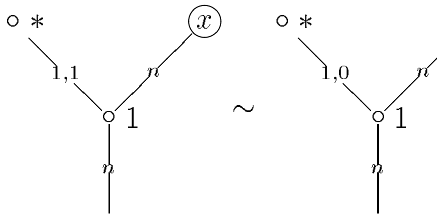


Similar the equivariance of the compositions is weakened up to coherent homotopies.

Furthermore the unit is not strict. Instead we get homotopies

$x \circ_i \mathrm{id} =$  $= x$

and

$\mathrm{id} \circ_1 x =$  $= x,$

which are compatible with the associating and permutating homotopies.

The homotopy invariance results 18.11 to 18.15 induce several homotopy invariance results for lax operads. We just formulate and prove one lemma, which regards the extension of a given strict C-familiy structure, i.e. actions of the symmetric groups, to C-operad structures.

Let $\Sigma_C \subset \mathbf{Op}_C$ be the T_C-suboperad generated by the permutations. Since each $\mathbf{Op}_C(\omega;\Gamma)$ with $\omega \in T_C$ and $\Gamma \in T_C^n$ is a discret set, the inclusion of $\Sigma_C(\omega;\Gamma)$ is a closed cofibration. Furthermore $\Sigma(\omega;\Gamma)$ is empty if $n \neq 1$. Hence the inclusion is obviously a Σ_n-invariant cofibration.

Definition 19.3. Let A and B be two C-collections. A *homotopy equivalence* $f : A \to B$ is a map of C-collections such that there exists a map $g : B \to A$ of C-collections and homotopies through maps of C-collections from id_A to $g \circ f$ and from id_B to $f \circ g$.

Lemma 19.4. *Let (B,η) be a lax C-operad with a strict action of the symmetric group, i.e. the restriction of η to $W\Sigma_C \subset W\mathbf{Op}_C$ factors through the augmentation ε_{Σ_C}.*

If A is a C-collection and if there exists a homotopy equivalence $f : A \to B$ of collections with inverse $g : B \to A$, then A is a lax operad with structure ψ and there exist two homotopy homomorphisms $(f,F) : (A,\psi) \to (B,\eta)$ and $(g,G) : (B,\eta) \to (A\psi)$ which are inverse to each other.

PROOF. Since f and g are homotopy inverse, there exists a homomorphism $H : W\mathbf{Iso}_{T_C} \to \mathbf{End}_{T_C}(A,B)$ by Proposition 18.12 such that $H_{A,B} = f$. Since A and B are C-collections they both are Σ_C-algebras. Furthermore f induces a homotopy homomorphism $(f, f \circ \varepsilon)$ of (homotopy) Σ_C-algebras, i.e. (homotopy) C-collections. Because f is a topological equivalence, Proposition 18.15 induces the existence of a homomorphism $H' : W\mathbf{Iso}_{\Sigma_C} \to \mathbf{End}_{T_C}(A,B)$ with $H'_{A,B} = f \circ \varepsilon$. The morphisms H'_A and H'_B are the given C-collection structures.

Proposition 18.14 induces the existence of a homomorphism $H'' : W\mathbf{Iso}_{\mathbf{Op}_C} \to \mathbf{End}_{T_C}(A,B)$ extending H' such that $H''_B = \eta$. Furthermore we obtain a lax operad structure on the collection A, extending H'_A and the C-collection structure of A.

By Proposition 18.13 the map $\bar{g} : B \to A$ of C-families underlying $H''_{B,A}$ is homotopy inverse to f and hence homotopic to g as a C-map. Following 18.11 there exists a homotopy homomorphism $(g, G) : (B, \eta) \to (A, \psi)$ such that G is homotopic to $H''_{B,A}$. Hence (g, G) is inverse to (f, F) in $\mathfrak{Map}_{\mathbf{Op}_C}$. □

EXAMPLE 19.5. Obviously the k-th space $C_n(k)$ of the little cubes operad is Σ_k-equivarantly homotopy equivalent to the space $F_n(k)$ of ordered configurations of k points in $(0,1)^n$. The map $\varphi_{n,k} : C_n(k) \to F_n(k)$ is given by $([a_1, b_1], \ldots, [a_k, b_k]) \mapsto (c_1, \ldots, c_k)$ where c_i is the center point of the cube $[a_i, b_i]$.

In the other direction we can "blow up" the points in $(0,1)^n$ to k distinct little cubes of the same size such that the interiors are pairwise disjoint. We can do it up to a maximal length $\frac{r}{\sqrt{2}}$ of the cubes, where r is the minimum of all distances between two points or one point and the boundary of I^n. Since r is contiuously determined by the configuration, we obtain a Σ_k-equivariant map $\psi_{n,k} : F_n(k) \to C_n(k)$.

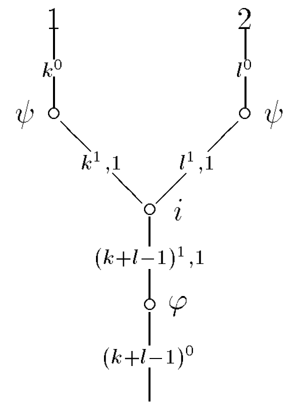
The composition $\varphi_{n,k} \circ \psi_{n,k}$ is the identity, and via the deformation of the given cubes into the "blow ups" of the centers we obtain a Σ_k-equivariant homotopy from the identity to $\psi_{n,k} \circ \varphi_{n,k}$.

By Lemma 19.4 the $\mathbb{N}$-collection $F_n(k)$ is a lax operad with a structure $\nu : W\mathbf{Op} \to \mathbf{End}_{\mathbb{N}}(F_n)$ extending the symmetric group actions. Furthermore there exist homotopy homomorphisms $(\varphi, \Phi) : (C_n, \eta) \to (F_n, \nu)$ and $(\psi, \Psi) : (F_n, \nu) \to (C_n, \eta)$, where η is the operad structure of the little cubes, extending $\varphi_{n,k}$ and $\psi_{n,k}$. These homotopy homomorphisms are inverse to each other.

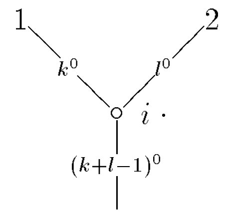
In addition to the induced structure ν on F_n we have "nice" multiplications, which are given by

$$(c_1, \ldots, c_k) \circ_i (d_1, \ldots, d_l) := \varphi_{n,k+l-1}\left(\psi_{n,k}(c_1, \ldots, c_k) \circ_i \psi_{n,l}(d_1, \ldots, d_l)\right).$$

This means we first blow up both configurations and then compose them in C_n. After that we shrink the cubes again to their center point. In terms of trees in $W\mathbf{Iso_{Op}}$, this operation is described by the following tree.



Here the secondary color 0 stands for the configurations and 1 for the little cubes. By shrinking the edges to length 0, we obtain a homotopy from this "nice" composition to the induced one, given by the tree



19.1. Algebras over Lax Operads. In the strict case an algebra over a C-operad A consists of a family $\{X_o\}_{o\in C}$ of spaces and a morphism $A \to \mathbf{End}_C(X)$ of C-operads. In the lax setting we do it in exactly the same way.

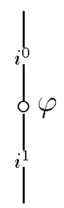
DEFINITION 19.6. An algebra $(X,(\varphi,\Phi))$ over a lax C-operad (A,η) is a family $\{X_o\}_{o\in C}$ together with a reduced homotopy homomorphism (φ,Φ) : $(A,\eta) \to \mathbf{End}_C(X)$.

NOTATION 19.7. In the following we will drop the index $\mathbf{Op}$ from the notation if we are working with the rectification $M_{\mathbf{Op}}A$ of a lax operad.

EXAMPLE 19.8. Let $\mathbf{Mon}$ be the operad of associative monoids, i.e. $\mathbf{Mon}(n) = \Sigma_n$. Then we can interpret $\mathbf{Mon}$ as a lax operad. An algebra over this lax operad is a space X together with a reduced homotopy homomorphism (φ,Φ) : $(\mathbf{Mon},\mu\circ\varepsilon) \to \mathbf{End}(X)$. The homomorphism $\Phi : W\mathbf{Mor_{Op}} \to \mathbf{End}_{\mathbb{N}}(\mathbf{Mon},\mathbf{End}(X))$ induces a multiplication $-\cdot- : X\times X \to X$, given by

$$\Phi(\varphi_2)(\mathrm{id}_2) \in \mathbf{End}(X)(2) = \mathfrak{Top}(X\times X \to X),$$

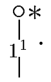
where φ_i is the tree



and id_2 the identity in Σ_2. Furthermore there exists a map

$$\Phi(*) : * \to \mathbf{End}(X)(0) = \mathfrak{Top}(*,X)$$

where $*$ is the tree

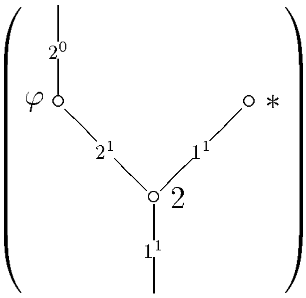


This map corresponds to an element $e \in X$.

For shorter (and less confusing) notation, we will omit the **Mon** argument, i.e. for a tree $T \in W\mathbf{Mor_{Op}}(o^1;\alpha^0)$ the term $\Phi(T)$ will denote both, a map $\mathbf{Mon}(\alpha) \to \mathbf{End}(X)(o)$ and the map $X^o \to X$, given by $\Phi(T)(\mathrm{id},\ldots,\mathrm{id})$.

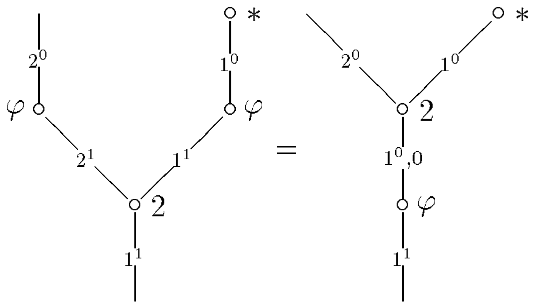
In this notation the product $x \cdot e$ is represented by

$$\Phi(\varphi_2)\,(x,\Phi(*)(*)) = \Phi$$



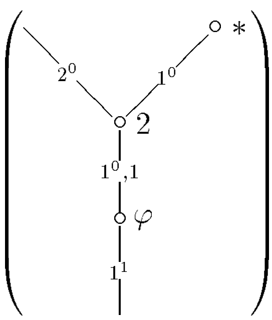
$$(x)$$

The last tree is equivalent to the following trees.



