On graded ideals over the exterior algebra with applications to hyperplane arrangements

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Dinh Phong Thieu

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Contents

Preface

The study of graded ideals in a polynomial ring is an important area in Commutative Algebra. It relates to the study of coordinate rings of algebraic varieties, Stanley-Reisner rings and many other topics in Algebra, Geometry, Combinatorics and Topology; see, e.g., the book by Bruns and Herzog [**12**] and Eisenbud [**21**]. As a tool, one usually uses graded ideals by the fact that every finitely generated graded algebra over a field is isomorphic to a quotient ring of a polynomial ring by a graded ideal. Since the work of Hilbert, many useful tools have been developed to study graded ideals over a polynomial ring. Many of these methods can be used in a similar way for graded ideals over a exterior algebra with some suitable adjustments. But there are still many notions and properties over the polynomial ring for which not much is known about their counterparts over the exterior algebra, e.g., regular elements (of degree greater than 1) or the Gorenstein property.

This thesis is concerned with structures and properties of graded ideals over the exterior algebra. We study minimal graded resolutions, Gröbner fans of graded ideals and variations of the Koszul property of standard graded algebras defined by graded ideals. We apply our results to Orlik-Solomon ideals of hyperplane arrangements and show in which way the exterior algebra is useful in the study of related combinatorial objects. For further applications of exterior algebra methods; see, e.g., [**1, 38, 45**].

Let K be a fixed field. The exterior algebra $E = K\langle e_1, \dots, e_n\rangle$ in n variables over K is a skew-commutative $\mathbb{Z}$-graded K-algebra, with $\deg e_i = 1$. We denote by $\mathcal{M}$ the category of finitely generated graded left and right E-modules M satisfying the equations $um = (-1)^{\deg u \deg m} mu$ for homogeneous elements $u \in E$, $m \in M$. A *graded ideal* of E is a graded submodule of E.

This thesis is divided in six chapters. Chapter 1 introduces definitions, notations and gives a short review on those facts which are relevant to next chapters. In particular, Section 1.2 collects necessary notions and properties related to free resolutions of a module $M \in \mathcal{M}$. One important invariant here is the *Castelnuovo-Mumford regularity* of $M \in \mathcal{M}$ given by $\operatorname{reg}_E(M) = \max\{j - i : \beta^E_{i,j}(M) \neq 0\}$ where $\beta^E_{i,j}(M) = \dim_K \operatorname{Tor}^E_i(K, M)_j$, for $i, j \in \mathbb{Z}$, are the *graded Betti numbers* of M.

The Cartan complex, which is the minimal free resolution of the residue field of E, is a very useful tool to investigate graded Betti numbers and the Castelnuovo-Mumford regularity of modules over the exterior algebra; see, e.g., Aramova and Herzog [**2**]. This complex has similar properties as well-known properties of the Koszul complex over the polynomial ring. In Chapter 2, after recalling some facts about Cartan homology and generic bases, we prove some properties for a special kind of generic bases, namely strongly generic bases. As applications, we provide formulas for computing the Castelnuovo-Mumford regularity of modules over the exterior algebra (see Theorem 2.3.2 and Corollary 2.3.3). These are similar to the classical results over the polynomial ring; see, Conca and Herzog [**15**, Proposition 1.2, Corollary 1.3].

We present in Chapter 3 an overview of Gröbner fans over the exterior algebra. Analogously to Gröbner fans over the polynomial ring defined by Mora and Robbiano [**44**], we see that the construction and many properties of Gröbner fans still work for the exterior algebra. See, e.g., [**10**], [**41**], [**61**] for more details on the Gröbner fan over the polynomial ring. However, we need some modifications to find a suitable definition for a special subfan of the Gröbner fan of a graded ideal in the exterior algebra corresponding to tropical varieties. Note that tropical varieties are the main objects of tropical geometry, and recently have received a lot of attention. See, e.g., [**43**] for more details. Using Definition 3.2.1, our subfan has some similar properties to known properties of the tropical variety over the polynomial ring. In particular, by considering the generic case, we prove in Theorem 3.3.2 and Theorem 3.3.7 the existence of the generic Gröbner fan of a graded ideal which are similar to a result of Römer and Schmitz [**54**] for the polynomial ring case.

Let $M \in \mathcal{M}$. We say that M has a *d-linear resolution* if $\beta^E_{i,i+j}(M) = 0$ for all i and all $j \neq d$. Following [**32**], M is called *componentwise linear* if the submodule $M_{\langle i \rangle}$ of M generated by M_i has an i-linear resolution for all $i \in \mathbb{Z}$. Furthermore, M is said to have *linear quotients* with respect to a system of homogeneous generators $m_1, \ldots, m_r$ if $(m_1, \ldots, m_{i-1}) :_E m_i$ is a linear ideal for $i = 1, \ldots, r$. Here a *linear ideal* is an ideal in E generated by linear forms. We say that M has *componentwise linear quotients* if each submodule $0 \neq M_{\langle i \rangle}$ of M for $i \in \mathbb{Z}$ has linear quotients w.r.t. some of minimal systems of homogeneous generators.

Chapter 4 is devoted to the study of the structure of minimal graded free resolutions of graded ideals in E. More precisely, we are interested in graded ideals which have d-linear resolutions, linear quotients and are componentwise linear. It is well-known that a graded ideal with linear quotients w.r.t. a minimal system of homogeneous generators is componentwise linear (see [**59**, Corollary 2.4] for the polynomial ring case and [**37**, Theorem 5.4.5] for the exterior algebra case). We give an another proof for this result in Corollary 4.2.5 by using Theorem 4.2.4 which states that if a graded ideal has linear quotients then it has componentwise linear quotients.

Motivated by a result of Conca and Herzog in [**15**, Theorem 3.1] that a product of linear ideals in the polynomial ring has a linear resolution, we study in Section 4.3 the question whether this result holds over the exterior algebra. At first, we have a positive answer in the case that the linear ideals are generated by variables (see Theorem 4.3.2). We continue in Section 4.4 with a discussion for the general case. More precisely, with additional assumptions we may prove the positive answer for the question above (see Theorem 4.4.7).

Koszul algebras are widely studied class of rings in algebra. They are standard graded K-algebras over which every finitely generated graded module has finite

Castelnuovo-Mumford regularity (see, e.g., [**5, 6**]). Let R be a standard graded K-algebra. R is called a *Koszul algebra* over K if K has a linear resolution over R. To ensure Koszulness of R, one shows for example that its defining ideal has a quadratic Gröbner basis w.r.t. some coordinate system of E_1 and some monomial order on E. In this case, one says that the algebra R is *G-quadratic*. For an updated survey on Koszul algebras, we refer the reader to Fröberg [**27**]. Another powerful tool to deduce Koszulness, namely Koszul filtrations, is introduced by Conca, Trung and Valla in [**18**]. Using this, one can define several variations of the Koszul property, e.g., universally Koszul, strongly Koszul and initially Koszul properties. See, e.g., [**9, 13, 14, 16, 17, 34**] for more details.

In Chapter 5, we study several variations of the Koszul property over the exterior algebra. Let $R = E/J$ be a standard graded K-algebra where $J \subset E$ is a graded ideal which does not contain linear forms. Assume that R is Koszul. Then J is generated in degree 2. We consider in Section 5.2 the universally Koszul property of algebras defined by edge ideals in the exterior algebra. It is shown that the classification of universally Koszul algebras defined by monomial ideals in the polynomial ring (see [**14**, Theorem 5]) still holds for the exterior algebra. For the convenience of the reader and for the proofs in Section 6.4 we reproduce this in Theorem 5.2.11. By slightly modifying the definition of the strongly Koszul property given in [**34**, Definition 1.1], we define in Section 5.3 the unconditioned strongly Koszul property and give an example to show that the universally Koszul and (unconditioned) strongly Koszul properties are distinct notions (see Example 5.3.3). Note that Conca, De Negri and Rossi also study the unconditioned strongly Koszul property in [**17**, Definition 3.11] using the name "strongly Koszul". Moreover, we prove in Proposition 5.3.5 that every algebra defined by a quadratic monomial ideal is unconditioned strongly Koszul (see [**17**, Theorem 3.15] for the proof in the polynomial ring case). We also give a necessary condition for elements of degree 2 in the exterior algebra to define unconditioned strongly Koszul algebras (see Proposition 5.3.6). In Section 5.4, we study standard graded K-algebras with Gröbner flags, namely initially Koszul algebras, over the exterior algebra. It is known that if a standard graded algebra over a polynomial ring has a Gröbner flag then it is G-quadratic (see [**9**, Proposition 2.3] and [**16**, Proposition 2.5]). This still holds for standard graded algebras over the exterior algebra (see Proposition 5.4.5).

Let $E = K\langle e_1, \dots, e_n\rangle$ be the exterior algebra over a field K with $\operatorname{char} K = 0$. For a set of indices $F = \{i_1, \dots, i_t\} \subseteq \{1, \dots, n\}$, we write $e_F = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_t}$ and $\partial\, e_F = \sum_{j=1}^{t} (-1)^{j-1} e_{F \setminus \{i_j\}}$. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential central hyperplane arrangement in $\mathbb{C}^l$ with the complement $\mathcal{X}(\mathcal{A}) = \mathbb{C}^l \setminus \bigcup_{H \in \mathcal{A}} H$. We say that a subset $\{H_i : i \in F\}$ of hyperplanes of $\mathcal{A}$ is *dependent* if their defining linear forms are linearly dependent. One important result of hyperplane arrangement theory, proved by Orlik and Solomon [**46**], is that the singular cohomology $H^\bullet(\mathcal{X}(\mathcal{A}); K)$ of $\mathcal{X}(\mathcal{A})$ with coefficients in K is isomorphic to the *Orlik-Solomon algebra* E/J where J is the *Orlik-Solomon ideal* of $\mathcal{A}$ generated by all elements ∂e_F such that $\{H_i : i \in F\}$ is dependent. See Orlik-Terao [**47**] and Yuzvinsky [**65**] for details. See also, e.g., [**1, 19, 22, 38, 56, 57**] for the study of Orlik-Solomon algebras via exterior algebra methods.

The motivation for Chapter 6 is to apply results in the previous chapters on Orlik-Solomon ideals and Orlik-Solomon algebras. More precisely, in Theorem 6.2.1 and Corollary 6.2.2 we characterize essential central hyperplane arrangements of rank ≤ 3 whose Orlik-Solomon ideals are componentwise linear. We also propose a conjecture to characterize componentwise linearity of Orlik-Solomon ideals in the general case (see Conjecture 6.3.1) and study exterior algebras with few number of variables and arrangements with small ranks. In Section 6.4, we classify completely Orlik-Solomon algebras which are universally Koszul as well as initially Koszul (see Theorem 6.4.3 and Theorem 6.4.5). More precisely, the Orlik-Solomon algebra E/J of an essential central hyperplane arrangement $\mathcal{A}$ is universally Koszul if and only if J has a 2-linear resolution; E/J has a Gröbner flag if and only if $\mathcal{A}$ is supersolvable.

Note: The content of Section 6.2 and 6.3 is contained in the preprint [**62**] with major changes in the presentation.

CHAPTER 1

Background

The purpose of this chapter is to introduce basic notions and methods that are used in the thesis. We assume that the reader is familiar with fundamental notions and results in commutative algebra. For more details of unexplained facts, we refer to the books by Matsumura [**42**], Eisenbud [**21**], and Bruns and Herzog [**12**]. We also use material of exterior algebras from the book by Herzog and Hibi [**32**].

1.1. Preliminaries

Let K be a field and V an n-dimensional K-vector space, where $n \geq 1$, with a fixed basis $e_1, \dots, e_n$. We denote by $E = K\langle e_1, \dots, e_n \rangle$ the exterior algebra of V. It is a standard graded K-algebra with defining relations $v \wedge v = 0$ for all $v \in V$ and graded components $E_i = \Lambda^i V$ by setting $\deg e_i = 1$. Elements of degree one in E are called *linear forms*. For linear forms v, w, one has $v \wedge w = -w \wedge v$. For a set of indices $F = \{i_1, \dots, i_t\} \subseteq [n] = \{1, \dots, n\}$ with $i_1 < i_2 < \ldots < i_t$, we denote by e_F the *monomial* $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_t}$ and write also $e_F = e_{i_1} \cdots e_{i_t}$ to simplify the notation. We also omit $\wedge$ in products of elements in E from now on.

An arbitrary element f in E can be written uniquely in the form $f = \sum_F a_F e_F$ which is a K-linear combination of monomials. The set $\operatorname{supp}(f) = \{e_F : a_F \neq 0\}$ is called the *support* of f. We identify sometimes $\operatorname{supp}(f)$ with $\{F : e_F \in \operatorname{supp}(f)\}$. We also define the *support* of a set $\mathcal{G}$ of elements in E by the union of all support-sets of elements in $\mathcal{G}$ if the union is a finite set.

The category of modules $\mathcal{M}$ considered in this thesis is the category of finitely generated graded left and right E-modules M satisfying the equations

$$um = (-1)^{\deg u \deg m} mu$$

for homogeneous elements $u \in E$, $m \in M$. For a graded E-module $M \in \mathcal{M}$ and $k \in \mathbb{Z}$, the set of all homogeneous elements of degree k belonging to M is denoted by M_k. A submodule of E is called an *ideal* of E. If an ideal is generated by monomials, then it is called a *monomial ideal*. Note that we only work with graded ideals which are always two-sided ideals and the only maximal graded ideal in E is $\mathfrak{m} = (e_1, \dots, e_n)$. For a graded ideal $J \subset E$, the graded E-module E/J admits a natural structure of a graded K-algebra.

We consider the duality functor $(-)^*$ on $\mathcal{M}$ given by $M^* = \operatorname{Hom}_E(M, E)$ for $M \in \mathcal{M}$. For a graded ideal $J \subset E$, let $(0 :_E J) = \{f \in E : fJ = 0\}$. One can prove that

$$(E/J)^* = \operatorname{Hom}_E(E/J, E) \cong (0 :_E J).$$

We use also the notation $\operatorname{ann}_E(J)$ for $(0 :_E J)$.

Note that every module $M \in \mathcal{M}$ has only finitely many non-zero homogeneous components. Let $d(M)$ denote the highest degree in which a non-zero module M is not zero, i.e., $d(M) = \max\{i \in \mathbb{Z} : M_i \neq 0\}$.

Since each homogeneous component of M is a finite dimensional K-vector space, one can define the *Hilbert function* $H(-, M)$ of M by $H(i, M) = \dim_K M_i$ for $i \in \mathbb{Z}$. The *Hilbert series* of M is $H_M(t) = \sum_{i\in\mathbb{Z}} H(i, M)t^i$. Note that it has only finitely many summands.

For $a \in \mathbb{Z}$, let $M(-a)$ be the module M *shifted in degrees* by a. More precisely, one has $M(-a)_i = M_{i-a}$. Observe that

$$d(M(-a)) = d(M) + a \text{ and } H_{M(-a)}(t) = t^a H_M(t).$$

Let $M \in \mathcal{M}$. Following [**1**], a linear form $v \in E_1$ is called *regular* on M (or M*-regular*) if the annihilator of v in M is the smallest possible submodule, i.e., $(0 :_M v) = vM$. Otherwise, v is called M*-singular*. The set of all M-singular elements is denoted by $V_E(M)$ and is called the *rank variety* of M. A sequence of linear forms $v_1, \dots, v_s$ is called an M*-regular sequence* if v_i is $M/(v_1, \dots, v_{i-1})M$-regular for $i = 1, \dots, s$ and $M/(v_1, \dots, v_s)M \neq 0$. It is shown in [**1**] that all maximal M-regular sequences have the same length. This length is called the *depth* of M over E and is denoted by $\operatorname{depth}_E(M)$.

Lemma 1.1.1. *Let $J \subset E$ be a graded ideal. A linear form $v \in E_1$ is E/J-regular if and only if $J :_E v = J + (v)$.*

Proof. v is E/J-regular if and only if $\operatorname{ann}_{E/J}(v) = v(E/J)$. The statement now follows from the fact that $\operatorname{ann}_{E/J}(v) = (J :_E v)/J$ and $vE/J = (J + (v))/J$. □

Next we sketch main features and facts of Gröbner basis theory over the exterior algebra. This theory is almost analogous to Gröbner basis theory over the polynomial ring. But it needs some suitable modifications since we have more zero divisors in the exterior algebra. See, e.g., the book by Herzog and Hibi [**32**, Section 5.2] for more details.

Definition 1.1.2. A *monomial order* on E is a total order $<$ on the set $\operatorname{Mon}(E)$ of all monomials of E such that:

- (i) $1 < u$ for all $1 \neq u \in \operatorname{Mon}(E)$;
- (ii) if $u, v \in \operatorname{Mon}(E)$ and $u < v$ then $uw < vw$ for all $w \in \operatorname{Mon}(E)$ such that $uw, vw \neq 0$.

Each monomial $u = e_F = e_{i_1} e_{i_2} \cdots e_{i_t} \in \operatorname{Mon}(E)$ has a corresponding squarefree monomial $u^* = x_F = x_{i_1} x_{i_2} \cdots x_{i_t}$ in the polynomial ring $S = K[x_1, \dots, x_n]$. For $f = \sum a_u u \in E$, we also denote $f^* = \sum a_u u^* \in S$. The reverse lexicographic order on E induced by $e_1 > e_2 > \ldots > e_n$ is defined as follows:

$$e_F <_{rlex} e_G \Leftrightarrow x_F <_{rlex} x_G.$$

Let $<$ be a monomial order on E, and $0 \neq f \in E$. Recall that the *initial monomial* of f with respect to $<$, denoted by $\operatorname{in}_<(f)$, is the largest monomial among the monomials belonging to $\operatorname{supp}(f)$. Let $J \subset E$ be a graded ideal. The *initial ideal*

$\mathrm{in}_<(J)$ of J is the monomial ideal in E generated by all monomials $\mathrm{in}_<(f)$ with $0 \neq f \in J$.

Definition 1.1.3. Let $0 \neq J \subset E$ be a graded ideal. A *Gröbner basis* of J with respect to $<$ is a finite set $g_1, \dots, g_s$ of generators of J such that

$$\mathrm{in}_<(J) = (\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \dots, \mathrm{in}_<(g_s)).$$

The Gröbner basis $\mathcal{G} = \{g_1, g_2, \dots, g_s\}$ of J is called *reduced* if for all i the coefficient of $\mathrm{in}_<(g_i)$ in the element g_i is 1, and for all i and j with $i \neq j$, $\mathrm{in}_<(g_i)$ divides none of the monomials of $\mathrm{supp}(g_j)$. A reduced Gröbner basis of J exists and is uniquely determined (see [**32**, Theorem 5.2.3]).

A finite subset of J is a *universal Gröbner basis* of J if it is a Gröbner basis of J with respect to every monomial order on E.

In a similar way for the polynomial ring, there is a division algorithm for the exterior algebra. We present here this result for the convenience of the reader.

Theorem 1.1.4 (**The division algorithm**, [**32**, Theorem 5.2.4])**.** *Let $g_1, \dots, g_s$, f be homogeneous non-zero elements of E and $<$ a monomial order on E. Then there exist homogeneous elements $h_1, \dots, h_s, r \in E$ such that f has a standard expression*

$$f = \sum_{i=1}^{s} h_i g_i + r,$$

with the property that no $v \in \mathrm{supp}(r)$ belongs to $(\mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_s))$, and whenever $h_i g_i \neq 0$, then $\mathrm{in}_<(h_i)^* \mathrm{in}_<(g_i)^* \leq \mathrm{in}_<(f)^*$.

Let $f \in E$ be a homogeneous non-zero element with the standard expression $f = \sum_{i=1}^{s} h_i g_i + r$ as above. If the remainder of f with respect to $g_1, \dots, g_s$ is 0, i.e., $r = 0$, then one says that f *reduces to* 0 with respect to $g_1, \dots, g_s$.

Analogously to the polynomial ring case, there are also exterior algebra versions of the Buchberger's criterion and the Buchberger's algorithm; see, e.g., [**32**, Theorem 5.2.6] for more details. Moreover, one has:

Lemma 1.1.5. *Let $0 \neq J \subset E$ be a graded ideal. Then:*

(i) *There are only finitely many distinct initial ideals of J.*
(ii) *There always exists a universal Gröbner basis for J.*
(iii) *For every monomial order $<$ on E, the monomials of E, which do not belong to $\mathrm{in}_<(J)$, form a vector space basis for E/J. In particular, J and $\mathrm{in}_<(J)$ have the same Hilbert series.*

Proof. (i), (ii): It is obvious that there are only finitely many monomials in E. Thus there are only finitely many distinct initial ideals of J. The union of the reduced Gröbner bases corresponding to the initial ideals of J is a universal Gröbner basis of J.

(iii): Let $\mathcal{G} = \{g_1, g_2, \dots, g_s\}$ be a Gröbner basis of J with respect to $<$, let $f \in E$ and r the remainder of f with respect to $g_1, \dots, g_s$. Then $f + J = r + J$ and by the division algorithm, we also have $\mathrm{supp}(r) \cap \mathrm{in}_<(J) = \emptyset$. Thus E/J is generated by the monomials $u \notin \mathrm{in}_<(J)$. Suppose that there exist monomials $u_1, \dots, u_m$ not in

$\mathrm{in}_<(J)$ which are linearly dependent modulo J, i.e., there exists $f = \sum_{i=1}^m a_i u_i \in J$ for some $a_i \in K$. Then no term of f belongs to $\mathrm{in}_<(J)$ but $\mathrm{in}_<(f) \in \mathrm{in}_<(J)$, a contradiction. This concludes the proof. □

Corollary 1.1.6. *Let $<, <'$ be monomial orders on E and $J, J' \subset E$ graded ideals.*

(i) *If $\mathrm{in}_<(J) \subseteq \mathrm{in}_{<'}(J)$, then $\mathrm{in}_<(J) = \mathrm{in}_{<'}(J)$.*
(ii) *If $J \subseteq J'$ and $\mathrm{in}_<(J) = \mathrm{in}_<(J')$, then $J = J'$.*

Proof. (i) This follows from Lemma 1.1.5 (iii), because the monomials of E not in $\mathrm{in}_<(J)$ ($\mathrm{in}_{<'}(J)$, respectively) form a basis for the same vector space E/J.

(ii) Since $\mathrm{in}_<(J) = \mathrm{in}_<(J')$, by Lemma 1.1.5 (iii) we get that J and J' have the same Hilbert series. Moreover, $J \subseteq J'$. Therefore, $J = J'$. □

1.2. Resolutions

We present in this section some homological properties of graded modules in $\mathcal{M}$ related to resolutions. Let $M \in \mathcal{M}$. As in the case of the polynomial ring, one can construct a graded free resolution of M. It is of the form

$$F_\bullet : \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where all F_i are finitely generated graded free E-modules. The complex $F_\bullet$ is called a *graded free resolution* of M. It is minimal if the chosen systems of generators are minimal in each step of its construction. This condition is equivalent to the requirement that all entries in the matrices representing the differential maps are elements in $\mathfrak{m}$. Moreover, it is known that two minimal resolutions of M are isomorphic as complexes. Thus the minimal graded free resolution of M is uniquely determined and it is an exact sequence of the form

$$\ldots \longrightarrow \bigoplus_{j\in\mathbb{Z}} E(-j)^{\beta^E_{1,j}(M)} \longrightarrow \bigoplus_{j\in\mathbb{Z}} E(-j)^{\beta^E_{0,j}(M)} \longrightarrow M \longrightarrow 0.$$

Note that $\beta^E_{i,j}(M) = \dim_K \mathrm{Tor}^E_i(K, M)_j$ for all $i, j \in \mathbb{Z}$. We call the numbers $\beta^E_{i,j}(M)$ the *graded Betti numbers* of M. The module M is said to have a *d-linear resolution* if $\beta^E_{i,i+j}(M) = 0$ for all i and $j \neq d$. This is equivalent to the condition that M is generated in degree d and all non-zero entries in the matrices representing the differential maps are of degree one.

Let $t(M)$ be the initial degree of M, i.e., $t(M) = \min\{i \in \mathbb{Z} : M_i \neq 0\}$. One has $\beta^E_{i,i+j}(M) = 0$ for all $j < t(M)$. The numbers $\beta^E_{i,i+t(M)}(M)$ describe the *linear strand* of the minimal graded free resolution of M, i.e., they count the number of linear syzygies appearing in the resolution.

To measure the growth rate of the Betti numbers of M, one use the *complexity* of M defined as

$$\mathrm{cx}\, M = \inf\{c \in \mathbb{N} : \beta^E_i(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\},$$

where $\beta^E_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(M)$ is the i-th *total Betti number* of M. Note that $\mathrm{cx}(0) = \mathrm{cx}(E) = -\infty$. In contrast to the polynomial ring, the projective dimension is not a very useful invariant for a module over the exterior algebra since a (minimal) graded projective resolution of the module has always infinite length unless the module is free. But one still can consider the (Castelnuovo-Mumford) *regularity* for a graded module $M \in \mathcal{M}$. It is given by

$$\operatorname{reg}_E(M) = \max\{j - i : \beta^E_{i,j}(M) \neq 0\} \text{ for } M \neq 0 \text{ and } \operatorname{reg}_E(0) = -\infty.$$

For every $0 \neq M \in \mathcal{M}$, one can show that $t(M) \leq \operatorname{reg}_E(M) \leq d(M)$ (see [**37**, Section 2.1]). So $\operatorname{reg}_E(M)$ is always a finite number for every $M \neq 0$.

Note that for a graded ideal $J \neq 0$, by the above definitions one has

$$\operatorname{cx}_E(E/J) = \operatorname{cx}_E(J) \text{ and } \operatorname{reg}_E(E/J) = \operatorname{reg}_E(J) - 1.$$

This can be seen indeed by the fact that if $F_\bullet \longrightarrow J$ is the minimal graded free resolution of J, then $F_\bullet \longrightarrow E \longrightarrow E/J$ is the minimal graded free resolution of E/J.

For a short exact sequence $0 \to M \to N \to P \to 0$ of non-zero modules in $\mathcal{M}$, there are relationships among the regularities of its modules by evaluating in Tor-modules in the long exact sequence

$$\begin{aligned} \ldots \longrightarrow \operatorname{Tor}^E_{i+1}(P,K)_{i+1+j-1} \longrightarrow \quad & \operatorname{Tor}^E_i(M,K)_{i+j} \longrightarrow \operatorname{Tor}^E_i(N,K)_{i+j} \longrightarrow \\ & \operatorname{Tor}^E_i(P,K)_{i+j} \longrightarrow \operatorname{Tor}^E_{i-1}(M,K)_{i-1+j+1} \longrightarrow \ldots \end{aligned}$$

More precisely, one has:

Lemma 1.2.1. *Let $0 \to M \to N \to P \to 0$ be a short exact sequence of non-zero modules in $\mathcal{M}$. Then:*

(i) $\operatorname{reg}_E(N) \leq \max\{\operatorname{reg}_E(M), \operatorname{reg}_E(P)\}$.
(ii) $\operatorname{reg}_E(M) \leq \max\{\operatorname{reg}_E(N), \operatorname{reg}_E(P) + 1\}$.
(iii) $\operatorname{reg}_E(P) \leq \max\{\operatorname{reg}_E(N), \operatorname{reg}_E(M) - 1\}$.

1.3. Generic initial ideals

In this section we recall some facts about generic initial ideals in an exterior algebra. The existence of generic initial ideals is proved by Aramova, Herzog and Hibi in [**3**, Theorem 1.6] and independently by Green in Chapter 5 of [**29**]. Most properties of generic initial ideals over a polynomial ring can be translated to generic initial ideals over an exterior algebra. Some of them are even better in the sense that they do not depend on the characteristic of the base field, e.g., the strongly stable property of generic initial ideals.

At first, we present briefly basis notions and properties of (strongly) stable ideals. Let $u = e_F \in E$ be a monomial where $F \subseteq [n]$. Denote by

$$\max(u) = \max\{i : i \in F\} \text{ and } \min(u) = \min\{i : i \in F\}.$$

Definition 1.3.1. A monomial ideal $J \subset E$ is called *stable* if $e_j \frac{u}{e_{\max(u)}} \in J$ for every monomial $u \in J$ and $j < \max(u)$. The ideal J is called *strongly stable* if $e_j \frac{u}{e_i} \in J$ for every monomial $u = e_F \in J$, $i \in F$ and $j < i$.

For a monomial ideal $J \subset E$, we denote by $G(J)$ the minimal set of monomial generators of J, and by $G(J)_j \subseteq G(J)$ the subset of generators of degree j in $G(J)$. Let $0 \neq J \subset E$ be a stable monomial ideal. One can compute several important invariants of J as follows (see, e.g., [**1**, Corollary 3.2], [**3**, Corollary 3.3] and [**37**, Lemma 3.1.4, 3.1.5] for more details):

Lemma 1.3.2. *Let $0 \neq J \subset E$ be a stable monomial ideal. Then:*
(i) $\beta^E_{i,i+j}(J) = \sum_{u \in G(J)_j} \binom{\max(u)+i-1}{\max(u)-1}$ *for all* $i \geq 0, j \in \mathbb{Z}$.
(ii) $\operatorname{cx} E/J = \max\{\max(u) : u \in G(J)\}$.
(iii) $\operatorname{reg}_E(J) = \max\{\deg u : u \in G(J)\}$.
(iv) $d(E/J) = n - \max\{\min(u) : u \in G(J)\}$ *if additionally J is strongly stable.*

Let $<$ be the reverse lexicographic order on E with $e_1 > e_2 > \ldots > e_n$. The initial ideal of a graded ideal $J \subset E$ with respect to this order is denoted by $\operatorname{in}(J)$. When the base field K is infinite, Aramova, Herzog and Hibi in [**3**, Theorem 1.6] proved the existence of a non-empty Zariski-open subset $U \subseteq \operatorname{GL}_n(K)$ such that all $\operatorname{in}(g(J))$ are the same monomial ideal for $g \in U$. This monomial ideal is called the *generic initial ideal* of J, denoted by $\operatorname{gin}(J)$. The generic initial ideal of a graded ideal is strongly stable by [**3**, Proposition 1.7]. This is independent of the characteristic of the base field K in contrast to the situation in a polynomial ring.

There are some expected relationships between a graded ideal in the exterior algebra and its (generic) initial ideal. For instance, the Hilbert functions of E/J and $E/\operatorname{in}(J)$ coincide for any graded ideal $J \subset E$. The Betti numbers of $E/\operatorname{in}(J)$ are not smaller than those of E/J, i.e.,

$$\beta^E_{i,j}(E/\operatorname{in}(J)) \geq \beta^E_{i,j}(E/J) \text{ for all } i, j;$$

see [**3**, Proposition 1.8]. The (Castelnuovo-Mumford) regularity of J and $\operatorname{gin}(J)$ coincide as follows:

Lemma 1.3.3 ([**2**, Theorem 5.3])**.** *Let $|K| = \infty$ and $0 \neq J \subset E$ be a graded ideal. One has*

$$\operatorname{reg}_E(J) = \operatorname{reg}_E(\operatorname{gin}(J)).$$

In particular, J has a d-linear resolution if and only if $\operatorname{gin}(J)$ *has a d-linear resolution.*

1.4. Simplicial complexes

Since the early work of Stanley and Hochster, it turned out to be very useful to use methods from commutative algebra to solve purely combinatorial problems related to simplicial complexes. Recall that a *simplicial complex* Δ on the ground set $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ such that if $G \in \Delta$ and $F \subset G$ then $F \in \Delta$. Elements of Δ are called *faces*. For a face $F \in \Delta$, the *dimension* of F is $\dim F = |F| - 1$. The *dimension* of Δ, write $\dim \Delta$, is the maximum of the dimensions of faces. The maximal faces under inclusion are called the *facets* of Δ. A simplicial complex Δ is called *pure* if all its facets have the same dimension. If Δ has only one facet then Δ is called a *simplex*.

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring. A famous algebra associated to each simplicial complex Δ was introduced by Stanley, namely its Stanley-Reisner ring. It is defined as follows: let I_Δ be the monomial ideal in S generated by all squarefree monomials x_F corresponding to non-faces of Δ, i.e., $I_\Delta = (x_F : F \notin \Delta)$.

The ideal I_Δ is called the *Stanley-Reisner ideal* and the quotient ring $K[\Delta] = S/I_\Delta$ is called the *Stanley-Reisner ring* of Δ.

Let $E = K\langle e_1, \dots, e_n\rangle$ be an exterior algebra. By the same way as in the case of polynomial rings, one can associate to a simplicial complex Δ the *exterior face ring* $K\{\Delta\} = E/J_\Delta$, where $J_\Delta = (e_F : F \notin \Delta)$ is the *exterior face ideal* of Δ; see, e.g., [**30**] for an overview.

To each simplicial complex Δ, one also can associate a dual simplicial complex called the *Alexander dual* of Δ. It is denoted by Δ^* and is defined by

$$\Delta^* = \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

Note that the facets of Δ^* are the complements of the minimal non-faces of Δ.

Example 1.4.1. Let $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$. Then Δ is a simplicial complex on [3]. It is a pure simplicial complex with $\dim \Delta = 1$. The Stanley-Reisner ring and the exterior face ring of Δ are:

$$K[\Delta] = K[x_1, x_2, x_3]/(x_2x_3) \text{ and } K\{\Delta\} = K\langle e_1, e_2, e_3\rangle/(e_2e_3), \text{ respectively.}$$

The Alexander dual of Δ is $\Delta^* = \{\emptyset, \{1\}\}$.

1.5. Resonance varieties

We present in this section basic notions and properties of resonance varieties which were defined firstly by Falk in [**24**] for Orlik-Solomon algebras of hyperplane arrangements. We will see in Chapter 6 that resonance varieties are very useful to study the ring structure of Orlik-Solomon algebras. For more details of resonance varieties, we refer to [**55**, Chapter 4].

Let $A = E/J$ be a graded algebra and $u \in A_1$, where J is a graded ideal of E. Since $u^2 = 0$, we have a cochain complex

$$(A, u) : \qquad 0 \longrightarrow A_0 \xrightarrow{\cdot u} A_1 \xrightarrow{\cdot u} \cdots \xrightarrow{\cdot u} A_r \xrightarrow{\cdot u} \dots$$

Its cohomology is denoted by $H^\bullet(A, u)$. For $p \geq 0$, the *degree-p resonance variety* of A is the set

$$R^p(A) = \{u \in A_1 : H^p(A, u) \neq 0\}.$$

It is known that $R^p(A)$ is an affine variety in the affine space $A_1 \cong K^m$ where $m = \dim_K A_1$.

Note that if $u \in R^p(A)$ then $\alpha u \in R^p(A)$ for all $\alpha \in K \setminus \{0\}$. Thus each resonance variety $R^p(A)$ is homogeneous and one can consider resonance varieties as projective varieties in the projective space $\mathbb{P}^{m-1}$. In other words, there exists a corresponding theory of projective resonance varieties in the projective setting. See [**24**], [**40**], [**55**, Chapter 4] for more details.

In the following we always assume that a graded ideal J is non trivial and contains no variable. The resonance varieties of $A = E/J$ can be computed by the following formulas:

$$(1) \quad R^1(A) = \{u \in E_1 : u = 0 \text{ or } \exists v \in E_1, 0 \neq uv \in J_2\},$$

$$(2) \quad R^p(A) = \{u \in E_1 : u = 0 \text{ or } \exists v \in E_p, v \notin J_p + uE_{p-1}, 0 \neq uv \in J_{p+1}\}.$$

In the case that J is a monomial ideal, the resonance varieties of E/J are completely determined by Papadima and Suciu (see [**48**, Theorem 5.5], [**49**, Theorem 3.8]). For the convenience of the reader we present a special case where the first resonance variety is irreducible. Concretely, by considering the resonance varieties of graded algebras defined by stable monomial ideals, we have:

Lemma 1.5.1. *Let $J \subset E$ be a stable monomial ideal with $J_2 \neq 0$. Then the first resonance variety $R^1(E/J)$ of E/J is irreducible.*

Proof. Let $t = \max\{\max(u) : u \in G(J)_2\}$. There exists an integer r with $1 \leq r < t$ such that $u = e_r e_t \in J_2$. We claim that $R^1(E/J) = \operatorname{span}_K\{e_1, \ldots, e_t\}$ which is irreducible.