The identity on X can be codified in the form

$$\Phi(\varphi_1) = \Phi\,(\varphi_1) = \Phi$$



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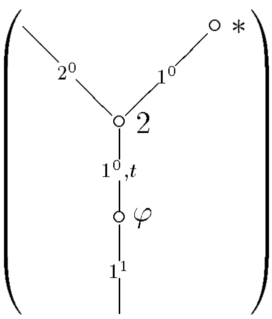
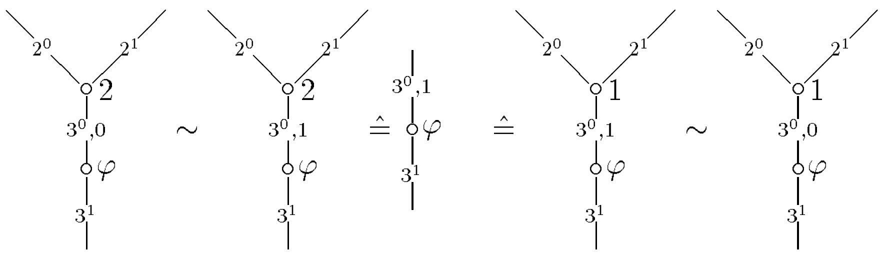
Hence the homotopy

$$\Phi$$



for $t \in [0,1]$ runs from the map $x \mapsto x \cdot e$ to the identity. Similarly, we obtain a homotopy from $x \mapsto e \cdot x$ to the identity. Therefore e is a homotopy unit of the multiplication on X.

In addition the multiplication is coherently homotopy associative. The homotopy from $x \cdot (y \cdot z)$ to $(x \cdot y) \cdot z$ is given by the following trees



For four arguments, we get a pentagon, subdivided into ten cubical models (cmp. Fig. 10). lax, the face $s = 1$ of each square collapses, and we get triangles. The parameter s is running radially toward the center, i.e. the border corresponds to $s = 0$. The parameter t is running from the center of the edges towards the vertices. In general we get the But since **Mon** is a strict operad interpreted as $(n-2)$-dimensional associahedron, if we examine the product of n elements.

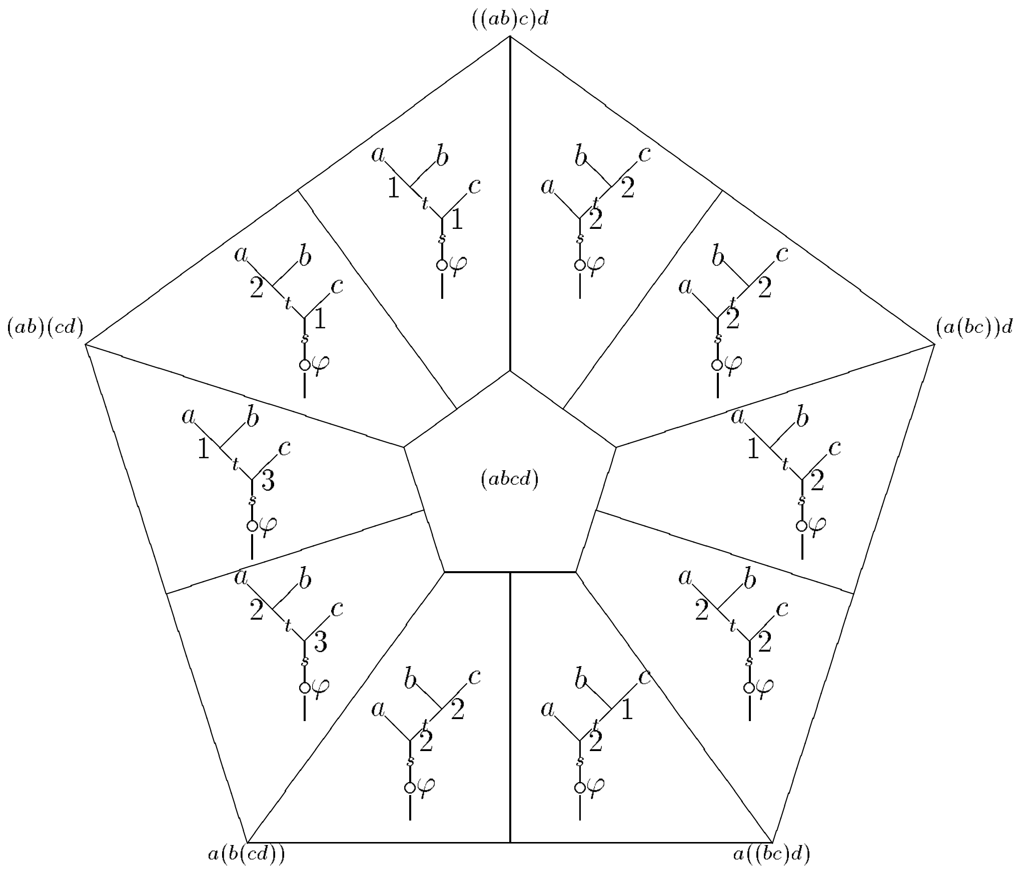
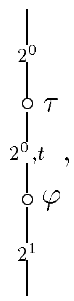


FIGURE 10. The pentagon for homotopy associative algebras

Example 19.9. Let **Com** be the operad of commutative monoids, i.e.

$$\mathbf{Com}(n) = *.$$

If we interpret **Com** as a lax operad, each algebra X over this lax operad is also an algebra over the lax version of **Mon**. Hence there exists a multiplication on X, which is coherently homotopy associative, and a homotopy unit. But furthermore there exists a commuting homotopy given by the trees



,

where $\tau \in \Sigma_2$ is the transposition. For $t = 0$ this tree induces the map $(x, y) \mapsto yx$ and for $t = 1$ the usual multiplication.

This commuting homotopy, together with the coherent homotopies for the associativity, imply maps from the permutoassociahedra of M. Kapranov (cmp. [**Kap93**]) into the endomorphism sets $\mathfrak{Top}(X^n, X)$. For three arguments the model is given by Figure 11.

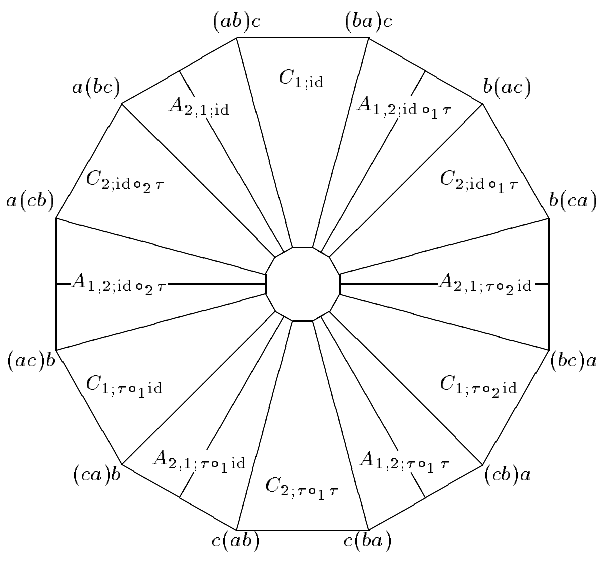
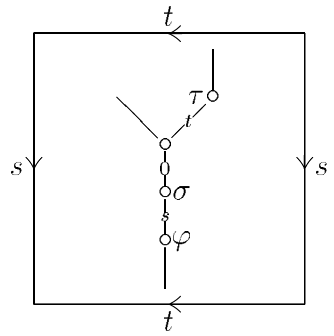
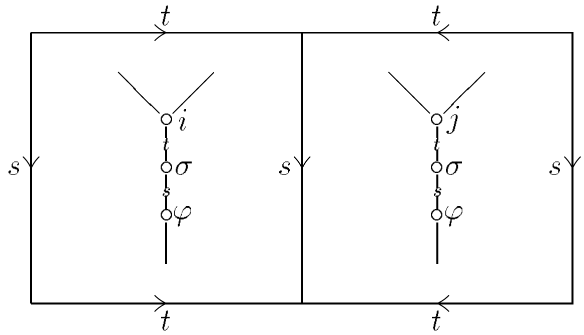


Figure 11. The permutoassociahedron

The commuting sections $C_{i;\sigma}$ with $i \in \{1,2\}$ and $\sigma \in \Sigma_3$ are given by squares of the type



while the sections $A_{i,j;\sigma}$ are given by the squares



In both cases the parameter s runs radially to the center. For $C_{i;\sigma}$, the parameter t runs anti-clockwise, and for $A_{i;\sigma}$ The right section becomes the section in clockwise direction.

For $s = 1$ the map $X^3 \to X$ induced by the tree is completely determined by the number of inputs the operation has. Hence the edge with $s = 1$ collapses and we obtain a triangular decomposition.

The correspondence of reduced homotopy homomorphisms with strict homomorphisms, described in Theorem 18.29, leads to the following

THEOREM 19.10. *Let X be a space and (A, η) a lax C-operad. The set X^A of A-algebra structures on X is bijective with the set X^{MA} of MA-structures on X. The bijection is given by*

$$(X, (\varphi, \Phi)) \mapsto (X, \bar{\varphi}),$$

where the homomorphism $\bar{\varphi} : MA \to \mathbf{End}(X)$ is induced by (φ, Φ).

According to the last theorem a possible choice for the category of algebras over a lax operad is $\mathfrak{Top}^{MA}$. The objects correspond precisely to our notion of algebra structures. But since we are interested in homotopy invariant notions, the better choice is $\mathfrak{Map}_{MA}$, i.e. the localization of the homotopy category $\mathfrak{HomTop}^{MA}$ along the topological equivalences.

REMARK 19.11. Another possibility would be the construction of a lax T_1-operad $\mathbf{Lax}_A$ such that a morphism between two algebras X and Y over the lax operad A, are codified by the reduced homotopy homomorphisms $\mathbf{Lax}_A \to \mathbf{End}(X,Y)$. It is possible to define an alternative, lax T_n-operad structure on $\mathbf{Mor}^n_{MA}$, which is a good candidate for this approach. But unfortunately we were not able to define a strict composition on these "morphisms". Instead we had to define a simplicial class consisting of the reduced homotopy homomorphisms $\mathbf{Lax}^n_A \to \mathbf{End}(X_0,\dots,X_n)$, which is a Δ-category in the sense of [**SV92**]. Using the techniques given in that paper it is possible to construct a category of algebras over the lax operad A and morphisms between them. But since we have no application of this construction, we believed that the result would not justify the effort.

If we choose $\mathfrak{Map}_{MA}$ as the category of algebras over a lax operad (A,η), then our notion is homotopy invariant, regarding the underlying spaces of the algebras. But what happens if we change the lax operad A up to topological equivalences?

Let $(g,G):(A,\eta_A)\to(B,\eta_B)$ be a morphism between two lax operads an $\bar{g}:MA\to MB$ a homomorphism representing the homotopy class $M(g,G)$. Then we have a functor $\mathfrak{Map}_{\bar{g}}:\mathfrak{Map}_{MB}\to\mathfrak{Map}_{MA}$. If $\bar{g}':MA\to MB$ is another representation of $M(g,G)$, then $\bar{g}$ and $\bar{g}'$ are homotopic through homomorphisms. Hence, following Lemma 18.42, the two functors $\mathfrak{Map}_{\bar{g}}$ and $\mathfrak{Map}_{\bar{g}'}$ are naturally isomorphic.

If $(h,H):(B,\eta_B)\to(C,\eta_C)$ is another morphism of lax operads, then the composition $\mathfrak{Map}_{\bar{g}}\circ\mathfrak{Map}_{\bar{h}}$ is naturally isomorphic to $\mathfrak{Map}_{\overline{hog}}$, where $\bar{h}$ is an representation of $M(h,H)$ and $\overline{h\circ g}$ one of $M((h,H)\square(g,G))$.

If g is a topological equivalence, then (g,G) is an isomorphism in $\mathfrak{LaxOp}=\mathfrak{Map}_{\mathbf{Op}}$, and hence $M(g,G)$ an isomorphism in $\mathfrak{HomTop}^{\mathbf{Op}}$. Therefore $\bar{g}$ is a homotopy equivalence between operads. By Theorem 18.43 the functor $\mathfrak{Map}_{\bar{g}}:\mathfrak{Map}_{MB}\to\mathfrak{Map}_{MA}$ is an equivalence of categories. This shows that algebras over lax operads are preserved, if the operad is changed by a topological equivalence.

20. Topological A_∞-categories

20.1. A_∞-categories. In section 16.6 we described categories with a given object set S as algebras over an operad $\mathbf{Cat}_S$, whose colors are pairs of objects. In addition we described functors as morphisms between two such algebras. We are now going to transfer this description to homotopy $\mathbf{Cat}_S$-algebras and homotopy homomorphisms. As we will see this leads to a model for the homotopy category of small, topological categories.

Recall that a small, topological category is a pair (A,S) with S the set of objects and A a $\mathbf{Cat}_S$-algebra. The functors are pairs $(F,\varphi):(A,S)\to(B,T)$ with $\varphi:S\to T$ a map of sets and $F:A\to\varphi B$ a morphism of $\mathbf{Cat}_S$-algebras. Here φB is the $(S\times S)$-family given by

$$\varphi B(a,b)=B\left(\varphi(a),\varphi(b)\right)\text{ for }a,b\in S.$$

The composition of two functors $(F,\varphi):(A,S)\to(B,T)$ and $(G,\psi):(B,T)\to(C,U)$ is given by

$$(G,\psi)\circ(F,\varphi)=(\varphi G\circ F,\psi\circ\varphi).$$

Using the notion of homotopy algebras, we can weaken the notion of categories up to coherent homotopies.

Definition 20.1. An A_∞*-category* $((A,\eta),S)$ consists of an object set S and a homotopy $\mathbf{Cat}_S$-algebra (A,η).

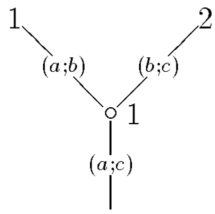
Remark 20.2. In the category of (graded) modules of a ring k, an A_∞-category C consists of a set S of objects, k-modules $C(a,b)$ for $a,b\in S$ and *compositions*

$$m_{a_1,\ldots,a_k} : C(a_1,a_2)\otimes C(a_2,a_3)\otimes\cdots\otimes C(a_{k-1},a_k)\to C(a_1,a_k)$$

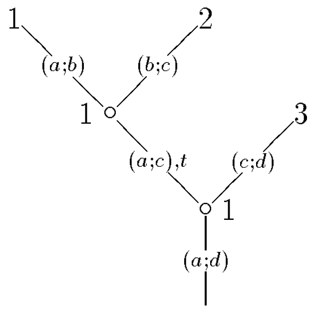
satisfying several coherent associativity conditions (see for example [**KF95**]). Or in shorter terms an A_∞-category C is an algebra over the S-colored analogue of the operad of A_∞-algebras, which is a cofibrant resolution of the operad of assocoative algebras. This corresponds precisely to our situation.

$\mathbf{Cat}_S$ is the S-colored analogue of the operad of associative monoids and $W\mathbf{Cat}_S$ the analogue of the algebraic operad of A_∞-algebras.

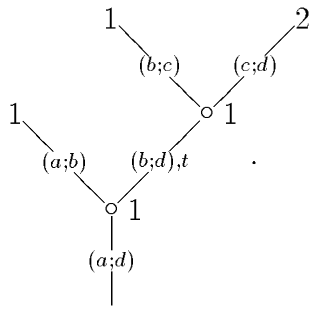
Before we proceed to describe A_∞-functors, we give a rough description of the structure of an A_∞-category A. First there exists a composition $A(a;b)\times A(b;c)\to A(a;c)$, given by the tree



in $W\mathbf{Cat}_S$. But this composition is only associative up to coherent homotopies. The homotopy is given by the trees



and



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For $t=0$ the induced operations $A(a,b)\times A(b,c)\times A(c,d)\to A(a,d)$ coincide. For $t=1$ the first tree corresponds to the composition $(f\circ g)\circ h$ and the second to $f\circ(g\circ h)$. If we look at compositions of four morphisms, we get a pentagon like the one in Figure 7. Each vertex corresponds to one bracketing of a word with four letters. In general we obtain the n-th associahedron for compositions of $n+2$ morphisms. In this sense, an A_∞-category is a category up to coherent homotopies.

In addition to this homotopy associative composition we have identites, given by the operations $*\to A(o;o)$, induced by the tree

.

But again, as in the case of lax operads, they are only an identity up to homotopy.

To define functors between A_∞-categories, we have to examine the transfer of colors in the setting of homotopy $\mathbf{Cat}_S$-algebras. Let $\varphi : S \to T$ be a map of sets and (A, η_A) a homotopy $\mathbf{Cat}_S$- and (B, η_B) a homotopy $\mathbf{Cat}_T$-algebra. Each tree R in $W\mathbf{Cat}_S$ induces a tree φR in $W\mathbf{Cat}_S$, which is given by replacing the edge colors $(a; b)$ by $(\varphi(a); \varphi(b))$. This induces a $W\mathbf{Cat}_S$-structure $\varphi\eta_B$ on φB, with $\varphi\eta_B(R) = \eta_B(\varphi R)$. More general, we obtain a n-simplex φH in $S\mathfrak{Map}_{\mathbf{Cat}_S}$ for each n-simplex H in $S\mathfrak{Map}_{\mathbf{Cat}_T}$, given by $\varphi H(R) = H(\varphi R)$, where φR is obtained from R, by applying φ to the components of the primary color. Since the underlying trees and all labels remain unchanged, these maps define a map of simplicial spaces and therefore a functor $\varphi^* : \mathfrak{Map}_{\mathbf{Cat}_T} \to \mathfrak{Map}_{\mathbf{Cat}_S}$.

DEFINITION 20.3. An *A_∞-functor* $((f, F), \varphi) : ((A, \eta_A), S) \to ((B, \eta_B), T)$ consists of a map $\varphi : S \to T$ of sets and a homotopy homomorphism $(f, F) : (A, \eta_A) \to \varphi^*(B, \eta_B)$.

Let $((f, F), \varphi) : ((A, \eta_A), S) \to ((B, \eta_B), T)$ and $((g, G), \psi) : ((B, \eta_B), S) \to ((C, \eta_C), U)$ be two A_∞-functors. Then their composition is given by

$$((g, G), \psi) \circ ((f, F), \varphi) = (\varphi^*(g, G)\Box(f, F), g_{\mathfrak{ob}} \circ f_{\mathfrak{ob}}) .$$

Since φ^* is a functor, this composition is associative. The identity of $((A, \eta), C)$ is the pair $(\mathrm{id}_{(A,\eta)}, \mathrm{id}_C)$. The category of A_∞-categories and -functors will be denoted with $\mathfrak{Map}_{\mathbf{Cat}}$.

REMARK 20.4. Of course the name $\mathfrak{Map}_{\mathbf{Cat}}$ is an abuse of notation, since there exists no operad $\mathbf{Cat}$ of categories. But since A_∞-categories and -functors are basically given by objects in the categories $\mathfrak{Map}_{\mathbf{Cat}_S}$ for all sets S, we think this notation is reasonable.

Now let $((f, F), \varphi) : ((A, \eta_A), S) \to ((B, \eta_B)), T)$ be an A_∞-functor such that φ is bijective and $(f, F) : (A, \eta_A) \to \varphi^*(B, \eta_B)$ is a topological equivalence. Then there exists an inverse $(g, G) : \varphi^*(B, \eta_B) \to (A, \eta_A)$ of (f, F). Furthermore let $\psi : D \to C$ be an inverse of φ. Then we have

$$\psi^* (\varphi^* B) (a, b) = \varphi^* B (\psi(a), \psi(b)) = B (\varphi \circ \psi(a), \varphi \circ \psi(b)) = (\varphi \circ \psi)^* B(a, b).$$

Therefore $\psi^*(g, G)$ is a homotopy homomorphism from $(B, \eta B)$ to $\psi^*(A, \eta_A)$ and we have an A_∞-functor $(\psi^*(g, G), \psi) : ((B, \eta_B T) \to ((A, \eta_A), S)$, which is inverse to $((f, F), \varphi)$. In fact we have

LEMMA 20.5. *An A_∞-functor $((f, F), \varphi) : ((A, \eta_A), S) \to ((B, \eta_B), T)$ is an isomorphism, if and only if φ is bijective and (f, F) is an isomorphism in $\mathfrak{Map}_{\mathbf{Cat}_S}$. The inverse is given by $(\psi^*(g, G), \psi)$, where ψ is an inverse of φ and (g, G) one of (f, F).*

PROOF. Let $((h, H), \psi)$ be an inverse of $((f, F), \varphi)$. Since we have

$$((h, H), \psi) \circ ((f, F), \varphi) = (\varphi^*(h, H)\Box(f, F), \psi \circ \varphi) ,$$

φ has to be bijective and (f, F) an isomorphism. On one hand, we have $\varphi^*(h, H)\Box(f, F) = \mathrm{id}_{(A,\eta_A)}$ and on the other $\psi^*(f, F)\Box(h, H) = \mathrm{id}_{(B,\eta_B)}$. The second equation implies

$$(f, F)\Box\varphi^*(h, H) = \varphi^* (\psi^*(f, F)\Box(h, H)) = \varphi^* \left(\mathrm{id}_{\varphi^*(B,\eta_B)}\right) = \mathrm{id}_{(B,\eta_B)}.$$

Therefore $(g, G) := \varphi^*(h, H)$ is an inverse of (f, F). Since $(h, H) = \psi^*(g, G)$, this implies the statement. $\square$

Now we are going to prove, that the category of A_∞-categories and -functors is equivalent to the "usual" homotopy category of $\mathfrak{TCat}$, given by homotopy classes of functors.

Definition 20.6. Let A and B be two categories and $\varphi : \mathfrak{ob}\, A \to \mathfrak{ob}\, B$ a map of the morphism sets. A family $F_t(o, i) : A(o, i) \to B(\varphi(o), \varphi(i))$ for $o, i \in \mathfrak{ob}\, A$ is called a homotopy through functors, if each family F_t is a functor. The two functors F_0 and F_1 are said to be *homotopic.*

It is well known that the relation "homotopic" is an equivalence relation. Therefore we have a category $\mathfrak{HomTCat}$ of small topological categories and homotopy classes of functors between them.

Similarly to the category of algebras over an operad, we have a notion of *homotopy equivalences*, which correspond to isomorphisms in the homotopy category $\mathfrak{HomTCat}$. We also have a notion of *topological equivalence*, i.e. functors, whose underlying maps are homotopy equivalences of spaces.