Observe that $e_i u/e_t \in J$ for all $i < t$ since J is stable. Hence

$$e_r e_i \in J_2 \text{ for } 1 \leq i \leq t \text{ and thus } e_r \sum_{i=1}^{t} \alpha_i e_i \in J_2 \text{ for every } \alpha_i \in K.$$

Using formula (1) we see that $\operatorname{span}_K\{e_1, \ldots, e_t\} \subseteq R^1(E/J)$. In particular, for $t = n$ we get that the claim holds, so assume $t < n$ in the following.

We consider an arbitrary element $0 \neq u = \sum_{i=1}^{n} \alpha_i e_i \in R^1(E/J)$. Suppose that there exists an integer s and $0 \neq v = \sum_{j=1}^{n} \beta_j e_j \in E_1$ with

$$t < s \leq n \text{ such that } \alpha_s \neq 0 \text{ and } 0 \neq uv \in J_2.$$

By the choice of t we see that

$$e_s e_j \notin J_2 \text{ for } j \in \{1, \ldots, n\} \setminus \{s\}.$$

Thus the monomial $e_s e_j$ does not appear in uv. So we get

$$\alpha_s \beta_j - \alpha_j \beta_s = 0 \text{ for all } j = 1, \ldots, n.$$

If $\beta_s = 0$ then $\beta_j = 0$ for $j = 1, \ldots, n$. This contradicts the fact that $v \neq 0$. So $\beta_s \neq 0$. This implies that $\alpha_j \neq 0$ if and only if $\beta_j \neq 0$ and in this case

$$\beta_j = \frac{\beta_s}{\alpha_s} \alpha_j \text{ for all } j = 1, \ldots, n.$$

Thus $v = ku$ where $k = \beta_s/\alpha_s$ and we see that $uv = 0$. This is also a contradiction to the choice of u and v. Hence $\alpha_s = 0$ for every integer s with $s > t$. Altogether we see that

$$R^1(E/J) \subseteq \operatorname{span}_K\{e_1, \ldots, e_t\} \text{ and then } R^1(E/J) = \operatorname{span}_K\{e_1, \ldots, e_t\}.$$

This concludes the proof. □

Observe that Lemma 1.5.1 motivates the problem whether the higher resonance varieties of algebras generated by stable monomial ideals are irreducible. For this problem, we have only little knowledge which is presented in Lemma 1.5.3 below. We need at first:

Lemma 1.5.2. *Let $J \subset E$ be a stable monomial ideal and let*

$$t_p = \max\{\max(u) : u \in G(J)_p\} \textit{ for } 1 \leq p \leq n.$$

Then $\operatorname{span}_K\{e_1, \ldots, e_{t_p}\} \subseteq R^{p-1}(E/J)$ *for* $1 \leq p \leq \max\{\deg u : u \in G(J)\}$.

Proof. Let $u = e_{i_1} \cdots e_{i_p} \in G(J)_p$ where $1 \le i_1 < \cdots < i_p = t_p$. Since $G(J)$ is the minimal set of generators of J and $u \in G(J)_p$, we have $u \notin \mathfrak{m}J_{p-1}$. Therefore,

$$e_{i_1} \cdots e_{i_{q-1}} \widehat{e_{i_q}} e_{i_{q+1}} \cdots e_{i_{p-1}} e_{i_p} \notin J_{p-1} \text{ for } 1 \le q \le p.$$

It follows from formula (2) that $e_{i_q} \in R^{p-1}(E/J)$ for $q = 1, \dots, p$.

Next we consider $i \in [t_p] \setminus \{i_1, \dots, i_p\}$. Since J is stable and $t_p = \max(u)$ we have

$$0 \ne e_i(u/e_{t_p}) \in J_p \text{ and } u/e_{t_p} \notin J_{p-1}.$$

Hence $e_i \in R^{p-1}(E/J)$. So $\{e_1, \dots, e_{t_p}\} \subseteq R^{p-1}(E/J)$. Let

$$0 \ne v = \sum_{j=1}^{t_p} \alpha_j e_j \in \operatorname{span}_K\{e_1, \dots, e_{t_p}\}$$

be an arbitrary element. Assume at first that $v \notin \operatorname{span}_K\{e_{i_1}, \dots, e_{i_p}\}$. This implies

$$0 \ne v(u/e_{t_p}) \in J_p.$$

So $v \in R^{p-1}(E/J)$. Next we assume that $v \in \operatorname{span}_K\{e_{i_1}, \dots, e_{i_p}\}$ and $\alpha_{i_q} \ne 0$ for some $1 \le q \le p$. Then $0 \ne v(u/e_{i_q}) = \alpha_{i_q} u \in J_p$. Again we see that $v \in R^{p-1}(E/J)$. Hence $\operatorname{span}_K\{e_1, \dots, e_{t_p}\} \subseteq R^{p-1}(E/J)$, as desired. □

Proposition 1.5.3. *Let $J \subset E$ be a stable monomial ideal generated in one degree $p \ge 2$. If the field K is algebraically closed, then the $(p-1)$-th resonance variety of E/J is maximal, i.e., $R^{p-1}(E/J) = V_E(E/J)$. In particular, $R^{p-1}(E/J)$ is irreducible.*

Proof. Let $t = \max\{\max(u) : u \in G(J)\}$. With Lemma 1.5.2 we see that

$$\operatorname{span}_K\{e_1, \dots, e_t\} \subseteq R^{p-1}(E/J).$$

In addition, by [**1**, Theorem 3.1 (2)] and Lemma 1.3.2 we know

$$\dim_K V_E(E/J) = \operatorname{cx}_E(E/J) = t.$$

Since $\operatorname{span}_K\{e_1, \dots, e_t\} \subseteq R^{p-1}(E/J) \subseteq V_E(E/J)$ and $\dim_K V_E(E/J) = t$, we get that $R^{p-1}(E/J) = V_E(E/J)$. □

Remark 1.5.4. The results of this section have an interpretation in algebraic combinatorics. Recall that a simplicial complex Δ on the vertex set $\{1, \dots, n\}$ is called a *shifted complex* if for every face $F \in \Delta$, $i \in F$ and $j > i$ it holds that $(F \setminus \{i\}) \cup \{j\} \in \Delta$. This combinatorial property corresponds to the strongly stable property of monomial ideals, i.e., the face ideal $J_\Delta = (e_F : F \notin \Delta)$ of Δ is strongly stable if and only if Δ is a shifted complex.

As shown above, we get from Lemma 1.5.1 that the first resonance variety of the exterior face ring E/J_Δ of a shifted complex Δ is irreducible.

1.6. Fans in $\mathbb{R}^n$

In this section we collect some facts about fans in $\mathbb{R}^n$ which are used in Chapter 3. One can find more details, e.g., in the books of Bruns and Gubeladze [**11**], Schrijver [**58**, Section 5.3] and Sturmfels [**61**, Chapter 2].

We consider $\mathbb{R}^n$ with fixed coordinates (where $n > 0$). For two vectors $u, w \in \mathbb{R}^n$ we write $u \cdot w = \langle u, w \rangle = \sum_{i=1}^n u_i w_i$ for the canonical scalar product. A *closed half space* of $\mathbb{R}^n$ is a set of the form $\{w \in \mathbb{R}^n : a \cdot w \leq \lambda\}$ where $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. If $\lambda = 0$ then the half space is said to be *linear*.

A subset $P \subset \mathbb{R}^n$ is called a *polyhedron* if it is the intersection of finitely many closed half spaces, i.e., there exist $a_1, \ldots a_m \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

$$P = \{w \in \mathbb{R}^n : a_i \cdot w \leq \lambda_i \text{ for } i = 1, \ldots, m\}.$$

In the case that all $\lambda_i = 0$, we call P a *polyhedral cone*. Thus P is a polyhedral cone if it is a finite intersection of linear closed half spaces. Note that every polyhedral cone contains the origin 0.

A bounded polyhedron is called a *polytope*. The *dimension* of a polyhedron P is the dimension of the smallest affine subspace containing P. A *face* of P is either the empty set or a non-empty subset of P which is the following set of maximizers of a linear form over P:

$$\text{face}_u(P) = \{v \in P : u \cdot v = \max\{u \cdot w : w \in P\}\}$$

where $u \in \mathbb{R}^n$. A *facet* of P is a face whose dimension is one smaller than the dimension of P.

Definition 1.6.1. A collection $\mathcal{C}$ of polyhedra in $\mathbb{R}^n$ is said to be a *polyhedral complex* if:

(i) all non-empty faces of a polyhedron $P \in \mathcal{C}$ are in $\mathcal{C}$;
(ii) the intersection of any $A, B \in \mathcal{C}$ is a common face of A and B.

The *support* of $\mathcal{C}$ is the union of its elements. A polyhedral complex is a *fan* if it consists only of polyhedral cones. A fan is *pure* if all its maximal cones have the same dimension.

Example 1.6.2. The set

$$P = \{w \in \mathbb{R}^3 : w_1, w_2, w_3 \leq 0\}$$

is a polyhedral cone in $\mathbb{R}^3$. It is the intersection of three closed half spaces. More precisely, $P = P_1 \cap P_2 \cap P_3$ where $P_i = \{w \in \mathbb{R}^3 : w_i \leq 0\}$ for $i = 1, 2, 3$. Let $u = (1, 2, 0) \in \mathbb{R}^3$. Then

$$\begin{aligned} \text{face}_u(P) &= \{w \in \mathbb{R}^3 : w_1, w_2, w_3 \leq 0 \text{ and } u \cdot w = w_1 + 2w_2 \text{ is maximal}\} \\ &= \{w \in \mathbb{R}^3 : w_1 = w_2 = 0, w_3 \leq 0\} \end{aligned}$$

is a face of P.

CHAPTER 2

Cartan homology and applications

The goal of this chapter is to study generic bases and Cartan homology. At first, we prove some properties for special generic bases, namely strongly generic bases. After that, as applications we provide formulas for computing the regularity of modules over the exterior algebra. We also consider some special cases to get similar results to known ones for modules over a polynomial ring.

2.1. Cartan homology

We recall in this section notions and properties of generic bases and the Cartan complex following the exposition of Aramova and Herzog in [**2**, Section 4] where they use the Cartan complex to investigate graded Betti numbers and Castelnuovo-Mumford regularity of modules over an exterior algebra.

We always assume that $E = K\langle e_1, \dots, e_n\rangle$ is the exterior algebra over an infinite field K with $\deg e_i = 1$ for $i = 1, \dots, n$. Recall that $\mathcal{M}$ is the category of finitely generated graded left and right E-modules M satisfying the equations $um = (-1)^{\deg u \deg m} mu$ for homogeneous elements $u \in E$, $m \in M$.

The Cartan complex over E has similar properties to known properties of the Koszul complex over the polynomial ring. It is defined as follows:

Let $\mathbf{v} = v_1, \dots, v_m$ be a sequence of linear forms in E_1. We denote by $C_\bullet(\mathbf{v}; E)$ the free divided power algebra $E\langle x_1, \dots, x_m\rangle$. So $C_\bullet(\mathbf{v}; E)$ is generated by the divided powers $x_i^{(j)}$ for $i = 1, \dots, m$ and $j \geq 0$ which satisfy the relations

$$x_i^{(j)} x_i^{(k)} = \binom{j+k}{j} x_i^{(j+k)}.$$

We see that $C_i(\mathbf{v}; E)$ is a free E-module with basis $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$, $a \in \mathbb{N}^m$, $|a| = i$. Moreover, there exists a complex structure on $C_\bullet(\mathbf{v}; E) = C_\bullet(v_1, \dots, v_m; E)$ given by the E-linear differential:

$$\partial_i : C_i(\mathbf{v}; E) \longrightarrow C_{i-1}(\mathbf{v}; E), \qquad x^{(a)} \mapsto \sum_{j \text{ with } a_j > 0} v_j x_1^{(a_1)} \cdots x_j^{(a_j - 1)} \cdots x_m^{(a_m)}.$$

Here one easily checks that $\partial \circ \partial = 0$.

Let $M \in \mathcal{M}$. Following [**2**], the *Cartan complex* of $\mathbf{v}$ with values in M is the complexes $C_\bullet(\mathbf{v}; M) = C_\bullet(\mathbf{v}; E) \otimes_E M$. Its homology $H_i(\mathbf{v}; M) = H_i(C_\bullet(\mathbf{v}; M))$ is called the *Cartan homology* of $\mathbf{v}$ with values in M.

For each $j = 1, \dots, m-1$, there exists an exact sequence of complexes

$$0 \longrightarrow C_\bullet(v_1, \dots, v_j; M) \xrightarrow{\iota} C_\bullet(v_1, \dots, v_{j+1}; M) \xrightarrow{\tau} C_{\bullet-1}(v_1, \dots, v_{j+1}; M)(-1) \longrightarrow 0,$$

where ι is the natural inclusion map and τ is given by

$$\tau(g_0 + g_1 x_{j+1} + \ldots + g_k x_{j+1}^{(k)}) = g_1 + g_2 x_{j+1} + \ldots + g_k x_{j+1}^{(k-1)},$$

where $g_i \in C_{k-i}(v_1, \ldots, v_j; M)$.

Then following [**2**, Propositions 4.1], there exists a long exact sequence of homology modules:

$$\ldots \longrightarrow H_i(v_1, \ldots, v_j; M) \xrightarrow{\alpha_i} H_i(v_1, \ldots, v_{j+1}; M) \xrightarrow{\beta_i} H_{i-1}(v_1, \ldots, v_{j+1}; M)(-1)$$

$$\xrightarrow{\delta_{i-1}} H_{i-1}(v_1, \ldots, v_j; M) \longrightarrow H_{i-1}(v_1, \ldots, v_{j+1}; M) \longrightarrow \ldots$$

for all $j = 1, \ldots, m-1$. Here α_i is induced by ι, β_i by τ and δ_{i-1} is the connecting homomorphism, which is defined as follows: if $z = g_0 + g_1 x_{j+1} + \ldots + g_{i-1} x_{j+1}^{(i-1)}$ is a cycle in $C_{i-1}(v_1, \ldots, v_{j+1}; M)$, then $\delta_{i-1}([z]) = [g_0 v_{j+1}]$.

Note that there exists a grading on the complex and its homologies which is induced by setting $\deg x_i = 1$ for $i = 1, \ldots, m$. Following [**1**, Remark 3.4(3)] or the proof of [**3**, Theorem 2.2], the Cartan complex $C_{\bullet}(v_1, \ldots, v_m; E)$ with values in E is exact if $v_1, \ldots, v_m$ are K-linearly independent. Hence it is a minimal graded free resolution of $H_0(v_1, \ldots, v_m; E) = E/(v_1, \ldots, v_m)$ over E. Using this fact, one can compute $\mathrm{Tor}_i^E(E/(v_1, \ldots, v_m), -)$. More precisely, there exist isomorphisms of graded E-modules: $\mathrm{Tor}_i^E(E/(v_1, \ldots, v_m), M) \cong H_i(\mathbf{v}; M)$ for all $i \geq 0$ and $M \in \mathcal{M}$ (see [**3**, Theorem 2.2]).

Given a linear form $v \in E_1$, we have a co-chain complex

$$(3) \qquad (M, v): \qquad \ldots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \ldots.$$

For $j \in \mathbb{Z}$ let $H^j(M, v)$ be the j-th cohomology module of (M, v). Then v is M-regular if and only if $H^j(M, v) = 0$ for all $j \in \mathbb{Z}$. Let $H(M, v) = \oplus_i H^i(M, v)$. We see that

$$H(M, v) = \frac{0 :_M v}{vM} \in \mathcal{M}.$$

2.2. Generic bases and strongly generic elements

We study in this section (strongly) generic bases and strongly generic elements for a module $M \in \mathcal{M}$ as considered firstly in [**2**].

Definition 2.2.1 ([**2**, Definition 4.7])**.** Let $1 \leq t \leq n$. A sequence $\mathbf{v} = v_1, \ldots, v_t$ in E_1 is called a *generic sequence* for M if the natural maps

$$\beta_i : H_i(v_1, \ldots, v_{j+1}; M) \longrightarrow H_{i-1}(v_1, \ldots, v_{j+1}; M)(-1)$$

are surjective for $j = 0, \ldots, t-1$ and all $i \gg 0$.

An element $v \in E_1$ is called a *generic element* if the sequence of one element v is generic. If a generic sequence $\mathbf{v}$ for M is a basis of E_1 then $\mathbf{v}$ is called a *generic basis* of E_1 for the E-module M.

In [**2**], Aramova and Herzog proved the existence of a generic basis by using a suitable extension of the base field. More precisely, one has:

Proposition 2.2.2 ([**2**, Proposition 4.5, Corollary 4.6])**.** *Let $v_1, \dots, v_n$ be a basis of E_1 and L/K a field extension containing algebraically independent elements a_{ij} over K, $i, j = 1, \dots, n$. Then $w_1, \dots, w_n$ with $w_j = \sum_{i=1}^{n} a_{ij} v_i$ for $j = 1, \dots, n$ is a generic basis of E'_1 for any E'-module $M' = L \otimes_K M$ where $E' = L \otimes_K E$ and $M \in \mathcal{M}$.*

Using this result, we have:

Lemma 2.2.3. *Let $M \in \mathcal{M}$. After a suitable field extension, there exists a generic basis $w_1, \dots, w_n$ for M as an E-module such that $[w_2], \dots, [w_n]$ is a generic basis for M/w_1M as an $E/(w_1)$-module where $[w_i] = w_i + (w_1) \in E/(w_1)$.*

Proof. Let $v_1, \dots, v_n$ be a basis of E_1 and L_1/K a field extension containing algebraically independent elements b_{ij} over K, $i, j = 1, \dots, n$. Let $E' = L_1 \otimes_K E$ and $u_j = \sum_{i=1}^{n} b_{ij} v_i \in E'_1$. Proposition 2.2.2 implies that $u_1, \dots, u_n$ is a generic basis for the E'-module $M' = L_1 \otimes_K M$.

Next we note that the elements $[u_2], \dots, [u_n]$ are a K-basis of $(E'/(u_1))_1$. Consider a field extension L/L_1 containing the algebraically independent elements c_{ij} over L_1 where $i, j = 2, \dots, n$. Let $w_j = \sum_{i=2}^{n} c_{ij} u_i$ for $j = 2, \dots, n$ and $w_1 = u_1$. By using Proposition 2.2.2 again we get that $[w_2], \dots, [w_n]$ is a generic basis for $(M'/w_1M') \otimes_{L_1} L$ as an $((E'/w_1E') \otimes_{L_1} L)$-module.

Let $M'' = M' \otimes_{L_1} L$ and $E'' = E' \otimes_{L_1} L$. Since $(M'/u_1M') \otimes_{L_1} L \cong M''/w_1M''$ and $(E'/u_1E') \otimes_{L_1} L \cong E''/(w_1)$, we get that $w_1, \dots, w_n$ is a desired basis for the E''-module M''. This concludes the proof. $\square$

Proposition 2.2.4. *Let $M \in \mathcal{M}$ and let $v_1, \dots, v_n$ be a generic basis of E_1 for the E-module M such that $[v_2], \dots, [v_n] \in E/(v_1)$ is a generic basis of $(E/(v_1))_1$ for the $E/(v_1)$-module M/v_1M. Then the permutation $v_2, \dots, v_n, v_1$ is also a generic basis of E_1 for the E-module M/v_1M.*

Proof. Let $E' = E/(v_1)$ and $M' = M/v_1M$. Since $[v_2], \dots, [v_n] \in E'$ is a generic basis of E'_1 for the E'-module M', the natural maps

$$H_i^{E'}([v_2], \dots, [v_{j+1}]; M') \xrightarrow{\beta'_i} H_{i-1}^{E'}([v_2], \dots, [v_{j+1}]; M')(-1)$$

are surjective for $j = 1, \dots, n-1$ and all $i \gg 0$. Note that we have an isomorphism of Cartan complexes

$$C_\bullet^E(v_2, \dots, v_j; M') \cong C_\bullet^{E'}([v_2], \dots, [v_j]; M') \text{ for } 2 \le j \le n.$$

Therefore, the homology modules are isomorphic and the maps

$$H_i^E(v_2, \dots, v_{j+1}; M') \xrightarrow{\beta_i} H_{i-1}^E(v_2, \dots, v_{j+1}; M')(-1)$$

are also surjective for $j = 1, \dots, n-1$ and all $i \gg 0$. So we only need to prove that the maps

$$H_i^E(v_2, \dots, v_n, v_1; M') \xrightarrow{\beta_i} H_{i-1}^E(v_2, \dots, v_n, v_1; M')(-1)$$

are surjective for all $i \gg 0$. By [**2**, Propositions 4.1], we have a long exact sequence of graded E-modules:

$$\dots \longrightarrow H_i^E(v_2, \dots, v_n, v_1; M') \xrightarrow{\beta_i} H_{i-1}^E(v_2, \dots, v_n, v_1; M')(-1)$$

$$\xrightarrow{\delta_{i-1}} H^E_{i-1}(v_2,\dots,v_n;M') \longrightarrow \dots,$$

where δ_{i-1} is the connecting homomorphism given by: if $z = g_0+g_1x_1+\dots+g_{i-1}x_1^{(i-1)}$ is a cycle in $C^E_{i-1}(v_2,\dots,v_n,v_1;M')$, then $\delta_{i-1}([z]) = [g_0v_1]$. Since $g_0v_1 = 0$ in M', we have that δ_{i-1} is the zero morphism. Hence the map β_i is surjective. This concludes the proof. □

Definition 2.2.5. A generic basis $\mathbf{v} = v_1,\dots,v_n$ of E_1 for the E-module M is called *strongly generic* if $[v_2],\dots,[v_n]$ is a generic basis of $(E/(v_1))_1$ for the $E/(v_1)$-module M/v_1M and $v_2,\dots,v_n,v_1$ is a generic basis of E_1 for the E-module M/v_1M. The first element v_1 of a strongly generic basis $\mathbf{v}$ is called a *strongly generic element*.

Remark 2.2.6. There always exists a strongly generic basis after a suitable field extension by Lemma 2.2.3 and Proposition 2.2.4. More precisely, let L_1/K be a field extension containing algebraically independent elements b_{ij} over K, $i,j = 1,\dots,n$ and L/L_1 a field extension containing the algebraically independent elements c_{ij} over L_1, $i = 1,\dots,n$ and $j = 2,\dots,n$ then $w_1 = \sum_{i=1}^n b_{i1}v_i$, $w_j = \sum_{i=1}^n c_{ij}\sum_{k=1}^n b_{ki}v_k$ for $j = 2,\dots,n$ is a strongly generic basis of E'_1 for any E'-module $M' = L\otimes_K M$ where $E' = L\otimes_K E$ and $M \in \mathcal{M}$.

2.3. Applications

Let M be an E-module in $\mathcal{M}$ and let $\mathbf{v} = v_1,\dots,v_n$ be a sequence in E_1. For $j = 1,\dots,n$ we denote by $M\langle j-1\rangle = M/(v_1,\dots,v_{j-1})M$, $H_i(j) = H_i(v_1,\dots,v_j;M)$ for $i > 0$ and $H_0(j) = H(M\langle j-1\rangle, v_j)$. For an E-module N, put $s(N) = \max\{i : N_i \neq 0\}$ if $N \neq 0$ and $s(0) = -\infty$. Set

$$r^E_j(M) = \max\{s(H_i(j)) - i : i \geq 1\} \text{ and } s^E_j(M) = s(H_0(j)) \text{ for } j = 1,\dots,n.$$

Given an E-module $M \in \mathcal{M}$, we sometimes write r_j, s_j instead of $r^E_j(M)$ and $s^E_j(M)$ (respectively) for short. Note that by [**2**, Page 705], we have: If $\mathbf{v}$ is a basis of E_1 and $M \in \mathcal{M}$, then $\operatorname{reg}_E(M) = \max\{r^E_n, s(M/\mathfrak{m}M)\}$.

Proposition 2.3.1. *Let $v \in E_1$ be a strongly generic element for M. Then*

$$\operatorname{reg}_E(M/vM) = \operatorname{reg}_{E/(v)}(M/vM).$$

Proof. Set $E' = E/(v)$ and $M' = M/vM$. By Definition 2.2.5 there exists a strongly generic basis $\mathbf{v} = (v, v_2\dots,v_n)$ for M such that $[v_2],\dots,[v_n] \in E'$ is a generic basis for the E'-module M' and $v_2,\dots,v_n,v$ is a generic basis for the E-module M'. By [**2**, Theorem 4.8] we have

$$\operatorname{reg}_E(M') = \max\{s^E_1(M'),\dots,s^E_{n-1}(M'), s^E_n(M'), s(M'/\mathfrak{m}M')\}$$

and

$$\operatorname{reg}_{E'}(M') = \max\{s^{E'}_1(M'),\dots,s^{E'}_{n-1}(M'), s(M'/\mathfrak{m}'M')\},$$

where $\mathfrak{m}' = \mathfrak{m}/(v) \subset E'$ is the maximal graded ideal of E'.

For $j = 1,\dots,n-1$ we have $s^E_j(M') = s^{E'}_j(M')$ because of the fact that

$$\frac{0 :_{M'\langle j-1\rangle} v_{j+1}}{v_{j+1}M'\langle j-1\rangle} \cong \frac{0 :_{M'\langle j-1\rangle} [v_{j+1}]}{[v_{j+1}]M'\langle j-1\rangle}$$

where $M'\langle j-1\rangle = M'/(v_2,\dots,v_j)M'$. Moreover, $M'/\mathfrak{m}M' \cong M'/\mathfrak{m}'M'$. So we only need to consider $s_n^E(M')$. We have

$$H(M'/(v_2,\dots,v_n)M',v) = H(M/\mathfrak{m}M,v) = M/\mathfrak{m}M.$$

Hence $s_n^E(M') = s(M/\mathfrak{m}M) = s(M'/\mathfrak{m}M') = s(M'/\mathfrak{m}'M')$. This concludes the proof. □

Next we prove the main result of this section which is similar to [**15**, Proposition 1.2] for modules and almost regular elements over the polynomial ring.

Theorem 2.3.2. *Let $v \in E_1$ be a strongly generic element for M. Then*

$$\operatorname{reg}_E(M) = \max\{\operatorname{reg}_E(M/vM), s(\frac{0:_M v}{vM})\}.$$

Proof. Let $E' = E/(v)$ and $M' = M/vM$. By Definition 2.2.5 there exists a strongly generic basis $\mathbf{v} = (v, v_2\dots,v_n)$ for M such that $u_1,\dots,u_{n-1} \in E'$ is a generic basis for the E'-module M' with $u_i = [v_{i+1}]$ for $i = 1,\dots,n-1$ and $v_2,\dots,v_n,v$ is a generic basis for the E-module M'.

By [**2**, Theorem 4.8] we have

$$\operatorname{reg}_E(M) = \max\{r_n, s(M/\mathfrak{m}M)\} = \max\{s_1, s_2,\dots,s_n, s(M/\mathfrak{m}M)\}.$$

Since $u_1,\dots,u_{n-1}$ is a generic basis for the E'-module M', we also have

$$\operatorname{reg}_{E'}(M') = \max\{s'_1,\dots,s'_{n-1}, s(M'/\mathfrak{m}'M')\} = \max\{s'_1,\dots,s'_{n-1}, s(M/\mathfrak{m}M)\},$$

where $s'_j = s(H(M'/(u_1,\dots,u_{j-1})M',u_j))$ for $j = 1,\dots,n-1$ and $M'/\mathfrak{m}'M' \cong M/\mathfrak{m}M$.

Note that $s'_j = s_{j+1}$ for $j = 2,\dots,n$ since

$$H(M'/(u_1,\dots,u_{j-1})M',u_j) \cong H(M/(v_1,v_2,\dots,v_j)M,v_{j+1}).$$

Combining the two equalities from above and the fact that $H_0(1) = H(M,v) = \frac{0:_M v}{vM}$, we get

$$\operatorname{reg}_E(M) = \max\{\operatorname{reg}_{E'}(M'), s(\frac{0:_M v}{vM})\}.$$

Proposition 2.3.1 implies that $\operatorname{reg}_E(M) = \max\{\operatorname{reg}_E(M/vM), s(\frac{0:_M v}{vM})\}$. □

From Theorem 2.3.2, we have following consequences.

Corollary 2.3.3. *Let $0 \neq J \subset E$ be a graded ideal and v a strongly generic element for E/J. Then*

$$\operatorname{reg}_E(J) = \max\{\operatorname{reg}_E(J+(v)), s(\frac{J:_E v}{J+(v)})+1\}.$$

Proof. The statement follows by applying Theorem 2.3.2 for E/J and the fact that $\operatorname{reg}_E(J) = \operatorname{reg}_E(E/J)+1$. □

A sequence $v_1,\dots,v_r$ in E_1 is called a *strongly generic sequence* for a graded module $M \in \mathcal{M}$ if v_i is a strongly generic element for the module $M/(v_1,\dots,v_{i-1})M$ for $i = 1,\dots,r$. Analogously to the result [**13**, Theorem 2.2] for modules over the polynomial ring, we can prove a similar one for modules over the exterior algebra.

Corollary 2.3.4. *Let $J \subset E$ be a graded ideal generated by a strongly generic sequence $v_1, \dots, v_r$ for an E-module $M \in \mathcal{M}$. Then* $\operatorname{reg}_E(JM) \le \operatorname{reg}_E(M) + 1$.

Proof. By Theorem 2.3.2, we have:

$$\operatorname{reg}_E(M/(v_1, \dots, v_{i-1})M) \ge \operatorname{reg}_E(M/(v_1, \dots, v_i)M) \text{ for } i = 1, \dots, r.$$

Therefore, $\operatorname{reg}_E(M/JM) \le \operatorname{reg}_E(M)$. Now by the short exact sequence

$$0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$$

and Lemma 1.2.1 we get that

$$\operatorname{reg}_E(JM) \le \max\{\operatorname{reg}_E(M/JM) + 1, \operatorname{reg}_E(M)\}.$$

Combining all inequalities together we conclude that $\operatorname{reg}_E(JM) \le \operatorname{reg}_E(M) + 1$. □

Remark 2.3.5. Without the hypothesis that $v_1, \dots, v_r$ is a strongly generic sequence for M, Corollary 2.3.4 may not hold. For example, let

$$J = (e_{123}, e_{134}, e_{125}, e_{256}) \subset E = K\langle e_1, \dots, e_6 \rangle.$$

By using Macaulay2 [**28**] we can check that $\operatorname{reg}_E(J) = 3$ but the ideal $P = (e_1, e_2)J = (e_{1234}, e_{1256})$ has $\operatorname{reg}_E(P) = 5 > 4$.

CHAPTER 3

Gröbner fans

The purpose of this chapter is to study Gröbner fans over the exterior algebra. At first, analogously to the polynomial ring case, we consider the Gröbner fan of a graded ideal over the exterior algebra. After that, we study this fan in the normal case as well as in the generic case. We also introduce one of possibilities of defining an analogue to tropical varieties over the exterior algebra as a subfan of the Gröbner fan and study how this subfan is stable in the generic case.

3.1. Gröbner fans

In [**44**], Mora and Robbiano defined the Gröbner fan for a graded ideal in a polynomial ring. This fan has been studied deeply with interesting applications in computational algebra and discrete geometry; see, e.g., [**10**], [**41**], [**61**]. Specially, its subfans, the tropical varieties, the main objects of tropical geometry, have recently received a lot of attention from mathematicians. The main goal of this section is to define the Gröbner fan of a graded ideal over the exterior algebra analogously to the one over the polynomial ring.

Definition 3.1.1. For $f = \sum_{F\subseteq[n]} a_F e_F \in E$ and $w \in \mathbb{R}^n$, the *w-degree* of a non-zero term $a_F e_F$ is $w \cdot F = \sum_{i\in F} w_i$. The *$w$-degree* of a non-zero element f is the maximum of all w-degrees of non-zero terms occurring in f. Let $\mathrm{in}_w(f)$ denote the *initial part* of f which consists of all non-zero terms $a_F e_F$ of f such that $w \cdot F$ is maximal. The element f is called *w-homogeneous* if $\mathrm{in}_w(f) = f$. The *initial ideal* of a graded ideal $J \subset E$ with respect to w is defined by

$$\mathrm{in}_w(J) = (\mathrm{in}_w(f) : f \in J).$$

If $J = \mathrm{in}_w(J)$ then we say that J is *w-homogeneous*. One can show that a graded ideal J is w-homogeneous if and only if J is generated by w-homogeneous elements.

Let $<$ be a monomial order on E and $w \in \mathbb{R}^n$. One has a new monomial order $<_w$ defined by: $e_F <_w e_T$ if $w \cdot F < w \cdot T$ or if $w \cdot T = w \cdot F$ and $e_F < e_T$. Note that $\mathrm{in}_<(J)$ is a monomial ideal while $\mathrm{in}_w(J)$ might not be. Let m be a monomial, $<$ a monomial order and $f \in E$. One has:

$$\text{if } m \wedge \mathrm{in}_<(f) \neq 0 \text{ then } m \wedge \mathrm{in}_<(f) = \mathrm{in}_<(m \wedge f).$$

For a graded ideal $J \subset E$, a monomial order $<$ on E and $w \in \mathbb{R}^n$, we define:

$$\begin{aligned} C_<(J) &= \{u \in \mathbb{R}^n : \mathrm{in}_u(J) = \mathrm{in}_<(J)\}, \\ C_w(J) &= \{u \in \mathbb{R}^n : \mathrm{in}_u(J) = \mathrm{in}_w(J)\}. \end{aligned}$$

Remark 3.1.2. (i) By Lemma 1.1.5, we note that for a graded ideal $J \subset E$, there are only finitely many sets $C_<(J)$.

(ii) Every initial ideal $\mathrm{in}_<(J)$ is of the form $\mathrm{in}_w(J)$ for some $w \in \mathbb{R}^n$; see Proposition 3.1.6. Consequently, every $C_<(J)$ is of the form $C_w(J)$.

In the following, we present sketches of results and their proofs which can be shown analogously to the polynomial ring case. One can see, e.g., [**41**] and [**61**] for more details.

Lemma 3.1.3. *Let $<$ be a monomial order on E, $w \in \mathbb{R}^n$ and $J \subset E$ be a graded ideal. Then $\mathrm{in}_{<_w}(J) = \mathrm{in}_<(\mathrm{in}_w(J))$. In particular, if $\mathcal{G}$ is a Gröbner basis of J w.r.t. $<_w$, then $\{\mathrm{in}_w(g) : g \in \mathcal{G}\}$ is a Gröbner basis of $\mathrm{in}_w(J)$ w.r.t. $<$.*

Proof. By the definition of $<_w$, we have $\mathrm{in}_<(\mathrm{in}_w(f)) = \mathrm{in}_{<_w}(f)$ for every $f \in J$. Therefore, $\mathrm{in}_{<_w}(J) \subseteq \mathrm{in}_<(\mathrm{in}_w(J))$.