Definition 20.7. A functor $f : A \to B$ of two topological categories, is called a *topological equivalence* if the map of object sets is bijective and each map $f(o, i) : A(o, i) \to B(f(o), f(i))$ is a homotopy equivalence.

There exists a functor

$$J : \mathfrak{HomTCat} \to \mathfrak{Map}_{\mathbf{Cat}}$$

given by $J(A, \eta) = J_S(A, \eta)$ and $J(F, \varphi) = (J_S F, \varphi)$, where $J_S : \mathfrak{HomTCat}_S = \mathfrak{HomTop}^{\mathbf{Cat}_S} \to \mathfrak{Map}_{\mathbf{Cat}_S}$ is the known functor. Since J_S maps topological equivalences to isomorphisms in $\mathfrak{Map}_{\mathbf{Cat}_S}$, the functor $J : \mathfrak{HomTCat} \to \mathfrak{Map}_{\mathbf{Cat}}$ maps topological equivalences of categories to isomorphisms. If Σ is the class of topological equivalences in $\mathfrak{HomTCat}$, we obtain a unique functor

$$\bar{J} : \mathfrak{HomTCat}\left[\Sigma^{-1}\right] \to \mathfrak{Map}_{\mathbf{Cat}}.$$

Theorem 20.8. *The functor* $\bar{J} : \mathfrak{HomTCat}[\Sigma^{-1}] \to \mathfrak{Map}_{\mathbf{Cat}}$ *and the projection* $\mathfrak{TCat}[\Sigma^{-1}] \to \mathfrak{HomTCat}[\Sigma^{-1}]$ *are equivalences of categories. Here* Σ *denotes the class of topological equivalences in* $\mathfrak{TCat}$ *as well as in* $\mathfrak{HomTCat}$.

This theorem will be proved in section 20.3.

20.2. The Grothendiek Construction. Before we prove Theorem 20.8 we describe the well-known Grothendiek construction (see for example [**Tho79**]) and examine their localization. Usually the Grothendiek construction is defined for covariant functors into the category of categories. We will adapt the construction to the contravariant case. For the remainder of this section, let $\mathcal{V}$ be an arbitrary but fixed category.

Let $F : \mathcal{V}^{op} \to \mathfrak{Cat}$ be a functor. We will denote the category $F(S)$ for an object S in $\mathcal{V}$ by F_S. Similar F_φ will denote the functor $F(\varphi) : F_S \to F_T$ for a morphism $\varphi : S \to T$ in $\mathcal{V}$.

DEFINITION 20.9. Let $\mathcal{V}$ be a category and $F : \mathcal{V}^{op} \to \mathfrak{Cat}$ a functor. The *Grothendiek construction* $F\int\mathcal{V}$ on F is the category with objects the pairs (X, S) with $S \in \mathfrak{ob}\,\mathcal{V}$ and $X \in \mathfrak{ob}\,F_S$, and with morphisms $(f, \varphi) : (X, S) \to (Y, T)$, where $\varphi : S \to T$ is one in $\mathcal{V}$ and $f : X \to F_\varphi(Y)$ one in F_S. The composition is defined by

$$(g, \psi) \circ (f, \varphi) = (F_\varphi(g) \circ f, \psi \circ \varphi) .$$

The composition is associative and the identity of (X, S) is the pair $(\mathrm{id}_X, \mathrm{id}_S)$. Furthermore each natural transformation $I : F \to F'$ of functors induces a functor $I\int\mathcal{V} : F\int\mathcal{V} \to F'\int\mathcal{V}$ with

$$I\int\mathcal{V}(X, S) = (I_S(X), S) \text{ and } I\int\mathcal{V}(f, \varphi) = (I_S(f), \varphi) .$$

NOTATION 20.10. In the following we differentiate between functors and natural transformations by using different brackets. The image of an object X under a functor F is written as $F(X)$, while the morphism $F(X) \to G(X)$ of a natural transformation $\eta : F \to G$ will be denoted with $\eta[X]$. The indexes are reserved for objects and morphisms of $\mathcal{V}$.

DEFINITION 20.11. Let $F, F' : \mathcal{V}^{op} \to \mathfrak{Cat}$ be two functors. A *lax natural transformation* $(I, \eta) : F \to F'$ consists of a family $\{I_S : F_S \to F'_S\}_{S \in \mathfrak{ob}\,\mathcal{V}}$ of functors and natural transformations $\eta_\varphi : I_S \circ F_\varphi \to F'_\varphi \circ I_D$ for each morphism $\varphi : S \to D$ in $\mathcal{V}$ such that

1. $\eta_{\mathrm{id}_S}[X] = \mathrm{id}_{I_S(X)}$ and
2. $F'_\varphi(\eta_\psi[X]) \circ \eta_\varphi[F_\psi(X)] = \eta_{\psi\circ\varphi}[X]$ for $\varphi : S \to T, \psi : T \to U$ and $X \in F_U$.

REMARK 20.12. Each strict natural transformation $I : F\ toF'$ can be interpreted as a lax natural transformation by setting

$$\eta_\varphi[X] = \mathrm{id}_{I_S \circ F_\varphi(X)} = \mathrm{id}_{F'_\varphi \circ I_D} .$$

LEMMA 20.13. *Let $F, F' : \mathcal{V}^{op} \to \mathfrak{Cat}$ be two functors and $(I, \eta) : F \to F'$ a lax natural transformation. Then there exists a functor $(I, \eta)\int\mathcal{V} : F\int\mathcal{V} \to F'\int\mathcal{V}$ given by*

$$(I, \eta)\int\mathcal{V}(X, S) = (I_S(X), S)$$

for $S \in \mathfrak{ob}\,\mathcal{V}$ and

$$(I, \eta)\int\mathcal{V}(f, \varphi) = (\eta_\varphi[Y] \circ I_S(f), \varphi)$$

for $(f, \varphi) : (X, S) \to (Y, T)$.

PROOF. For each object S of $\mathcal{V}$ we have

$$\begin{aligned}(I, \eta)\int\mathcal{V}(\mathrm{id}_X, \mathrm{id}_S) &= (\eta_{\mathrm{id}_S}[X] \circ I_S(\mathrm{id}_X), \mathrm{id}_S) \\ &= \left(\mathrm{id}_{I_S(X)}, \mathrm{id}_S\right) .\end{aligned}$$

For $(f,\varphi):(X,S)\to(Y,T)$ and $(g,\psi):(Y,t)\to(Z,U)$ we have

$$\begin{aligned}(I,\eta)\int\mathcal{V}(g,\psi)\circ I\int\mathcal{V}(f,\varphi) &= (\eta_\psi[Z]\circ I_T(g),\psi)\circ(\eta_\varphi[Y]\circ I_S(f),\varphi)\\ &= \left(F'_\varphi(\eta_\psi[Z]\circ I_T(g))\circ\eta_\varphi[Y]\circ I_S(f),\psi\circ\varphi\right)\\ &= \left(F'_\varphi(\eta_\psi[Z])\circ\eta_\varphi[F_\psi(Z)]\circ I_S(F_\varphi(g))\circ I_S(f),\psi\circ\varphi\right)\\ &= (\eta_{\psi\circ\varphi}[Z]\circ I_S(F_\varphi(g)\circ f),\psi\circ\varphi)\\ &= (I,\eta)\int\mathcal{V}(F_\varphi(g)\circ f,\psi\circ\varphi)\\ &= (I,\eta)\int\mathcal{V}((f,\varphi)\circ(g,\psi)).\end{aligned}$$

□

LEMMA 20.14. *Let $(I,\eta):F\to F'$ and $(J,\mu):F'\to F''$ be two lax natural transformations. Then there exists a lax natural transformation $(J\circ I,\kappa):F\to F''$, given by $(I\circ J)_S=I_S\circ J_S$ and*

$$\kappa_\varphi[X]=\mu_\varphi[I_T(X)]\circ J_S(\eta_\varphi[X])$$

for $\varphi:S\to T$ and $X\in F_T$. This natural transformation is called the composition of (I,φ) and (J,ψ).

The proofs of this and the following lemma are straightforward and left to the reader.

LEMMA 20.15. *Let $(I,\eta):F\to F'$ and $(J,\mu):F'\to F''$ be two lax natural transformations. Then we have*

$$(J,\mu)\int\mathcal{V}\circ(I,\eta)\int\mathcal{V}=((J,\mu)\circ(I,\eta))\int\mathcal{V}.$$

COROLLARY 20.16. *Let $F:\mathcal{V}^{op}\to\mathfrak{Cat}$ be a functor and A an arbitrary category. If there exist functors $I_S:F_S\to A$ for each object S in $\mathcal{V}$, and natural transformations $\eta_\varphi:I_S\circ F_\varphi\to I_T$ for each morphism $\varphi:S\to T$ in $\mathcal{V}$ such that*

$$\eta_{id_S}[X]=id_{I_S(X)} \text{ and } \eta_\psi[Z]\circ\eta_\varphi[F_\psi(Z)]=\eta_{\psi\circ\varphi}[Z]$$

for $\varphi:S\to T,\psi:T\to U,X\in\mathfrak{ob}\,F_S$ and $Z\in\mathfrak{ob}\,F_U$, then there exists a functor $\bar{K}:F\int\mathcal{V}\to X$.

PROOF. There exists a functor $A:\mathcal{V}^{op}\to\mathfrak{Cat}$, mapping each object of $\mathcal{V}$ to A and each morphism to the identity of A. The conditions induce a lax natural transformation $(I,\eta):F\to A$ and therefore a functor $(I,\eta)\int\mathcal{V}:F\int\mathcal{V}\to A\int\mathcal{V}$. The composition with the projection functor $P_A:A\int\mathcal{V}\to A$, given by $(X,S)\mapsto A$ and $(f,\varphi)\mapsto f$ induces a functor $\bar{K}:=P_A\circ(K,\eta)\int\mathcal{V}$. □

DEFINITION 20.17. Let $F,G:\mathcal{V}^{op}\to\mathfrak{Cat}$ be functors and $(I,\eta),(J,\mu):F\to G$ be lax natural transformations. A *lax natural 2-transformation* $\nu:(I,\eta)\to(J,\mu)$ is a family of natural transformations $\nu_S:I_S\to J_S$ for each object S of $\mathcal{V}$ such that

$$G_\varphi(\nu_D[Y])\circ\eta_\varphi[Y]=\mu_\varphi[Y]\circ\nu_S[F_\varphi(Y)]$$

for each morphism $\varphi : S \to T$ of $\mathcal{V}$ and each object Y of F_T.

LEMMA 20.18. *Let $F, G : \mathcal{V}^{op} \to \mathfrak{Cat}$ be functors, $(I, \eta), (J, \mu) : F \to G$ lax natural transformations and $\nu : (I, \eta) \to (J, \mu)$ an lax natural 2-transformation. Then there exists a natural transformation $\nu\!\int\!\mathcal{V} : (I, \eta)\!\int\!\mathcal{V} \to (J, \mu)\!\int\!\mathcal{V}$ given by*

$$\nu\int\mathcal{V}[X, S] = (\nu_S[X], id_S)\,.$$

Again the proof is left to the reader.