Now let $a = \mathrm{in}_<(g)$, where $g \in \mathrm{in}_w(J)$, be a minimal monomial generator of $\mathrm{in}_<(\mathrm{in}_w(J))$. Since $\mathrm{in}_w(\mathrm{in}_w(J)) = \mathrm{in}_w(J)$, we note that $\mathrm{in}_w(J)$ is w-homogeneous. So we may assume that g is w-homogeneous and $g = \sum_i m_i \wedge \mathrm{in}_w(g_i)$, where $m_i \in E$ are monomials and $g_i \in J$ are homogeneous elements. It is clear that $m_i \wedge \mathrm{in}_w(g_i) \neq 0$. Therefore,

$$m_i \wedge \mathrm{in}_w(g_i) = \mathrm{in}_w(m_i \wedge g_i) \text{ and } g = \sum_i \mathrm{in}_w(m_i \wedge g_i).$$

Since the elements $\mathrm{in}_w(m_i \wedge g_i)$ are w-homogeneous of the same w-degree, we have

$$g = \sum_i \mathrm{in}_w(m_i \wedge g_i) = \mathrm{in}_w(\sum_i m_i \wedge g_i).$$

Let $f = \sum_i m_i \wedge g_i \in J$. Then

$$a = \mathrm{in}_<(g) = \mathrm{in}_<(\mathrm{in}_w(f)) = \mathrm{in}_{<_w}(f) \in \mathrm{in}_{<_w}(J).$$

Thus $\mathrm{in}_{<_w}(J) \supseteq \mathrm{in}_<(\mathrm{in}_w(J))$. Hence $\mathrm{in}_{<_w}(J) = \mathrm{in}_<(\mathrm{in}_w(J))$.

In particular, if $\mathcal{G}$ is a Gröbner basis of J with respect to $<_w$, then we have $\mathrm{in}_{<_w}(J) = \mathrm{in}_<(\mathrm{in}_w(J))$. Therefore,

$$(\mathrm{in}_<(\mathrm{in}_w(g)) : g \in \mathcal{G}) = (\mathrm{in}_{<_w}(g) : g \in \mathcal{G}) = \mathrm{in}_{<_w}(J) = \mathrm{in}_<(\mathrm{in}_w(J)).$$

This concludes the proof. □

Corollary 3.1.4. *Let $J \subset E$ be a graded ideal and $w \in \mathbb{R}^n$. Then J and $\mathrm{in}_w(J)$ have the same Hilbert series.*

Proof. Let $<$ be a monomial order on E. By Lemma 1.1.5, we have the following equalities of Hilbert series:

$$H_J(t) = H_{\mathrm{in}_{<_w}(J)}(t) \text{ and } H_{\mathrm{in}_w(J)}(t) = H_{\mathrm{in}_<(\mathrm{in}_w(J))}(t).$$

By Lemma 3.1.3, $\mathrm{in}_{<_w}(J) = \mathrm{in}_<(\mathrm{in}_w(J))$. Thus J and $\mathrm{in}_w(J)$ have the same Hilbert series. □

Example 3.1.5. Let $E = K\langle e_1, \dots, e_6\rangle$. For $w = (1, 2, 3, 2, 1, 0) \in \mathbb{R}^6$, we have $\mathrm{in}_w(e_{245} - e_{134}) = e_{134}$. Let $J = (e_{12} - e_{34} + e_{25}, e_{13} - e_{15}, e_{245} - e_{134}) \subset E$. Then

$$\mathrm{in}_w(J) = (e_{13}, e_{34}, e_{245}, e_{125} - e_{145}),$$

Let $<$ be the reverse lexicographic order on E with $e_1 > e_2 > \ldots > e_6$. We have

$$\mathrm{in}_{<_w}(J) = \mathrm{in}_<(\mathrm{in}_w(J)) = (e_{13}, e_{34}, e_{125}, e_{245}).$$

Lemma 3.1.6 ([**41**, Proposition 2.4.4])**.** *Let $J \subset E$ be a graded ideal. For any monomial order $<$ on E, there exists a weight vector $w \in \mathbb{R}^n$ such that*

$$\mathrm{in}_w(J) = \mathrm{in}_<(J).$$

Proof. Let $\mathcal{G} = \{g_1, \dots, g_r\}$ be the reduced Gröbner basis of J with respect to $<$. Write $g_i = \sum_j a_{ij} e_{F_{ij}}$, where $\mathrm{in}_<(g_i) = a_{i1} e_{F_{i1}}$. Let $C_<$ be the set of all vectors $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(g_i) = a_{i1} e_{F_{i1}}$, i.e.,

$$C_< = \{w \in \mathbb{R}^n : w \cdot F_{i1} > w \cdot F_{ij} \text{ for } i = 1, \dots, r \text{ and all } j \neq 1\}.$$

By using the same proof as in the case of polynomial rings, see [**61**, Proposition 1.11], we get that $C_< \neq \emptyset$. Now for any weight vector $w \in C_<$, we have $\mathrm{in}_<(J) \subseteq \mathrm{in}_{<_w}(J)$. By Corollary 1.1.6, we conclude that $\mathrm{in}_<(J) = \mathrm{in}_w(J)$. □

Proposition 3.1.7. *Let $<$ be a monomial order on E and $J \subset E$ be a graded ideal. Let $w \in \mathbb{R}^n$ and $\mathcal{G}$ be the reduced Gröbner basis of J with respect to $<_w$. Then for $w' \in \mathbb{R}^n$, $\mathrm{in}_{w'}(J) = \mathrm{in}_w(J)$ if and only if $\mathrm{in}_{w'}(g) = \mathrm{in}_w(g)$ for all $g \in \mathcal{G}$.*

Proof. ($\Leftarrow$): Assume that $\mathrm{in}_{w'}(g) = \mathrm{in}_w(g)$ for all $g \in \mathcal{G}$. Since $\{\mathrm{in}_w(g) \in \mathcal{G}\}$ is a Gröbner basis of $\mathrm{in}_w(J)$ with respect to $<$, it is also a system of generators of $\mathrm{in}_w(J)$. This implies that $\mathrm{in}_w(J) \subseteq \mathrm{in}_{w'}(J)$. Taking initial ideals with respect to $<$, we get $\mathrm{in}_<(\mathrm{in}_w(J)) \subseteq \mathrm{in}_<(\mathrm{in}_{w'}(J))$. By Lemma 3.1.3, this implies that $\mathrm{in}_{<_w}(J) \subseteq \mathrm{in}_{<_{w'}}(J)$. This inclusion cannot be proper by Corollary 1.1.6, so $\mathrm{in}_<(\mathrm{in}_w(J)) = \mathrm{in}_<(\mathrm{in}_{w'}(J))$. Thus $\mathrm{in}_{w'}(J) = \mathrm{in}_w(J)$ since the Hilbert series of all considered ideals coincide.

($\Rightarrow$): Assume that $\mathrm{in}_{w'}(J) = \mathrm{in}_w(J)$. By Lemma 3.1.3, $\mathrm{in}_w(\mathcal{G}) = \{\mathrm{in}_w(g) \in \mathcal{G}\}$ is also a Gröbner basis of $\mathrm{in}_{w'}(J)$ with respect to $<$. Let $g \in \mathcal{G}$ and $e_F = \mathrm{in}_{<_w}(g)$. Since $\mathcal{G}$ is the reduced Gröbner basis of J with respect to $<_w$, we note that e_F is the only monomial occurring in g which is divisible by a leading term with respect to $<_w$ of an element in $\mathcal{G}$. We have that $\mathrm{in}_<(\mathrm{in}_{w'}(g))$, which is also a monomial of g, is divisible by a leading term w.r.t. $<_w$ of an element of $\mathcal{G}$. The only possibility is $\mathrm{in}_<(\mathrm{in}_{w'}(g)) = e_F$. Let $\mathrm{in}_w(g) = e_F + h$, and $\mathrm{in}_{w'}(g) = e_F + h'$. Since $\mathcal{G}$ is the reduced Gröbner basis, h and h' both are sums of terms not in $\mathrm{in}_{<_w}(J)$. Note that $\mathrm{in}_w(g) - \mathrm{in}_{w'}(g) = h - h' \in \mathrm{in}_w(J)$. So $\mathrm{in}_<(h - h') \in \mathrm{in}_<(\mathrm{in}_w(J)) = \mathrm{in}_{<_w}(J)$. Thus $h - h' = 0$. Hence $\mathrm{in}_w(g) = \mathrm{in}_{w'}(g)$ for all $g \in \mathcal{G}$, as desired. □

Proposition 3.1.8. *Let $J \subset E$ be a graded ideal and $w \in \mathbb{R}^n$. Then $C_w(J)$ is the relative interior of a polyhedral cone in $\mathbb{R}^n$.*

Proof. Let $\mathcal{G} = \{g_1, \dots, g_r\}$ be the reduced Gröbner basis of J with respect to $<_w$. For $g_i \in \mathcal{G}$, write $g_i = \sum_j a_{ij} e_{F_{ij}} + \sum_k b_{ik} e_{T_{ik}}$ as the sum of non-zero terms with $\mathrm{in}_w(g_i) = \sum_j a_{ij} e_{F_{ij}}$.

By Proposition 3.1.7, we get that

$$\begin{aligned} C_w(J) &= \{u \in \mathbb{R}^n : \mathrm{in}_u(g_i) = \sum_j a_{ij} e_{F_{ij}} \text{ for } i = 1, \dots, r\} \\ &= \{u \in \mathbb{R}^n : u \cdot F_{ij} = u \cdot F_{it} > u \cdot T_{ik} \text{ for } i = 1, \dots, r \text{ and all } j, k, t\}. \end{aligned}$$

This is the relative interior of a polyhedral cone in $\mathbb{R}^n$. $\square$

Lemma 3.1.9. *Let $<$ be a monomial order on E and $J \subset E$ be a graded ideal. Let $w \in \mathbb{R}^n$ and $\mathcal{G}$ be the reduced Gröbner basis of J with respect to $<_w$. Then for $u \in \overline{C_w(J)}$, we have*

$$\mathrm{in}_u(J) = (\mathrm{in}_u(g) : g \in \mathcal{G}).$$

Proof. Let $I = (\mathrm{in}_u(g) : g \in \mathcal{G})$. Since $\mathrm{in}_u(g) \in \mathrm{in}_u(J)$ for all $g \in \mathcal{G}$, it is clear that $I \subseteq \mathrm{in}_u(J)$.

Now using the same notations as in Proposition 3.1.8, we note that

$$\overline{C_w(J)} = \{v \in \mathbb{R}^n : v \cdot F_{ij} = v \cdot F_{it} \geq v \cdot T_{ik} \text{ for } i = 1, \dots, r \text{ and all } j, k, t\}.$$

This implies that

$$\mathrm{in}_w(g) = \mathrm{in}_w(\mathrm{in}_u(g)) \text{ for all } g \in \mathcal{G}.$$

Therefore, by Lemma 3.1.3 we have

$$\mathrm{in}_w(J) \subseteq \mathrm{in}_w(I) \subseteq \mathrm{in}_w(\mathrm{in}_u(J)).$$

By Lemma 3.1.4, we note that $\mathrm{in}_w(J)$ and $\mathrm{in}_w(\mathrm{in}_u(J))$ have the same Hilbert series. Hence

$$\mathrm{in}_w(J) = \mathrm{in}_w(I) = \mathrm{in}_w(\mathrm{in}_u(J)).$$

Taking initial ideals w.r.t. $<$, we get $\mathrm{in}_<(\mathrm{in}_w(I)) = \mathrm{in}_<(\mathrm{in}_w(\mathrm{in}_u(J)))$. Then by Lemma 3.1.3, we have $\mathrm{in}_{<_w}(I) = \mathrm{in}_{<_w}(\mathrm{in}_u(J))$. Recall that $I \subseteq \mathrm{in}_u(J)$. Using Corollary 1.1.6, we can conclude that $I = \mathrm{in}_u(J)$. $\square$

With the above properties, we also have an analogous result to the case of polynomial rings, which ensures the existence of Gröbner fans. We recall the result in the following:

Proposition 3.1.10 ([**41**, Proposition 2.4.9])**.** *Let $J \subset E$ be a graded ideal. Then the set $\mathrm{GF}(J) = \{\overline{C_w(J)} : w \in \mathbb{R}^n\}$ forms a polyhedral fan.*

Proof. We need to prove the following statements:

(i) For $u \in \overline{C_w(J)}$, then $\overline{C_u(J)}$ is a face of $\overline{C_w(J)}$.
(ii) The intersection of two polyhedral cones $\overline{C_w(J)}$ and $\overline{C_{w'}(J)}$ is a common face of them.

For (i), let $<$ be a monomial order on E and $\mathcal{G}$ be the reduced Gröbner basis of J w.r.t. $<_w$. By Lemma 3.1.9 and Lemma 3.1.3, we have

$$\mathrm{in}_{<_w}(J) = \mathrm{in}_{<_w}(\mathrm{in}_u(J)) = \mathrm{in}_{(<_w)_u}(J).$$

Thus $\mathcal{G}$ is also the reduced Gröbner basis of J w.r.t. $(<_w)_u$. By Proposition 3.1.7, we get that

$$C_u(J) = \{u' \in \mathbb{R}^n : \mathrm{in}_{u'}(g) = \mathrm{in}_u(g) \text{ for all } g \in \mathcal{G}\}.$$

Now using the concrete description as a set of $C_w(J)$ and $C_u(J)$ in Proposition 3.1.8 and taking their closure, we see that $\overline{C_u(J)}$ is a face of $\overline{C_w(J)}$.

For (ii), assume that two polyhedral cones $\overline{C_w(J)}$ and $\overline{C_{w'}(J)}$ do not contain each other. Then by (i), we get that $\overline{C_w(J)} \cap \overline{C_{w'}(J)}$ is the union of common faces of these two polyhedral cones. Since this intersection is convex, it must in fact be one face. This concludes the proof. □

Definition 3.1.11. The polyhedral fan $\mathrm{GF}(J) = \{\overline{C_w(J)} : w \in \mathbb{R}^n\}$ is called the *Gröbner fan* of the graded ideal J.

Example 3.1.12. Let $E = K\langle e_1, \ldots, e_6\rangle$ and $J = (e_{12} - e_{34}, e_{13} - e_{15}, e_{24} - e_{36})$ a graded ideal in E. Let $w = (1, 2, 3, 2, 1, 0) \in \mathbb{R}^6$. Then $\mathrm{in}_w(J) = (e_{34}, e_{13}, e_{24}, e_{145})$. Let $<$ be the reverse lexicographic order on E with $e_1 > e_2 > \ldots > e_6$. The reduced Gröbner basis of J with respect to $<_w$ is $\mathcal{G} = \{e_{12} - e_{34}, e_{13} - e_{15}, e_{24} - e_{36}, e_{145}\}$.

By Proposition 3.1.7, we get that

$$C_w(J) = \{u \in \mathbb{R}^6 : u_3 + u_4 > u_1 + u_2, u_3 > u_5, u_2 + u_4 > u_3 + u_6\}.$$

By replacing the strict inequalities in the above set by non-strict inequalities we obtain the closure $\overline{C_w(J)}$. Using Macaulay2 [**28**], we can compute the Gröbner fan $\mathrm{GF}(J)$ of J. It is a pure fan with f-vector $(1, 6, 41, 27)$ and has 16 rays.

3.2. Exterior algebra analogues to tropical varieties

Over the polynomial ring, tropical varieties have recently received a lot of attention in the context of tropical geometry; see, e.g., [**43**], [**61**] for more details. The goal of this section is to find a suitable definition of such kind of sets over the exterior algebra and study properties of these sets.

Definition 3.2.1. Let $J \subset E$ be a graded ideal. Recall that $\mathrm{Mon}(J)$ is the set of all monomials of J. We let $T(J)$ be the set of all $w \in \mathbb{R}^n$ such that

$$\mathrm{Mon}(J) = \mathrm{Mon}(\mathrm{in}_w(J)).$$

Example 3.2.2. (i) Let $J = (e_{12} - e_{34}, e_{13} - e_{25}) \subset K\langle e_1, \ldots, e_5\rangle$. Then we have

$$\mathrm{Mon}(J) = \{e_F : F \subseteq [5], |F| \geq 3\} \setminus \{e_{145}\}.$$

Let $w \in \mathbb{R}^5$. Then $w \in T(J)$ if and only if $\mathrm{in}_w(e_{12} - e_{34}) = e_{12} - e_{34}$ and $\mathrm{in}_w(e_{13} - e_{25}) = e_{13} - e_{25}$. Therefore,

$$T(J) = \{w \in \mathbb{R}^5 : w_1 + w_2 = w_3 + w_4, w_1 + w_3 = w_2 + w_5\}$$

and it is a 3-dimensional vector subspace of $\mathbb{R}^5$.

(ii) Let $I = (e_{12}, e_{34}, e_{13} - e_{25}) \subset K\langle e_1, \ldots, e_5\rangle$. Then

$$\mathrm{Mon}(I) = \{e_{12}, e_{34}\} \cup \{e_F : F \subseteq [5], |F| \geq 3\} \setminus \{e_{145}\}.$$

This implies that $w \in T(I)$ if and only if $\mathrm{in}_w(e_{13} - e_{25}) = e_{13} - e_{25}$. So we have $T(I) = \{w \in \mathbb{R}^5 : w_1 + w_3 = w_2 + w_5\} \subset \mathbb{R}^5$ and it is a 4-dimensional vector space.

Note that for the polynomial ring case, if $J \subset I$ then $T(I) \subset T(J)$. But this is not true for the exterior algebra case with respect to our definition. More precisely, the above examples show that $J \subset I$ but $T(I) \not\subset T(J)$.

Remark 3.2.3. Observe that every graded ideal $J \subset E$ contains the monomial $e_1e_2 \dots e_n$. One can ask if there is another possibility of defining analogues to tropical varieties for the exterior algebra. Indeed, we could define that

$$T(J) = \{w \in \mathbb{R}^n : \mathrm{Mon}(\mathrm{in}_w(J)) = \{e_1e_2 \dots e_n\}\}.$$

With this alternative definition the class of relevant ideals would be too small since for example, for every $f \in J$, we have $e_F \in J$ where F is the set of all variables appearing in the standard expression of f. So Definition 3.2.1 allows a richer theory.

Remark 3.2.4. Let $J \subset E$ be a graded ideal. Then $T(J)$ is always non-empty since the vector $w = (1, \dots, 1) \in T(J)$.

Proposition 3.2.5. *The set $T(J)$ of a graded ideal $J \subset E$ is a subfan of the Gröbner fan of J.*

Proof. By Definition 3.2.1, we see that

$$T(J) = \bigcup_{w \in T(J)} C_w(J) \text{ as sets.}$$

More precisely, $T(J)$ is the union of $C_w(J)$ such that $\mathrm{in}_w(J)$ contains only the monomials in $\mathrm{Mon}(J)$. Hence $T(J) \subset |\mathrm{GF}(J)|$. We need to prove that if $w \in T(J)$ then $\overline{C_w(J)} \subset T(J)$. Let $u \in \overline{C_w(J)}$. By the proof of Lemma 3.1.9, we have

$$\mathrm{in}_w(J) = \mathrm{in}_w(\mathrm{in}_u(J)).$$

So a monomial of $\mathrm{in}_u(J)$ is also a monomial of $\mathrm{in}_w(J)$. Since $w \in T(J)$ we have that $\mathrm{Mon}(\mathrm{in}_w(J)) = \mathrm{Mon}(J) \subseteq \mathrm{Mon}(\mathrm{in}_u(J))$. Therefore, $\mathrm{Mon}(\mathrm{in}_w(J)) = \mathrm{Mon}(\mathrm{in}_u(J))$ and then $u \in T(J)$. By the same argument to Proposition 3.1.10, we get that $T(J)$ is also a fan. This concludes the proof. □

3.3. The generic case

The generic Gröbner fans and the generic tropical varieties over the polynomial ring are very useful tool in characterizing algebraic properties of coordinate rings of algebraic varieties, e.g., the Cohen-Macaulayness, the depth and the multiplicity; see [**53**] for more details. In [**54**], Römer and Schmitz proved the existence of generic Gröbner fans and generic tropical varieties over the polynomial ring S. In fact, they showed that for every graded ideal $I \subset S$, there exists a non-empty Zariski-open subset $U \subset \mathrm{GL}_n(K)$ such that the Gröbner fan and the tropical variety of I are constant under the actings of all $g \in U$. This motivates us to consider in this section the problem whether for a graded ideal $J \subset E$, there exist fans gGF, gT and a non-empty Zariski-open subset $U \subset \mathrm{GL}_n(K)$ such that for all $g \in U$ we always have that $\mathrm{GF}(g(J)) = gGF$ and $T(g(J)) = gT$.

We always assume in this section that the base field K is infinite. At first, we will specify the meaning of the term *generic* as follows:

Definition 3.3.1. Let $\mathcal{Y} = \{y_{ij} : i, j = 1, \dots, n\}$ be a set of n^2 independent variables over K and let $K' = K(\mathcal{Y})$ be the quotient field of $K[\mathcal{Y}]$. In the following we denote by y the K-algebra homomorphism

$$y : K\langle e_1, \dots, e_n\rangle \longrightarrow K'\langle e_1, \dots, e_n\rangle, \qquad e_i \longmapsto \sum_{j=1}^{n} y_{ij} e_j.$$

For each $g = (g_{ij}) \in \mathrm{GL}_n(K)$, substituting y_{ij} by g_{ij}, we have a K-algebra automorphism on $K\langle e_1, \dots, e_n\rangle$. We identify g with the induced automorphism and use the notation g for both of them. Note that for a graded ideal $J \subset E$, $y(J)$ might not be an ideal in $K'\langle e_1, \dots, e_n\rangle$ but $g(J)$ is always a graded ideal in $K\langle e_1, \dots, e_n\rangle$ since g is an automorphism.

We prove next the main result of this section:

Theorem 3.3.2. *Let $0 \neq J \subset E$ be a graded ideal. There exists a Zariski-open subset $\emptyset \neq U \subset \mathrm{GL}_n(K)$ such that all $g(J)$ have the same support for all $g \in U$.*

Proof. Choose a K-vector space basis for each $J_d \neq 0$, $d \in \mathbb{Z}$ and let $\mathcal{H} = \{h_1, \dots, h_s\}$ be the union of all these bases. Then $\mathcal{H}$ is a finite set and every element of J is a K-linear combination of finitely many elements in $\mathcal{H}$.

For each $J_d \neq 0$, $d \in \mathbb{Z}$, there exists a subset of $\mathcal{H}$, for simplicity say $\{h_1, \dots, h_r\}$ with $r \leq s$, which is a K-basis of J_d. For $i = 1, \dots, r$, write $y(h_i) = \sum_{j=1}^{m} c_{ji}(y) e_{F_j}$ where F_j runs over all $m = \binom{n}{d}$ subsets of d elements in $[n]$ and $c_{ji}(y) \in K[\mathcal{Y}]$. We denote by $A(y) = (c_{ji}(y))_{m\times r}$ the matrix with the entries $c_{ji}(y)$ and by $A_L(y)$ the induced matrix by all rows $j \in L$ of $A(y)$ where $L \subseteq [m]$.

Note that a support of an element in E_d is of the form $\{e_{F_j} : j \in F\}$ where $F \subset [m]$ and $F_j \subset [n]$ with $|F_j| = d$. We identify this support with F and use the notation F for both of them.

Now let F be a support of an element in E_d and assume that there exist $f \in J_d$ and $g \in \mathrm{GL}_K(n)$ such that $\mathrm{supp}(g(f)) = F$. Write $f = \sum_{i=1}^{r} \alpha_i h_i$ with $\alpha_i \in K$ for $i = 1, \dots, r$. Then

$$y(f) = \sum_{i=1}^{r} \alpha_i y(h_i) = \sum_{j=1}^{m} \pm e_{F_j} \sum_{i=1}^{r} c_{ji}(y)\alpha_i.$$

Thus $\mathrm{supp}(g(f))$ depends on whether the polynomial $\sum_{i=1}^{r} c_{ji}(y)\alpha_i$ is non-zero at g for $j = 1, \dots, m$.

Let $T = [m] \setminus F$ and $(\alpha_i) = (\alpha_i)_{r\times 1}$. Then we have

$$\begin{cases} A_{\{j\}}(g)(\alpha_i) \neq 0 \text{ for all } j \in F, \\ A_T(g)(\alpha_i) = 0. \end{cases}$$

Therefore,

$$\begin{cases} A_{T\cup\{j\}}(g)(\alpha_i) \neq 0 \text{ for all } j \in F, \\ A_T(g)(\alpha_i) = 0. \end{cases}$$

Let $V(L)$ be the K-vector space of the solutions of $A_L(g)(x_i)_{r\times 1} = 0$ for $L \subset [m]$. Then we have

$$(\alpha_i) \in V(T) \text{ and } (\alpha_i) \notin V(T \cup \{j\}) \text{ for all } j \in F.$$

Since $V(T \cup \{j\}) \subseteq V(T)$ for all $j \in F$, we get that

$$\dim_K V(T) > \dim_K V(T \cup \{j\}) \text{ for all } j \in F.$$

Hence $\operatorname{rank}(A_{T\cup\{j\}}(g)) > \operatorname{rank}(A_T(g))$ for all $j \in F$. This is equivalent to the condition of being non-zero of finitely many determinants of size $(\operatorname{rank}(A_T(g))+1)$-minors of the matrices $A_{T\cup\{j\}}(g)$. More precisely, there is a finite set of polynomials in $K[\mathcal{Y}]$, which are determinants of size $(\operatorname{rank}(A_T(g))+1)$-minors of the matrices $A_{T\cup\{j\}}(y)$, such that the existence of (α_i) depends only on being non-zero of such polynomials at g. Therefore, the existence of (α_i), i.e., f, is an open condition in g. Let $U_F \subset \mathrm{GL}_n(K)$ be the Zariski-open subset such that this open condition is fulfilled. So when F runs over the subsets of $[m]$, if $U_F \neq \emptyset$ then for each $g \in U_F$ there exists $f \in J$ such that $\operatorname{supp}(g(f)) = F$.

Let $U_d = \bigcap_F U_F$ where F runs over subsets $[m]$ with $U_F \neq \emptyset$. Note that this intersection is finite and there are finitely many U_d. Set $U = \bigcap_{U_d \neq \emptyset} U_d$. Then U is a non-empty Zariski-open subset of $\mathrm{GL}_n(K)$. We claim that

$$\operatorname{supp}(g_1(J)) = \operatorname{supp}(g_2(J)) \text{ for all } g_1, g_2 \in U.$$

Let $F \in \operatorname{supp}(g_1(J))$. Then $\emptyset \neq U \subset U_F$ and there exists $f_1 \in J$ such that $\operatorname{supp}(g_1(f_1)) = F$. Since $g_2 \in U_F$ and by the definition of U_F, there exists also $f_2 \in J$ such that $\operatorname{supp}(g_2(f_2)) = F$. Thus $F \in \operatorname{supp}(g_2(J))$ and $\operatorname{supp}(g_1(J)) \subseteq \operatorname{supp}(g_2(J))$. By the same argument, we get $\operatorname{supp}(g_1(J)) \supseteq \operatorname{supp}(g_2(J))$. Hence $\operatorname{supp}(g_1(J)) = \operatorname{supp}(g_2(J))$ for all $g_1, g_2 \in U$. So all $g(J)$ have the same support for all $g \in U$. More precisely, this support is given by

$$\operatorname{supp}(g(J)) = \{\{e_{F_j} : j \in F\} : U_F \neq \emptyset\}.$$

This concludes the proof. □

We get a direct consequence of Theorem 3.3.2 as an alternative proof of the existence of generic initial ideals in the following (see, e.g., [**3**, Theorem 1.6] and [**29**] for the first proof).

Corollary 3.3.3. *Let $0 \neq J \subset E$ be a graded ideal. Then the generic initial ideal* $\operatorname{gin}(J)$ *exists.*

Proof. Let $<$ be the reverse lexicographic order on E. Let $\emptyset \neq U$ be the Zariski-open subset of $\mathrm{GL}_n(K)$ as chosen in Theorem 3.3.2. Then $\operatorname{supp}(g(J))$ is constant for all $g \in U$. This implies that $\operatorname{in}_<(g(J))$ is also constant for all $g \in U$, say $I = \operatorname{in}_<(g(J))$. Therefore, we can use U as the non-empty Zariski-open subset of $\mathrm{GL}_n(K)$ to compute $\operatorname{gin}(J)$, i.e., $\operatorname{gin}(J) = I$. This concludes the proof. □

Next we prove the existence of a generic Gröbner fan. We have:

Theorem 3.3.4. *Let $0 \neq J \subset E$ be a graded ideal. Then there exists a non-empty Zariski-open subset $U \subset \mathrm{GL}_n(K)$ such that every ideal $g(J)$ has the same Gröbner fan for every $g \in U$.*

Proof. Let $\emptyset \neq U$ be the Zariski-open subset of $\mathrm{GL}_n(K)$ as chosen in Theorem 3.3.2. Then $\operatorname{supp}(g(J))$ is constant for all $g \in U$.

Let $g_1, g_2 \in U$. We need to prove that for all $w \in \mathbb{R}^n$, the equality of polyhedral cones $C_w(g_1(J)) = C_w(g_2(J))$ holds. Let $<$ be a monomial order on E and $\mathcal{G}_1$ be the reduced Gröbner basis of $g_1(J)$ w.r.t. $<_w$. Assume that $\mathcal{G}_1 = \{g_1(f_1), \ldots, g_1(f_r)\}$ where $f_i \in J$ for $i = 1, \ldots, r$. Since $\operatorname{supp}(g_1(J)) = \operatorname{supp}(g_2(J))$, there exist elements, say $f'_1, \ldots, f'_r \in J$, such that

$$\operatorname{supp}(g_1(f_i)) = \operatorname{supp}(g_2(f'_i)) \text{ for } i = 1, \ldots, r.$$

Let $\mathcal{G}_2 = \{g_2(f'_1), \ldots, g_2(f'_r)\}$. Then we get $\operatorname{supp}(\mathcal{G}_1) = \operatorname{supp}(\mathcal{G}_2)$. Since

$$\operatorname{supp}(g_1(J)) = \operatorname{supp}(g_2(J)),$$

we note that $\operatorname{in}_{<_w}(g_1(J)) = \operatorname{in}_{<_w}(g_2(J))$. Thus

$$\operatorname{in}_{<_w}(g_2(J)) = (\operatorname{in}_{<_w}(g_2(f'_i)) : i = 1, \ldots, r).$$

Hence $\mathcal{G}_2$ is the reduced Gröbner basis of $g_2(J)$ w.r.t. $<_w$. Now by Proposition 3.1.7, $C_w(g_1)$ and $C_w(g_2)$ depend only on $\operatorname{supp}(\mathcal{G}_1) = \operatorname{supp}(\mathcal{G}_2)$. Thus $C_w(g_1) = C_w(g_2)$. This concludes the proof. $\square$

Since every non-empty Zariski-open subset is dense in $\mathrm{GL}_n(K)$, the following definition makes sense.

Definition 3.3.5. Let $J \subset E$ be a graded ideal. The *generic Gröbner fan of* J, denote by $\mathrm{gGF}(J)$, is the unique polyhedral fan that equals to $\mathrm{GF}(g(J))$ for all g in a non-empty Zariski-open subset of $\mathrm{GL}_n(K)$. Note that by Theorem 3.3.2 this fan always exists.

As explained above the set $T(J)$ is a subfan of the Gröbner fan of J and thus closely related to initial ideals of J. The existence of the exterior generic Gröbner fan leads to a question, whether there exists a generic set $T(J)$ over the exterior algebra analogously to the generic tropical varieties over the polynomial ring (see [**54**]) and how it looks like if it does exist. Next we present an explicit answer of this question.

Definition 3.3.6. Let $J \subset E$ be a graded ideal. If the fan $T(g(J))$ is the same fan for all g in a Zariski-open subset $\emptyset \neq U \subset \mathrm{GL}_n(K)$, then we denote this fan by $\mathrm{gT}(J)$.

Theorem 3.3.7. *Let* $0 \neq J \subset E$ *be a graded ideal. Then* $\mathrm{gT}(J)$ *exists.*

Proof. Let U be the non-empty Zariski-open subset of $\mathrm{GL}_n(K)$ as chosen in Theorem 3.3.2. Then $\operatorname{supp}(g(J))$ is constant for all $g \in U$. Observe that

$$\operatorname{Mon}(g(J)) = \operatorname{Mon}(\operatorname{supp}(g(J))) \text{ and } \operatorname{Mon}(\operatorname{in}_w(g(J))) = \operatorname{Mon}(\operatorname{in}_w(\operatorname{supp}(g(J))))$$

for $w \in \mathbb{R}^n$. Therefore, by Definition 3.2.1 we get that $T(g(J))$ depends only on $\operatorname{supp}(g(J))$ for $g \in U$. So $T(g(J))$ is constant for all $g \in U$. Thus $\mathrm{gT}(J)$ exists. $\square$

We conclude this section with some examples in which we consider $\mathrm{gGF}(J)$ and $\mathrm{gT}(J)$ where J is generated by a homogeneous element $f \in E$. For this, we need the following observation:

Lemma 3.3.8. *Let $0 \neq f \in E$ be a homogeneous element with $\deg(f) = d$. Then there exists a non-empty Zariski-open set $U \subset \mathrm{GL}_n(K)$ such that for all $g \in U$, $\mathrm{supp}(g(f))$ contains all monomials e_F with $|F| = d$.*

Proof. Let $f = \sum_{T \subset [n]} a_T e_T \in E_d$ with $|T| = d$. Then

$$g(f) = \sum_{a_T \neq 0} a_T g(e_T) = \sum_{a_T \neq 0} a_T \prod_{i \in T} (\sum_{j=1}^{n} g_{ij} e_j) = \sum_{F \subset [n], |F| = d} c_F(g) e_F.$$

Here

$$c_F(g) = \sum_T (a_T \sum_{\sigma_F} \pm \prod_{i \in T} g_{i\sigma(i)})$$

where σ_F runs over the set of bijective maps from T to F.

Observe that $c_F(g)$ is a polynomial function in g_{ij}. We can choose U to be a non-empty Zariski-open set of $\mathrm{GL}_n(K)$ such that $c_F(g) \neq 0$ for all $g \in U$. This concludes the proof. □

Proposition 3.3.9. *Let $0 \neq f \in E$ be a homogeneous element with $\deg(f) = d$ and $J = (f)$. We have:*

(i) *For $w \in \mathbb{R}^n$ with*

$$w_{i_1} \geq \ldots \geq w_{i_{d-k-1}} > w_{i_{d-k}} = \ldots = w_{i_{d+t}} > w_{i_{d+t+1}} \geq \ldots \geq w_{i_n},$$

where $\{i_1, \ldots, i_n\} = [n]$ and $k, t \geq 0$, the relative open cone $C[w]$ of $\mathrm{gGF}(J)$ is a subset of the set

$$\{u \in \mathbb{R}^n : u_{i_1} \geq \ldots \geq u_{i_{d-k-1}} > u_{i_{d-k}} = \ldots = u_{i_{d+t}} > u_{i_{d+t+1}} \geq \ldots \geq u_{i_n}\}.$$

(ii) $\mathrm{gT}(J)$ *is a subset of the set*

$$\mathrm{gGF}(J) \setminus \bigcup_{\{i_1, \ldots, i_n\} = [n]} \{u \in \mathbb{R}^n : u_{i_1} \geq \ldots \geq u_{i_d} > u_{i_{d+1}} \geq \ldots \geq u_{i_n}\}.$$

Proof. Let $w \in \mathbb{R}^n$. There always exists an ordering of the coordinates of w with

$$w_{i_1} \geq \ldots \geq w_{i_{d-k-1}} > w_{i_{d-k}} = \ldots = w_{i_{d+t}} > w_{i_{d+t+1}} \geq \ldots \geq w_{i_n},$$

where $\{i_1, \ldots, i_n\}$ is a permutation of $[n]$ and $k, t \geq 0$.

Let $\emptyset \neq U' \subset \mathrm{GL}_n(K)$ be the Zariski-open subset chosen as in Lemma 3.3.8 and $\emptyset \neq U'' \subset \mathrm{GL}_n(K)$ be the Zariski-open subset chosen as in Theorem 3.3.2. Let $U = U' \cap U''$. Then $U \neq \emptyset$ and we only need to consider the sets $\mathrm{GF}(g(J))$ and $T(g(J))$ for $g \in U$. We have that $g(f)$ contains all monomials e_F of degree d or all $g \in U$. Let $h = \sum_{F \subset [n], |F| = d} e_F$, $C_w(h) = \{u \in \mathbb{R}^n : \mathrm{in}_u(h) = \mathrm{in}_w(h)\}$ and $T(h) = \{u \in \mathbb{R}^n : \mathrm{in}_u(h) \text{ is not a monomial}\}$. Then $\mathrm{supp}(g(f)) = \mathrm{supp}(h)$.

For (i), let $\Gamma = \{F \subset \{i_1, \ldots, i_{d+t}\} : |F| = d \text{ and } F \supset \{i_1, \ldots, i_{d-k-1}\}\}$. Then $w \cdot F$ is constant for all $F \in \Gamma$. Moreover, $w \cdot F > w \cdot T$ for all $T \subset [n]$, $|T| = d$ and $T \notin \Gamma$. Thus $\mathrm{in}_w(h) = \sum_{F \in \Gamma} e_F$. Therefore,

$$\begin{aligned} C_w(h) &= \{u \in \mathbb{R}^n : \mathrm{in}_u(h) = \mathrm{in}_w(h)\} = \{u \in \mathbb{R}^n : \mathrm{in}_u(h) = \sum_{F \in \Gamma} e_F\} \\ &= \{u \in \mathbb{R}^n : u_{i_1} \geq \ldots \geq u_{i_{d-k-1}} > u_{i_{d-k}} = \ldots = u_{i_{d+t}} > u_{i_{d+t+1}} \geq \ldots \geq u_{i_n}\}. \end{aligned}$$

Let $u \in C[w]$. Then $\text{in}_u(g(J)) = \text{in}_w(g(J))$. Thus $\text{in}_u(g(f)) = \text{in}_w(g(f))$. Since $\text{supp}(g(f)) = \text{supp}(h)$, we get that $\text{in}_w(h) = \text{in}_u(h)$. Hence $C[w] \subseteq C_w(h)$.

For (ii), let $u \in \mathbb{R}^n$ such that $u \notin T(h)$. We may assume further that $u_{i_1} \geq u_{i_2} \geq \ldots \geq u_{i_n}$, where $\{i_1, \ldots, i_n\}$ is a permutation of $[n]$. Then $u \notin T(h)$ if and only if $\text{in}_u(h) = e_{i_1 \ldots i_d}$. This is the case only when $u_{i_d} > u_{i_{d+1}}$. So we get

$$T(h) = \mathbb{R}^n \setminus \bigcup_{\{i_1,\ldots,i_n\}=[n]} \{u \in \mathbb{R}^n : u_{i_1} \geq \ldots \geq u_{i_d} > u_{i_{d+1}} \geq \ldots \geq u_{i_n}\}.$$

Since

$$\text{supp}(g(f)) = \text{supp}(h) \text{ and } T(g(f)) \subseteq \{u \in \mathbb{R}^n : \text{in}_u(g(f)) \text{ is not a monomial}\},$$

we get that $T(g(f)) \subseteq T(h)$. Moreover, $T(g(f)) \subseteq \text{gGF}(J)$. Hence

$$\text{gT}(J) \subseteq T(g(f)) \subseteq \text{gGF}(J) \cap T(h).$$

This concludes the proof. □

CHAPTER 4

Linear resolutions and componentwise linearity

The goal of this chapter is to study graded ideals with linear resolutions and componentwise linear ideals in the exterior algebra. We use an extension of the notion of linear quotients to give another proof of the well-known result that an ideal with linear quotients is componentwise linear. We also study ideals whose product has a linear resolution.

4.1. Preliminaries

In this section, we recall some facts about componentwise linear ideals and linear quotients over the exterior algebra. Componentwise linearity was defined for ideals over the polynomial ring by Herzog and Hibi in [**33**] to characterize a class of simplicial complexes, namely, sequentially Cohen-Macaulay simplicial complexes. Such ideals have received a lot of attention in several articles; see, e.g., [**4**], [**31**], [**36**], [**59**]. In this section we follow the presentation in the book of Herzog and Hibi (see [**32**, Chapter 8]) and Kämpf's dissertation (see [**37**, Section 5.3, 5.4]).

Definition 4.1.1. Let $M \in \mathcal{M}$ be a finitely generated graded E-module. Recall that M has a *d-linear resolution* if $\beta^E_{i,i+j}(M) = 0$ for all i and all $j \neq d$. Following [**32**] we call M *componentwise linear* if the submodule $M_{\langle i \rangle}$ of M generated by M_i have an i-linear resolution for all $i \in \mathbb{Z}$.

Note that a componentwise linear module which is generated in one degree has a linear resolution. A module that has a linear resolution is componentwise linear. We illustrate these properties by an example of componentwise linear ideals: stable monomial ideals. Indeed, if $J \subset E$ is a stable ideal then each $J_{\langle i \rangle}$ is also stable. By Lemma 1.3.2(iii), a stable ideal has a linear resolution if it is generated in one degree. Hence stable ideals are componentwise linear.

Next we list some properties of componentwise linear ideals used in this thesis.

Lemma 4.1.2. *Let $0 \neq J \subset E$ be a graded ideal. Then we have:*

(i) ([**4**, Theorem 2.1]). *If $|K| = \infty$ then J is componentwise linear if and only if J and* $\operatorname{gin}(J)$ *have the same graded Betti numbers (independent of* $\operatorname{char}(K)$*).*

(ii) ([**37**, Theorem 5.3.7]). *J is componentwise linear if and only if*

$$\operatorname{reg}(J_{\leq d}) \leq d \text{ for all } d \in \mathbb{Z}.$$

(iii) ([**37**, Corollary 5.3.8]). *If J is componentwise linear then*

$$\operatorname{reg}(J) = \max\{j : \beta^E_{0j}(J) \neq 0\}.$$

Lemma 4.1.3 ([**37**, Lemma 5.3.4]). *Let $0 \neq J \subset E$ be a graded ideal. If J has a d-linear resolution, then $\mathfrak{m}J$ has a $(d+1)$-linear resolution.*

Next we recall some facts about ideals with linear quotients over the exterior algebra. For more details, one can see [**37**, Section 5.4].

Definition 4.1.4. Let $J \subset E$ be a graded ideal with homogeneous generators $u_1, \dots, u_r$. If $(u_1, \dots, u_{i-1}) :_E u_i$ is generated by linear forms for $i = 1, \dots, r$ then J is said to have *linear quotients* w.r.t. $u_1, \dots, u_r$. If there exists a minimal system of homogeneous generators such that J has linear quotients w.r.t. this system then we say that J has *linear quotients*.

Remark 4.1.5. (i) For the definition of linear quotients over the exterior algebra, we need the condition that $0 :_E u_1$ has to be generated by linear forms, i.e., u_1 is a product of linear forms.

(ii) Definition 4.1.4 depends on the order of generators. For example, the ideal $J = (e_1, e_{123} + e_{345})$ has linear quotients w.r.t. the given order of the generators since $0 :_E e_1 = (e_1)$ and $(e_1) :_E (e_{123} + e_{345}) = (e_1, e_3, e_4, e_5)$. However, J does not have linear quotients w.r.t. the reversed order on the generators since $e_{123} + e_{345}$ is not a product of linear forms.

(iii) Let J be a graded ideal with linear quotients w.r.t. a minimal system of homogeneous generators $u_1, \dots, u_r$. Then

$$\deg u_i \geq \min\{\deg u_1, \dots, \deg u_{i-1}\} \text{ for } 1 \leq i \leq r.$$

Indeed, assume that $\deg u_i < \min\{\deg u_1, \dots, \deg u_{i-1}\}$. Then there exists a non-zero K-linear combination of u_j, where $j = 1, \dots, i-1$, belonging to (u_i) since $(u_1, \dots, u_{i-1}) :_E u_i$ is generated by linear forms. Hence, we can omit one u_k in $\{u_1, \dots, u_{i-1}\}$ to get a smaller system of generators. This contradicts the fact that $u_1, \dots, u_r$ is a minimal system of generators of J.

The regularity of a graded ideal with linear quotients behaves like the regularity of a stable ideal (see Lemma 1.3.2 (iii)) as we can see in the following proposition:

Proposition 4.1.6 ([**37**, Theorem 5.4.3]). *Let $J \subset E$ be a graded ideal with linear quotients w.r.t. homogeneous generators $u_1, \dots, u_r$. Then*

$$\operatorname{reg}_E(J) = \max\{\deg u_1, \dots, \deg u_r\}.$$

In particular, if J has linear quotients and J is generated in one degree, then J has a linear resolution.

4.2. Criteria and examples

The main goal of this section is to present an alternative proof of the result that graded ideals with linear quotients are componentwise linear; see [**37**, Theorem 5.4.5] for the first proof of this fact. For this, we use a so-called notion of componentwise linear quotients which is defined for monomial ideals over the polynomial ring by Jahan and Zheng in [**36**]. We also review matroidal ideals over the exterior algebra as important examples of monomial ideals with linear quotients.

Let $J \subset E$ be a graded ideal with linear quotients w.r.t. homogeneous generators $u_1, \dots, u_r$. For a permutation $\{i_1, \dots, i_r\} = [r]$, we say that the order $u_{i_1}, \dots, u_{i_r}$ of $u_1, \dots, u_r$ is *degree increasing* if $\deg u_{i_1} \le \dots \le \deg u_{i_r}$. By using exterior algebra's methods, we have the following lemmas which are similar to the ones for monomial ideals over the polynomial ring in [**36**, Lemma 2.1, 2.5]. Note that we prove them here for graded ideals.

Lemma 4.2.1. *Let $J \subset E$ be a graded ideal with linear quotients w.r.t. a minimal system of homogeneous generators $u_1, \dots, u_r$. Then J also has linear quotients w.r.t. a degree increasing order of this system.*

Proof. We prove the statement by induction on r. The case $r = 1$ is trivial. Assume $r > 1$. Observe that the ideal $(u_1, \dots, u_{r-1})$ has linear quotients w.r.t. homogeneous generators $u_1, \dots, u_{r-1}$. By the induction hypothesis, we may assume that $\deg u_1 \le \dots \le \deg u_{r-1}$.

If $\deg u_r \ge \deg u_{r-1}$ then we are done. Assume next that $\deg u_r < \deg u_{r-1}$ and let i be the smallest integer such that $\deg u_{i+1} > \deg u_r$. It is clear that $i + 1 \ne 1$ since $\deg u_1 = \min\{\deg u_1, \dots, \deg u_r\}$ by Remark 4.1.5. We now claim that J has linear quotients w.r.t. the degree increasing order $u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{r-1}$. For this, we have to prove that

$$(u_1, \dots, u_i) :_E u_r \text{ and } (u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{j-1}) :_E u_j \text{ for } j = i+1, \dots, r-1$$

are generated by linear forms.

At first, we claim that $(u_1, \dots, u_i) :_E u_r = (u_1, \dots, u_{r-1}) :_E u_r$ which is generated by linear forms since J has linear quotients w.r.t. the homogeneous elements $u_1, \dots, u_r$. The inclusion "$\subseteq$" is clear. Now let f be a linear form in $(u_1, \dots, u_{r-1}) :_E u_r$. Then $fu_r \in (u_1, \dots, u_{r-1})$. Write

$$fu_r = g + h, \text{ where } g \in (u_1, \dots, u_i) \text{ and } h \in (u_{i+1}, \dots, u_{r-1}).$$

Let $\deg u_r = d$. Then $\deg fu_r = d + 1$ and $\deg u_j \ge d + 1$ for $j = i + 1, \dots, r - 1$. If $h \ne 0$, we may assume that $\deg g = \deg h = d + 1$ since the ideals are homogeneous. This implies that h is a linear combination of some of $u_{i+1}, \dots, u_{r-1}$ and this linear combination is in $(u_1, \dots, u_i, u_r)$ since $h = fu_r - g \in (u_1, \dots, u_i, u_r)$. This contradicts the fact that $u_1, \dots, u_r$ is a minimal system of generators of J. Hence $h = 0$ and then $fu_r = g \in (u_1, \dots, u_i)$. Thus $f \in (u_1, \dots, u_i) :_E u_r$. So $(u_1, \dots, u_i) :_E u_r = (u_1, \dots, u_{r-1}) :_E u_r$ is generated by linear forms.

Next let $i + 1 \le j \le r - 1$. We claim that

$$(u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{j-1}) :_E u_j = (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}) :_E u_j$$

which is generated by linear forms by the assumption. The inclusion "$\supseteq$" is clear. Let $f \in (u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{j-1}) :_E u_j$. We have

$$fu_j = g + hu_r, \text{ where } g \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}) \text{ and } h \in E.$$

Then $fu_j - g = hu_r$. Therefore, $hu_r \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}, u_j)$ and thus

$$h \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}, u_j) :_E u_r = (u_1, \dots, u_i) :_E u_r.$$

The equality here holds since $(u_1, \dots, u_{r-1}) :_E u_r = (u_1, \dots, u_i) :_E u_r$. Hence $hu_r \in (u_1, \dots, u_i)$ and $fu_j \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1})$. So we get that

$$f \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}) :_E u_j.$$

This concludes the proof. □

Similar to Lemma 4.1.3, for ideals with linear quotients we have:

Lemma 4.2.2. *Let $J \subset E$ be a graded ideal. If J has linear quotients, then $\mathfrak{m}J$ has linear quotients.*

Proof. By Lemma 4.2.1, we may assume that J has linear quotients w.r.t. a minimal system of homogeneous generators $u_1, \dots, u_r$ and the order $u_1, \dots, u_r$ is degree increasing. We prove the assertion by induction on r.

The case $r = 0$ is trivial. Now let $r \geq 1$, consider the ordered set

$$B = \{u_1e_1, \dots, u_1e_n, u_2e_1, \dots, u_2e_n, \dots, u_re_1, \dots, u_re_n\}.$$

Then B is a system of generators of $\mathfrak{m}J$. Note that B is usually not minimal.

For $1 \leq i \leq r, 1 \leq j \leq n$, denote by

$$J_{i,j} = \mathfrak{m}(u_1, \dots, u_{i-1}) + (u_ie_1, \dots, u_ie_{j-1}),$$

$$I_{i,j} = (u_1, \dots, u_{i-1}) :_E u_i + (e_1, \dots, e_j).$$

Note that $I_{i,j}$ is generated by linear forms since $(u_1, \dots, u_{i-1}) :_E u_i$ is generated by linear forms. By removing elements u_ie_j in B from the left side to the right side of B w.r.t. the given order on B if $u_ie_j \in J_{i,j}$, we get a linearly independent set

$$B' = \{u_ie_j \in B : u_ie_j \notin J_{i,j}\}.$$

Consider the order on B' defined as follows: $u_{i_1}e_{j_1}$ comes before $u_{i_2}e_{j_2}$ if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$. By the choice of the degree increasing order and the definition of B', we see that B' is a minimal system of homogeneous generators of $\mathfrak{m}J$. We claim now that $\mathfrak{m}J$ has linear quotients w.r.t. B'.

By the induction hypothesis, we have that $\mathfrak{m}(u_1, \dots, u_{r-1})$ has linear quotients w.r.t. the following system of generators

$$B'' = \{u_ie_j \in B' : i = 1, \dots, r-1\} \subset B'.$$

Thus $J_{i,j} :_E u_ie_j$ is generated by linear forms for $i < r$ and $u_ie_j \in B'$. Hence we only need to prove that $J_{r,j} :_E u_re_j$ is generated by linear forms for $u_re_j \in B'$. For this, we claim that $J_{r,j} :_E u_re_j = I_{r,j}$.

Let $f = g + h \in I_{r,j}$, where $h \in (e_1, \dots, e_j)$ and $g \in (u_1, \dots, u_{r-1}) :_E u_r$. Then $h(u_re_j) \in (u_re_1, \dots, u_re_{j-1}) \subseteq J_{r,j}$. We see also that

$$g(u_re_j) = \pm e_j(gu_r) \in \mathfrak{m}(u_1, \dots, u_{r-1}) \subseteq J_{r,j}.$$

So we get $I_{r,j} \subseteq J_{r,j} :_E u_re_j$.

Next let $f \in J_{r,j} :_E u_re_j$. Then $f(u_re_j) \in J_{r,j}$. Hence $fe_j \in J_{r,j} :_E u_r$. To ensure that $f \in I_{r,j}$ we prove the following:

(i) $J_{r,j} :_E u_r \subseteq I_{r,j-1}$,
(ii) $I_{r,j-1} :_E e_j = I_{r,j}$.

For (i), let $g \in J_{r,j} :_E u_r$. Then $gu_r = h_1 + h_2 u_r \in J_{r,j}$, where $h_1 \in \mathfrak{m}(u_1, \dots, u_{r-1})$ and $h_2 \in (e_1, \dots, e_{j-1})$. This implies that $(g - h_2)u_r \in (u_1, \dots, u_{r-1})$. Thus $g - h_2 \in (u_1, \dots, u_{r-1}) :_E u_r$. So we get $g \in I_{r,j-1}$ since $h_2 \in (e_1, \dots, e_{j-1})$. Thus $J_{r,j} :_E u_r \subseteq I_{r,j-1}$.

For (ii), we claim that $e_j \notin I_{r,j-1}$. Indeed, if $e_j \in I_{r,j-1}$, then

$$e_j u_r \in (u_1, \dots, u_{r-1}) + (e_1, \dots, e_{j-1})u_r.$$

It follows that

$$e_j u_r \in \mathfrak{m}(u_1, \dots, u_{r-1}) + (e_1, \dots, e_{j-1})u_r = J_{r,j}$$

since $\deg e_j u_r \geq \deg u_i + 1$ for $i = 1, \dots, r-1$. This contradicts the fact that $e_j u_r \notin J_{r,j}$ because of the choice of B'. Since $I_{r,j-1}$ is generated by linear forms and $e_j \notin I_{r,j-1}$, we get $I_{r,j-1} :_E e_j = I_{r,j}$. This concludes the proof. $\square$

Remark 4.2.3. (i) The converse of the above lemma is not true. For instance, consider the ideal $J = (e_{12}, e_{34}) \subset K\langle e_1, e_2, e_3, e_4\rangle$. Then $\mathfrak{m}J = (e_{123}, e_{124}, e_{134}, e_{234})$ has linear quotients w.r.t. the given order of monomial generators. But J does not have linear quotients since J is generated in degree 2 and J does not have a 2-linear resolution (see Proposition 4.1.6).

(ii) We cannot replace $\mathfrak{m}$ in the above lemma by a subset of variables. So we see that the product of two graded ideals with linear quotients might not have linear quotients. For example, let us recall the example from Remark 2.3.5: let $J = (e_{123}, e_{134}, e_{125}, e_{256})$. We can check that J has linear quotients w.r.t. the given order of monomial generators but $P = (e_1, e_2)J = (e_{1234}, e_{1256})$ does not have linear quotients by Proposition 4.1.6 because of the fact that P is generated in one degree and it does not have a linear resolution.

Next we prove the main result of this section. For this, we say that a graded ideal $J \subset E$ has *componentwise linear quotients* if all component ideals $0 \neq J_{\langle d\rangle}$ have linear quotients.

Theorem 4.2.4. *Let $J \subset E$ be a graded ideal. If J has linear quotients, then J has componentwise linear quotients.*

Proof. Assume that J has linear quotients w.r.t. a minimal system of homogeneous generators $u_1, \dots, u_m$. By Lemma 4.2.1, we may assume further that the given order on this system is degree increasing, i.e., $\deg(u_1) \leq \ldots \leq \deg(u_m)$. Let $d = \deg(u_1)$, $d + t = \deg(u_m)$ and $1 = r_{d-1} \leq r_d < r_{d+1} < \ldots < r_{d+t} = m$ such that $\deg(u_j) = d + i$ if $r_{d+i-1} + 1 \leq j \leq r_{d+i}$ for $i = 0, \dots, t$. Observe that for $i > t$, $J_{d+i} = \mathfrak{m}^{i-t} J_{\langle d+t\rangle}$. By Lemma 4.2.2, we only need to prove that $J_{\langle d+i\rangle}$ has linear quotients for $i = 0, \dots, t$. We prove this by induction on i.

It is clear that $J_{\langle d\rangle} = (u_1, \dots, u_{r_d})$ and $J_{\langle d\rangle}$ has linear quotients w.r.t. the minimal system of homogeneous generators $u_1, \dots, u_{r_1}$. So the case $i = 0$ is true.

Assume $i > 0$. By the induction hypothesis we have that $J_{\langle d+i-1\rangle}$ has linear quotients. Then by Lemma 4.2.2, the ideal $\mathfrak{m}J_{\langle d+i-1\rangle}$ has linear quotient w.r.t. a minimal system of homogeneous generators, say $w_1, \dots, w_s$. For simplicity, write

$p = r_{d+i-1}$ and $v_1, \dots, v_q$ instead of $u_{p+1}, \dots, u_{r_{d+i}}$ where $q = r_{d+i} - r_{d+i-1}$. Then

$$J_{\langle d+i \rangle} = \mathfrak{m} J_{\langle d+i-1 \rangle} + (v_1, \dots, v_q) = (w_1, \dots, w_s, v_1, \dots, v_q).$$

Observe that $w_1, \dots, w_s, v_1, \dots, v_q$ is a minimal system of homogeneous generators of $J_{\langle d+i \rangle}$. We claim that $J_{\langle d+i \rangle}$ has linear quotients w.r.t. this system. For this, we only need to check that $(w_1, \dots, w_s, v_1, \dots, v_{j-1}) :_E v_j$ is generated by linear forms for $1 \le j \le q$. Indeed, we claim that

$$(w_1, \dots, w_s, v_1, \dots, v_{j-1}) :_E v_j = (u_1, \dots, u_p, v_1, \dots, v_{j-1}) :_E v_j, \tag{4}$$

which is generated by linear forms since J has linear quotients w.r.t. the homogeneous generators $u_1, \dots, u_m$.

The inclusion "$\subseteq$" is clear. Next let $f \in (u_1, \dots, u_p, v_1, \dots, v_{j-1}) :_E v_j$, then $f v_j \in (u_1, \dots, u_p, v_1, \dots, v_{j-1})$. Write $f v_j = g + h$, where $g \in (u_1, \dots, u_p)$ and $h \in (v_1, \dots, v_{j-1})$. Since $\deg f v_j \ge d + i + 1$, we have that $\deg g \ge d + i + 1$. Moreover, $\deg u_k \le d + i - 1$ for $k = 1, \dots, p$. This implies that

$$g \in (u_1, \dots, u_p)_{\ge d+i+1} \subseteq \mathfrak{m} J_{\langle d+i-1 \rangle} = (w_1, \dots, w_s).$$

Hence

$$f v_i \in (w_1, \dots, w_s, v_1, \dots, v_{i-1}) \text{ and thus } f \in (w_1, \dots, w_s, v_1, \dots, v_{i-1}) :_E v_i.$$

This concludes the proof. □

Next by using Theorem 4.2.4, we present an alternative proof for the result of Kämpf in [**37**, Theorem 5.4.5]. Note that this result is analogous to a result over the polynomial ring of Sharifan and Varbaro in [**59**, Corollary 2.4].

Corollary 4.2.5. *If $J \subset E$ is a graded ideal with linear quotients, then J is componentwise linear.*

Proof. By Theorem 4.2.4, J has componentwise linear quotients. It follows that every component ideal $0 \ne J_{\langle d \rangle}$ of J has linear quotients. Thus $J_{\langle d \rangle}$ has a linear resolution by Proposition 4.1.6. This concludes the proof. □

Definition 4.2.6. Let J be a graded ideal with componentwise linear quotients. We say that J has *strongly componentwise linear quotients* if at each degree d, the component ideal $0 \ne J_{\langle d \rangle}$ has linear quotients w.r.t. a minimal system of homogeneous generators $w_1, \dots, w_s, v_1, \dots, v_q$ such that $w_1, \dots, w_s$ is a minimal system of homogeneous generators of $\mathfrak{m} J_{\langle d-1 \rangle}$ and $\mathfrak{m} J_{\langle d-1 \rangle}$ has linear quotients w.r.t. this system.

The converse of Theorem 4.2.4 is still unknown. However, we can prove the following:

Proposition 4.2.7. *Let $J \subset E$ be a graded ideal with strongly componentwise linear quotients. Then J has linear quotients.*

Proof. Let d be the initial degree of J and $d+t$ be the maximal degree of minimal generators of J. We prove the statement by induction on t.

The case $t = 0$ is trivial since $J_{\langle d \rangle}$ has linear quotients and J is generated in one degree, i.e., $J = J_{\langle d \rangle}$.

Assume $t \geq 1$. Recall that we denote by $J_{\leq k} = (f \in J : \deg f \leq k)$. Since J has componentwise linear quotients, we have that $J_{\leq d+t-1}$ has componentwise linear quotients. By the induction hypothesis, we get that $J_{\leq d+t-1}$ has linear quotients w.r.t. a minimal system of homogeneous generators, say $u_1, \ldots, u_p$. Since J has strongly componentwise linear quotients, $J_{\langle d+t \rangle}$ has linear quotients w.r.t. a minimal system of homogeneous generators, say $w_1, \ldots, w_s, v_1, \ldots, v_q$, such that $w_1, \ldots, w_s$ is a minimal system of homogeneous generators of $\mathfrak{m}J_{\langle d+t-1 \rangle}$ and $\mathfrak{m}J_{\langle d+t-1 \rangle}$ has linear quotients w.r.t. this system. Observe that $u_1, \ldots, u_p, v_1, \ldots, v_q$ is a minimal system of homogeneous generators of J. We claim that J has linear quotients w.r.t. this system. Indeed, we only need to prove that for $j = 1, \ldots, q$ we have

$$(u_1, \ldots, u_p, v_1, \ldots, v_{j-1}) :_E v_j = (w_1, \ldots, w_s, v_1, \ldots, v_{j-1}) :_E v_j,$$

which is generated by linear forms. This follows from the same argument as in the proof of Theorem 4.2.4. More precisely, this is exactly the equation (4). So J has linear quotients. □

To conclude this section, we present a class of monomial ideals with linear quotients, which will be used in the next section.

Example 4.2.8. A monomial ideal $J \subset E$ is said to be *matroidal* if it is generated in one degree and if it satisfies the following exchange property:

For all $u, v \in G(J)$, and all i with $i \in \operatorname{supp}(u) \setminus \operatorname{supp}(v)$, there exists an integer j with $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u)$ such that $(u/e_i)e_j \in G(J)$.

Analogously to the polynomial rings case one sees that matroidal ideals have linear quotients. Thus a matroidal ideal is a componentwise linear ideal generated in one degree and has a linear resolution. For the convenience of the reader we present a proof of this property (over the exterior algebra) following the corresponding proof in [**15**, Proposition 5.2] for polynomial rings.

Proof. Let $J \subset E$ be a matroidal ideal. We claim that J has linear quotients with respect to the reverse lexicographical order of the generators.

Let $u \in G(J)$ and let J_u be the ideal generated by all $v \in G(J)$ with $v > u$ in the reverse lexicographical order. Denote by $\gcd(u, v)$ the greatest common divisor of u and v. Then we get

$$J_u :_E u = (v/\gcd(u, v) : v \in J_u) + \operatorname{ann}(u).$$

We need to prove that $J_u :_E u$ is generated by linear forms. Note that $\operatorname{ann}(u)$ is generated by variables appearing in u. So we only need to check that for each $v \in G(J)$ and $v > u$, there exists a variable $e_j \in J_u :_E u$ such that e_j divides $v/[v, u]$.

Let $u = e_1^{a_1} \ldots e_n^{a_n}$ and $v = e_1^{b_1} \ldots e_n^{b_n}$, where $0 \leq a_i, b_j \leq 1$ and $\deg u = \deg v$. Since $v > u$, there exists an integer i such that $a_i > b_i$ and $a_k = b_k$ for $k = i+1, \ldots, n$. Moreover, J is a matroidal ideal and $i \in \operatorname{supp}(u) \setminus \operatorname{supp}(v)$. Hence there exists an integer j such that $b_j > a_j$ and $u' = e_j(u/e_i) \in G(J)$ since $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u)$. Then $ue_j = u'e_i$. Since $j < i$, we get $u' > u$ and $u' \in J_u$. Hence $e_j \in J_u :_E u$. Moreover, $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u) = \operatorname{supp}(v/\gcd(u, v))$, so e_j divides $v/\gcd(u, v)$. This concludes the proof. □

4.3. Linear resolution of products of ideals

Motivated by a result of Conca and Herzog in [**15**] that a product of linear ideals over the polynomial ring has a linear resolution, we study in this section the following related problem:

Question 4.3.1. Let $J_1, \dots, J_d \subseteq E$ be linear ideals. Is it true that the product $J = J_1 \dots J_d$ has a linear resolution?

At first, by modifying methods of Conca and Herzog in [**15**] for the exterior algebra, we get a positive answer for the above question in the case that J_i is generated by variables for $i = 1, \dots, d$.

Theorem 4.3.2. *A product of linear ideals which are generated by variables has a linear free resolution.*

Proof. Let $J_1, \dots, J_d \subseteq E$ be linear ideals generated by variables and $J = J_1 \dots J_d$. If $J = 0$, then the statement is trivial. We prove the statement for $J \neq 0$ by two ways. One uses properties of matroidal ideals and the other is a more conceptual proof.
First proof: Recall that a monomial ideal J is matroidal if it is generated in one degree such that for all $u, v \in G(J)$, and all i with $i \in \operatorname{supp}(u) \setminus \operatorname{supp}(v)$, there exists an integer j with $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u)$ such that $(u/e_i)e_j \in G(J)$. For the convenience of the reader, we present next the fact (following the proof of Conca and Herzog [**15**] in the polynomial ring case) that a product of two matroidal ideals over the exterior algebra is also a matroidal ideal. In fact, let I, J be matroidal ideals, $u, u_1 \in G(I)$ and $v, v_1 \in G(J)$ such that $uv, u_1v_1 \neq 0$ and $uv, u_1v_1 \in G(IJ)$. Let $i \in \operatorname{supp}(u_1v_1) \setminus \operatorname{supp}(uv)$. We claim that there exists an integer $j \in \operatorname{supp}(uv) \setminus \operatorname{supp}(u_1v_1)$ such that $(u_1v_1/e_i)e_j \in G(IJ)$.

Since $\operatorname{supp}(u_1v_1) = \operatorname{supp}(u_1) \cup \operatorname{supp}(v_1)$, without loss of generality, we may assume that $i \in \operatorname{supp}(u_1)$. Then $i \in \operatorname{supp}(u_1) \setminus \operatorname{supp}(u)$. Since I is a matroidal ideal, there exists $j_1 \in \operatorname{supp}(u) \setminus \operatorname{supp}(u_1)$ such that $u_2 = (u_1/e_i)e_{j_1} \in G(I)$. Then there are two cases:

Case 1: If $j_1 \notin \operatorname{supp}(v_1)$, then

$$j_1 \in \operatorname{supp}(uv) \setminus \operatorname{supp}(u_1v_1) \text{ and } 0 \neq (u_1v_1/e_i)e_{j_1} = u_2v_1 \in G(IJ).$$

So we can choose $j = j_1$.

Case 2: If $j_1 \in \operatorname{supp}(v_1)$, then $j_1 \notin \operatorname{supp}(v)$ because $j_1 \in \operatorname{supp}(u)$ and $uv \neq 0$. So $j_1 \in \operatorname{supp}(v_1) \setminus \operatorname{supp}(v)$. Since J is matroidal, there exists $k_1 \in \operatorname{supp}(v) \setminus \operatorname{supp}(v_1)$ such that $v_2 = (v_1/e_{j_1})e_{k_1} \in G(J)$. Note that $k_1 \neq i$ since $i \notin \operatorname{supp}(v)$ but $k_1 \in \operatorname{supp}(v)$.

If $k_1 \notin \operatorname{supp}(u_2) \setminus \operatorname{supp}(u)$, then $k_1 \notin \operatorname{supp}(u_1)$ since $u_2 = (u_1/e_i)e_{j_1}$. We get

$$k_1 \in \operatorname{supp}(uv) \setminus \operatorname{supp}(u_1v_1)$$

and

$$0 \neq (u_1v_1/e_i)e_{k_1} = (u_1/e_i)e_{j_1}(v_1/e_{j_1})e_{k_1} = u_2v_2 \in G(IJ).$$

So we are done because we can choose $j = k_1$.

Otherwise $k_1 \in \operatorname{supp}(u_2) \setminus \operatorname{supp}(u)$. Since I is matroidal, there exists j_2 such that

$$j_2 \in \operatorname{supp}(u) \setminus \operatorname{supp}(u_2) \text{ with } 0 \neq u_3 = (u_2/e_{k_1})e_{j_2} \in G(I).$$

Observe that $j_2 \neq i$ since $j_2 \in \operatorname{supp}(u)$ and $i \notin \operatorname{supp}(u)$. Then we get

$$0 \neq (u_1v_1/e_i)e_{j_2} = ((u_1/e_i)e_{j_1}/e_{k_1})e_{j_2}(v_1/e_{j_1})e_{k_1} = u_3v_2 \in G(IJ)$$

and we can choose $j = j_2$. Hence the product of two matroidal ideals is also matroidal.

It is obvious that J_i is a matroidal ideal for $i = 1, \dots, d$. Thus by induction on d we see that J is also a matroidal ideal. So J has a linear resolution by the fact a matroidal ideal has a linear resolution; see Example 4.2.8.

Second proof: Let I_i be the square free monomial ideals in the polynomial ring $S = K[x_1, \dots, x_n]$ corresponding to J_i. Then $I = I_1 \dots I_d$ has a linear resolution by the result of Conca and Herzog [**15**, Theorem 3.1]. Let $I_{[d]}$ be the ideal generated by the squarefree monomials of degree d belonging to I. Then $I_{[d]}$ has a linear resolution by [**32**, Proposition 8.2.17]. Hence, J has a linear resolution by a result of Aramova, Avramov and Herzog ([**1**, Corollary 2.2]). □

Considering a product of two linear ideals, we have:

Corollary 4.3.3. *Let I, J be linear ideals such that $IJ \neq 0$. Then IJ has a 2-linear free resolution.*

Proof. Since I, J are linear ideals, we can assume that $I + J = \mathfrak{m}$, otherwise I, J are in a smaller exterior algebra, in which we can modulo a regular sequence to get $I + J = \mathfrak{m}$. By changing the coordinate and choosing suitable generators, we may assume further that

$$I = (e_1, \dots, e_r, \dots, e_s) \text{ and } J = (e_r, \dots, e_s, \dots, e_n),$$

where $1 \leq r \leq s \leq n$ and $I_1 \cap J_1 = \operatorname{span}_K\{e_r, \dots, e_s\}$. Then I, J are linear ideals generated by variables. By Proposition 4.3.2, we get that IJ has a 2-linear free resolution. □

Next we consider one more special case of products of ideals: powers of ideals. In [**35**], Herzog, Hibi and Zheng proved that if a monomial ideal I in the polynomial ring S has a 2-linear resolution, then every power of I has a linear resolution. We have the same result for the exterior algebra:

Proposition 4.3.4. *Let $J \subset E$ be a non-zero monomial ideal in E. If J has a 2-linear resolution, then every power of J has a linear resolution.*

Proof. Let $I \subset S$ be the ideal in the polynomial ring S corresponding to J. Then I is a squarefree ideal with a 2-linear resolution by [**1**, Corollary 2.2]. We only need to consider the case $J^m \neq 0$ for an integer m. We have that I^m has a linear resolution by [**35**, Theorem 3.2]. By [**32**, Proposition 8.2.17], the squarefree monomial ideal $(I^m)_{[2m]}$ has also a linear resolution. Note that $(I^m)_{[2m]}$ corresponds to J^m in E, so using [**1**, Corollary 2.2] again, we conclude that J^m has a linear resolution. □

Remark 4.3.5. Observe the following:

(i) A linear form f is E/J-regular but it might not be E/J^2-regular. This is a difference between the polynomial ring and the exterior algebra. For instance, let $J = (e_{12} + e_{34}, e_{13}, e_{23})$. Then e_4 is E/J-regular since $J :_E e_4 = J + (e_4)$. But e_4 is not E/J^2-regular since

$$J^2 = (e_{1234}) \text{ and } J^2 :_E (e_4) = (e_{123}) + (e_4) \supsetneq J^2 + (e_4).$$

(ii) Let $J \subset E$ be a monomial ideal such that J and $(E/J)^*$ have linear projective resolutions. Then J reduces to a power of the maximal ideal modulo some maximal E/J-regular sequence of linear forms of E by [**22**, Theorem 3.4]. But it might not hold for powers of J. This can be seen in Example 4.3.6.