Now we are going to apply these results to localization problems. For the remainder of this section let $F : \mathcal{V}^{op} \to \mathfrak{TCat}$ be an arbitrary, fixed functor. We assume that there exists a space Σ_S of morphisms in F_S for each object S in $\mathcal{V}$ such that for each morphism $\varphi : S \to T$ in $\mathcal{V}$ and for each $f \in \Sigma_T$ the image $F_\varphi(f)$ is an element of Σ_S or an isomorphism in F_S.

The universal property of the localization $F_c[\Sigma_c^{-1}]$ implies that each morphism $\varphi : S \to T$ induces a unique functor

$$F_\varphi[\Sigma^{-1}] : F_T[\Sigma_T^{-1}] \to F_S[\Sigma_S^{-1}]$$

such that the diagram

$$\begin{tikzcd} F_S \arrow[r, "f_\varphi"] \arrow[d, "P_S"'] & F_T \arrow[d, "P_T"] \\ F_S[\Sigma_S^{-1}] \arrow[r, "F_\varphi[\Sigma^{-1}]"'] & F_T[\Sigma_T^{-1}] \end{tikzcd}$$

commutes. More general, we obtain a functor $F[\Sigma^{-1}] : \mathcal{V}^{op} \to \mathfrak{TCat}$, with

$$F[\Sigma^{-1}](c) = F_S[\Sigma_S^{-1}].$$

The space Σ of morphisms $F\!\int\!\mathcal{V}$ is the subspace of all morphisms $(f, \varphi) : (X, S) \to (Y, T)$ such that φ is an isomorphism in $\mathcal{V}$ and such that f is an element of Σ_S. We can form the localization $F\!\int\!\mathcal{V}[\Sigma^{-1}]$.

THEOREM 20.19. *The localization $F\!\int\!\mathcal{V}[\Sigma^{-1}]$ is isomorphic to the category $F[\Sigma^{-1}]\!\int\!\mathcal{V}$.*

PROOF. The functors $P_S : F_S \to F_S[\Sigma_S^{-1}]$ induce a natural transformation from F to $F[\Sigma^{-1}]$ and a functor

$$P\int\mathcal{V} : F\int\mathcal{V} \to F[\Sigma^{-1}]\int\mathcal{V}.$$

If $(f, \varphi) : (X, S) \to (Y, T)$ is an element of Σ, i.e. φ is an isomorphism and f an element of Σ_S, then we have

$$P\int\mathcal{V}(f, \varphi) = (P_S(f), \varphi)\,,$$

which is an isomorphism in $F[\Sigma^{-1}]\!\int\!\mathcal{V}$. Hence $P\!\int\!\mathcal{V}$ induces a uniquely determined functor

$$J : F\int\mathcal{V}[\Sigma^{-1}] \to F[\Sigma^{-1}]\int\mathcal{V}$$

such that $P\!\int\!\mathcal{V} = J \circ P$.

On the other hand we have functors $K_S : F_S \to F\int\mathcal{V}[\Sigma^{-1}]$, given by

$$K_S(X) = (X, S) \text{ and } K_S(f) = (f, \mathrm{id}_S).$$

Furthermore we define

$$\eta_\varphi[X] := \big(\mathrm{id}_{F_\varphi}(X), \varphi\big) : (F_\varphi(X), S) \to (X, T)$$

for each morphism $\varphi : S \to T$ and each object X in F_T. It is easy to see that the $\eta_\varphi[X]$ form natural transformations η_φ from $K_S \circ F_\varphi$ to K_T, and that they satisfy the conditions of Corollary 20.16. The induced functor $K : F\int\mathcal{V} \to F\int\mathcal{V}[\Sigma^{-1}]$ is the projection P, given by the universal property of the localization.

Since K_S maps each morphism in Σ_S to an isomorphism in $F\int\mathcal{V}[\Sigma^{-1}]$, there exist uniquely determined functors $\bar{K}_S : F_S[\Sigma_S^{-1}] \to F\int\mathcal{V}[\Sigma^{-1}]$ such that $\bar{K}_S \circ P_S = K_S$. By Lemma 1.2 in chapter 1.1 of [**GZ67**] we have a uniquely determined natural transformations $\bar{\eta}_\varphi : \bar{K}_S \circ F_\varphi[\Sigma^{-1}] \to \bar{K}_T$ such that

$$\bar{\eta}_\varphi\,[P_T(X)] = \eta_\varphi[X].$$

Therefore $\bar{\eta}_\varphi$ satisfies the conditions of Corollary 20.16 and there exists a functor

$$\bar{K} : F[\Sigma^{-1}]\int\mathcal{V} \to F\int\mathcal{V}[\Sigma^{-1}]$$

such that $\bar{K} \circ P\int\mathcal{V} = K = P$.

We obtain $\bar{K} \circ J \circ P = \bar{K} \circ P\int\mathcal{V} = P$ and therefore

$$\bar{K} \circ J = \mathrm{id}_{F\int\mathcal{V}[\Sigma^{-1}]}.$$

On the other hand we have $J \circ \bar{K} \circ P\int\mathcal{V} = J \circ P = P\int\mathcal{V}$. If we can prove, that this implies, that $J \circ \bar{K}$ is the identity, we are done.

$J \circ \bar{K}$ induces functors $(J \circ \bar{K})_c : F_c[\Sigma_c^{-1}] \to F[\Sigma^{-1}]\int\mathcal{V}$, given by

$$(J \circ \bar{K})_S(x) = J \circ \bar{K}(X, S) \text{ and } (J \circ \bar{K})_S(f) = J \circ \bar{K}(f, \mathrm{id}_S).$$

Similarly we have functors

$$\left(J \circ \bar{K} \circ P\int\mathcal{V}\right) : F_S \to F[\Sigma^{-1}]\int\mathcal{V}.$$

This functors are equal to the compositions $(J \circ \bar{K})_S \circ P_S$ and to $I_S \circ P_S$, where $I_S : F_S[\Sigma_c^{-1}] \to F[\Sigma^{-1}]\int\mathcal{V}$ is the functor induced by the identity. The universal property of the localization $F_S[\Sigma_C^{-1}]$ implies $(J \circ \bar{K})_S = I_S$. Now we can apply the decomposition

$$(f, \varphi) = \big(\mathrm{id}_{F_\varphi(Y)}, \varphi\big) \circ (f, \mathrm{id}_S)$$

for each morphism $(f,\varphi):(X,S)\to(Y,T)$ and obtain

$$\begin{aligned}
J\circ\bar K(f,\varphi) &= \left(J\circ\bar K\left(\mathrm{id}_{F_\varphi(Y)},\varphi\right)\right)\circ\left(J\circ\bar K(f,\mathrm{id}_S)\right)\\
&= \left(J\circ\bar K\circ P\int\mathcal{V}\left(\mathrm{id}_{F_\varphi(Y)},\varphi\right)\right)\circ\left((J\circ\bar K)_S(f)\right)\\
&= P\int\mathcal{V}\left(\mathrm{id}_{F_\varphi(Y)},\varphi\right)\circ I_S(f)\\
&= \left(\mathrm{id}_{F_\varphi(Y)},\varphi\right)\circ(f,\mathrm{id}_S)\\
&= (f,\varphi).
\end{aligned}$$

Here we used the fact that the identites in $F_S[\Sigma_S^{-1}]$ are the images of the identities in F_S under the projection P_S. Since $P\int\mathcal{V}$ is the identity on the objects, we also have $J\circ\bar K(X,S)=(X,S)$. $\square$

20.3. The proof of Theorem 20.8. For the remainder of this section let $F:\mathfrak{Sets}^{op}\to\mathfrak{Cat}$ be the functor, which assigns to each set S the category $F_S=\mathfrak{Map}_{\mathbf{Cat}_S}$ and to each map $\varphi:S\to T$ of sets functor $F_\varphi=\varphi^*:\mathfrak{Map}_{\mathbf{Cat}_T}\to\mathfrak{Map}_{\mathbf{Cat}_S}$. Furthermore let $G:\mathfrak{Sets}^{op}\to\mathfrak{Cat}$ be the functor given by $G_S=\mathfrak{HomTCat}_S$ and $G_\varphi=\varphi^*:\mathfrak{HomTCat}_T\to\mathfrak{HomTCat}_S$. Directly from the definitions follows that $\mathfrak{Map}_{\mathbf{Cat}}$ is the category $F\int\mathfrak{Sets}$ and that $\mathfrak{HomTCat}$ is $G\int\mathfrak{Sets}$.

Let Σ be the class of topological equivalences in $\mathfrak{HomTCat}$, i.e. the elements of Σ are the pairs $(f,\varphi):(X,S)\to(Y,T)$ with $\varphi:S\to T$ a bijection of sets and $f:X\to G_\varphi(Y)$ a topological equivalence, i.e. an element in Σ_S. Then the classes Σ_S and Σ fit into the preliminaries of Theorem 20.19. Hence we obtain an isomorphism

$$G\int\mathfrak{Sets}\left[\Sigma^{-1}\right]\simeq G[\Sigma^{-1}]\int\mathfrak{Sets}$$

where $\bar G:=G[\Sigma^{-1}]:\mathfrak{Sets}^{op}\to\mathfrak{Cat}$ is the functor given by

$$\bar G_S=G_S[\Sigma_S^{-1}]\text{ and }\bar G_\varphi=G_\varphi[\Sigma_S^{-1}].$$

Following Corollary 18.36 the functor $J_S:\mathfrak{HomTCat}_S\to\mathfrak{Map}_{\mathbf{Cat}_S}$ is right adjoint to the functor $M_S:\mathfrak{Map}_{\mathbf{Cat}_S}\to\mathfrak{HomTCat}_S$. If Σ_S is the class of topological equivalences in $\mathfrak{HomTCat}_S$ then, by Theorem 18.38, these two functors induce an equivalence

$$\bar M_S:F_S=\mathfrak{Map}_{\mathbf{Cat}_S}\leftrightarrows\mathfrak{HomTCat}_S[\Sigma_S^{-1}]=\bar G_S:\bar J_S$$

of categories. For each map $\varphi:S\to T$ of sets the image $G_\varphi(f)$ of an element in Σ_T is an element of Σ_S, since the underlying maps of morphism spaces still are homotopy equivalences.

Since J_S does not change the underlying spaces of $\mathbf{Cat}_S$-algebras and the underlying maps of morphisms between them, the J_S form a natural transformation from G to F. Hence we obtain a functor

$$J\int\mathfrak{Sets}:G\int\mathfrak{Sets}\to F\int\mathfrak{Sets}$$

given by $\bar{J}\int\mathfrak{Sets}(X,S) = (J_S(X),S)$ and $\bar{J}\int\mathfrak{Sets}(f,\varphi) = (J_S(f),\varphi)$. The functor $\bar{J} : \mathfrak{Hom}\mathfrak{TCat}[\Sigma^{-1}] \to \mathfrak{Map}_{\mathbf{Cat}}$ given in Theorem 20.8 is precisely the functor induced by $J\int\mathfrak{Sets}$. The isomorphism

$$G\int\mathfrak{Sets}\left[\Sigma^{-1}\right] \simeq \bar{G}\int\mathfrak{Sets}$$

translates $\bar{J}$ into the functor

$$\bar{J}\int\mathfrak{Sets} : \bar{G}\int\mathfrak{Sets} \to F\int\mathfrak{Sets},$$

which again is given by the natural transformation from $\bar{G}$ to F induced by the $\bar{J}_S$. Hence it suffices to prove that $\bar{J}\int\mathfrak{Sets}$ is an equivalence of categories.