Example 4.3.6. Let $J = (e_{12} + e_{34}, e_{12} + e_{35}, e_{23}, e_{24}, e_{25}, e_{45}) \subset K\langle e_1, \dots, e_5\rangle$. Then e_1 is E/J-regular and J reduces to $(e_2, \dots, e_5)^2$ modulo e_1. By [**22**, Theorem 3.4], we have that J and $(E/J)^*$ have linear projective resolutions. However,

$$J^2 = (e_{1234}, e_{1235}, e_{1245}, e_{2345})$$

has a linear resolution while $(E/J^2)^*$ does not have a linear resolution because it is not generated in one degree since $(E/J^2)^* \cong \operatorname{ann}(J^2) = (e_2, e_{34}, e_{35}, e_{45})$. Note that we use Macaulay2 [**28**] here to compute $\operatorname{ann}(J^2)$.

For the case of monomial ideals, we have:

Proposition 4.3.7. *Let $J \subset E$ be a monomial ideal with* $\operatorname{depth}(E/J) = 0$ *such that J and $(E/J)^*$ have linear projective resolutions. Then J^m and $(E/J^m)^*$ have linear projective resolutions for every power $0 \neq J^m$ of J.*

Proof. Since J and $(E/J)^*$ have linear projective resolutions, J reduces to a power of the maximal ideal modulo some maximal E/J-regular sequence of linear forms by [**22**, Theorem 3.4]. Moreover, $\operatorname{depth}(E/J) = 0$. Therefore, J itself must be a power of the maximal ideal. So does every power of J. Thus J^m and $(E/J^m)^*$ have linear projective resolutions for every power $0 \neq J^m$ of J by [**22**, Theorem 3.4]. □

4.4. Discussion for the general case

We discuss in this section some tools which could be useful to study Question 4.3.1 in the general case. By using properties of strongly generic bases and adding some further assumptions, we prove results on products of linear ideals in the exterior algebra which are similar to known results of Conca and Herzog [**15**] in the polynomial ring case. The base field K is considered in this section with $\operatorname{char}(K) = 0$.

Let $I_1, \dots, I_d \subset E$ be linear ideals such that $J = I_1 \cdots I_d \neq 0$. Denote by

$$J_i = I_1 \cdots I_{i-1} I_{i+1} \cdots I_d \text{ for } i = 1, \dots, d.$$

Definition 4.4.1. Let $v \in E_1$ be a strongly generic element for the E-module E/J. The element v is said to be a *strongly stable generic* element for E/J if

$$(vJ_1 \cap vJ_2 \cap \cdots \cap vJ_d)_d \subseteq J.$$

In many computations, the notions of strongly generic element and strongly stable generic element were simultaneously satisfied. Thus, we propose the following:

Question 4.4.2. Let J be a non-zero product of linear ideals in E and v a strongly generic element for the E-module E/J. Then v is also a strongly stable generic element for E/J.

Remark 4.4.3. In the special case that $I_1, \dots, I_d$ are principal linear ideals, we get a positive answer for Question 4.4.2. Indeed, the following stronger statement is true:

Let $I_1, \dots, I_d$ be linear ideals and $J = I_1 \dots I_d \neq 0$. Let v be a linear form in $I_1 + \cdots + I_d$ and $f \in vJ_1 \cap \cdots \cap vJ_d$ with $\deg f = d$ such that

$$0 \neq f = vu_{12}u_{13} \cdots u_{1d} = vu_{21}u_{23} \cdots u_{2d} = \cdots = vu_{d1} \cdots u_{d(d-1)},$$

where u_{ij} are linear forms in I_j for $i, j = 1, \dots, d$ and $i \neq j$. Then $f \in J$.

Note that this statement is stronger than the first claim above since if $I_1, \dots, I_d$ are principal linear ideals and $f \in vJ_i$ with $\deg f = d$ then $f = \alpha v u_1 \dots \widehat{u_i} \dots u_d$ where $\alpha \in K$ and $u_i \in E_1$ such that $I_i = (u_i)$ for $i = 1, \dots, d$.

Proof. At first, we claim the following:

Let $\{u_1, \dots, u_d\}$ and $\{v_1, \dots, v_d\}$ be sets of linearly independent elements in E_1 where $1 \leq d \leq n$. Then the following statements are equivalent:

(i) $\alpha u_1 \cdots u_d = v_1 \cdots v_d$, where $0 \neq \alpha \in K$;

(ii) $\operatorname{span}_K\{u_1, \dots, u_d\} = \operatorname{span}_K\{v_1, \dots, v_d\}$.

In fact, (ii) $\Rightarrow$ (i) is clear. For (i) $\Rightarrow$ (ii), we assume that $u_i \notin \operatorname{span}_K\{v_1, \dots, v_d\}$ for some $1 \leq i \leq d$. Then the set $\{u_i, v_1, \dots, v_d\}$ is linearly independent. Note that a product of linearly independent elements in E_1 is non-zero. Thus $u_i v_1 \cdots v_d \neq 0$. So $u_i \alpha u_1 \cdots u_d \neq 0$. This contradicts the fact that $u_i u_1 \cdots u_d = 0$.

For a linear form $u = \sum_{i=1}^n \alpha_i e_i \in E_1$, we set $S(u) = \{i : \alpha_i \neq 0\}$. If C is a set of linear forms, then we write $S(C) = \bigcup_{u \in C} S(u)$.

Without loss of generality, we may assume that $v = e_1$ and $u_{1j} = e_j$ for $j = 2, \dots, d$. Let $T = \{1, 2, \dots, d\}$. Then $f = e_T$. By the equivalent statements from above, we have $u_{ij} \in \operatorname{span}_K\{e_1, \dots, e_d\}$ and $u_{ij} \in I_j$ for $1 \leq i, j \leq d$ and $i \neq j$.

For a set of indices $F \subseteq [d]$, we define $C(F) = \bigcup_{i \in F, j \neq i} \operatorname{supp}(u_{ji})$. Put $F_0 = \{1\}$. Let $C_1 = C(F_0)$ and $F_1 = S(C_1)$.

If $1 \notin F_1$, we define $C_2 = C(F_1 \cup \{1\})$ and $F_2 = S(C_2)$. Continuing in this way, whenever $1 \notin F_i$ we define $C_{i+1} = C(F_i \cup \{1\})$ and $F_{i+1} = S(C_{i+1})$ for $i = 1, 2, \dots$ until $1 \in F_i$. Observe that

$$C_1 \subseteq C_2 \subseteq \cdots \text{ and } F_1 \subseteq F_2 \subseteq \cdots .$$

Since $F_i \subset T$ for all $i = 1, 2, \dots$, there exists an integer $1 \leq k$ such that the process ends at F_k, i.e., $1 \in F_k$, or $F_k = F_{k+1}$ and in this case we also have that $C_k = C_{k+1}$. Write $F = F_k$ and $C = C_k$. Then we have $C = C(F \cup \{1\})$ and $F = S(C)$.

There are two possibilities for F:

Case 1: $1 \in F$. By the definition of F, there exist $j_{k-1} \in F_{k-1} \setminus F_{k-2}$ and $i_{k-1} \neq j_{k-1}$ such that $\alpha_{k-1} e_1$ is a non-zero term of $u_{i_{k-1} j_{k-1}}$ with $0 \neq \alpha_{k-1} \in K$.

Observe that

$$f = e_T = \alpha_{k-1}^{-1} u_{i_{k-1}j_{k-1}} e_{T\setminus\{1\}}.$$

Since $j_{k-1} \in F_{k-1} \setminus F_{k-2}$, there exists $j_{k-2} \in F_{k-2} \setminus F_{k-3}$ and $i_{k-2} \neq j_{k-2}$ such that $\alpha_{k-2} e_{j_{k-1}}$ is a non-zero term of $u_{i_{k-2}j_{k-2}}$ with $0 \neq \alpha_{k-2} \in K$. Then we have

$$f = e_T = \alpha_{k-1}^{-1}\alpha_{k-2}^{-1} u_{i_{k-1}j_{k-1}} u_{i_{k-2}j_{k-2}} e_{T\setminus\{1,j_{k-1}\}}.$$

Continuing with this process we get that there exist $0 \neq \alpha_t \in K$ and $j_t \in F_t \setminus F_{t-1}$ and $i_t \neq j_t$ for $t = 0, \ldots, k$ such that $\alpha_t e_{j_{t+1}}$ is a non-zero term of $u_{i_t j_t}$. Note that here $j_0 = 1$ and we set $F_{-1} = \emptyset$. Therefore, we have an expression of f as the product:

$$f = \alpha_{k-1}^{-1} \cdots \alpha_0^{-1} u_{i_{k-1}j_{k-1}} \cdots u_{i_1 j_1} u_{i_0 1} e_{T\setminus\{1,j_{k-1},\ldots,j_1\}}.$$

Since $j_{k-1}, \ldots, j_1$ are pairwise distinct, $e_i \in I_i$ and $u_{i_t j_t} \in I_{j_t}$, where $t = 0, \ldots, k-1$ and $1 \leq i \leq d$, we get that $f \in J$. Hence the statement holds.

Case 2: $1 \notin F$. Set $G = T \setminus (F \cup \{1\})$ and $l = |F|$. We claim that $G = \emptyset$. In fact, if $G \neq \emptyset$ then since $F = S(C(F \cup \{1\}))$ we get that $\{u_{ij} : j \in F \cup \{1\}\}$ is a set of $l+1$ linear forms in $K\langle e_t : t \in F\rangle$ for $i \in G$. Hence $\prod_{j\in F\cup\{1\}} u_{ij} = 0$ for $i \in G$. This is impossible because $f \neq 0$. Hence $G = \emptyset$, i.e., $F = \{2, \ldots, d\}$. This means $e_1 \notin \operatorname{supp}(u_{ij})$ for all u_{ij}.

Observe that $\{u_{i1}, \ldots, u_{i(i-1)}, u_{i(i+1)}, \ldots, u_{id}\}$ is a linearly independent set of $(d-1)$ linear forms in $K\langle e_2, \ldots, e_d\rangle$ for $i = 1, \ldots, d$. By the equivalent statements from the beginning of the proof, we get that

$$0 \neq g_i = \prod_{j\neq i} u_{ij} = a_i e_F \text{ with } a_i \in K \setminus \{0\} \text{ for all } i = 1, \ldots, d.$$

Since $e_1 = v \in I_1 + \cdots + I_d$, we can write $e_1 = v_1 + \cdots + v_d$ with $v_i \in I_i$ for $i = 1, \ldots, d$. Note that $e_F = a_i^{-1} g_i \in J_i$ for $i = 1, \ldots, d$. Hence $f = e_1 e_F = \sum_{i=1}^d v_i e_F \in \sum_{i=1}^d I_i J_i = J$. This concludes the proof. □

Next we claim that if the answer of Question 4.4.2 is positive, i.e., every strongly generic element is strongly stable generic, then following analogously methods of Conca and Herzog [**15**] in the exterior algebra settings we can prove that a non-zero product of linear ideals in E has a linear resolution. This can seen by the following results:

Lemma 4.4.4. *Let $0 \neq I_1, \ldots, I_d \subset E$ be linear ideals such that $I_k \subsetneq \mathfrak{m}$ for $k = 1, \ldots, d$. Then after a suitable field extension L/K, there exists a strongly generic element v for any E'-module of the form $M' = L \otimes_K M$ such that $v \notin I'_k = L \otimes_K I_k$ for $k = 1, \ldots, d$, where $E' = L \otimes_K E$ and $M \in \mathcal{M}$.*

Proof. Let L_1/K be a field extension containing algebraically independent elements b_{ij} over K where $i, j = 1, \ldots, n$. Let $u_j = \sum_{i=1}^n b_{ij} e_i$ for $j = 1, \ldots, n$. We denote by $I_k^* = L_1 \otimes_K I_k$ for $k = 1, \ldots, d$. Since $I_k \subsetneq \mathfrak{m}$, we note that $\bigcup_{k=1}^d I_k \subsetneq \mathfrak{m}$. Thus $\bigcup_{k=1}^d I_k^* \subsetneq \mathfrak{m}^*$ where $\mathfrak{m}^* = L_1 \otimes_K \mathfrak{m}$. Hence there exists $1 \leq j \leq n$ such that $u_j \notin \bigcup_{k=1}^d I_k^*$. Without loss of generality, we may assume that $u_1 \notin \bigcup_{k=1}^d I_k^*$.

Next we use again the second field extension as in the proof of Lemma 2.2.3. More precisely, let L/L_1 be a field extension containing the algebraically independent elements c_{ij} over L_1 where $i, j = 2, \ldots, n$. Let $w_j = \sum_{i=2}^{n} c_{ij} u_i$ for $j = 2, \ldots, n$ and choose $v = u_1$. By Lemma 2.2.3 and Proposition 2.2.4, we get that $v, w_2, \ldots, w_n$ is a strongly generic basis of E'_1 for any E'-module $M' = L \otimes_K M$ where $E' = L \otimes_K E$ and $M \in \mathcal{M}$. In particular, v is a strongly generic element for M'. Moreover, since $u_1 \notin \bigcup_{k=1}^{d} I_k^*$, we also have $v = u_1 \notin L \otimes_{L_1} I_k^* = L \otimes_K I_k = I'_k$ for $k = 1, \ldots, d$, as desired. □

Lemma 4.4.5. *Let $I_1, \ldots, I_d \subset E$ be linear ideals with $J = I_1 \cdots I_d \neq 0$. Let $I_A = \sum_{j \in A} I_j$ and $P_A = \prod_{j \in A} I_j$ where A runs over the subsets of $[d]$. If every strongly generic element for the E-modules E/P_A is strongly stable generic, then*

$$J = \bigcap_{A} I_A^{|A|}.$$

Note that then also $J = \mathfrak{m}^d \cap \bigcap_{A, I_A \neq \mathfrak{m}} I_A^{|A|}$.

Recall that, for an E-module $N \in \mathcal{M}$, we denote $s(N) = \max\{i : N_i \neq 0\}$ if $N \neq 0$ and $s(0) = -\infty$. Let $0 \neq J \subset E$ be a graded ideal and $v \notin J$ be a strongly generic element for the E-module E/J. Note that by Lemma 4.4.4, there always exists such the element v after a suitable field extension. We set $\operatorname{sat}(J, v) = s(\frac{J :_E v}{J + (v)}) + 1$ for short. Using Lemma 4.4.5, we have:

Corollary 4.4.6. *Let $I_1, \ldots, I_d \subset E$ be linear ideals with $J = I_1 \ldots I_d \neq 0$ and $d \in \mathbb{N}$. Let $P_A = \prod_{j \in A} I_j$ where A runs over the subsets of $[d]$. If every strongly generic element for the E-modules E/P_A is strongly stable generic, then after a suitable field extension there exists a strongly generic element $v \in E_1$ for the E-modules E/P_A such that* $\operatorname{sat}(J, v) \leq d$.

Proof of 4.4.6. Let $I_A = \sum_{j \in A} I_j$ where A runs over the subsets of $[d]$. Considering the linear ideals I_A and the E-modules E/P_A applying Lemma 4.4.4 we may assume (after a suitable field extension) that there exists a strongly generic element $v \in E_1$ for all E-modules E/P_A such that $v \notin I_A$ if $I_A \neq \mathfrak{m}$ where A runs over the subsets of $[d]$. Then for each linear ideal I_A with $v \notin I_A$, we note that v is E/I_A^k-regular for all $k > 0$. Thus for these ideals we have $I_A^k :_E v = I_A^k + (v)$.

We may assume that $I_{[d]} = I_1 + \cdots + I_d = \mathfrak{m}$ because otherwise we have $v \notin I_A$. Then v is E/J-regular. Thus $J :_E v = J + (v)$. Hence $\operatorname{sat}(J, v) < d$ and this proves the statement in this case.

Let $f \in J :_E v$ with $\deg f \geq d$. We need to prove that $f \in J + (v)$. By Lemma 4.4.5 we have $J = \bigcap_A I_A^{|A|}$. Thus $f \in I_A^{|A|} :_E v = I_A^{|A|} + (v)$ for $A \subset [d]$ such that $I_A \neq \mathfrak{m}$. Hence

$$f \in \mathfrak{m}^d \cap \bigcap_{A, I_A \neq \mathfrak{m}} (I_A^{|A|} + (v)). \tag{5}$$

Without loss of generality we may assume that $E = K\langle v, e_2, \ldots, e_n \rangle$. Let $\tilde{\mathfrak{m}}, \tilde{J}, \tilde{I_A}, \tilde{P_A}, \tilde{f}$ be the images of $\mathfrak{m}, J, I_A, P_A, f$ in $E/(v) = \tilde{E}$, respectively. We use also the notion $\tilde{I_A}, \tilde{P_A}$ for the extended ideal $\tilde{I_A}E, \tilde{P_A}E$ in E w.r.t. the embedding $\tilde{E} \hookrightarrow E$. Observe that the image of I_A^k in $\tilde{E}$ is $\tilde{I_A}^k$ for $k \geq 1$. So we have

$I_A^k + (v) = \tilde{I_A}^k + (v)$ in E. Using this in (5) we get

$$f \in \mathfrak{m}^d \cap \bigcap_{A, I_A \neq \mathfrak{m}} (\tilde{I_A}^{|A|} + (v)).$$

Thus we have

$$\tilde{f} \in \tilde{\mathfrak{m}}^d \cap \bigcap_{A, I_A \neq \mathfrak{m}} \tilde{I_A}^{|A|} \subseteq \tilde{\mathfrak{m}}^d \cap \bigcap_{A, \tilde{I_A} \neq \tilde{\mathfrak{m}}} \tilde{I_A}^{|A|}.$$

Note that every strongly generic element for the $\tilde{E}$-modules $\tilde{E}/\tilde{P_A}$ is also strongly stable generic. Applying Lemma 4.4.5 to the linear ideals $\tilde{I}_i$, where $i = 1, \dots, d$, in the exterior algebra $\tilde{E}$, we have that

$$\tilde{J} = \tilde{I}_1 \dots \tilde{I}_d = \tilde{\mathfrak{m}}^d \cap \bigcap_{A, \tilde{I_A} \neq \tilde{\mathfrak{m}}} \tilde{I_A}^{|A|}.$$

Thus $\tilde{f} \in \tilde{J}$. Hence $f \in \tilde{J} + (v)$ since $f \in (\tilde{f}, v)$ in E. Note that

$$\begin{aligned} \tilde{J} + (v) &= \tilde{I}_1 \cdots \tilde{I}_d + (v) = (\tilde{I}_1 + (v)) \dots (\tilde{I}_d + (v)) + (v) \\ &= (I_1 + (v)) \dots (I_d + (v)) = I_1 \dots I_d + (v) \\ &= J + (v). \end{aligned}$$

So we get $f \in J + (v)$. This concludes the proof. □

Next we prove Lemma 4.4.5:

Proof of 4.4.5. We prove the statement by induction on d and on n. Set $J_i = I_1 \cdots I_{i-1} I_{i+1} \cdots I_d$ for $i = 1, \dots, d$. By induction on d, we only need to prove that

$$J = J_1 \cap \dots \cap J_d \cap (\sum_{i=1}^{d} I_i)^d. \tag{6}$$

The case $d = 1$ is trivial. Assume $d > 1$. The inclusion "$\subseteq$" is clear. For the inclusion "$\supseteq$", we may assume that $\sum_{i=1}^d I_i = \mathfrak{m}$ because otherwise all the ideals I_i live in a smaller exterior algebra.

Let $v \in E_1$ be a strongly generic element for E-modules E/P_A as chosen in the proof of Corollary 4.4.6 after a suitable field extension. Note that v is a also strongly stable generic element for $E/P_{[d]} = E/J$. Moreover, we also have that 6 holds in E if and only if the extended equation holds after the field extension. Since J_i is a product of $(d-1)$ linear ideals, by the induction hypothesis on d we note that Corollary 4.4.6 holds (for $d-1$). So we have $\operatorname{sat}(J_i, v) \leq d - 1$ for $i = 1, \dots, d$.

Now we use a second induction on n to prove (6). The case $n = 1$ is trivial. Assume $n > 1$. Let $f \in J_1 \cap \dots \cap J_d$ with $\deg f \geq d$. We must show that $f \in J$.

Observe that the ideal $J + (v)/(v)$ of $E/(v)$ is the product of the linear ideals $I_i + (v)/(v)$. By the induction hypothesis on n, we get that

$$(J_1 + (v)/(v)) \cap \dots \cap (J_d + (v)/(v)) \cap \mathfrak{m}^d/(v) = J + (v)/(v).$$

In other words,

$$J + (v) = (J_1 + (v)) \cap \dots \cap (J_d + (v)) \cap \mathfrak{m}^d.$$

Thus $f \in J + (v)$. Let $f = f' + vg$ where $f' \in J$ and $g \in E$. Replacing f by $f - f'$, we may assume that $f = vg$. Then we have $vg \in J_1 \cap \cdots \cap J_d$. This implies that $vg \in J_i$ for all i. Hence $g \in J_i :_E v$.

Observe that $\deg g \geq d - 1$ because $\deg f \geq d$. Since $\operatorname{sat}(J_i, v) \leq d - 1$ for $i = 1, \dots, d$, i.e., $s(\frac{J_i :_E v}{J_i + (v)}) \leq d - 2$, we get that $g \in J_i + (v)$ and $vg \in v(J_i + (v)) = vJ_i$ for $i = 1, \dots, d$. If $\deg f \geq d + 1$, then $\deg g \geq d$ and

$$g \in (J_1 + (v)) \cap \cdots \cap (J_d + (v)) \cap \mathfrak{m}^d = J + (v).$$

This implies that $f = vg \in v(J + (v)) = vJ \subset J$. It remains to consider the case that $\deg f = d$. Here we have

$$f = vg \in (vJ_1 \cap \cdots \cap vJ_d)_d \subset J$$

since v is also strongly stable generic. Thus $f \in J$. This concludes the proof. □

We prove now the main result of this section:

Theorem 4.4.7. *Let $I_1, \dots, I_d \subset E$ be linear ideals with $J = I_1 \cdots I_d \neq 0$. Let $P_A = \prod_{j \in A} I_j$ where A runs over the subsets of $[d]$. If every strongly generic element for the E-modules E/P_A is strongly stable generic, then J has a d-linear resolution.*

Proof. We prove the statement by induction on n. The case $n = 1$ is trivial. Assume $n > 1$. Let v be a strongly generic element as chosen in the proof of Corollary 4.4.6 after a suitable field extension. Note that J has a d-linear resolution if and only if the extended ideal after the field extension has a d-linear resolution. By Lemma 2.3.3, we have

$$\operatorname{reg}_E(J) = \max\{\operatorname{reg}_E(J + (v)), \operatorname{sat}(J, v)\}. \tag{7}$$

By Corollary 4.4.6, we also have that $\operatorname{sat}(J, v) \leq d$. The critical case is if $J \not\subseteq (v)$ holds, i.e., $J + (v)/(v) \neq 0$. Otherwise, $\operatorname{reg}_E(J + (v)) = \operatorname{reg}_E((v)) = 1 \leq d$. Thus $\operatorname{reg}_E(J) \leq d$. Hence $\operatorname{reg}_E(J) = d$ since J is generated in degree d.

Assume now $J \not\subseteq (v)$. Observe that $J + (v)/(v)$ is the product of the linear ideals $I_i + (v)/(v)$ in $E/(v)$. By the induction hypothesis we get

$$\operatorname{reg}_{E/(v)}(J + (v)/(v)) = d.$$

Since $E/(J + (v)) \cong (E/(v))/(J + (v)/(v))$, we have

$$\operatorname{reg}_{E/(v)}(E/(J + (v))) = d - 1.$$

Applying Proposition 2.3.1 to E/J, we get that

$$\operatorname{reg}_E((E/J)/(vE/J)) = \operatorname{reg}_{E/(v)}((E/J)/(vE/J)).$$

Observe that $E/(J + (v)) \cong (E/J)/(vE/J)$. Thus

$$\operatorname{reg}_E(E/(J + (v))) = \operatorname{reg}_{E/(v)}(E/(J + (v))) = d - 1.$$

So we get $\operatorname{reg}_E(J + (v)) = d$. Recall that $\operatorname{sat}(J, v) \leq d$. Using these facts in (7) we see that $\operatorname{reg}_E(J) = d$. Note that J is generated in degree d. Hence J has a d-linear resolution. □

To conclude this section, we propose the following:

Open problem 4.4.8.

(i) Prove the results in this section without the condition that every strongly generic element is strongly stable generic.

(ii) A product of linear ideals generated by variables always has linear quotients since it is matroidal (see Example 4.2.8 and Theorem 4.3.2). Is it true that a product of arbitrary linear ideals always has linear quotients?

CHAPTER 5

On the Koszul property over the exterior algebra

Let K be an infinite field, $E = K\langle e_1, \dots, e_n\rangle$ an exterior algebra over K with the standard grading $\deg e_i = 1$ for $i = 1, \dots, n$ and $\mathfrak{m} = (e_1, \dots, e_n)$ the maximal graded ideal of E. Let $R = E/J$ be a standard graded K-algebra where J is a graded ideal of E. The goal of this chapter is to study several variations of the Koszul property over the exterior algebra. More precisely, we study universally Koszul, (unconditioned) strongly Koszul and initially Koszul properties of R. These topics are motivated from results of Blum [**9**] and Conca [**13**], [**14**].

5.1. Preliminaries

We present in this section basis facts about Koszul algebras over the exterior algebra. For more details, we refer to the survey by Fröberg [**27**], the paper of Conca [**13**] and the book by Ene and Herzog [**23**, Section 6.1].

Definition 5.1.1. A standard graded K-algebra R over E is said to be *Koszul* if the R-module $K = R/\mathfrak{m}$ has a linear free resolution over R.

Example 5.1.2. (i) The exterior algebra E is Koszul since the Cartan complex is the linear free resolution of K over E (see Section 2.1).

(ii) The K-algebra E/J defined by a quadratic monomial ideal $J \subset E$ is Koszul (see, e.g., Theorem 5.1.5).

We collect some well-known facts in the following lemma whose proof is trivial:

Lemma 5.1.3. *Let R be a standard graded K-algebra over E. The following statements are equivalent:*

(i) *R is a Koszul algebra;*
(ii) $\operatorname{reg}_R(K) = 0$*;*
(iii) $\operatorname{Tor}_i^R(K, K)_j = 0$ *for all $i \neq j$.*

Remark 5.1.4. From Lemma 5.1.3, one can deduce a well-known necessary condition for the Koszulness of the K-algebra E/J where $J \subset E$ is a graded ideal: If E/J is Koszul, then J is generated in degrees ≤ 2. From now on, we always assume that J does not contain linear forms. In other words, we consider the Koszul property of $R = E/J$ only in the case that J is generated in degree 2. We also identify $e_i \in E$ with $[e_i] \in R$ for $i = 1, \dots, n$.

A well-known sufficient condition is the following result (see, e.g., [**27**]):

Theorem 5.1.5. *Let $J \subset E$ be a graded ideal which has a quadratic Gröbner basis with respect to some monomial order on E. Then E/J is a Koszul algebra.*

Note that, the converse of Theorem 5.1.5 is false in the polynomial ring case. However, no counter example is known over an exterior algebra. We propose a possible counter example in Example 5.2.2.

5.2. Universally Koszul property

In this section, we present the universally Koszul property for standard graded K-algebras over the exterior algebra. We also study this property for the quotient rings of edge ideals of graphs. Most of the results in this section are similar to known facts in the polynomial ring case; see [**13**] for more details. At first, we recall the notion of Koszul filtrations as follows:

Definition 5.2.1 (Conca, Trung, Valla [**18**])**.** Let R be a standard graded K-algebra. A *Koszul filtration* of R is a family $\mathcal{F}$ of ideals of R such that:

(i) Every ideal $0 \neq I \in \mathcal{F}$ is generated by linear forms;
(ii) $\mathcal{F}$ contains the ideal 0 and the maximal graded ideal $\mathfrak{m}$ of R;
(iii) For $0 \neq I \in \mathcal{F}$, there exists $J \in \mathcal{F}$ such that $J \subset I$, I/J is cyclic and $J :_R I \in \mathcal{F}$.

Example 5.2.2. (i) Let $R = K\langle e_1, e_2, e_3\rangle/(e_{12})$. Then the collection

$$\mathcal{F} = \{0, (e_1), (e_1, e_2), (e_1, e_2, e_3)\}$$

is a Koszul filtration of R since $0 :_R (e_1) = (e_1, e_2)$, $(e_1) :_R (e_1, e_2) = (e_1, e_2)$ and $(e_1, e_2) :_R (e_1, e_2, e_3) = (e_1, e_2, e_3)$.

(ii) Let $J = (e_{12} - e_{34}, e_{13} - e_{24}) \subset E = K\langle e_1, \dots, e_4\rangle$ and $R = E/J$. At first, we see that R has a Koszul filtration so R is Koszul by Remark 5.2.6. Indeed, considering the following family of ideals in R:

$$\mathcal{F} = \{(0_R), (e_1 + e_4), (e_1 + e_4, e_2 + e_3), (e_1 + e_4, e_2 + e_3, e_3), (e_1 + e_4, e_2 + e_3, e_3, e_4)\}.$$

We have that:

$$\begin{aligned} 0 :_R (e_1 + e_4) &= (e_1 + e_4, e_2 + e_3), \\ (e_1 + e_4) :_R (e_1 + e_4, e_2 + e_3) &= (e_1 + e_4, e_2 + e_3), \\ (e_1 + e_4, e_2 + e_3) :_R (e_1 + e_4, e_2 + e_3, e_3) &= (e_1 + e_4, e_2 + e_3, e_3, e_4), \\ (e_1 + e_4, e_2 + e_3, e_3) :_R (e_1 + e_4, e_2 + e_3, e_3, e_4) &= (e_1 + e_4, e_2 + e_3, e_3, e_4). \end{aligned}$$

Then $\mathcal{F}$ is a Koszul filtration of R.

Next we claim that J does not have a quadratic Gröbner basis w.r.t. the coordinate $e_1, \dots, e_4$ and any monomial order on E. Indeed, assume the contrary that J has a quadratic Gröbner basis with respect to some orders $<$ on E. Then $\mathrm{in}_<(J)$ is one of the following monomial ideals: (e_{12}, e_{13}), (e_{12}, e_{24}), (e_{34}, e_{13}) and (e_{34}, e_{24}). Observe that non of these four ideals contain all monomials of degree 3. This contradicts to the fact that J contains all monomials of degree 3.

So J does not have a quadratic Gröbner basic w.r.t. the natural coordinate of E, but $R = E/J$ is Koszul.

Analogously to [**18**, Proposition 1.2], one can prove that:

Proposition 5.2.3. *Let $\mathcal{F}$ be a Koszul filtration of R. Then for every $I \in \mathcal{F}$, the quotient R/I has a linear R-free resolution.*

Denote by

$$\mathcal{L}(R) = \{I \subset R : I \text{ is an ideal generated by linear forms}\}.$$

Definition 5.2.4. A standard graded K-algebra R over E is called *universally Koszul* if $\mathcal{L}(R)$ is a Koszul filtration of R.

The universally Koszul property has the following characterizations which can be proved over an exterior algebra analogously to [**13**, Proposition 1.4]:

Proposition 5.2.5. *Let R be a standard graded K-algebra over E. The following statements are equivalent:*

(i) *R is universally Koszul;*
(ii) *For every ideal $I \in \mathcal{L}(R)$ the quotient R/I has a linear R-free resolution;*
(iii) *For every $I \in \mathcal{L}(R)$ one has $\operatorname{Tor}_2^R(R/I, K)_j = 0$ for $j > 2$;*
(iv) *For every $I \in \mathcal{L}(R)$ and $x \in R_1 \setminus I$ one has $I :_R (x) \in \mathcal{L}(R)$.*

Remark 5.2.6. Observe the following:

(i) Since every Koszul filtration contains the maximal graded ideal $\mathfrak{m}_R$ of R, by Proposition 5.2.3 we observe that if R has a Koszul filtration then $\mathfrak{m}$ has a linear R-free resolution. Hence R is a Koszul algebra. Thus the universally Koszul property implies the Koszul property of R.

(ii) If R is universally Koszul and $J \subset R$ is a graded ideal generated by linear forms then R/J is universally Koszul (see [**13**, Lemma 1.6] for the polynomial ring case).

Example 5.2.7. (i) The exterior algebra $E = K\langle e_1, \dots, e_n\rangle$ is universally Koszul since one can check that the condition (iv) in Proposition 5.2.5 is fulfilled. Indeed, let $I \in \mathcal{L}(E)$ and $x \in E_1 \setminus I$. By changing the coordinate, we may assume that $x = e_s$ and $I = (e_1, \dots, e_{s-1})$ for some $1 \le s \le n$. Then $I :_E (e_s) = (e_1, \dots, e_s) \in \mathcal{L}(E)$.

(ii) Let $f \in E_2$. If f is a product of two linear forms, then $R = E/(f)$ is universally Koszul. Indeed, after a suitable change of coordinates, f is a quadratic monomial. Thus we may assume that $f = e_{12}$. Now let $I \in \mathcal{L}(R)$ and $x \in R_1 \setminus I$. Let $J \in E$ be the corresponding linear ideal to I, i.e., $I = (J + (e_{12}))/(e_{12})$.

If $e_{12} \in J$, then $(J + (e_{12})) :_E (x) = J :_E (x)$ is generated by linear forms. So $I :_R (x)$ is also generated by linear forms. If $e_{12} \notin J$, then we may assume that $J = (e_3, \dots, e_s)$ for some $3 \le s \le n$. If $(x) \in (e_1, \dots, e_s)$, then $(J + (e_{12})) :_E (x) = (e_1, \dots, e_s)$. Otherwise, we may assume that $x = e_{s+1}$. Then

$$(J + (e_{12})) :_E (x) = (e_{12}, e_3, \dots, e_s) :_E (e_{s+1}) = (e_{12}, e_3, \dots, e_s, e_{s+1})$$

Thus $I :_R (x) = I + (x)$ is generated by linear forms. By Proposition 5.2.5, we conclude that R is universally Koszul.

(iii) In the polynomial ring case, following [**13**, Proposition 3.1] we have that a quadratic hypersurface ring defined by an irreducible quadric is universally Koszul. But this is not true in the exterior algebra case. For example, let $f = e_{12} + e_{34}$ in $E = K\langle e_1, \dots, e_4\rangle$. Then $R = E/(f)$ is not Koszul. More precisely, using Macaulay2

[**28**] we get that $\beta^R_{3,4}(K) = 5 \neq 0$. Thus K does not have a linear free resolution over R and then R is not Koszul.

By many computations, we always see that $R = E/(f)$ is not Koszul if f is an irreducible quadric. Thus we propose the following open question:

Question 5.2.8. Let $f \in E_2$. Then $E/(f)$ is Koszul if and only if f is reducible (i.e., f is a product of two linear forms)?

Using the same method as in [**14**], we can classify all graphs such that the algebras defined by their edge ideals are universally Koszul over the exterior algebra. We recall first some facts. If $R = K\langle e_1, \dots, e_n\rangle/I$, we set $R\langle e\rangle = K\langle e_1, \dots, e_n, e\rangle/I$ and consider this with its natural grading. Let $A = K\langle e_1, \dots, e_n\rangle/I$ and $B = K\langle f_1, \dots, f_m\rangle/J$. The *fiber product* of A and B is $K\langle e_1, \dots, e_n, f_1, \dots, f_m\rangle/P$ where

$$P = I + J + (e_i f_j : i = 1, \dots, n \text{ and } j = 1, \dots, m).$$

Analogously to [**13**, Lemma 1.6], one has:

Lemma 5.2.9. *Let R, A and B be standard graded algebras over E. One has:*

(i) *The extension $R\langle e\rangle$ of R is universally Koszul if and only if R is universally Koszul.*

(ii) *The fiber product of A and B is universally Koszul if and only if both A and B are universally Koszul.*

Proof. (i) Let $R' = R\langle e\rangle$. We need to prove that $I :_{R'} x \in \mathcal{L}(R')$ for every $I \in \mathcal{L}(R')$ and $x \in R'_1 \setminus I$. We have following cases:

Case 1: The generators of I belong to R. Then $I = JR' = J + Je$ where $J \in \mathcal{L}(R)$.

If $x \in R_1$, we claim that $I :_{R'} x = (J :_R x)R' \in \mathcal{L}(R')$ since $J :_R x \in \mathcal{L}(R)$. Indeed, it is clear that $(J :_R x)R' \subseteq I :_{R'} x$. Let $f = f_1 + f_2 e \in I :_{R'} x$ where $f_1, f_2 \in R$. Then $f_1 x + f_2 x e \in J + Je$. Thus $f_1 x, f_2 x \in J$. Hence $f \in (J :_R x)R'$. So we have $I :_{R'} x = (J :_R x)R' \in \mathcal{L}(R')$.

If $x \notin R_1$, we may assume that $x = z + e$, where $z \in R_1$. We claim that $I :_{R'} x = I + (x) \in \mathcal{L}(R')$. Indeed, it is clear that $I + (x) \subseteq I :_{R'} x$. Let $f = f_1 + f_2 e \in I :_{R'} x$ where $f_1, f_2 \in R$. Then $f = f_1 - f_2 z + f_2 x$. Since $x \in I :_{R'} x$, we may assume that $f \in R$. Then $fz + fe \in I = J + Je$. Thus $f \in J \subset I$. Hence $I :_{R'} x = I + (x) \in \mathcal{L}(R')$.

Case 2: Some of the generators of I do not belong to R. Then we may decompose I as $JR' + (y + e)$ where $J \in \mathcal{L}(R)$ and $y \in R_1$. One can check that $I \cap R = J$.

If $x \in R_1$, we claim that $I :_{R'} x = (J :_R x)R' + (y + e) \in \mathcal{L}(R')$. Indeed, it is clear that $(J :_R x)R' + (y + e) \subseteq I :_{R'} x$. Let $f \in I :_{R'} x$. We may assume that $f \in R$ since $(y + e) \in I :_{R'} x$. Then $fx \in I \cap R = J$. Thus $f \in (J :_R x)$. Hence

$$I :_{R'} x = (J :_R x)R' + (y + e) \in \mathcal{L}(R').$$

It remains to consider $x = z + e$, where $z \in R_1$. Since $x \notin I$, we have $z - y \notin J$. We claim that

$$I :_{R'} x = (J :_R (z - y))R' + (x) + (y + e) \in \mathcal{L}(R').$$

Indeed, one can check that $(J :_R (z-y))R' + (x) + (y+e) \subseteq I :_{R'} x$. Let $f \in I :_{R'} x$. Then $fx \in I = JR' + (y+e)$. We may assume that $f \in R$ since $(y+e) \in I :_{R'} x$. Observe that $fx = fz + fe = fz - fy + f(y+e)$, so we get $f(z-y) = fx - f(y+z) \in I \cap R = J$. Since $z - y \notin J$ and R is universally Koszul, we have $f \in J :_R (z-y) \in \mathcal{L}(R)$. This concludes the proof.

(ii) The proof is verbatim the same as in [**13**, Lemma 1.6]. □

We also have an exterior algebra version of [**14**, Lemma 4] as follow:

Lemma 5.2.10. *Let $J \subset E = K\langle x, y, z, t\rangle$ be a quadratic monomial ideal. Then $R = E/J$ is not universally Koszul if J is one of the following ideals:*

(i) (xy, zt),

(ii) (xy, yz, zt).

Proof. For both of the two cases, we claim that $0 :_R (y+z)$ has a minimal generator of degree 2. Indeed, it is clear that $xt \in 0 :_R (y+z)$. Since $\text{span}_K\{y, z\}$ is the vector space of all elements of degree 1 in $(xy, zt) :_E (y+z)$, $(xy, yz, zt) :_E (y+z)$, and $xt \notin (y, z)$, we get that xt is a minimal generator of $0 :_R (y+z)$. Thus E/J is not universally Koszul. □

Using the above lemmas, one can classify universally Koszul algebras defined by monomial ideals over the exterior algebra. For the convenience of the reader we reproduce here an exterior algebra version of [**14**, Theorem 5]:

Theorem 5.2.11. *Let $R = E/J$ where $J \subset E$ is a quadratic monomial ideal. The following statements are equivalent:*

(i) *R is universally Koszul;*

(ii) *The restriction of J to any subset of 4 variables is not one of the types in Lemma 5.2.10.*

Proof. (i) $\Rightarrow$ (ii): The assertion follows directly from Remark 5.2.6 (ii) and Lemma 5.2.10.

(ii) $\Rightarrow$ (i): Let $E = K\langle e_1, \dots, e_n\rangle$. We prove the statement by induction on n. The case $n = 1$ is trivial. Consider the case $n > 1$. Let $U = \{e_1, \dots, e_n\}$ and let $V = \{v_1, \dots, v_r\}$ be a maximal subset of U such that for all $v_i, v_j \in V$ with $i \neq j$ one has $v_iv_j \notin J$. Let $W = U \setminus V$ and $G_i = \{x \in U : xv_i \in J\}$. Then we have $W = \bigcup_{i=1}^r G_i$. We claim that for $1 \le i < j \le r$ then either $G_i \subseteq G_j$ or $G_j \subseteq G_i$. Indeed, if there exist $x \in G_i \setminus G_j$ and $y \in G_j \setminus G_i$ then one has $xv_i, yv_j \in J$. Moreover, by the definition of the sets V, G_i, G_j, we note that J does not contain v_iv_j, xv_j and yv_i. This is a contradiction since the restriction of J to $\{x, y, v_i, v_j\}$ would be either (xv_i, yv_j) or (xv_i, yv_j, xy) which are one of the types (i), (ii) in Lemma 5.2.10. Hence by a suitable renumbering if needed, we may assume that $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_r = W$.

By the same argument as above, we also note that if $x \in G_i$ and $y \in U \setminus G_i$ then $xy \in J$ for $i = 1, \dots, r$.

If $G_1 = \emptyset$ then v_1 does not appear in the minimal set of generators of J. Let J' be the ideal in $E' = E/(v_1)$ generated by the same minimal set of generators of J. Then by the induction hypothesis, we have $R' = E'/J'$ is universally Koszul. Therefore, $R = R'\langle v_1\rangle$ is universally Koszul by Lemma 5.2.9 (i).

If $G_1 \neq \emptyset$ then for $x \in G_1$ and $y \in U \setminus G_1$ we have $xy \in J$. Let J_1, J_2 denote the restriction of J to G_1 and $\overline{G_1} = U \setminus G_1$, respectively. Set $A = K\langle G_1\rangle/J_1$ and $B = K\langle\overline{G_1}\rangle/J_2$. Observe that, R is the fiber product of A and B. By the induction hypothesis, we note that A and B are universally Koszul algebras. Hence by Lemma 5.2.9 (ii), R is also a universally Koszul algebra. □

Recall that for a graph G with the vertex set $V(G)$, the edge ideal $J(G)$ of G is defined by

$$J(G) = (e_{ij} : i, j \in V(G) \text{ and } (i, j) \text{ is an edge of } G).$$

As direct consequences of Theorem 5.2.11, we have:

Corollary 5.2.12. *Let G be a graph with the edge ideal $J(G) \subset E$. Then the algebra $E/J(G)$ is universally Koszul if and only if every subgraph of 4 vertices in G is not one of the forms in Figure 1.*

Figure 1: Subgraphs of 4 vertices

Corollary 5.2.13. *Let $J \subset E$ be a monomial ideal and $I \subset K[x_1, \dots, x_n] = S$ the corresponding squarefree monomial ideal. Then*

$$S/I \text{ is universally Koszul} \Leftrightarrow E/J \text{ is universally Koszul.}$$

Proof. The assertion follows directly from [**14**, Theorem 5] and Theorem 5.2.11. □

5.3. Strongly Koszul and unconditioned strongly Koszul properties

The strongly Koszul property over the polynomial ring was introduced by Herzog, Hibi and Restuccia [**34**, Definition 1.1]. Analogously, we consider in this section the strongly Koszul property over the exterior algebra. We also define a slightly different property of the Koszul property, namely unconditioned strongly Koszul property, and prove a necessary condition for elements of degree 2 to define unconditioned strongly Koszul algebras. Note that the unconditioned Koszul property is also studied by Conca, De Negri and Rossi in [**17**] with the shorter name “strongly Koszul”.

Let R be a standard graded K-algebra over E and $\mathbf{u} = \{u_1, \dots, u_n\} \subset R_1$ a minimal system of generators of the maximal graded ideal $\mathfrak{m}_R$ of R.

Definition 5.3.1.

(i) We say that R is *strongly Koszul* w.r.t. $\mathbf{u}$ if the ideal $(u_{i_1}, \dots, u_{i_{j-1}}) :_R u_{i_j}$ is generated by a subset of $\mathbf{u}$ for every $1 \le i_1 \le \dots \le i_j \le n$.

(ii) The algebra R is called *unconditioned strongly Koszul* w.r.t. $\mathbf{u}$ if for every subset $F \subseteq [n]$ and every $j \notin F$, we have $(u_i : i \in F) :_R u_j$ is generated by a subset of $\mathbf{u}$.

Remark 5.3.2. Observe the following:

(i) Denote by $\mathcal{L}(\mathbf{u}) = \{0, (u_{i_1}, \dots, u_{i_j}),$ where $1 \le i_1 < \cdots < i_j \le n\}$. If R is (unconditioned) strongly Koszul w.r.t. $\mathbf{u}$, then $\mathcal{L}(\mathbf{u})$ is a Koszul filtration of R.

(ii) The unconditioned strongly Koszul property implies the strongly Koszul property. We also note that the (unconditioned) strongly Koszul property and the universally Koszul property are not the same. This can be seen by the example below.

(iii) As in the case of the universally Koszul property, we also have that a strongly Koszul algebra is a Koszul algebra. Therefore, if E/J is strongly Koszul where $J \subset E$ is a graded ideal, then J must be generated in degree 2.

Example 5.3.3. Let $E = K\langle e_1, \dots, e_4\rangle$ and $\mathbf{u} = \{e_1, e_2, e_3, e_4\}$.

(i) Let $J = (e_{12}, e_{34}) \subset E$ and $R = E/J$. Then R is not universally Koszul by Lemma 5.2.10. However, one can check easily that R is unconditioned strongly Koszul w.r.t. $\mathbf{u}$.

(ii) Let $f = e_{13} + e_{14} + e_{23} \in E$ and set $R = E/(f)$. Then R is not unconditioned strongly Koszul w.r.t. $\mathbf{u}$ since $(e_2) :_R e_1$ is not generated by a subset of $\mathbf{u}$. Indeed, we have $e_3 + e_4 \in (e_2) :_R e_1$ since $e_1(e_3 + e_4) = -e_{23}$ in R. But $e_3, e_4 \notin (e_2) :_R e_1$ because $e_{13}, e_{14} \notin (e_2, f)$.

Next we claim that every algebra defined by a quadratic monomial ideal is unconditioned strongly Koszul. At first, we need:

Lemma 5.3.4. *Let $R = E/J$ where $J \subset E$ is a quadratic monomial ideal. Then the following statements are equivalent:*

(i) *R is unconditioned strongly Koszul w.r.t. $e_1, \dots, e_n$;*
(ii) *$0 :_R e_1$ is generated by linear forms and $R/(e_1)$ is unconditioned strongly Koszul w.r.t. $e_2, \dots, e_n$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that R is unconditioned strongly Koszul with respect to $e_1, \dots, e_n$. Then $0 :_R e_1$ is generated by linear forms. Let $F \subseteq [n] \setminus \{1\}$ and $j \notin F \cup \{1\}$. We have $(e_1, e_i : i \in F) :_R e_j$ is generated by a subset of $\{e_1, \dots, e_n\}$. Thus the same is true for $(e_i : i \in F) :_{R/(e_1)} e_j$. Hence $R/(e_1)$ is unconditioned strongly Koszul w.r.t. $e_2, \dots, e_n$.

(ii) $\Rightarrow$ (i): Let $F \subseteq [n] \setminus \{1\}$ and $j \notin F \cup \{1\}$. Since $R/(e_1)$ is unconditioned strongly Koszul w.r.t. $e_2, \dots, e_n$, we have $(e_1, e_i : i \in F) :_R e_j = (e_1) + I$, where I is a linear ideal generated by a subset of $\{e_2, \dots, e_n\}$. Since J is a quadratic monomial ideal, we also get that $(e_i : i \in F) :_R e_j = (e_1) + I$ if $e_1 e_j \in J$ and $(e_i : i \in F) :_R e_j = I$ otherwise. Therefore, to ensure the unconditioned strongly Koszul property of R, we need to decide whether $(e_i : i \in F) :_R e_1$ is generated by linear forms. Since J is a monomial ideal, we have

$$((e_i : i \in F) + J) :_E e_1 = (e_i : i \in F) + J + (e_j : e_1 e_j \in J).$$

Thus $(e_i : i \in F) :_R e_1$ is generated by linear forms. This concludes the proof. $\square$

Proposition 5.3.5. *Every standard graded algebra over E defined by a quadratic monomial ideal is unconditioned strongly Koszul.*

Proof. Let $R = E/J$ where $J \subset E$ is a quadratic monomial ideal. We prove the assertion by induction on $\dim_K R_1$. The case $\dim_K R_1 = 1$ is trivial.

If $\dim_K R_1 = 2$, we only have one case $R = K\langle e_1, e_2\rangle/(e_{12})$. Then R is unconditioned strongly Koszul since

$$0 :_R e_1 = 0 :_R e_2 = (e_1, e_2) \text{ and } (e_1) :_R e_2 = (e_2) :_R e_1 = (e_1, e_2).$$

For the case $\dim_K R_1 > 2$, by the induction hypothesis and Lemma 5.3.4, we only need to check that $0 :_R e_1$ is generated by linear forms. Let $e_F \in E$ be a monomial such that $e_F \notin J$. Assume that $[e_F] \in 0 :_R e_1$. Then $e_F e_1 \in J$. This implies that there exists $i \in F$ such that $e_1 e_i \in J$. So $e_i \in 0 :_R e_1$ and e_i divides e_F. Thus $0 :_R e_1$ is generated by linear forms. □

To conclude this section, we present a necessary condition for quadratic elements to define unconditioned strongly Koszul algebras in the natural coordinates. Let $f \in E_2$. We consider the graph $G(f)$ with the vertex set

$$V(f) = \{j : e_j \text{ divides some elements in } \operatorname{supp}(f)\}$$

and the edge set $E(f) = \{(j, k) : e_j e_k \in \operatorname{supp}(f)\}$. We have:

Proposition 5.3.6. *Let $f \in E_2$ and $R = E/(f)$. If R is an unconditioned strongly Koszul algebra w.r.t. $\mathbf{u} = \{e_1, \dots, e_n\}$, then $G(f)$ is the union of its complete subgraphs.*

Proof. Assume the contrary. Then there exists a connected subgraph of $G(f)$ which is not complete and has at least 3 indices. Thus there exist $i, j, k \in V(f)$ such that $(i, j), (i, k) \in E(f)$ and $(j, k) \notin E(f)$. Hence $e_i e_j, e_i e_k$ are in $\operatorname{supp}(f)$ but $e_j e_k \notin \operatorname{supp}(f)$. Let $I = (e_l : l \in [n] \setminus \{i, j, k\})$. Write

$$f = e_i(\alpha e_j + \beta e_k) + g \text{ where } g \in I \text{ and } 0 \neq \alpha, \beta \in K. \tag{8}$$

Observe that $I :_R e_i = I + (e_i, \alpha e_j + \beta e_k)$ is not generated by a subset of $\mathbf{u}$ since $e_j, e_k \notin I :_R e_i$. Indeed, this follows from the fact that $e_i e_j, e_i e_k \notin I + (f)$ which can be seen by using (8). This is a contradiction since R is unconditioned strongly Koszul. □

Note that the converse of Proposition 5.3.6 is false:

Example 5.3.7. Let $f = e_{12} + e_{13} + e_{14} + e_{23} - e_{24} + e_{34} \in E = K\langle e_1, \dots, e_4\rangle$. It is clear that $G(f)$ is a complete graph. Let $R = E/(f)$. Note that

$$f = e_1(e_2 + e_3 + e_4) + (e_3 - e_2)(e_4 - e_2).$$

Thus

$$h = (e_2 + e_3 + e_4)(e_3 - e_2) \in 0 :_R e_1 \text{ since } e_1 h = f(e_3 - e_2) \in (f).$$

Observe that

$$h = e_{23} + (e_4 - e_2)(e_3 - e_2) \notin (f) + (e_1),$$

since otherwise $e_{23} \in (f) + (e_1)$ which is impossible. Now if R is unconditioned strongly Koszul, then $0 :_R e_1 = (e_1)$ since $e_2, e_3, e_4 \notin 0 :_R e_1$. Hence $0 :_E e_1 = (e_1) + (f)$. This is a contradiction since $h \in 0 :_R e_1$ and $h \notin (e_1) + (f)$. So R is not unconditioned strongly Koszul.

Remark 5.3.8. For the polynomial ring case, we have the same result as Proposition 5.3.6 and it is not only a necessary condition but also a sufficient condition. In other words, one can classify quadratic polynomials in the polynomial ring which define unconditioned strongly Koszul algebras (see [**63**] for more details).

5.4. Initially Koszul property

We study in this section standard graded K-algebras over the exterior algebra with the initially Koszul property. The content of this topic is an analogue to the work of Blum in [**9**] and Conca, Rossi and Valla in [**16**] for standard graded K-algebras over the polynomial ring.

Definition 5.4.1. Let R be a standard graded K-algebra over E and let

$$F : V_0 = 0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = R_1$$

be a complete flag of R_1, where V_i is a subspace of dimension i for $i = 1, \ldots, n$. We say that F is a *Gröbner flag* of R if the ideals (V_i) form a Koszul filtration of R, i.e., for $i = 1, \ldots, n$, there exists j_i such that $(V_{i-1}) :_R (V_i) = (V_{j_i})$. If R has a Gröbner flag, following [**9**], R is said to be an *initially Koszul* algebra.

Remark 5.4.2. Note that the universally Koszul property is equivalent to the existence of a Koszul filtration which is as large as possible, and the existence of a Gröbner flag is equivalent to the existence of a Koszul filtration which is as small as possible. More precisely, if R has a Gröbner flag then there exists an ordered system of generators $u_1, \ldots, u_n$ of R_1 such that $\{0, (u_1, \ldots, u_j) \text{ for } 1 \le j \le n\}$ is a Koszul filtration of R, i.e., for every $i = 1, \ldots, n$, we have

$$(u_1, \ldots, u_{i-1}) :_R u_i = (u_1, u_2, \ldots, u_{j_i}) \text{ for some } j_i \le n.$$

Note that

$$(u_1, \ldots, u_{i-1}) :_R u_i \supseteq (u_1, u_2, \ldots, u_i).$$

Thus $j_i \ge i$ for $i = 1, \ldots, n$. We denote by $j(F)$ the sequence of numbers $j_1, j_2, \ldots, j_n$.

Similarly to results of Conca, Rossi and Valla in [**16**], we present next some properties of the standard graded algebra $R = E/J$ with a Gröbner flag F, where $J \subset E$ is a graded ideal. At first we have:

Proposition 5.4.3. *Let $J \subset E$ be a graded ideal such that $R = E/J$ is initially Koszul with a Gröbner flag F. Then for $i = 0, \ldots, n$, the Hilbert series of $R/(V_i)$ depends only on $j(F)$.*

Proof. Let $F : V_0 = 0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = R_1$. For $i = 1, \ldots, n$ we have short exact sequences

$$0 \longrightarrow R/(V_{j_i})[-1] \xrightarrow{u_i} R/(V_{i-1}) \longrightarrow R/(V_i) \longrightarrow 0.$$

Thus

$$H_{R/(V_{i-1})}(t) = H_{R/(V_i)}(t) + tH_{R/(V_{j_i})}(t) \text{ for } i = 1, \ldots, n. \tag{9}$$

Note that $j_i \ge i$ for $i = 1, \ldots, n$, and $H_{R/(V_n)}(t) = 1$. Hence

$$j_n = n \text{ and } H_{R/(V_{n-1})}(t) = 1 + t.$$

By induction on i using (9), we get that for every i, the Hilbert series of $R/(V_i)$ depends only on $j(F)$. This concludes the proof. □

Example 5.4.4. Let $J = (e_{12}, e_{13}, e_{14}, e_{23}) \subset K\langle e_1, \dots, e_5\rangle$. Then R/J is initially Koszul. Indeed, the flag $F : V_0 = 0 \subset V_1 \subset \dots \subset V_5 = R_1$, where $V_i = \text{span}_K\{e_1, \dots, e_i\}$ for $i = 1, \dots, 5$, is a Gröbner flag of R since $0 :_R (V_1) = (V_4)$, $(V_1) :_R (V_2) = (V_3)$, $(V_2) :_R (V_3) = (V_3)$, $(V_3) :_R (V_4) = (V_4)$ and $(V_4) :_R (V_5) = (V_5)$. Thus $j(F) = (4, 3, 3, 4, 5)$. By Proposition 5.4.3 we get that

$$\begin{aligned}
H_{R/(V_5)}(t) &= H_K(t) = 1,\\
H_{R/(V_4)}(t) &= H_{R/(V_5)}(t) + tH_{R/(V_5)}(t) = 1 + t,\\
H_{R/(V_3)}(t) &= H_{R/(V_4)}(t) + tH_{R/(V_4)}(t) = 1 + 2t + t^2,\\
H_{R/(V_2)}(t) &= H_{R/(V_3)}(t) + tH_{R/(V_3)}(t) = 1 + 3t + 3t^2 + t^3,\\
H_{R/(V_1)}(t) &= H_{R/(V_2)}(t) + tH_{R/(V_3)}(t) = 1 + 4t + 5t^2 + 2t^3,\\
H_{R/(V_0)}(t) &= H_R(t) = H_{R/(V_1)}(t) + tH_{R/(V_4)}(t) = 1 + 5t + 6t^2 + 2t^3.
\end{aligned}$$

Recall that a graded algebra $R = E/J$, where $J \subset E$ is a graded ideal, is G-*quadratic* if J has a quadratic Gröbner basis with respect to some coordinate system of E_1 and some monomial order $<$ on E. Conca, Rossi and Valla in [**16**] and Blum in [**9**] obtained a characterization of the algebras which have Gröbner flags. We sketch an exterior algebra version of [**9**, Proposition 2.3], [**16**, Proposition 2.5] as follows:

Proposition 5.4.5. *Let R be a standard graded K-algebra over E. The following statements are equivalent:*

(i) *R has a Gröbner flag;*
(ii) *there exists a presentation of R, say $R \cong E/J$, such that if $<$ is the reverse lexicographic order induced by the total order $e_1 > e_2 > \dots > e_n$, then $\text{in}_<(J)$ is a quadratic monomial ideal and if $e_ie_j \in \text{in}_<(J)$ with $i < j$ then $e_ke_j \in \text{in}_<(J)$ for all $i < k < j$.*

In particular, if R has a Gröbner flag, then R is G-quadratic.

As direct consequences of Proposition 5.4.5, analogously to [**9**, Corollary 3.2] one has:

Corollary 5.4.6. *Let $J \subset E$ be a stable monomial ideal generated in degree 2. Then E/J is initially Koszul.*

Let $f \in E_2$. Recall that

$$\text{supp}(f) = \{e_ie_j : c_{ij}e_ie_j \text{ is a term of } f \text{ with } 0 \neq c_{ij} \in K\}.$$

Consider the reverse lexicographic order $<$ on E with $e_1 > e_2 > \dots > e_n$. For the case where $J = (f)$ is a principal ideal, we have:

Proposition 5.4.7. *Let $J = (f)$ be a principal ideal where $f \in E_2$. If $R = E/J$ is initially Koszul, then $\text{in}_<(f) = e_ie_{i+1}$ for some $i \in \{1, \dots, n\}$.*

Proof. Let $\mathrm{in}_<(f) = e_ie_j$ for some $1 \le i < j \le n$. Suppose that $j > i+1$. Let $j_1 < \cdots < j_r \in [n]$ be all indices such that $e_ie_{j_k} \in \mathrm{supp}(f)$ and $j_r = j$. Then

$$(e_1, \ldots, e_{i-1}, f) :_R e_i = (e_1, \ldots, e_{i-1}, e_i, e_{j_1} + \cdots + e_{j_r}, f).$$

Since R is initially Koszul, we get that $r = 1$. But then

$$e_{i+1} \notin (e_1, \ldots, e_{i-1}, e_i, e_j, f).$$

This contradicts the definition of an initially Koszul algebra. So $j = i+1$, as desired. □

Remark 5.4.8. A special class of graded algebras with Gröbner flags are the so-called universally initially Koszul algebras defined by Blum [**9**] as follow: A standard graded K-algebra $R = E/J$ over E is called *universally initially Koszul*, write *u-i-Koszul* for short, if every filtration

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = R_1$$

is a Gröbner flag, where V_i is a subspace of dimension i of R_1.

Analogously to [**9**, Proposition 4.10] one can classify all monomial ideals J which define u-i-Koszul algebras. More precisely, a standard graded K-algebra $R = E/J$, where $0 \neq J \subset E$ be a monomial ideal, is u-i-Koszul if and only if $J = \mathfrak{m}^2$.

CHAPTER 6

Orlik-Solomon algebras and ideals

The goal of this chapter is to study classes of essential central hyperplane arrangements whose Orlik-Solomon ideals are componentwise linear or whose Orlik-Solomon algebras satisfy variations of the Koszul property. We relate the first property to resonance varieties of the corresponding Orlik-Solomon algebras. At first, we prove that if J is an Orlik-Solomon ideal of an essential central hyperplane arrangement, then its first resonance variety is irreducible if and only if the subideal of J generated by all degree 2 elements has a 2-linear resolution. As an application we characterize those hyperplane arrangements of rank ≤ 3 where J is componentwise linear. For the general situation, we suggest a conjecture to characterize componentwise linearity of Orlik-Solomon ideals. We also prove complete classifications of Orlik-Solomon algebras which are universally Koszul or have Gröbner flags. Most of the content of this chapter is contained in the preprint [**62**].

6.1. Hyperplane arrangements and resonance varieties

In this section we review some algebraic aspects of hyperplane arrangements with particular attention to their Orlik-Solomon algebras and resonance varieties. For more details, we refer to [**55**, Section 1.4, Chapter 4] and the book by Orlik-Solomon [**46**, Chapter 3].

We always assume that $\mathcal{A} = \{H_1, \dots, H_n\}$ is an essential central hyperplane arrangement in $\mathbb{C}^l$ with the complement $\mathcal{X}(\mathcal{A}) = \mathbb{C}^l \setminus \bigcup_{H \in \mathcal{A}} H$. We say that a set of hyperplanes $\{H_{i_1}, \dots, H_{i_t}\}$ is *dependent* if the set of their defining linear forms is linearly dependent. Let $E = K\langle e_1, \dots, e_n\rangle$ be the standard graded exterior algebra over a field K with $\deg e_i = 1$, $i = 1, \dots, n$ and $\operatorname{char} K = 0$. Let $\partial : E \longrightarrow E$ be the K-linear map on E defined by $\partial\, e_i = 1$ for $i = 1, \dots, n$ and

$$\partial\, e_F = \sum_{j=1}^{t} (-1)^{j-1} e_{i_1} \dots \widehat{e_{i_j}} \dots e_{i_t} \text{ for } F = \{i_1, \dots, i_t\} \subseteq [n] \text{ with } t \geq 2. \tag{10}$$

One can show that ∂ is a differential map on E satisfying the graded Leibniz formula. Moreover, for a set of indices $F = \{i_1, \dots, i_t\} \subseteq [n]$ with $1 \notin F$ one can also check by formula (10) that

$$\partial e_F = (e_{i_2} - e_{i_1}) \dots (e_{i_t} - e_{i_1}) = \sum_{j=1}^{t} (-1)^{j-1}\, \partial\, e_{F \setminus \{i_j\} \cup \{1\}}. \tag{11}$$

In the last decades, many properties of hyperplane arrangements have been studied using the so-called the *Orlik-Solomon algebra* of $\mathcal{A}$. This algebra is the quotient ring E/J where J is the *Orlik-Solomon ideal* of $\mathcal{A}$ given by

$$J = (\partial\, e_F : \{H_i : i \in F\} \text{ is dependent}).$$

Orlik and Solomon [**46**] showed that the cohomology ring of $\mathcal{X}(\mathcal{A})$ is entirely determined by the intersection lattice

$$L(\mathcal{A}) = \{ \bigcap_{H \in \mathcal{A}'} H | \mathcal{A}' \subseteq \mathcal{A}\}$$

of $\mathcal{A}$. More precisely, the singular cohomology $H^\bullet(\mathcal{X}(\mathcal{A}); K)$ of $\mathcal{X}(\mathcal{A})$ with coefficients in K is isomorphic to the Orlik-Solomon algebra of $\mathcal{A}$. See Orlik-Terao [**47**] and Yuzvinsky [**65**] for details. See also, e.g., [**1, 19, 22, 38, 45, 56, 57**] for the study of Orlik-Solomon algebras via exterior algebra methods and algebraic properties of arbitrary modules over E.

Next we collect some facts and results about the intersection lattice and resonance varieties used in the next sections. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential central hyperplane arrangement in $\mathbb{C}^l$ with the intersection lattice $L(\mathcal{A})$. Let J be the Orlik-Solomon ideal and $A = E/J$ be the Orlik-Solomon algebra of $\mathcal{A}$. We denote by $J_{\leq i}$ the ideal generated by all homogeneous elements of degree $\leq i$ of J.

Observe that $L(\mathcal{A})$ is a partially ordered set whose elements are the linear subspaces of $\mathbb{C}^l$ obtained as intersections of sets of hyperplanes from $\mathcal{A}$ and ordered by reverse inclusion. The intersection lattice $L(\mathcal{A})$ is a ranked poset. Indeed, rank(X) is the codimension of X in $\mathbb{C}^l$ for $X \in L(\mathcal{A})$ and rank$(\mathcal{A})$ is the maximal value of $\{\text{rank}(X) : X \in L(\mathcal{A})\}$. See [**47**, Section 2.1] or [**55**, Section 1.2] for details. Note that if $X = H_{i_1} \cap \dots \cap H_{i_t}$ and rank$(X) < t$ then $\{H_{i_1}, \dots, H_{i_t}\}$ is a dependent set. In particular, if rank$(\mathcal{A}) = r$ then all sets of more than r hyperplanes are dependent sets and then $J_{\leq r} = J$.

Since every set of two hyperplanes is independent, we have $J_1 = 0$. Recall that the resonance varieties of an Orlik-Solomon algebra A can be computed by the formulas (1) and (2) (on page 18), i.e.,

$$\begin{aligned} R^1(A) &= \{u \in E_1 : u = 0 \text{ or } \exists v \in E_1, 0 \neq uv \in J_2\}, \\ R^p(A) &= \{u \in E_1 : u = 0 \text{ or } \exists v \in E_p, v \notin J_p + uE_{p-1}, 0 \neq uv \in J_{p+1}\}. \end{aligned}$$

As shown by Falk in [**24**] or by Libgober-Yuzvinsky in [**40**] we know that $R^1(A)$ is an algebraic variety in the affine space $E_1 = K^n$ and each component of $R^1(A)$ is a linear subspace of K^n. Moreover, two distinct irreducible components meet only at 0 and if u, v belong to the same irreducible component of $R^1(A)$, then $uv \in J_2$ (see [**24**], [**40**] for more details). If these two properties hold for the first resonance variety of a graded algebra $A = E/J$, we say that A satisfies *property* $(*)$. Falk also proved that, for each $X \in L_2(\mathcal{A})$ which is the intersection of more than two hyperplanes, there is a corresponding irreducible component of $R^1(A)$, called the *local component* which is defined by

$$\Gamma_X = \{(x_i) \in E_1 = K^n : x_i = 0 \text{ if } X \not\subseteq H_i \text{ and } \sum_{H_i \supseteq X} x_i = 0\}.$$

For higher resonance varieties $R^p(A)$ with $p > 1$, Libgober and Yuzvinsky proved in [**40**] that they are also the union of linear subspaces, but these subspaces can have none-zero intersection. Moreover, the results in [**55**, Theorem 4.46, Corollary 4.49] and [**1**, Theorem 3.1] imply that

$$R^p(A) \subseteq R^q(A) \text{ for } p < q \leq \operatorname{rank}(\mathcal{A}).$$

The rank variety $V_E(A)$ of A is the set of all linear forms in E which are not A-regular elements. This implies that

$$R^p(A) \subseteq V_E(A) \text{ for all } 1 \leq p.$$

Moreover, $V_E(A)$ is a linear subspace of E_1 and $\dim_K V_E(A) = \operatorname{cx}_E(A)$. We refer to the paper of Aramova, Avramov, and Herzog [**1**] for more details.

To conclude this section, let us recall the notion of matroids used in the next sections. Given a ground set $[n] = \{1, \dots, n\}$, a *matroid* on $[n]$ is a collection $\mathcal{C}$ of subsets of $[n]$, called *circuits*, such that:

(i) $\emptyset \notin \mathcal{C}$;
(ii) If $F \in \mathcal{C}$ then $G \notin \mathcal{C}$ for every $G \subsetneq F$;
(iii) If $F, T \in \mathcal{C}$ with $F \neq T$ and $i \in F \cap T$, then $(F \cup T) \setminus \{i\}$ contains a circuit.

A subset of $[n]$ is called a *dependent set* if it contains a circuit. Otherwise, it is called an *independent set*.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential central hyperplane arrangement in $\mathbb{C}^l$. A *circuit* of $\mathcal{A}$ is a subset $\{i_1, \dots, i_t\}$ of $[n]$ such that the set $\{H_{i_1}, \dots, H_{i_t}\}$ is a minimal dependent sets of hyperplanes. Let $M(\mathcal{A})$ be the collection of all circuits of $\mathcal{A}$. One can check that $M(\mathcal{A})$ satisfies the axioms of a matroid as above. We say that $M(\mathcal{A})$ is the *underlying matroid* of $\mathcal{A}$ (or the *matroid* of $\mathcal{A}$ for short).