We need an inverse. Unfortunately the functors $\bar{M}_S$ do not induce a natural transformation from F to $\bar{G}$. But we can define a lax natural transformation $(E,\eta) : F \to \bar{G}$, which induces a functor

$$(E,\eta)\int\mathfrak{Sets} : F\int\mathfrak{Sets} \to \bar{G}\int\mathfrak{Sets}.$$

As seen above, we have an equivalence $\bar{M}_S : F_S \leftrightarrows \bar{G}_S : \bar{J}_S$, whose unit is given by functors

$$I_S : \mathrm{id}_{F_S} \to \bar{J}_S \circ \bar{M}_S$$

and whose counit is given by functors

$$E_S : \bar{M}_S \circ \bar{J}_S \to \mathrm{id}_{\bar{G}_S}.$$

For each map $\varphi : S \to T$ of sets and each object $X \in F_T$ let $\eta_\varphi[X]$ be the morphism

$$\eta_\varphi[X] := E_S\left[\bar{G}_\varphi(\bar{M}_T(X))\right] \circ \bar{M}_S\left(F_\varphi(I_T[X])\right) : \bar{M}_S\left(F_\varphi(X)\right) \to G_\varphi\left(\bar{M}_T(X)\right).$$

Here we use again the convention that terms in parentheses are arguments for functors, and that terms in square brackets relate to natural transformations. The indices are reserved for sets and maps of sets.

These morphisms $\eta_\varphi[X]$ induce natural transformations

$$\eta_\varphi : \bar{M}_S \circ F_\varphi \to \bar{G}_\varphi \circ \bar{M}_T$$

for each map $\varphi : S \to T$, because we have for each $f : X \to Y$ in F_T

$$\begin{aligned}
\eta_\varphi[Y] \circ \bar{M}_S\left(F_\varphi(f)\right) &= E_S\left[\bar{G}_\varphi\left(\bar{M}_T(Y)\right)\right] \circ \bar{M}_S\left(F_\varphi\left(I_T[Y]\right) \circ F_\varphi(f)\right) \\
&= E_S\left[\bar{G}_\varphi\left(\bar{M}_T(Y)\right)\right] \circ \bar{M}_S\left(F_\varphi\left(\bar{J}_T\left(\bar{M}_T(f)\right) \circ I_T[X]\right)\right) \\
&= E_S\left[\bar{G}_\varphi\left(\bar{M}_T(Y)\right)\right] \circ \bar{M}_S\left(\bar{J}_S\left(\bar{G}_\varphi\left(\bar{M}_T(f)\right)\right) \circ F_\varphi\left(I_T[X]\right)\right) \\
&= \bar{G}_\varphi\left(\bar{M}_T(f)\right) \circ E_S\left[\bar{G}_\varphi\left(\bar{M}_T(X)\right)\right] \circ \bar{M}_S\left(F_\varphi\left(I_T[X]\right)\right) \\
&= \bar{G}_\varphi\left(\bar{M}_T(f)\right) \circ \eta_\varphi[X].
\end{aligned}$$

In this series of equations we only used the naturality of several morphisms and the fact that $F_\varphi \circ \bar{J}_T = \bar{J}_S \circ \bar{G}_\varphi$.

Furthermore we have

$$\eta_{\mathrm{id}_S}[X] = E_S\left[\bar{M}_S(X)\right] \circ \bar{M}_S\left(I_S[X]\right)$$

which is the identity, because I_S and E_S are the unit and counit of the adjunction of $\bar{J}_S$ and $\bar{M}_S$. We also obtain

$$\eta_{\psi\circ\varphi}[X] = \bar{G}\left(\eta_\psi[X]\right) \circ \eta_\varphi\left[F_\psi(X)\right]$$

for $\varphi : S \to T, \psi : T \to U$ and X in F_U. Hence (E, η) is a lax natural transformation from F to $\bar{G}$ and we obtain a functor

$$(E,\eta)\int \mathfrak{Sets} : F\int \mathfrak{Sets} \to \bar{G}\int \mathfrak{Sets}.$$

Using the several naturalities, the fact that I_S and E_S are the unit and counit of the adjunction $\bar{M}_S : F_S \leftrightarrows \bar{G}_S : \bar{J}_S$ and the fact that $F_\varphi \circ \bar{J}_T = \bar{J}_S \circ \bar{G}_\varphi$ we can prove that there exist lax natural 2-transformations

$$I : (\mathrm{id}, \mathrm{id}) \to (\bar{J}, \mathrm{id}) \circ (\bar{M}, \eta) = \big(\bar{J}_S \circ \bar{M}_S, \bar{J}_S(\eta_\varphi[X])\big)$$

given by

$$I_S : \mathrm{id}_{F_S} \to \bar{J}_S \circ \bar{M}_S$$

and

$$E : (\bar{M}, \eta) \circ (\bar{J}, \mathrm{id}) = \big(\bar{M}_S \circ \bar{J}_S, \eta_\varphi[\bar{J}_T(X)]\big) \to (\mathrm{id}, \mathrm{id})$$

given by

$$E_S : \bar{M}_S \circ \bar{J}_S \to \mathrm{id}_{\bar{G}_S}$$

which induce two natural transformations

$$I\int \mathfrak{Sets} : \mathrm{id}_{F\int\mathfrak{Sets}} \to (\bar{J}, \mathrm{id}) \circ (\bar{M}, \eta)\int \mathfrak{Sets}$$

and

$$E\int \mathfrak{Sets} : (\bar{M}, \eta) \circ (\bar{J}, \mathrm{id})\int \mathfrak{Sets} \to \mathrm{id}_{\bar{G}\int\mathfrak{Sets}}$$

which are given by

$$I\int \mathfrak{Sets}[X, S] = (I_S[X], \mathrm{id}_S) \quad \text{and} \quad E\int \mathfrak{Sets}[X, S] = (E_S[X], \mathrm{id}_S).$$

Since $E_S[X]$ is an isomorphism in $\mathfrak{Hom}\mathfrak{T}\mathfrak{Cat}_S[\Sigma_S^{-1}]$ and $I_S[X]$ one in $\bar{M}ap_{\mathbf{Cat}_S}$, the functors $(\bar{J}, \mathrm{id})\int\mathfrak{Sets}$ and $(\bar{M}, \eta)\int\mathfrak{Sets}$ are equivalences of categories. This completes the proof of Theorem 20.8.

The Milgram Non-Operad

In [**Mil66**] R.J. Milgram introduced geometric models J^n for the iterated loop-space operads $\Omega^n\Sigma^n$. Later J.M. Boardman, R.M. Vogt and P. May proved that n-fold loop spaces are closely related to E_n operads in general, and thus to the little cube operads (see [**BV68**], [**BV73**], [**May72**]). Hence the question arises if the operad structure of $\Omega^n\Sigma^n$ translates to the geometric model J^n.

In [**BFSV98a**] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R.M. Vogt construct an operad $\mathcal{M}_n$, which codifies n-fold monoidal categories, a categorial analog of n-fold loop spaces. They observe that an equivalent *preoperad* is embedded in $\mathcal{M}_n$, whose free space is of the same homotopy type as J^nX . For $n = 2$ the spaces are even homeomorphic (see [**BFSV98a**, 3.12 – 14]).

Due to the underlying polytopes, the permutohedra, the Milgram-construction J^n is of some importance for the examination of coherent homotopy-commutativity. Similar to the associahedra introduced by Stasheff in [**Sta63**], Williams uses the permutohedra and the Milgram-construction in [**Wil69**] to define his notion of C_n-spaces, which is used in several subsequent papers, and is occurring in papers of McGibbon and Hemmi (see for example [**McG89**] and [**Hem91**]).

In fact there exists an operad structure with the permutohedra as underlying spaces (this was pointed out to me by Clemens Berger and Zig Fiedorowicz), simply by using the convex extension of the permutation operad. But since the permutohedra are contractible this is an E_∞ operad in the sense of Boardman and Vogt (i.e. the symmetric group action does not need to be free). Therefore an algebra of this operad is homotopy equivalent to an infinite loop space and hence its associated monad can not be of the same homotopy type as the Milgram construction J^n which is just an n-fold loop space.

In [**Ber96**] and [**Ber97**] C. Berger conjectured an E_n operad structure of the permutohedra, whose associated monad is the Milgram construction J^n.

I will show that Berger's construction does not work. The first observation is that the would-be operad bears a structure far too strong, namely that of strictly abelian monoids. This collapse of structure is then used to show that the suggested structure does not define an operad at all. In fact the proof shows that the multiplication defined by Berger does not respect the degeneration conditions.

Nonetheless the construction defines preoperads $J^{(n)}$, which are homotopy equivalent to the little n-cubes. In particular, the k-th space $J_k^{(n)}$ is Σ_n-equivariantly homotopy equivalent to the real configuration space $F(\mathbb{R}^n, k)$.

Furthermore the "non-monad" associated to the preoperad $J^{(n)}$ is an alternative description of the Milgram construction J_n.

I would like to thank Zig Fiedorowicz and especially Clemens Berger for some helpful and enlightening discussions, and James McClure for the discovery of some mistakes and errors in an earlier version of this paper.

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21. The Permutohedra and Berger's construction

Let Λ be the category of finite sets $\mathbf{n} = \{1, \ldots, n\}$ and injective maps. Each injective map $\varphi : \mathbf{n} \to \mathbf{m}$ has an unique decomposition of the form $\varphi = \varphi^{inc} \circ \varphi^{\natural}$, such that $\varphi^{inc} : \mathbf{n} \to \mathbf{m}$ is increasing and $\varphi^{\natural} : \mathbf{n} \to \mathbf{n}$ is a permutation.

DEFINITION 21.1. The *Permutation preoperad* $\Sigma : \Lambda^{op} \to \mathfrak{Top}$ is the functor with $\Sigma(\mathbf{n}) = \Sigma_n$ and $\varphi^* := \Sigma(\varphi) : \Sigma_m \to \Sigma_n$ given by $\sigma \mapsto (\sigma \circ \varphi)^{\natural}$ for $\varphi \in \Lambda(\mathbf{n}, \mathbf{m})$

The multiplications of the *Permutation Operad* Σ are given by

$$\begin{aligned}\gamma^{\Sigma}_{n;i_1,\ldots,i_n} : \Sigma_n \times \Sigma_{i_1} \times \cdots \times \Sigma_{i_n} &\to \Sigma_{i_1+\cdots+i_n}\\ (\sigma; \tau_1, \ldots, \tau_n) &\mapsto \sigma(i_1, \ldots, i_n) \circ (\tau_1 \oplus \cdots \oplus \tau_n)\end{aligned}$$

where $\sigma(i_1, \ldots, i_n)$ permutes the blocks (see [**Ber97**, 1.15.(a)]) and $- \oplus - : \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ is the canonical product of permutations.