6.2. Orlik-Solomon ideals with irreducible resonance varieties

The goal of this section is to present results related to the question in which cases the resonance varieties of hyperplane arrangements are irreducible. At first we consider the first resonance variety and we get the following main result of this section:

Theorem 6.2.1. *Let $\mathcal{A}$ be an essential central hyperplane arrangement with Orlik-Solomon ideal J and Orlik-Solomon algebra $A = E/J$. The following statements are equivalent:*

(i) *The first resonance variety $R^1(A)$ of A is irreducible;*
(ii) *The ideal $J_{\langle 2\rangle}$ has a 2-linear resolution.*

In particular, if J is componentwise linear, then $R^1(A)$ is irreducible.

Proof. (i) $\Rightarrow$ (ii): Assume that $R^1(A) \neq \{0\}$ is irreducible. By property $(*)$ of A, we get that $J_2 \neq 0$. Since elements of $L_2(\mathcal{A})$, which are intersections of more than two hyperplanes, correspond to the local components of $R^1(A)$ as noted above (see [**24**]), there is exactly one element X in $L_2(\mathcal{A})$ which is an intersection of more than two hyperplanes. We choose a maximal integer s with $3 \leq s \leq n$ such that X

is the intersection of s hyperplanes of the arrangement. Without loss of generality we assume that $\mathcal{A} = \{H_1, \dots, H_s, H_{s+1}, \dots, H_n\}$ and $X = H_1 \cap H_2 \cap \dots \cap H_s$.

Let $F = \{i, j, k\} \subseteq \{1, \dots, s\}$ with $|F| = 3$. Since

$$2 = \operatorname{rank}(H_1 \cap H_2 \cap \dots \cap H_s) \geq \operatorname{rank}(H_i \cap H_j \cap H_k) \geq 2,$$

we get that $H_i \cap H_j \cap H_k = H_1 \cap H_2 \cap \dots \cap H_s$. Thus F is a dependent set of $\mathcal{A}$. Next we assume that $G = \{i, j, k\} \subseteq \{1, \dots, n\}$ with $|G| = 3$ and for example $i \geq s + 1$. If G is dependent, then $H_i \cap H_j \cap H_k$ would have rank 2 which implies by our assumption on $L_2(\mathcal{A})$ that $H_i \cap H_j \cap H_k = X$. But then it would follow that

$$X = H_1 \cap H_2 \cap \dots \cap H_s = H_1 \cap H_2 \cap \dots \cap H_s \cap H_i \cap H_j \cap H_k$$

which is a contradiction to the choice of s. Hence

$$\begin{aligned} J_{\langle 2 \rangle} &= (\partial e_F : F \text{ is dependent}, |F| = 3) \\ &= ((e_i - e_k)(e_j - e_k) : \{i, j, k\} \text{ is dependent for } 1 \leq i < j < k \leq s) \\ &= ((e_i - e_1)(e_j - e_1) : \{1, i, j\} \text{ is dependent for } 2 \leq i < j \leq s) \\ &= (e_2 - e_1, \dots, e_s - e_1)^2. \end{aligned}$$

Note that we used at the third equation formula (11) from page 68. We get that $J_{\langle 2 \rangle}$ is a square of a linear ideal, say I, so $J_{\langle 2 \rangle}$ has a 2-linear resolution. In fact, after an appropriate change of coordinates, we may assume that I is generated by variables, say $I = (e_1, \dots, e_{s-1})$. Then every power of I is a stable monomial ideal of E which is generated in one degree and has a linear resolution (see, e.g., [**3**, Corollary 3.4 (a)]).

(ii) $\Rightarrow$ (i): Since $J_{\langle 2 \rangle}$ has a 2-linear resolution, $J_{\langle 2 \rangle}$ and $\operatorname{gin}(J_{\langle 2 \rangle})$ have the same graded Betti numbers (see [**4**, Theorem 2.1]). Thus

$$\operatorname{reg}_E(J_{\langle 2 \rangle}) = \operatorname{reg}_E(\operatorname{gin}(J_{\langle 2 \rangle})) = 2.$$

Hence $\operatorname{gin}(J_{\langle 2 \rangle})$ is a strongly stable monomial ideal generated in degree 2. So we have $G(\operatorname{gin}(J_{\langle 2 \rangle}))_2 = G(\operatorname{gin}(J_{\langle 2 \rangle}))$. By Lemma 1.3.2 (ii), we get that

$$\beta^E_{i,i+2}(J_{\langle 2 \rangle}) = \beta^E_{i,i+2}(\operatorname{gin}(J_{\langle 2 \rangle})) = \sum_{u \in G(\operatorname{gin}(J_{\langle 2 \rangle}))} \binom{\max(u) + i - 1}{\max(u) - 1}.$$

We consider the polynomial function

$$P \colon \mathbb{Q} \to \mathbb{Q}, \quad i \mapsto P(i) = \sum_{u \in G(\operatorname{gin}(J_{\langle 2 \rangle}))} \binom{\max(u) + i - 1}{\max(u) - 1}.$$

Observe that $\deg P = t - 1$ where $t = \max\{\max(u) : u \in G(\operatorname{gin}(J_{\langle 2 \rangle}))\}$. It is a consequence of [**56**, Theorem 4.3] that $\deg P = \dim R^1(A) - 1$. Recall that we consider $R^1(A)$ as an affine variety in $E_1 = K^n$ while in [**56**] this space is viewed as a projective variety. It follows $\dim R^1(A) = t$. As noted above $R^1(A)$ is the union of linear components Γ_j. There exists one linear component, say Γ_p, of $R^1(A)$ such that $\dim \Gamma_p = t$. By [**56**, Theorem 5.6] we have for $i \gg 0$ that

$$\beta^E_{i,i+2}(J_{\langle 2\rangle}) \geq \sum_{\Gamma_j \text{ component of } R^1(\mathcal{A})} (i+1)\binom{\dim \Gamma_j + i}{i+2} \geq (i+1)\binom{t+i}{i+2}.$$

Using the fact that $\operatorname{gin}(J_{\langle 2\rangle})$ and $(e_1,\dots,e_t)^2$ are both strongly stable monomial ideals generated in degree 2 and by the definition of t we get

$$G(\operatorname{gin}(J_{\langle 2\rangle})) \subseteq G((e_1,\dots,e_t)^2).$$

Then we see with Lemma 1.3.2 (ii) and a direct computation (see, e.g., [**38**, Proposition 6.12]) that

$$\beta^E_{i,i+2}(\operatorname{gin}(J_{\langle 2\rangle})) \leq \beta^E_{i,i+2}((e_1,\dots,e_t)^2) = (i+1)\binom{t+i}{i+2} \text{ for all } i \geq 0.$$

Using all inequalities together we get that

$$\beta^E_{i,i+2}(J_{\langle 2\rangle}) = \beta^E_{i,i+2}(\operatorname{gin}(J_{\langle 2\rangle})) = (i+1)\binom{t+i}{i+2} \text{ for } i \gg 0.$$

Using again [**56**, Theorem 5.6] this implies that $R^1(A)$ has exactly one irreducible component. Thus $R^1(A)$ is irreducible. □

If the rank of the arrangement is small, we get:

Corollary 6.2.2. *Let $\mathcal{A}$ be an essential central hyperplane arrangement such that* $\operatorname{rank}(\mathcal{A}) \leq 3$ *with Orlik-Solomon ideal J and Orlik-Solomon algebra $A = E/J$. The following statements are equivalent:*

(i) *The first resonance variety $R^1(A)$ of A is irreducible;*
(ii) *J is componentwise linear.*

Proof. (i) ⇒ (ii): Since $R^1(A)$ is irreducible, we get that $J_{\leq 2} = J_{\langle 2\rangle}$ has a 2-linear resolution. Thus $\operatorname{reg}(J_{\leq 2}) = 2$.

We have $J = J_{\leq 3}$ because $\operatorname{rank}(\mathcal{A}) \leq 3$. It follows from [**38**, Corollary 6.7] that

$$\operatorname{reg}(J_{\leq 3}) = \operatorname{reg}(J) = \operatorname{reg}(E/J) + 1 \leq 3.$$

Moreover, $\operatorname{reg}(J_{\leq k}) \leq 3 \leq k$ for $k \geq 3$. Using [**37**, Theorem 5.3.7] we see that J is componentwise linear.

(ii) ⇒ (i): If J is componentwise linear, then $J_{\langle 2\rangle}$ has a 2-linear resolution. Hence Theorem 6.2.1 implies that $R^1(A)$ is irreducible. □

Note that a graded ideal which has a linear resolution is also componentwise linear. But the converse is not true even for the case the ideal is an Orlik-Solomon ideal, i.e., there exist Orlik-Solomon ideals which are componentwise linear and do not have linear resolutions as the following example shows.

Example 6.2.3. Let $\mathcal{A}$ be an essential central hyperplane arrangement in $\mathbb{C}^3$ defined by the equation

$$Q = xy(x-y)z(2x+y-z)(x+3y+z).$$

Let $E = K\langle e_1, \dots, e_6\rangle$ be the exterior algebra where each e_i corresponds to the i-th factor in the polynomial. The Orlik-Solomon ideal of $\mathcal{A}$ is

$$J = (\partial e_{123}) + (\partial e_{ijkl} : \{i, j, k, l\} \subseteq [6]).$$

We see that $L_2(\mathcal{A})$ has only one element $X = H_1 \cap H_2 \cap H_3$ which is an intersection of more than two hyperplanes. Hence $R^1(A) = \Gamma_X = \operatorname{span}_K\{(e_2 - e_1), (e_3 - e_1)\}$ is irreducible. By Corollary 6.2.2, the ideal J is a componentwise linear ideal. We observe that the elements ∂e_{ijkl} are not redundant for all $1 \le i, j, k, l \le 6$, so $J \neq J_{\langle 2\rangle}$. This implies that J is not generated in one degree. Hence J does not have a linear resolution.

We saw that the componentwise linear property of an Orlik-Solomon ideal can be characterized in terms of data of the hyperplane arrangement if the rank is small. We wonder if a similar statement can be proved for arbitrary essential central hyperplane arrangements. Note that a characterization for Orlik-Solomon ideals to have a linear resolution is given in [**22**, Corollary 3.6]; see also [**38**, Theorem 6.11] which is a first step to such a result.

We ask ourself:

Question 6.2.4. Assume that the Orlik-Solomon ideal J of an essential central hyperplane arrangement $\mathcal{A}$ is componentwise linear. Are then all resonance varieties $R^p(A)$, where $0 \le p \le \operatorname{rank}(\mathcal{A})$, irreducible?

Recall that the simplest matroids are the *uniform matroids* $U_{p,q}$ with $p \le q$. They are matroids over the ground set $[q]$ whose independent sets are all subsets of $[q]$ of cardinality $\le p$. Supporting Question 6.2.4, we prove that if the Orlik-Solomon ideal has a linear resolution, then resonance varieties are irreducible.

Proposition 6.2.5. *Let $\mathcal{A}$ be an essential central hyperplane arrangement with Orlik-Solomon ideal J and Orlik-Solomon algebra $A = E/J$ such that J has a d-linear resolution for $2 \le d \le n$. Then Question 6.2.4 has an affirmative answer. More precisely, we have*

$$R^p(A) = 0 \text{ for } 0 \le p \le d-2 \text{ and } R^{d-1}(A) = V_E(A) \text{ is irreducible.}$$

Proof. Assume that J has a d-linear free resolution. Then $J_0 = \ldots = J_{d-1} = \{0\}$ and $J = (J_d)$. Hence $R^p(A) = 0$ for $0 \le p \le d-2$. By [**38**, Theorem 6.11 (iii)], the matroid $M(\mathcal{A})$ of $\mathcal{A}$ is $M(\mathcal{A}) = U_{d,n-f} \oplus U_{f,f}$ where $U_{d,n-f}$, $U_{f,f}$ are uniform matroids. Therefore,

$$\begin{aligned} J = J_{\langle d\rangle} &= (\partial e_F : F \in U_{d,n-f}, |F| = d+1) \\ &= ((e_{i_2} - e_{i_1}) \ldots (e_{i_{d+1}} - e_{i_1}) : F = \{i_1, \ldots, i_{d+1}\} \subseteq [n-f]) \\ &= ((e_{i_2} - e_1) \ldots (e_{i_{d+1}} - e_1) : F = \{1, i_2, \ldots, i_{d+1}\} \subseteq [n-f]). \end{aligned}$$

Note that we used at the third equation formula (11) (on page 68). Thus by formula (2), we get

$$R^{d-1}(A) = \operatorname{span}_K\{e_i - e_1 : 2 \le i \le n-f\}.$$

This implies already that $R^{d-1}(A)$ is irreducible and $\dim_K R^{d-1}(A) = n - f - 1$.

Note that $U_{f,f} = \bigoplus_{i=1}^{f} U_{1,1}$. So $M(\mathcal{A})$ has $(f+1)$ connected components following [**38**, Page 200]. By [**1**, Theorem 3.1, 3.2] and [**38**, Corollary 6.5], we get that

$$\dim_K V_E(A) = \mathrm{cx}_E(A) = n - \mathrm{depth}_E(A) = n - f - 1 = \dim_K R^{d-1}(A).$$

Since $R^{d-1}(A) \subseteq V_E(A)$, we conclude that $R^{d-1}(A) = V_E(A)$. □

Remark 6.2.6. We have some evidences based on computations that the converse of Proposition 6.2.5 is true. So we can ask assuming that $R^p(A) = 0$ for $0 \leq p \leq d-2$ and $R^{d-1}(A) = V_E(A)$ is irreducible, if then J has a d-linear resolution. Moreover, if $J \subset E$ is an arbitrary graded ideal with a d-linear resolution, one interesting problem could be whether $R^{d-1}(E/J)$ is always maximal (i.e., $R^{d-1}(E/J) = V_E(E/J)$) or at least irreducible.

Next we present another corollary of Theorem 6.2.1. For this, we recall the following conjecture of Suciu and Schenck in [**57**, Conjecture B, page 2271].

Conjecture 6.2.7. *Let $\mathcal{A}$ be an essential central hyperplane arrangement with Orlik-Solomon ideal $J \subset E$. Then for $i \gg 0$, the graded Betti numbers of the linear strand of E/J are given by*

$$\beta^E_{i,i+1}(E/J) = i \sum_{r \geq 1} h_r \binom{r+i-1}{i+1},$$

where h_r is the number of r-dimensional components of $R^1(E/J)$ in the space K^n.

Supporting this conjecture, we have:

Corollary 6.2.8. *Let $\mathcal{A}$ be an essential central hyperplane arrangement with Orlik-Solomon ideal J and Orlik-Solomon algebra $A = E/J$ such that $J_{\langle 2 \rangle}$ has a 2-linear resolution. Then Conjecture 6.2.7 is true for E/J.*

Proof. Let $t = \max\{\max(u) : u \in G(\mathrm{gin}(J_{\langle 2 \rangle}))\}$. In the proof of Theorem 6.2.1 we showed that

$$\beta^E_{i,i+2}(J) = \beta^E_{i,i+2}(J_{\langle 2 \rangle}) = (i+1)\binom{t+i}{i+2} \text{ for } i \gg 0.$$

We know also that $h_r = 1$ for $r = \dim R^1(E/J) = t$ and $h_r = 0$ for $r \neq t$ since $R^1(E/J)$ is irreducible. Combining with the fact that $\beta^E_{i+1,i+2}(E/J) = \beta^E_{i,i+2}(J)$, we can conclude the proof. □

As we have seen in Corollary 6.2.8, in the case of hyperplane arrangements, the irreducibility of the first resonance ensures that Conjecture 6.2.7 is true. In the following, we present a counter example in which the irreducibility of the first resonance does not imply the formula for the Betti numbers as in Conjecture 6.2.7.

Example 6.2.9. Let $E = K\langle e_1, \dots, e_5 \rangle$ be the exterior algebra over a field K. Let $J = (e_{12}, e_{13}, e_{14}, e_{15}, e_{234}) \subset E$. We see that J is a strongly stable monomial ideal. By Lemma 1.5.1 we get

$$R^1(E/J) = \mathrm{span}_K\{e_1, \dots, e_5\} \text{ and } R^1(E/J) \text{ is irreducible.}$$

Note that E/J does not have property $(*)$ since $e_2, e_5 \in R^1(E/J)$ but $e_2e_5 \notin J_2$.

By Lemma 1.3.2 (ii), for $i \geq 0$, we have

$$\beta^E_{i,i+2}(J) = \sum_{u \in G(J)_2} \binom{\max(u)+i-1}{\max(u)-1} = \binom{i+1}{1} + \binom{i+2}{2} + \binom{i+3}{3} + \binom{i+4}{4}.$$

By induction, one can proves that

$$\binom{i+1}{1} + \binom{i+2}{2} + \binom{i+3}{3} + \binom{i+4}{4} < (i+1)\binom{i+5}{i+2}, \text{ for } i \geq 0.$$

Then we get

$$\beta^E_{i+1,i+2}(E/J) = \beta^E_{i,i+2}(J) < (i+1)\binom{i+5}{i+2} = (i+1)\sum_{r\geq 1} h_r \binom{r+i}{i+2}$$

where h_r is the number of components of $R^1(E/J)$ which have dimension r in the affine space $E_1 = K^n$. Here $h_r = 0$ for $r \neq 5$ and $h_5 = 1 \neq 0$. Thus we do not get the same formula of graded Betti numbers as in the statement of Conjecture 6.2.7.

However, a monomial ideal could satisfy property $(*)$ and Conjecture 6.2.7 still makes sense in this case. An example for this is in the following remark.

Remark 6.2.10. Let G be a graph on a finite vertex set V_G and with edge set E_G. For a vertex $v \in V_G$ let $\deg v$ denote the number of edges incidents to v. Recall that a graph G is a disjoint union of complete graphs if there exist complete graphs G_i such that the vertex sets V_{G_i} of G_i are disjoint, $|V_{G_i}| \geq 2$, the vertex set V_G of G is $V_G = \bigcup_i V_{G_i}$ and the edge set E_G of G is $E_G = \bigcup_i E_{G_i}$. Let $n = |V_G|$ and E be the exterior algebra on n exterior variables $e_1, \ldots, e_n$ over a field K. The edge ideal $J(G)$ of G is defined as $J(G) = (e_ie_j : \{\, i,j\} \in E_G)$. Then we claim that $E/J(G)$ satisfies property $(*)$ and the graded Betti numbers in the linear strand of $J(G)$ are given by

$$\beta^E_{i,i+2}(J(G)) = (i+1)\sum_{r=2}^{n} h_r \binom{r+i}{i+2},$$

where h_r is the number of r-dimensional components of $R^1(E/J)$ in the affine space K^n. So Conjecture 6.2.7 is true for edge ideals of disjoint unions of complete graphs. To prove this, we need the following lemmas.

Lemma 6.2.11. *Let G be a disjoint union of complete graphs and $n = |V_G|$. Then $R^1(E/J(G))$ is a union of linear subspaces and $E/J(G)$ satisfies property $(*)$.*

Proof. Let G be the disjoint union of complete graphs $G_1, \ldots, G_t$. Let $r_i = |V_{G_i}|$ and so $n = \sum_{i=1}^t r_i$. Consider the edge ideals $J(G)$ and $J(G_i)$ in the exterior algebra E. It is clear that $J(G) = \sum_{i=1}^t J(G_i)$. The first resonance variety of $E/J(G)$ can be computed as

$$R^1(E/J(G)) = \{u \in E_1 : u = 0 \text{ or } \exists v \in E_1 \text{ such that } 0 \neq u \wedge v \in J(G)\}. \tag{12}$$

Let $V_{G_i} = \{i_j : j = 1, \dots, r_i\} \subseteq [n]$. Because of $e_{i_p} \wedge e_{i_q} \in J(G_i)$ for $1 \leq p, q \leq r_i$ and Equation (12) we have

$$\operatorname{span}_K\{e_{i_1}, \dots, e_{i_{r_i}}\} \subseteq R^1(E/J(G)).$$

We claim that the irreducible components of $R^1(E/J(G))$ are exactly the vector spaces $\operatorname{span}_K\{e_{i_1}, \dots, e_{i_{r_i}}\}$ for $1 \leq i \leq t$. Assume that there exists an irreducible component which is not of this form. Then there exist linear forms $u, v \in E_1$ such that

$$0 \neq u \wedge v \in J(G) \text{ and } u \notin \operatorname{span}_K\{e_{i_1}, \dots, e_{i_{r_i}}\} \text{ for all } 1 \leq i \leq t.$$

Let $u = \sum_{k=1}^n \alpha_k e_k$ and $v = \sum_{k=1}^n \beta_k e_k$ for $\alpha_k, \beta_k \in K$.

Now we show that $\operatorname{supp}(u) = \operatorname{supp}(v)$. For this let $k_1 \in \operatorname{supp}(v)$ be arbitrary and choose i such that $k_1 \in V_{G_i}$. Since $\operatorname{supp}(u)$ is not contained in V_{G_i} there exists $k_2 \in \operatorname{supp}(u)$ with $k_2 \notin V_{G_i}$. Observe that $e_p \wedge e_q \in J(G) = \sum_{i=1}^t J(G_i)$ if and only if there is $1 \leq i \leq t$ such that $p, q \in V_{G_j}$ for some j. So $e_{k_1} \wedge e_{k_2} \notin J(G)$. It follows that $\alpha_{k_1}\beta_{k_2} - \alpha_{k_2}\beta_{k_1} = 0$ because $u \wedge v \in J(G)$. Hence

$$k_1 \in \operatorname{supp}(u),\ k_2 \in \operatorname{supp}(v) \text{ and } \alpha_{k_2}/\beta_{k_2} = \alpha_{k_1}/\beta_{k_1}.$$

In particular, we see that $\operatorname{supp}(v) \subseteq \operatorname{supp}(u)$ and $\operatorname{supp}(v)$ is not contained in one of the V_{G_j}. With the same arguments we get that

$$\operatorname{supp}(u) \subseteq \operatorname{supp}(v) \text{ and then } \operatorname{supp}(u) = \operatorname{supp}(v).$$

Moreover, we also get that for every $k, k' \in \operatorname{supp}(u) = \operatorname{supp}(v)$, if k and k' do not live in the same V_{G_i} for some $1 \leq i \leq t$, then $\alpha_k/\beta_k = \alpha_{k'}/\beta_{k'}$. Since $\operatorname{supp}(u)$ is not contained in one of the V_{G_j}, we can conclude that α_k/β_k is the same constant for every $k \in \operatorname{supp}(u) = \operatorname{supp}(v)$. But then we get the contradiction that $u \wedge v = 0$.

So we see that all irreducible components of $R^1(E/J(G))$ are induced by the complete subgraphs of G. More precisely,

$$R^1(E/J(G)) = \bigcup_{i=1}^t \operatorname{span}_K\{e_{i_1}, \dots, e_{i_{r_i}}\}.$$

We also get that $R^1(E/J(G))$ satisfies property($*$) on Page 2. □

Lemma 6.2.12. *Let i, r be integers with $i, r \geq 0$. Then we have*

$$\sum_{j=0}^{i} \binom{i}{j}\binom{r}{j+2} = \binom{r+i}{i+2}.$$

Proof. Considering the polynomial $f(x) = (1+x)^{r+i}$ in the polynomial ring $K[x]$, we get

$$(1+x)^{r+i} = (1+x)^i(1+x)^r = (\sum_{j=0}^{i}\binom{i}{j}x^j)(\sum_{t=0}^{r}\binom{r}{t}x^t).$$

This implies that the coefficient of x^{r-2} is

$$\sum_{j=0}^{i}\binom{i}{j}\binom{r}{r-2-j} = \sum_{j=0}^{i}\binom{i}{j}\binom{r}{j+2}.$$

Moreover, $(1+x)^{r+i} = \sum_{j=0}^{r+i} \binom{r+i}{j} x^j$. Thus the coefficient of x^{r-2} in this equation is $\binom{r+i}{r-2} = \binom{r+i}{i+2}$. Hence we conclude that $\sum_{j=0}^{i} \binom{i}{j}\binom{r}{j+2} = \binom{r+i}{i+2}$. □

We are ready to prove the claim in Remark 6.2.10:

Proof. Let G be the disjoint union of complete graphs $G_1, \dots, G_t$ with the vertex sets $V_{G_i} = \{i_1, \dots, i_{r_i}\}$, $i = 1, \dots, t$. By Lemma 6.2.11 we see that

$$R^1(E/J(G)) = \bigcup_{i=1}^{t} \operatorname{span}_K\{e_{i_1}, \dots, e_{i_{r_i}}\}.$$

Let $k_i(G)$ be the number of complete subgraph on i vertices of G. Observe that a disjoint union of complete graphs has no induced 4-cycles. It follows from [**51**, Proposition 2.4] that

$$\begin{aligned}
\beta^S_{i,i+2}(I(G)) &= \sum_{v \in V_G} \binom{\deg v}{i+1} - k_{i+2}(G) = \sum_{j=1}^{t} \sum_{v \in V_{G_j}} \binom{\deg v}{i+1} - k_{i+2}(G) \\
&= \sum_{j=1}^{t} \sum_{v \in V_{G_j}} \binom{r_j - 1}{i+1} - \sum_{j=1}^{t} k_{i+2}(G_j) = \sum_{j=1}^{t} r_j \binom{r_j - 1}{i+1} - \sum_{j=1}^{t} \binom{r_j}{i+2} \\
&= \sum_{r=1}^{t} r \cdot h_r \binom{r-1}{i+1} - \sum_{r=1}^{t} h_r \binom{r}{i+2} = (i+1) \sum_{r=1}^{t} h_r \binom{r}{i+2}.
\end{aligned}$$

Here $S = K[x_1, \dots, x_n]$ is the polynomial ring over K, the ideal $I(G) = (x_i x_j : \{i,j\} \in E_G)$ is the edge ideal of G over S and $\beta^S_{i,j}(I(G))$ denote the graded Betti numbers of $I(G)$ over S. Note that $I(G)$ is a so-called squarefree S-module in the sense of [**64**, Definition 2.1]. Then it follows from [**52**, Corollary 1.3] that

$$\begin{aligned}
\beta^E_{i,i+2}(J(G)) &= \sum_{j=0}^{i} \binom{i+1}{j+1} \beta^S_{j,j+2}(I(G)) = \sum_{j=0}^{i} \binom{i+1}{j+1} (j+1) \sum_{r=1}^{t} h_r \binom{r}{j+2} \\
&= (i+1) \sum_{j=0}^{i} \binom{i}{j} h_r \sum_{r=1}^{t} \binom{r}{j+2} = (i+1) \sum_{r=1}^{t} h_r \sum_{j=0}^{i} \binom{i}{j} \binom{r}{j+2} \\
&= (i+1) \sum_{r=1}^{t} h_r \binom{r+i}{i+2},
\end{aligned}$$

where we get the last equality from Lemma 6.2.12. Since

$$\beta^E_{i,i+2}(J(G)) = \beta^E_{i+1,i+2}(E/J(G)),$$

Conjecture 6.2.7 holds for $E/J(G)$. □

6.3. Componentwise linear Orlik-Solomon ideals

The goal of this section is to investigate cases in which Orlik-Solomon ideals are componentwise linear. We suggest a conjecture characterizing componentwise linearity of Orlik-Solomon ideals and study exterior algebras with few number of variables or arrangements with small ranks.

Let $\mathcal{A}$ be an essential central hyperplane arrangement with Orlik-Solomon ideal J, Orlik-Solomon algebra $A = E/J$. For an element X in the intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$, we denote $|X| = s$ if X can be expressed as an intersection of maximally s hyperplanes and in this section we identify X with the sequence of indices of s hyperplanes whose intersection equals to X, i.e., if $|X| = s$ and $X = H_{i_1} \cap \ldots \cap H_{i_s}$ then we write $X = \{i_1, \ldots, i_s\}$ up to a permutation of indices. We always have $|X| \geq \text{rank}(X)$. Recall that X is a *dependent element* in $L(\mathcal{A})$ if $|X| > \text{rank}(X)$ and otherwise X is an *independent element*. If every subset of $\leq \text{rank}(X)$ elements of X is independent, then X is said to be *uniform*, otherwise we call X is *non-uniform*. We also denote by $\overline{L_i}$ the subset of $L(\mathcal{A})$ containing all dependent elements of rank i in $L(\mathcal{A})$. A set $\{j_1, \ldots, j_t\} \subseteq [n]$ is said to be a *circuit* of $\mathcal{A}$ if $\{H_{j_1}, \ldots, H_{j_t}\}$ is a minimal dependent set of hyperplanes.

Let $X \in L(\mathcal{A})$ with $\text{rank}(X) = r$ and $|X| = s \geq r+1$. Observe that every subset of $r+1$ elements of X is dependent. Let J_X be the ideal generated by

$$J_X = (\partial\, e_F : F \subseteq X, |F| = r+1).$$

We suggest and study the following conjecture:

Conjecture 6.3.1. *Let $\mathcal{A}$ be an essential central hyperplane arrangement of* rank m *with Orlik-Solomon ideal J. The following statements are equivalent:*

(i) *J is componentwise linear;*

(ii) *There exist dependent elements X_i of rank i in $L(\mathcal{A})$ for $2 \leq i \leq m$ such that*

$$X_2 \subset X_3 \subset \cdots \subset X_m$$

and every circuit of $\mathcal{A}$ of rank i is a subset of X_i.

Let us illustrate the above notions and Conjecture 6.3.1 by the following example:

Example 6.3.2. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^4$ with the defining polynomial

$$Q = xy(x+y)z(x+z)t.$$

Let E be the exterior algebra of variables $e_1, \ldots, e_6$ where e_i responds to i-th factor in the equation of $\mathcal{A}$. Then $\overline{L_2} = \{\{1,2,3\},\{1,4,5\}\}$, $\overline{L_3} = \{\{1,2,3,4,5\}\}$, $\overline{L_4} = \{\{1,2,3,4,5,6\}\}$. Moreover, $\{1,2,3\}$ and $\{1,4,5\}$ are uniform elements, $\{1,2,3,4,5\}$ is a non-uniform element because it contains a dependent proper subset $\{1,2,3\}$.

Given an element X in $L(\mathcal{A})$, we can compute its associated ideal J_X. For instance for $X_3 = \{1,2,3,4,5\} \in \overline{L_3}$ we have

$$J_{X_3} = (\partial\, e_F : F \subset [5], |F| = 4).$$

Now we get

$$J = (\partial\, e_{123}, \partial\, e_{145}, \partial\, e_{ijkl}) \text{ and } J_{\langle 3\rangle} = \mathfrak{m}(\partial\, e_{123}, \partial\, e_{145}) + J_{X_3},$$

where $\{i,j,k,l\}$ runs over all four-tuples of indices $1,2,3,4,5$. Since $\overline{L_2}$ has two elements, $R^1(A)$ has 2 local components. So $R^1(A)$ is not irreducible. Thus $J_{\langle 2\rangle}$

does not have a 2-linear resolution by Theorem 6.2.1. Hence J is not componentwise linear. The condition in Conjecture 6.3.1 (ii) is also not fulfilled. So Conjecture 6.3.1 is true in this example.

Remark 6.3.3. We list some cases where Conjecture 6.3.1 is true:

(i) The number of hyperplanes of $\mathcal{A}$ is less than or equal to 6 as we will see in Example 6.3.8.
(ii) $\operatorname{rank}(\mathcal{A}) \leq 3$ by Theorem 6.2.1 and Corollary 6.2.2.
(iii) $\operatorname{rank}(\mathcal{A}) = 4$ and all dependent elements of rank 3 in $L(\mathcal{A})$ are non-uniform by Theorem 6.3.7.

We can prove the direction "(ii)⇒ (i)" of Conjecture 6.3.1.

Theorem 6.3.4. *The implication "(ii) ⇒ (i)" of Conjecture 6.3.1 is true.*

To prove this theorem, we need the following lemma:

Lemma 6.3.5. *Let $X = \{i_1, \ldots, i_m\}$ be a dependent element of rank r in $L(\mathcal{A})$. Then J_X has a system of minimal generators of the form*

$$G(J_X) = \{\partial\, e_F : F \subset X, |F| = r+1 \text{ and } i_1 \in F\}.$$

Proof. At first we prove that $G(J_X)$ is a linearly independent set over K. This follows from the fact that $e_{i_1} G(J_X)$ is a set of disjoint monomials since $e_{i_1}\partial\, e_F = e_F$ because F contains i_1.

Moreover, for $T \subset X, |T| = r+1, i_1 \notin T$, by Equation (11) we get that $\partial\, e_T \in (G(J_X))$. Since

$$J_X = (\partial\, e_T : T \subset X, |T| = r+1)$$

we have $G(J_X)$ is a set of generators. This concludes the proof. □

Now we are ready to prove Theorem 6.3.4:

Proof. Since $J = (\partial\, e_F : F \text{ is a circuit })$, we have $J = \sum_{i=1}^{r} J_{X_i}$. Without loss of generality, we can assume that $X_i = \{1, \ldots, s_i\}$ for $i = 2, \ldots, m+1$, where $s_i = |X_i|$. Note that $s_2 < s_3 < \ldots < s_{m+1}$. By Lemma 6.3.5, for $i = 3, \ldots, m+1$, we can choose a minimal system of generators of J_{X_i} whose generators are of the form $\partial\, e_F$ where $F \subset X$ and $1 \in F$. Then by changing the coordinates

$$\varphi : E_1 \longrightarrow E_1, e_i \longmapsto \begin{cases} e_1 \text{ if } i = 1, \\ e_i + e_1 \text{ if } i \geq 2, \end{cases}$$

and the fact that $\varphi(\partial\, e_F) = e_{F\setminus\{1\}}$ for $1 \in F$, we get that $\varphi(J_{X_i})$ is a strongly stable monomial ideal with respect to the ordering $e_2 > \ldots > e_n > e_1$ and it is generated in one degree for $i = 2, \ldots, m+1$. Now it is clear that the sum of strongly stable monomial ideals is a strongly stable monomial ideal. Therefore, $\varphi(J)$ is strongly stable monomial ideal. This implies that $\varphi(J)$ is componentwise linear. Hence J is componentwise linear. □

In Remark 6.3.3, we saw that Conjecture 6.3.1 is true for $\operatorname{rank}(\mathcal{A}) \leq 3$. One of the next interesting cases is $\operatorname{rank}(\mathcal{A}) = 4$. For this, let $\overline{L_3} = \{X_{31}, X_{32}, \ldots, X_{3s}\}$

be the set of all dependent elements of rank 3. We have $J_{\langle 3\rangle} = \mathfrak{m}J_{\langle 2\rangle} + \sum_{j=1}^{s} J_{X_{3j}}$, where $\mathfrak{m} = (e_1, \dots, e_n)$. For the case all elements of $\overline{L_3}$ are non-uniform, we have the following results:

Proposition 6.3.6. *Suppose that* $\operatorname{rank}(\mathcal{A}) = 4$ *and all elements of* $\overline{L_3}$ *are non-uniform. If there exist two elements of* $\overline{L_3}$*, say* $X_{31}, X_{32} \in \overline{L_3}$*, such that* X_{31}, X_{32} *contain the same element of rank 2 of* $L(\mathcal{A})$*, say* X_2*, and* $|X_{31}|, |X_{32}| \geq |X_2| + 2$*, then* J *is not componentwise linear.*

Proof. If $L(\mathcal{A})$ has more than two dependent elements of rank 2 then $R^1(A)$ is not irreducible. Hence by Theorem 6.2.1, we get that $J_{\langle 2\rangle}$ does not have a 2-linear resolution and then J is not componentwise linear. Since every dependent element of rank 3 in $\overline{L_3}$ is non-uniform, $\overline{L_2} \neq \emptyset$. It remains to consider the case that $L(\mathcal{A})$ has exactly one dependent element of rank 2, say $X_2 = \{1, \dots, t\}$ where $t \geq 3$. Then $X_2 \subset X$ and $1 \in X$ for all $X \in \overline{L_3}$. By Lemma 6.3.5 applied to X_2 and $X \in \overline{L_3}$, we get that J_{X_2} and J_X have systems of minimal generators of the forms

$$\{\partial\, e_T : T \subset X_2, 1 \in T, |T| = 3\} \text{ and } \{\partial\, e_F : F \subset X, 1 \in F, |F| = 4\},$$

respectively. Note that for $T \subseteq [n]$ with $1 \in T$, we have

$$e_1\, \partial\, e_T = e_T \text{ and } (e_i - e_1)\, \partial\, e_T = \begin{cases} 0 \text{ if } i \in T, \\ \partial\, e_{T\cup\{i\}} \text{ if } i \notin T. \end{cases}$$

Since $J_{\langle 3\rangle} = \mathfrak{m}J_{X_2} + \sum_{X\in\overline{L_3}} J_X$ and $\mathfrak{m} = (e_1, e_2 - e_1, \dots, e_n - e_1)$ we get that $J_{\langle 3\rangle}$ has a system of generators of the form

$$\{\partial\, e_F, e_T : F, T \text{ are dependent and } 1 \in F, 1 \in T, |F| = 4, |T| = 3\}.$$

By changing the coordinates using φ as in the proof of Theorem 6.3.4, i.e., $\varphi(e_1) = e_1$ and $\varphi(e_i) = e_i + e_1$ for $2 \leq i \leq n$, and using the fact that $\varphi(\partial\, e_F) = e_{F\setminus\{1\}}$ and $\varphi(e_T) = e_T$ we get that $\varphi(J_{\langle 3\rangle})$ is a monomial ideal generated in degree 3 with a system of generators $G = \{g_1, \dots, g_r\}$, where $g_i = e_{F\setminus\{1\}}$ or $g_i = \varphi(e_T)$ with the same assumptions for T, F as above.