The product $\Sigma_{i_1} \oplus \cdots \oplus \Sigma_{i_n} \subset \Sigma_{i_1+\cdots+i_n}$ will be denoted with $\Sigma(i_1, \ldots, i_n)$.

For more details about (pre)operads in general the reader is referred to [**Ber96**], [**Ber97**] or [**May72**].

In contrary to Clemens Berger I will use the left action of the symmetric group on $\mathbb{R}^n$, which seems to be the more common description.

DEFINITION 21.2. For $n \geq 1$ the symmetric group Σ_n acts on $\mathbb{R}^n$ from the left by permuting the coordinates in the following way.

$$\sigma(x_1, \ldots, x_n) = \big(x_{\sigma^{-1}(1)}, \ldots x_{\sigma^{-1}(n)}\big).$$

The *n-th Permutohedron* $P_n \subset \mathbb{R}^n$ is the convex hull of the orbit of $(1, 2, \ldots, n) \in \mathbb{R}^n$ under this operation.

The convex hull of the orbit of $\Sigma(i_1, \ldots, i_r) \subset \Sigma_n$ will be denoted with $P(i_1, \ldots, i_r) \subset P_n$.

The point $\sigma(1, \ldots, n) \in P_n$ will be denoted with σ.

REMARK 21.3. The notation of σ for the point $(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))$ seems somewhat confusing. But since we will extend the permutation operad to the permutohedra, we can calculate the vertices right from the permutations, without applying it to the $\mathbb{R}^n$.

The geometric and simplicial properties of these polygons were examined in [**Mil66**], [**Wil69**] and [**Bau80**]. Here I will give only a rough sketch of the few details I will use.

P_1 consists of only one point, P_2 is homeomorphic to the unit interval in $\mathbb{R}$ and P_3 to the hexagon in $\mathbb{R}^2$. In general P_n is a $(n-1)$ dimensional polytope.

Obviously there exists a left Σ_n-action on P_n. The vertices are mapped to vertices, and for each $\sigma \in \Sigma_n$ the map $P_n \to P_n$ with $x \mapsto \sigma x$ is a homeomorphism. But unfortunately this action is not free, since the barycenter of each permutohedron is a fixed point.

An arbitrary point $x \in P_n$ will be denoted by a linear combination of permutations

$$x = \sum_{\sigma \in \Sigma_n} t_\sigma \sigma \qquad \text{with} \qquad \sum_{\sigma \in \Sigma_n} t_\sigma = 1.$$

If $x = \sum_{\sigma \in \Sigma_n} s_\sigma \sigma$ and $y = \sum_{\tau \in \Sigma_m} t_\tau \tau$ are points of P_n resp. P_m, the point $x \oplus y \in P_{n+m}$ is given by

$$x \oplus y = \sum_{\sigma \in \Sigma_n} \sum_{\tau \in \Sigma_m} s_\sigma t_\tau \sigma \oplus \tau.$$

In [**Mil66**] Milgram defined maps $I_k : P_k \times P_{n-k} \to P_n$ given by $(x, y) \mapsto x \oplus y$ mapping the product of two permutohedra into certain faces of a higher dimensional permutohedron. More general the codimension $(r-1)$ faces of P_n are in one-to-one correspondence with the ordered partitions of $\{1, \dots, n\}$ of type $(i_1, \dots, i_r)$ with $i_1 + \cdots + i_r = n, i_k \geq 1$.

Remark 21.4. Each partition of type $(i_1, \dots, i_r)$ can be interpreted as a permutation of $i_1 + \cdots + i_r$ elements. If the classes are given by

$$\{j_1, \dots, j_{i_1}\}, \{j_{i_1+1}, \dots, j_{i_1+i_2}\}, \dots, \{j_{i_1+\cdots+i_{r-1}+1}, \dots, j_{i_1+\cdots+i_r}\}$$

with $j_k < j_l$ for $i_m \leq k < l < i_{m+1}$, then the corresponding permutation is given by $k \mapsto j_k$ for $1 \leq k \leq i_1 + \cdots + i_r$.

The converse does not hold, since the same permutation can be associated to different partitions. For example the identity in Σ_3 corresponds to the partition $\{1\}, \{2, 3\}$ and to $\{1, 2\}, \{3\}$.

The vertices of the codimension $(r-1)$ face, corresponding to a partition of type $(i_1, \dots, i_r)$, with associated permutation $\sigma \in \Sigma_n$, are given by the coset $\sigma\Sigma(i_1, \dots, i_r)$. In addition there is a homeomorphism $I_\sigma : P_{i_1} \times \cdots \times P_{i_r} \to \sigma P(i_1, \dots, i_r) \subset P_n$ with

$$(x_1, \dots, x_r) \mapsto \sigma(x_1 \oplus \cdots \oplus x_r).$$

Using the *(right) weak Bruhat order* on the symmetric groups, the 1-skeleton of these faces can be oriented.

Definition 21.5. The *inversion index* $inv(\sigma)$ of a permutation $\sigma \in \Sigma_n$ is the number of ordered pairs $(i, j), 1 \leq i < j \leq n$, whose orders are inverted by σ, i.e. $\sigma(i) > \sigma(j)$.

The *(right) Weak Bruhat Order* of Σ_n is the partial order generated by $\sigma < \tau$, if τ is the composition of σ and a transposition of two subsequent numbers, that is $\tau = \sigma \circ (i, i+1)$, and $inv(\sigma) < inv(\tau)$.

REMARK 21.6. Since we use the left action of Σ_n on $\mathbb{R}^n$, we have to use the right weak Bruhat order, instead of the left weak Bruhat order, as Clemens Berger did.

EXAMPLE 21.7. Σ_3 is given by the Poset

$$\begin{tikzcd}
& (123) \arrow[dl] \arrow[dr] & \\
(213) \arrow[d] & & (132) \arrow[d] \\
(231) \arrow[dr] & & (312) \arrow[dl] \\
& (321) &
\end{tikzcd}$$

Here and in the following the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$$

will be denoted with $(\sigma(1), \sigma(2), \dots, \sigma(n))$ (the commas will be left, if unnecessary).

REMARK 21.8. If the vertices of the poset (i.e. the permutations) are interpreted as points in $\mathbb{R}^3$, it does not seem to describe the border of P_3 - the points $(2,3,1)$ and $(3,1,2)$ have to be exchanged. But since we use the left action of Σ_n on $\mathbb{R}^n$, the permutation (231) corresponds to the point $(3,1,2)$ and the permutation (312) to the point $(2,3,1)$. In fact the correspondence holds for all vertices of P_3 and the poset Σ_3.

Applying this partial order to the permutohedra the edges will be oriented. The face corresponding to a certain partition of type $(i_1, \dots, i_r)$ and associated permutation σ has exactly one initial vertex, given by σ and a unique terminal vertex, given by the permutation that turns the classes of the partition "upside-down", i.e.

$$l \mapsto k_{l'} \quad \text{with} \quad l' = 2(i_1 + \dots i_l) + i_{l+1} - l + 1 \quad \text{if} \quad i_l < l \leq i_{l+1}.$$

DEFINITION 21.9. An interval $[\sigma_1, \sigma_2]$ in Σ_n with the weak Bruhat order is called *admissible*, if it is the vertex set of some coset $\sigma\Sigma(i_1, \dots, i_r)$. Hence σ_1 is the initial and σ_2 the terminal vertex of a face of P_n.

The *geodesic* of an arbitrary interval $[\sigma_1, \sigma_2]$ of the weak Bruhat order in Σ_n, is the average of all oriented edge-paths between σ_1 and σ_2 in P_n. The *barycenter* of $[\sigma_1, \sigma_2]$ is the barycenter of its geodesic.

The barycenter of an admissible interval coincides with the barycenter of the corresponding cell. For example the geodesic of $[(123), (213)] \subset \Sigma_3$ is the corresponding edge of P_3, and its barycenter the barycenter of the interval. The geodesic of $[(123), (321)] \subset \Sigma_3$ is the line between (123) and (321) and its barycenter is the point B. The geodesic of the non-admissible interval

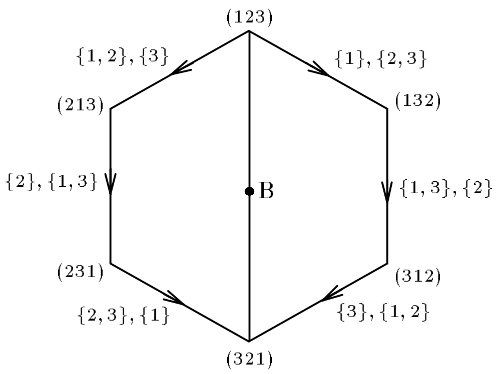

FIGURE 12. The geodesic and barycenter of P_3

$[(123),(231)] \subset \Sigma_3$ consists of the two edges $[(123),(213)]$ and $[(213),(231)]$ of P_3. Hence its barycenter is the point $(213) \in P_3$ (see Fig. 12).

To define an operad-multiplication on the permutohedra we have to define maps $P_r \times P_{i_1} \times \cdots \times P_{i_r} \to P_{i_1+\cdots+i_r}$ which satisfy the associativity conditions. But the intention to formulate an operad, whose associated monad is the Milgram construction gives certain additional restrictions to the choice of the multiplication.

The operad structure has to extend the permutation operad Σ. This can be done very easily by mapping the vertices of the product $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the corresponding vertices of $P_{i_1+\cdots+i_r}$, given by Σ. But the extension of this map can be done in two different ways.

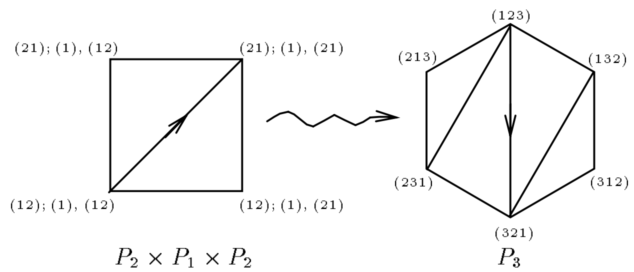

FIGURE 13. The convex multiplication

The first possibility is the *convex* extension of the permutation operad, by mapping $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the convex hull of the image vertices (see Fig. 13). This construction does in fact define an operad whose n-th space is P_n. It is E_∞ in the sense of Boardman and Vogt, i.e. its underlying spaces are contractible but the actions of the symmetric groups does not need to be free. Thus its algebras are homotopy equivalent to infinite loop spaces (cmp. [**BV73**] section VI.3). But since the Milgram construction is only a model for n-fold loop spaces the associated monad of the convex extension is of the wrong homotopy type.

Berger tried to get the correct operad by application of two changes to the convex extension of the permutation operad. First he deformed the multiplication such that it respects the relations on the borders of the permutohedra, given in Milgram's construction. But this does not affect the homotopy type of the spaces. In a second step he made the action of the symmetric group free, which would give the correct homotopy type. 1 In the Milgram construction the border of $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ has to be mapped to the border of $P_{i_1+\cdots+i_r}$. Berger did this by using a *cubical* extension of the symmetric operad instead of the convex one. He defined the operad-multiplication $\gamma^P_{n;i_1,\ldots,i_n} : P_n \times P_{i_1} \times \cdots \times P_{i_n} \to P_{i_1+\cdots+P_{i_n}}$ as the affine extension of the permutation operad, such that the barycenter of any interval in $\Sigma_{i_1+\cdots+i_n}$ is mapped to the barycenter of its geodesics in $P_{i_1+\cdots+i_n}$ (see Fig. 14). Hence for $n = 2$ and $i_1 + i_2 = 3$ the squares A,B,C and D of $P_2 \times P_1 \times P_2$, resp. $P_2 \times P_2 \times P_1$ are mapped to the corresponding segments of P_3.

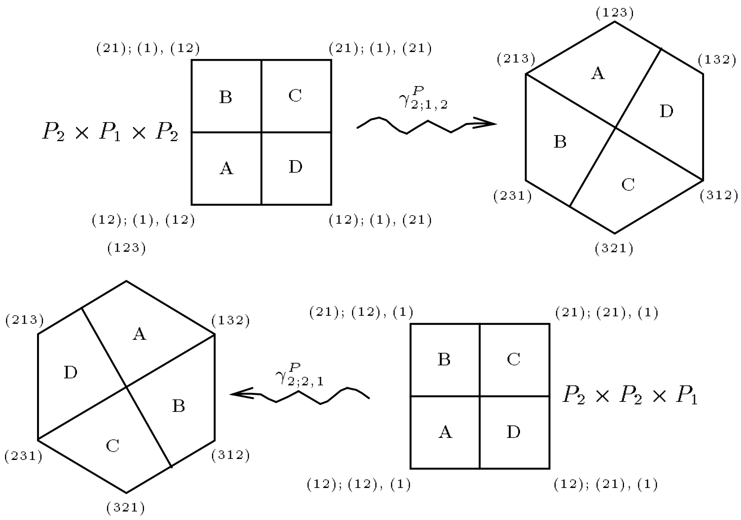

FIGURE 14. The cubical multiplication

The second step, in which the symmetric group action is made free, is done in the definition of the would-be operad. For each permutation in Σ_n a copy of P_n^k is added and an appropriate quotient of the space $P_n^k \times \Sigma_n$ is taken to be the $n - th$ space of the new E_k operad.

Here I will only describe the suggested construction for the operad $J^{(2)}$ corresponding to J_2, i.e. 2-fold loop spaces.

DEFINITION 21.10. (cmp. [**Ber97**, 2.12]) Let $J_n^{(2)}$ be the quotient space of $P_n \times \Sigma_n$ under the relation

$$(\tau x; \sigma) \sim (x; \tau\sigma)$$

for any partition $\tau \in \Sigma_n$ of type $(i_1, \ldots, i_r)$, $i_1 + \cdots + i_r = n$, $x \in P\,(i_1, \ldots, i_r)$ and $\sigma \in \Sigma_n$.

The action of $\varphi \in \Lambda(\mathbf{n}, \mathbf{n})$ is induced by

$$\begin{aligned} \varphi^* : P_m \times \Sigma_m &\to P_n \times \Sigma_n \\ (x; \sigma) &\mapsto \big(x(\sigma\varphi)^{inc}; (\sigma\varphi)^{\natural}\big). \end{aligned}$$

REMARK 21.11. Since I use the left instead of the right action, the relation given by Berger has to be changed slightly.

Following [**Ber97**, 2.14] the Λ structure and the maps $\gamma^P_{i_1,\dots,i_r} \times \gamma^\Sigma_{i_1,\dots,i_r}$ induce an E_2-operad structure on $J^{(2)}_n$.

22. The commutativity of Berger's construction

In the following we assume that Berger's construction defines an operad. If $(X,*)$ is a $J^{(2)}$-algebra, there exist maps $F_n : P_n \times \Sigma_n \times X^n \to X$. These fulfill certain conditions, induced by the operad structure and the relations on $J^{(2)}_n$. Used here are

1. the associativity condition
$$\begin{aligned} F_n\big(s; F_{i_1} &= (t_1; x^1_1, \dots, x^1_{i_1}), \dots, F_{i_n}(t_n; x^n_1, \dots, x^n_{i_n})\big) \\ &= F_{i_1+\dots+i_n}\big(\gamma_{n;i_1,\dots,i_n}(s; t_1, \dots, t_n); x^1_1, \dots x^n_{i_n}\big), \end{aligned}$$
2. the degeneration relations on the associated monad, induced by the two maps $\mathbf{1} \to \mathbf{2}$ $F_2(s; x, *) = F_1(*; x) = x = F_2(s; *, x)$
3. and the Σ_n-equivariance: $F_n(s\sigma; x_1, \dots, x_n) = F_n\big(s; x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}\big)$.

REMARK 22.1. Since only the identities of Σ_n are used, in the future this coordinate will be dropped.

There is a multiplication on X, given by $xy = F_2\big((12); x, y\big)$. The associativity of the permutation operad shows the associativity of this multiplication. Since the degeneration relations hold $*$ is a unit. Therefore X is a associative monoid with strict unit.

Obviously the map $\varphi : I \to P_2$ with $t \mapsto (12)(1-t) + (21)t$ is a homeomorphism. Thus there exists a homotopy $h_t : X \times X \to X$ with
$$h_t(x, y) = F_2\big((12)(t-1) + (21)t; x, y\big),$$
running from $h_0(x,y) = xy$ to $h_1(x,y) = F_2\big((21); x, y)\big) = F_2\big((12); y, x\big) = yx$.

Via F_3 and the degenerations one gets even stricter conditions for the commuting homotopy h_t. The maps $\gamma^P_{2;1,2}$ and $\gamma^P_{2;2,1}$ are homeomorphisms. Therefore there exist $s_i, t_i \in P_2, i = 1, 2$ for each $r \in P_3$ such that
$$\gamma^P_{2:2,1}\big(s_2; t_2, (1)\big) = r = \gamma^P_{2;1,2}\big(s_1; (1), t_1\big).$$
Hence for $x, y \in X$ one gets
$$F_2(s_2; x, y) = F_3(r; *, x, y) = F_2(t_1; x, y)$$
In the first case the homotopy h_t is mapped to the edges $[(123), (132)]$ and $[(231), (321)]$ (cmp. Fig. 15). In the second case h_t is mapped to the geodesics of the intervals $[(123), (312)]$ and $[(213), (321)]$, such that the center point of the homotopy $h_{\frac{1}{2}}(x, y)$ is mapped to (132) and (231). Thus the homotopy needs to be equal to xy on its first half and equal to yx on the second half (In the first case the edge $[(213), (231)]$ is mapped to xy and the edge $[(132), (312)]$ to yx). Thus $(X, *)$ must be an abelian monoid.

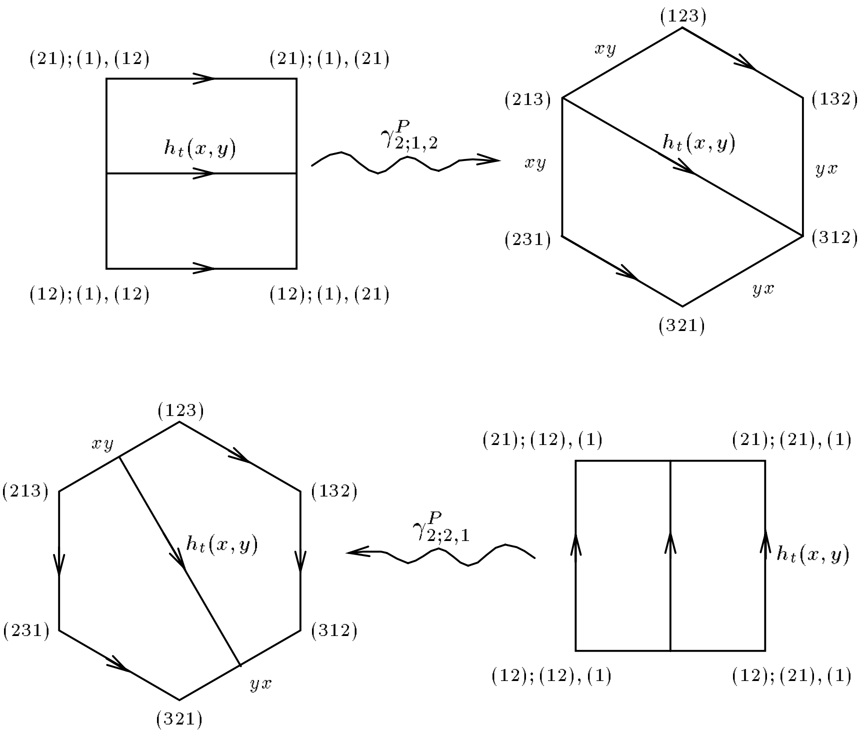

Figure 15. The commuting homotopy

Remark 22.2. Since the permutation coordinate wasn't used, the freeness of the symmetric group action is not involved in the failure of the suggested construction. In fact the cubical extension of the multiplication does not fulfill the needed degeneration properties.

Now let X be a 2-connected CW-complex with non-degenerate base point $*$ (i.e. the inclusion $* \hookrightarrow X$ is a closed cofibration). Then the two-fold Moore loop space $Y := \Omega^2_M X$ of X is a connected CW-complex. The canonical evaluation map $e : \Sigma^2\Omega^2_M X \to X$ induces a homomorphism of monoids $\Omega^2_M e : \Omega^2_M \Sigma^2 Y \to Y$.

If $J^{(2)}$ is an E_2 operad, whose associated monad is the Milgram construction, there exists a homomorphism of monoids $\psi : J^{(2)}Y \to \Omega^2_M\Sigma^2 Y$ (the map is given in [**Mil66**, 5.2.]). Therefore we get a homomorphism $\varphi := \Omega^2_M e \circ \psi : J^{(2)}Y \to Y$. From the construction of ψ in [**Mil66**] one can see that the diagram

$$\begin{tikzcd} J^{(2)}Y \arrow[r, "\psi"] & \Omega^2_M\Sigma^2 Y \arrow[r, "\Omega^2_M e"] & Y \\ & Y \arrow[ul, hook] \arrow[u, hook] \arrow[ur, equal] & \end{tikzcd}$$

with the inclusion $i : Y \hookrightarrow \Omega^2_M\Sigma^2 Y$ given by

$$(i(y))(s,t) = y \wedge s \wedge t,$$

commutes. Therefore The homomorphism $\varphi = \Omega^2_M\psi$ is an extension of the identity and hence surjective.

Since $J^{(2)}Y$ is the free $J^{(2)}$-algebra of the space Y, it is an abelian monoid. The surjectivity of φ now shows that $Y = \Omega^2_M X$ is strictly commutative, too. But obviously this is wrong. Therefore $J^{(2)}$ can not be an operad whose associated monad is the Milgram construction.

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