Let $\{f_1, \dots, f_r\}$ be the free generators of $\bigoplus_{i=1}^{r} E(-3)$ such that f_i is mapped to g_i in the minimal graded free resolution of $\varphi(J_{\langle 3\rangle})$. Then the kernel of this map,

$$U = \{\sum_{i=1}^{r} a_i f_i : a_i \in E, \sum_{i=1}^{r} a_i g_i = 0\}$$

is the first syzygy module of $\varphi(J_{\langle 3\rangle})$.

Suppose that J is componentwise linear. Then $J_{\langle 3\rangle}$ has a 3-linear resolution and so does $\varphi(J_{\langle 3\rangle})$. This implies that U is minimally generated by certain $h_k = \sum_{i=1}^{r} \alpha_{ki} f_i$ with $\alpha_{ki} \in E_1$.

Since X_{31}, X_{32} contain two more elements outside X_2, without loss of generality we can assume that

$$\{i_1, i_2\} \subseteq X_{31} \setminus X_2 \text{ and } \{j_1, j_2\} \subseteq X_{32} \setminus X_2.$$

Observe that $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$. Otherwise if $i \in \{i_1, i_2\} \cap \{j_1, j_2\}$ then $X_2 \cup \{i\}$ is a subset of both X_{31}, X_{32} and has rank 3, hence $X_{31} = X_{32}$ since

$$\bigcap_{j \in X_{31}} H_j = \bigcap_{j \in X_2 \cup \{i\}} H_j = \bigcap_{j \in X_{32}} H_j,$$

which is a contradiction. Since $\{1, 2, i_1, i_2\} \subset X_{31}$ and $\{1, 2, j_1, j_2\} \subset X_{32}$ are circuits, we may assume that

$$g_1 = \varphi(\partial\, e_{12i_1i_2}) = e_{2i_1i_2} \text{ and } g_2 = \varphi(\partial\, e_{12j_1j_2}) = e_{2j_1j_2}.$$

Then we have $e_{j_1j_2}g_1 - e_{i_1i_2}g_2 = 0$. Therefore, $e_{j_1j_2}f_1 - e_{i_1i_2}f_2 \in U$. Hence this relation has a representation

$$e_{j_1j_2}f_1 - e_{i_1i_2}f_2 = \sum_k \beta_k h_k = \sum_k \sum_{i=1}^{r} \beta_k \alpha_{ki} f_i$$

where $\beta_k \in E_1$. Then $e_{j_1j_2} = \sum_k \beta_k \alpha_{k1}$ since the f_i are free generators. This implies that there exists k such that e_{j_1} or e_{j_1} belong to $\operatorname{supp}(\alpha_{k1})$. Assume that $e_{j_1} \in \operatorname{supp}(\alpha_{k1})$.

Since $\alpha_{k1}g_1 + \ldots + \alpha_{kr}g_r = 0$ and $\alpha_{k1}g_1$ contains the monomial $e_{2i_1i_2j_1}$ which cannot appear in $\alpha_{k2}g_2$, the sum $\sum_{i=3}^{r} \alpha_{ki}g_i$ contains $e_{2i_1i_2j_1}$. Assume that $\alpha_3 g_3$ contains $e_{2i_1i_2j_1}$. Let $g_3 = e_{F\setminus\{1\}}$ where F is a dependent set, $1 \in F$ and $|F| = 4$. Then $F \setminus \{1\} \subset \{2, i_1, i_2, j_1\}$. Since $\{1, 2, i_1, i_2\} \subset X_{31}$, we have that $F \cap X_{31}$ has an independent subset of three indices. Moreover, $\operatorname{rank}(F) = \operatorname{rank}(X_{31}) = 3$. This implies that $F \subset X_{31}$ and then $F = \{1, 2, i_1, i_2\}$. Hence $g_3 = e_{2i_1i_2} = g_2$, which is a contradiction. So we can conclude that $\varphi(J_{\langle 3\rangle})$ does not have a 3-linear resolution and then so does $J_{\langle 3\rangle}$. Hence J is not componentwise linear. □

Theorem 6.3.7. *Suppose that* $\operatorname{rank}(\mathcal{A}) = 4$ *and all elements of* $\overline{L_3}$ *are non-uniform. Then Conjecture 6.3.1 holds.*

Proof. By Theorem 6.3.4, we only need to prove the implication "(i) ⇒ (ii)" in Conjecture 6.3.1. Suppose that J is componentwise linear. By Theorem 6.2.1, we get that $L(\mathcal{A})$ has exactly one dependent element of rank 2, say X_2. By Proposition 6.3.6, there is at most one dependent element of rank 3 in $\overline{L_3}$, say X_3, such that $|X_3| \geq |X_2| + 2$. If there exists such X_3, then one can show that every circuit of rank 3 must contain two indices outside X_2 which can only be chosen in $X_3 \setminus X_2$. Moreover, for $i_1, i_2 \in X_2, j_1, j_2 \in X_3 \setminus X_2$, the set $\{i_1, i_2, j_1, j_2\}$ is a circuit. Thus the condition (ii) of Conjecture 6.3.1 is fulfilled. Hence Conjecture 6.3.1 holds if $\operatorname{rank}(\mathcal{A}) = 4$ and all elements of $\overline{L_3}$ are non-uniform. □

Next we present examples where Conjecture 6.3.1 is true.

Example 6.3.8. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential central hyperplane arrangement with Orlik-Solomon ideal J.

Case 1: n=1, 2. Since every set of two hyperplane is an independent set, we have $J = (0)$.

Case 2: n=3. If $\{H_1, H_2, H_3\}$ is a dependent set, then $J = (\partial\, e_{123})$ has a 2-linear resolution. Hence it is componentwise linear.

Case 3: n=4. If there is only one dependent set of three hyperplanes, say $\{H_1, H_2, H_3\}$, then we have the same situation as in case 2. Indeed $J = (\partial\, e_{123})$ is componentwise linear.

If there exist more than one dependent set of three hyperplanes, then all sets of three hyperplanes are dependent. Therefore,

$$J = (\partial\, e_{123}, \partial\, e_{124}, \partial\, e_{134}, \partial\, e_{234}) = (e_2 - e_1, e_3 - e_1, e_4 - e_1)^2$$

has a 2-linear resolution.

If there is only one dependent set of hyperplanes and every set of three hyperplane is independent, then we get that $\{H_1, H_2, H_3, H_4\}$ is dependent. Therefore, $J = (\partial\, e_{1234})$ has a 3-linear resolution.

Case 4: n=5. We have the following cases for J up to a permutation of indices of hyperplanes:

(i) All possibilities of case 3.
(ii) $J = (\partial\, e_{12345})$. Then J has a 4-linear resolution.
(iii) $J = (\partial\, e_F; F \subset [5], |F| = 4)$. Then J has a 3-linear resolution.
(iv) $J = (\partial\, e_F : F \subset [5], |F| = 3)$. Then J has a 2-linear resolution.
(v) $J = (\partial\, e_{123}, \partial\, e_{145})$. Then J does not have a 2-linear resolution by Theorem 6.2.1.

Case 5: n=6. We have the following cases of J up to a permutation of indices of hyperplanes:

(i) All possibilities of case 4.
(ii) $J = (\partial\, e_{123456})$. Then J has a 5-linear resolution.
(iii) $J = (\partial\, e_F : F \subset [6], |F| = 5)$. Then J has a 4-linear resolution.
(iv) $J = (\partial\, e_F : F \subset [6], |F| = 4)$. Then J has a 3-linear resolution.
(v) $J = (\partial\, e_F : F \subset [6], |F| = 3)$. Then J has a 2-linear resolution.
(vi) $J = (\partial\, e_{123}, \partial\, e_F : F \subset [5], |F| = 4)$. Then J does not have a linear resolution but it is still componentwise linear.
(vii) $J = (\partial\, e_{123}, \partial\, e_F : F \subset [6], |F| = 4)$. Then J does not have a linear resolution but it is still componentwise linear.
(viii) $J = (\partial\, e_T, \partial\, e_F : T \subset [4], F \subset [6], |T| = 3, |F| = 4)$. Then J does not have a linear resolution but it is still componentwise linear.
(ix) $J = (\partial\, e_{1234}, \partial\, e_F : F \subset [6], |F| = 5)$. Then J does not have a linear resolution but it is still componentwise linear.
(x) $J_{\langle 3\rangle} = (\partial\, e_{1234}, \partial\, e_{1256})$ or $J_3 = (\partial\, e_{1234}, \partial\, e_{1256}, \partial\, e_{3456})$. Then $J_{\langle 3\rangle}$ does not have a 3-linear resolution. Hence J is not componentwise linear.
(xi) $J_{\langle 2\rangle} = (\partial\, e_{123}, \partial\, e_{456})$ or $J_{\langle 2\rangle} = (\partial\, e_{123}, \partial\, e_{145}, \partial\, e_{256})$ or

$$J_{\langle 2\rangle} = (\partial\, e_{123}, \partial\, e_{145}, \partial\, e_{256}, \partial\, e_{346}).$$

Then $J_{\langle 2\rangle}$ does not have a 2-linear resolution. Hence J is not componentwise linear.

To conclude this example, we note that Conjecture 6.3.1 holds for all above cases. More precisely, in case 4 (v), case 5 (x), (xi), the Orlik-Solomon ideals are not componentwise linear since circuits of arrangements of rank 2 (or rank 3) are not subsets of the same dependent element in the intersection lattice.

Remark 6.3.9. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be an essential central hyperplane arrangement in $\mathbb{C}^l$ with Orlik-Solomon ideal J, matroid $M(\mathcal{A})$. Let $<$ be a monomial order on E with $e_1 > e_2 > \ldots > e_n$. Since $\beta^E_{i,j}(E/\mathrm{in}_<(J)) \geq \beta^E_{i,j}(E/J)$ (see [**3**, Proposition 1.8]), the linearity of the free resolution of $\mathrm{in}_<(J)$ implies the linearity of the free resolution of J. More precisely, if the initial ideal $\mathrm{in}_<(J)$ has an m-linear resolution then so does J. The reverse direction in this situation is also true. Indeed, if J has an m-linear resolution, then the matroid of $\mathcal{A}$ is $M(\mathcal{A}) = U_{m,n-f} \oplus U_{f,f}$ (see [**38**, Theorem 6.10]). By renumbering if needed, we get that

$$J = (\partial\, e_F : F \subseteq [n-f], |F| = m+1).$$

Thus

$$\mathrm{in}_<(J(\mathcal{A})) = (e_T : T \subseteq \{2, \ldots, n-f\} \text{ and } |T| = m) = (e_2, \ldots, e_{n-f})^m.$$

This implies that $\mathrm{in}_<(J)$ has an m-linear resolution.

Such statements are not true for componentwise linearity, i.e., there exists an arrangement with Orlik-Solomon ideal J such that $\mathrm{in}_<(J)$ is componentwise linear but J is not componentwise linear (see the below example).

From now to the end of this section, we always assume that $<$ is a monomial order on E with $e_1 > e_2 > \ldots > e_n$. Recall that a circuit of $M(\mathcal{A})$ is a minimal dependent set of $M(\mathcal{A})$ and a *broken circuit* w.r.t. $<$ is the set received from a circuit by deleting its smallest element w.r.t. $<$. Note that the smallest element w.r.t. $<$ of a circuit $C \in M(\mathcal{A})$ corresponds to the largest index in C. It is known (see, e.g., [**50**, Theorem 4.1, Proposition 4.2]) that the initial ideal $\mathrm{in}_<(J)$ is the monomial ideal generated by broken circuits. Therefore, corresponding to $\mathrm{in}_<(J)$, we have a simplicial complex, namely the *broken circuit complex* $\Delta(M(\mathcal{A}))$. It is the collection of all subsets of $[n]$ which do not contain a broken circuit. Then the face ideal of $\Delta(M(\mathcal{A}))$ is $J(\Delta(M(\mathcal{A}))) = \mathrm{in}_<(J)$.

Example 6.3.10. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^3$ defined by the following equation:

$$Q(x, y, z) = x(x+y)yz(x+z).$$

Then $\mathcal{A} = \{H_1, \ldots, H_5\}$ where the hyperplanes are ordered according to the order of factors in $Q(x, y, z)$ above. The circuits of the matroid $M(\mathcal{A})$ of $\mathcal{A}$ are: $\{1,2,3\}, \{1,4,5\}, \{2,3,4,5\}$. Thus the Orlik-Solomon ideal of $\mathcal{A}$ is

$$J(\mathcal{A}) = (\partial\, e_{123}, \partial\, e_{145}, \partial\, e_{2345}).$$

The broken circuits w.r.t. $<$ of $M(\mathcal{A})$ are $\{1,2\}, \{1,4\}, \{2,3,4\}$. Thus the broken circuit complex $\Delta(M(\mathcal{A}))$ has the facets $\{(1,3,5), (2,3,5), (2,4,5), (3,4,5)\}$. The face ideal of $\Delta(M(\mathcal{A}))$ is $J_{\Delta(M(\mathcal{A}))} = (e_{12}, e_{13}, e_{234})$. We can check that $J_{\Delta(M(\mathcal{A}))}$ is componentwise linear. But J is not componentwise linear since $J_{\langle 2\rangle}$ does not have a 2-linear resolution by the same argument as in Example 6.3.2.

Motivated by the problem whether the Orlik-Solomon ideal $J(\mathcal{A})$ is componentwise linear, we consider the question when the ideal $J_{\Delta(M(\mathcal{A}))}$ is componentwise linear. We observe the following:

Remark 6.3.11. (i) The supersolvable property of $\mathcal{A}$ (or equivalently, A is G-quadratic) does not ensure the componentwise linear property of $J_{\Delta(M(\mathcal{A}))}$. For instance, considering the braid arrangement $\mathcal{A}_3$ in Example 6.4.6, we have

$$J_{\Delta(M(\mathcal{A}_3))} = (e_{23}, e_{45}, e_{46}, e_{56}).$$

Using Macaulay2 [**28**], one can check that $\beta_{1,4}(J_{\Delta(M(\mathcal{A}_3))}) = 3 \neq 0$. Thus $J_{\Delta(M(\mathcal{A}_3))}$ does not have a 2-linear resolution. Since $J_{\Delta(M(\mathcal{A}_3))}$ is generated in one degree and does not have a linear resolution, we get that $J_{\Delta(M(\mathcal{A}_3))}$ is not componentwise linear.

(ii) The componentwise linear property of $J_{\Delta(M(\mathcal{A}))}$ depends on the order on the ground set $[n]$. For example, considering the order $<'$ on $E = K\langle e_1, \dots, e_6\rangle$ with $e_1, e_6 < e_2, \dots, e_5$. The face ideal of the broken circuit complex (w.r.t. $<'$) corresponding to $\mathcal{A}_3$ is

$$J_{\Delta(M(\mathcal{A}_3))} = \mathrm{in}_{<'}(J(\mathcal{A}_3)) = (e_{23}, e_{25}, e_{34}, e_{45}) = (e_2, e_4)(e_3, e_5)$$

which has a 2-linear resolution by Theorem 4.3.2. Since $J_{\Delta(M(\mathcal{A}_3))}$ is generated in one degree and has a linear resolution, we get that $J_{\Delta(M(\mathcal{A}_3))}$ is componentwise linear.

Example 6.3.12. There exist arrangements such that the face ideals of broken circuits complexes corresponding to their matroids are not componentwise linear for every order on the ground set. For example, let $\mathcal{A}$ be the arrangement in $\mathbb{C}^3$ defined by

$$Q(x, y, z) = xy(x + y)(x + z)(y + z)(x + y + 2z).$$

Then $\mathcal{A} = \{H_1, \dots, H_6\}$ where the hyperplanes are ordered according to the order of factors in $Q(x, y, z)$ above. The set of 3-circuits of the matroid $M(\mathcal{A})$ of $\mathcal{A}$ is: $\{(1, 2, 3), (4, 5, 6)\}$. Thus $J(\mathcal{A})_{\langle 2\rangle} = (\partial\, e_{123}, \partial\, e_{456})$. Hence for every order on the ground set $\{1, \dots, 6\}$, we get that $(\mathrm{in}_< J(\mathcal{A}))_{\langle 2\rangle} = (e_{ij}, e_{pq})$, where $i, j \in \{1, 2, 3\}$ and $p, q \in \{4, 5, 6\}$. Therefore, $(J_{\Delta(M)})_{\langle 2\rangle} = (\mathrm{in} J(\mathcal{A}))_{\langle 2\rangle}$ does not have an 2-linear resolution since (e_{ij}, e_{pq}) always has a non-linear first syzygy defined by the relation $e_{ij}e_{pq} - e_{pq}e_{ij} = 0$. Hence $J_{\Delta(M(\mathcal{A}))}$ is not componentwise linear.

Open problem 6.3.13. *Let $\mathcal{A}$ be an essential central hyperplane arrangement with matroid $M(\mathcal{A})$ such that two arbitrary circuits of $M(\mathcal{A})$ have non-empty intersection. Is it true that there exists a monomial order on E such that with respect to this order $J_{\Delta(M(\mathcal{A}))}$ is componentwise linear?*

6.4. On the Koszul property of Orlik-Solomon algebras

In this section, we study classes of Orlik-Solomon algebras satisfying variations of the Koszul property. More precisely, we classify hyperplane arrangements whose Orlik-Solomon algebras are universally Koszul as well as initially Koszul. Note that the universally Koszul, initially Koszul properties imply the Koszul property. Thus a necessary condition for an Orlik-Solomon algebra A to be universally Koszul or initially Koszul is that A is quadratic, i.e., the corresponding Orlik-Solomon ideal is generated by quadrics.

We recall first some facts about arrangements and their intersection lattices. We use here and in the following the notation of elements in an intersection lattice $L(\mathcal{A})$ which we introduced on page 77.

Lemma 6.4.1. *Let $\mathcal{A}$ be an essential central hyperplane arrangement with intersection lattice $L(\mathcal{A})$ and Orlik-Solomon ideal J. Suppose that elements in $\overline{L_2}$ are disjoint. Then there exists a change of coordinate $\varphi \in \mathrm{GL}_n(K)$ such that $\varphi(J_{\langle 2\rangle})$ is a monomial ideal.*

Proof. Assume that $\overline{L_2} = \{X_1, \ldots, X_r\}$ where $X_i = \{i_1, \ldots, i_{s_i}\}$ are dependent elements of rank 2 in $L(\mathcal{A})$ such that $X_i \cap X_j = \emptyset$ for $i, j \in \{1, \ldots, r\}$, $i \neq j$. Consider the change of coordinate $\varphi : E_1 \to E_1$ defined by: $e_{i_1} \mapsto e_{i_1}$ and $e_{i_k} \mapsto e_{i_k} + e_{i_1}$ for $k = 2, \ldots, s_i$ and $i = 1, \ldots, r$. Then

$$\varphi(\partial\, e_{i_1 i_k i_t}) = \varphi((e_{i_k} - e_{i_1})(e_{i_t} - e_{i_1})) = e_{i_k} e_{i_t} \text{ for } k, t = 2, \ldots, s_i \text{ and } i = 1, \ldots, r.$$

By Lemma 6.3.5 we get that $\varphi(J_{X_i}) = (e_{i_k} e_{i_t} : k, t = 2, \ldots, s_i)$.

Since $J_{\langle 2\rangle} = J_{X_1} + \cdots + J_{X_r}$, we get that $\varphi(J_{\langle 2\rangle})$ is a monomial ideal. □

We illustrate the above lemma by the following example:

Example 6.4.2. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$ defined by the equation:

$$Q = xy(x+y)(x+3y+z)(x+4y+2z)(y+z).$$

Let $E = K\langle e_1, \ldots, e_6\rangle$ where e_i responds to i-th factor in the equation of $\mathcal{A}$ for $i = 1, \ldots, 6$. Then $\overline{L_2} = \{\{1,2,3\}, \{4,5,6\}\}$ and $J_{\langle 2\rangle} = (\partial\, e_{123}, \partial\, e_{456})$. Consider the change of coordinate $\varphi : E_1 \to E_1$ defined by $\varphi(e_1) = e_1$, $\varphi(e_2) = e_2 + e_1$, $\varphi(e_3) = e_3 + e_1$, $\varphi(e_4) = e_4$, $\varphi(e_5) = e_5 + e_4$, $\varphi(e_6) = e_6 + e_4$. Then we have:

$$\varphi(J_{\langle 2\rangle}) = (e_2 e_3, e_4 e_5).$$

Next we classify classes of Orlik-Solomon algebras which are universally Koszul:

Theorem 6.4.3. *Let A be an Orlik-Solomon algebra of an essential central hyperplane arrangement $\mathcal{A}$. The following statements are equivalent:*

(i) *A is universally Koszul;*
(ii) *The matroid of $\mathcal{A}$ is $M(\mathcal{A}) = U_{2,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n$;*
(iii) *The Orlik-Solomon ideal J has a 2-linear free resolution.*

Proof. (ii) ⇔ (iii): See [**38**, Theorem 6.10].

(iii) ⇒ (i): Suppose that J has a 2-linear free resolution. By the proof of Theorem 6.2.1, we have that J is of the form

$$J = (e_2 - e_1, \ldots, e_s - e_1)^2 \text{ for some } 3 \leq s \leq n.$$

Then by changing the coordinates:

$$e_i \mapsto \begin{cases} e_i + e_1 \text{ if } 2 \leq i \leq s, \\ e_i, \text{ otherwise,} \end{cases}$$

the ideal J becomes a monomial ideal of the form $(e_2, \ldots, e_s)^2$. Since $E/(e_2, \ldots, e_s)^2$ is universally Koszul by Theorem 5.2.11, $A = E/J$ is also universally Koszul.

(i) $\Rightarrow$ (ii): Suppose that A is universally Koszul. Note that $\overline{L_2} \neq \emptyset$ since $0 \neq J$ is generated in degree 2. We claim that the set $\overline{L_2}$ of dependent elements of rank 2 in $L(\mathcal{A})$ has only one element. Assume the contrary, i.e., $|\overline{L_2}| \geq 2$. Then we consider two cases as follows:

Case 1: All elements of $\overline{L_2}$ are disjoint. Let $\overline{L_2} = \{X_1, \dots, X_r\}$ where $r \geq 2$ and $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq r$. Then after the change of coordinate φ as in Lemma 6.4.1, all $\varphi(J_{X_i})$ are monomial ideals and so does $\varphi(J)$. Moreover, since $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq r$, we get that the restriction of $\varphi(J)$ to the set of variables $\{e_{i_2}, e_{i_3}, e_{j_2}, e_{j_3}\}$ is the monomial ideal $(e_{i_2}e_{i_3}, e_{j_2}e_{j_3})$ for $i, j \in \{1, \dots, r\}$, $i \neq j$. Observe that, $(e_{i_2}e_{i_3}, e_{j_2}e_{j_3})$ is a monomial ideal of the type as in Lemma 5.2.10 (i). By Theorem 5.2.11, E/J is not universally Koszul which is a contradiction to our assumption (i).

Case 2: There exist $X_1, X_2 \in \overline{L_2}$ such that $X_1 \neq X_2$ and $X_1 \cap X_2 \neq \emptyset$. Since $X_1 \neq X_2$, we see that $X_1 \cap X_2$ is a set of one element, say $X_1 \cap X_2 = \{1\}$. Note that $|X_1|, |X_2| \geq 3$ since they are dependent elements of $L(\mathcal{A})$. Without loss of generality, we may assume that $2, 3 \in X_1$ and $4, 5 \in X_2$. Then $C_1 = \{1, 2, 3\}$ and $C_2 = \{1, 4, 5\}$ are two 3-circuits of $\mathcal{A}$.

Observe that we have no more 3-circuit of three indices in $[5] = \{1, \dots, 5\}$. Otherwise, let $C \subset [5]$ be a 3-circuit and $C \neq C_1, C_2$. Then $|C \cap C_1| = 2$ or $|C \cap C_2| = 2$. If, say $|C \cap C_1| = 2$, then $|C \cap X_1| \geq 2$. Thus $C \subset X_1$ and $X_1 \cap X_2 \supseteq (C \cup C_1) \cap C_2$. This is a contradiction since $(C \cup C_1) \cap C_2$ has at least two elements but $|X_1 \cap X_2| = 1$. Hence there exist only the 3-circuits C_1, C_2 in [5].

Write $J = (\partial e_{123}, \partial e_{145}, \partial e_{F_1}, \dots, \partial e_{F_r})$ where F_i are 3-circuits of $\mathcal{A}$ for $1 \leq i \leq r$. Let $u = e_2 - e_1 + e_4 - e_1 = \partial e_{12} + \partial e_{14}$. We claim that

$$J :_E (u) = J + (u).$$

Since A is universally Koszul, $0 :_A (u)$ is generated by linear forms in A. We only need to prove that $(J :_E (u))_1 = (J + (u))_1$.

For a linear form $w = \sum_{i=1}^{n} \alpha_i e_i \in E_1$ and a set of indices $F \subset [n]$, we denote by $w_F = \sum_{i \in F} \alpha_i e_i$. It is obvious that $(J + (u))_1 \subset (J :_E (u))_1$ since $u \in J :_E (u)$. Assume that there exists $v \in E_1$ such that $0 \neq uv \in J$. Then by [**25**, Corollary 3.2] (see also [**55**, Corollary 4.9]), we have that for every $X \in \overline{L_2}$, either $\partial u_X = \partial v_X = 0$, or $v_X = \alpha u_X$ where $\alpha \in K$ since we have already $|X| \geq 3$. Since $\partial u_{X_1} = \partial u_{X_2} = -1$ and $e_1 \in \operatorname{supp}(u_{X_1}) \cap \operatorname{supp}(u_{X_2})$, we get that

$$v_{X_1} = cu_{X_1} = ce_2 - 2ce_1 \text{ and } v_{X_2} = cu_{X_2} = ce_4 - 2ce_1, \text{ where } c \in K.$$

This implies that $v = ce_2 + ce_4 - 2ce_1 + v' = cu + v'$ where $v' \in E_1$ such that $e_1, e_2, e_4 \notin \operatorname{supp}(v')$. Note that $\operatorname{supp}(v') \neq \emptyset$ since $uv \neq 0$. Let $k \in \operatorname{supp}(v')$. Since $uv \in J$ and $e_1e_k \in \operatorname{supp}(uv)$, there exists $X \in \overline{L_2}$ with $|X| \geq 3$ such that $\{1, k\} \subset X$. If $\partial u_X \neq 0$, then $v_X = \alpha u_X$, where $\alpha \in K$, by [**25**, Corollary 3.2]. This is impossible since $k \in \operatorname{supp}(v) \setminus \operatorname{supp}(u)$. So $\partial u_X = 0$. This implies that $\{1, 2, 4\} \subset X$. Thus $\{1, 2, 4\} \subset [5]$ is a 3-circuit. This contradicts the fact that we have only two 3-circuits $C_1, C_2 \subset [5]$. Hence such a v chosen as above can not exist and this implies $(J :_E (u))_1 = (J + (u))_1$. So we have $J :_E (u) = J + (u)$.

Let $f = \partial e_{135} = e_{13} - e_{15} + e_{35}$. By formula (11) from Page 67, one can check that $\partial e_{T_1} \partial e_{T_2} = \partial e_{T_1 \cup T_2}$ for $T_1, T_2 \subset [n]$ and $T_1 \cap T_2 = \{1\}$. Using this equation, we have

$$uf = (\partial e_{12} + \partial e_{14}) \partial e_{135} = \partial e_{1235} + \partial e_{1345} = \partial e_{15} \partial e_{123} + \partial e_{13} \partial e_{145} \in J.$$

Thus $f \in J :_E (u) = J + (u)$. Hence, there is a representation

$$\partial e_{135} = g + uh = g + (\partial e_{12} + \partial e_{14})h, \text{ where } g \in J \text{ and } h \in E_1. \tag{13}$$

Since $e_{35} \in \operatorname{supp}(\partial e_{135})$ and e_{35} does not occur in $(\partial e_{12} + \partial e_{14})h$ for every $h \in E_1$, we get that $e_{35} \in \operatorname{supp}(g)$. Thus there exists F_i such that e_{35} occurs in ∂e_{F_i}. This implies that $\{3, 5\} \subset F_i$. Let $X \in \overline{L_2}$ with $F_i \subset X$. Note that the sets $\{1, 3, 5\}, \{2, 3, 5\}, \{3, 4, 5\}$ are not circuits. Hence, $1, 2, 4 \notin X$.

Let $I = (e_j : j \notin X)$ and $P = (e_{i_1} e_{i_2} : i_1, i_2 \in X)$. Then $e_1, e_2, e_4 \in I$, $e_{35} \in P_2$ and $J_X \subset P$. Observe that $I_2 \cap (J_X)_2 = \{0\}$ since $I_2 \cap P_2 = \{0\}$. Moreover, $J \subset J_X + I$ since $\partial e_{F_j} \in I$ for every $F_j \not\subset X$. Recall from equation (13) that $f = \partial e_{135} = g + uh$. Write $g = g_1 + g_2$ where $g_1 \in (J_X)_2$ and $g_2 \in I_2$. It follows that

$$e_{13} - e_{15} + e_{35} = g_1 + g_2 + (\partial e_{12} + \partial e_{14})h.$$

Thus

$$e_{35} - g_1 = g_2 - e_{13} + e_{15} + (\partial e_{12} + \partial e_{14})h \in I_2.$$

Since $e_{35} - g_1 \in P_2$ and $g_2 - e_{13} + e_{15} + (\partial e_{12} + \partial e_{14})h \in I_2$, we get that $e_{35} = g_1 \in (J_X)_2$ because $I_2 \cap P_2 = \{0\}$. So $e_{135} \in J_X$. This is only possible if $\{1, 3, 5\} \subset X$, i.e., $\{1, 3, 5\}$ is a circuit. This is a contradiction since there exist only the 3-circuits $C_1, C_2 \subset [5]$.

Combining case 1 and case 2 we get that $\overline{L_2}$ has indeed only one element, say $\overline{L_2} = \{X\}$ where $X = \{i_1, \ldots, i_s\}$. Since J is generated in degree 2, we have $J = J_X$. Thus every 3-circuit of $M(\mathcal{A})$ is a subset of X. Since $\mathcal{A}$ is an essential central hyperplane arrangement, $M(\mathcal{A})$ does not have any circuit of ≤ 2 elements. Hence we get that $M(\mathcal{A}) = U_{2,s} \oplus U_{n-s,n-s}$, as desired. ☐

Let $\mathcal{A}$ be an essential central hyperplane arrangement with the complement $\mathcal{X}(\mathcal{A})$ and its fundamental group $\pi_1(\mathcal{X}(\mathcal{A}))$. Let

$$Z = Z_1 = \pi_1(\mathcal{X}(\mathcal{A})), Z_2 = [Z_1, Z], \ldots, Z_{i+1} = [Z_i, Z], \ldots$$

be the *lower central series* (LCS for short) and set $\varphi_i = \operatorname{rank}(Z_i/Z_{i+1})$. There is a lot of attention in [**26**], [**39**], [**50**], [**56**], [**57**], [**60**] to a special formula, called *LCS formula*, which states that

$$\prod_{j=1}^{\infty} (1 - t^j)^{\varphi_j} = H_A(-t).$$

It was proved by Shelton and Yuzvinsky in [**60**] that the formula holds if and only if the algebra A is Koszul. From the classification of Orlik-Solomon algebras satisfying the universally Koszul property, we compute the LCS formula in this as follows:

Corollary 6.4.4. *Let A be an Orlik-Solomon algebra of an essential central hyperplane arrangement $\mathcal{A}$ such that A is universally Koszul. Then the matroid $M(\mathcal{A})$ of $\mathcal{A}$ is $U_{2,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n$ and we have:*

$$\prod_{j=1}^{\infty}(1-t^j)^{\varphi_j} = 1 - nt + \cdots + (-1)^k(\binom{f+1}{k} + (n-f-1)\binom{f+1}{k-1})t^k + \cdots$$

Proof. By Theorem 6.4.3, we have already that $M(\mathcal{A}) = U_{2,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n$. Therefore, $J = (\partial\, e_{ijk} : 1 \leq i,j,k \leq n-f)$. Let $<$ be the lexicographic order on E with $e_1 > e_2 > \ldots > e_n$. We only need to compute the Hilbert function of A, i.e., the number of monomials of one degree in E which do not belong to $\operatorname{in}_<(J)$. Note that the broken circuits of $M(\mathcal{A})$ generate the initial ideal of J w.r.t. $<$ (see, e.g., [**7**], [**50**, Theorem 4.1]). So we have $\operatorname{in}_<(J) = (e_{ij} : 1 \leq i,j \leq n-f-1)$. Thus $e_F \notin \operatorname{in}_<(J)$ if and only if $F \cap \{1,\ldots,n-f-1\}$ has at most 1 element. Therefore,

$$H(A,k) = \binom{f+1}{k} + (n-f-1)\binom{f+1}{k-1} \text{ for } k = 1,\ldots,n.$$

This concludes the proof. □

In the hyperplane arrangement theory, there is an important characterization of supersolvable arrangement as follows: let $\mathcal{A}$ be an arrangement with its Orlik-Solomon algebra A. Then we have

$$\mathcal{A} \text{ is supersolvable} \Longleftrightarrow A \text{ is G-quadratic.}$$

As an application of Section 5.4, we get another characterization of supersolvable arrangements. We have:

Theorem 6.4.5. *Let $\mathcal{A}$ be an arrangement with Orlik-Solomon algebra A. The following statements are equivalent:*

(i) *$\mathcal{A}$ is supersolvable;*
(ii) *A is G-quadratic;*
(iii) *A is initially Koszul.*

Proof. (i) ⇔ (ii) see, e.g., [**50**, Theorem 4.3]. (iii) ⇒ (ii) follows from Proposition 5.4.5.

(i)⇒ (iii): Suppose that $\mathcal{A}$ is supersolvable with the Orlik-Solomon ideal J. By [**8**, Theorem 2.8 (5)], there exists a partition $[n] = F_1 \cup \cdots \cup F_r$ such that for any two distinct indices $x, y \in F_i$, there is $z \in F_j$ with $j < i$ such that $\{x,y,z\}$ is a circuit. By a suitable change of indices, we may assume that for $s < r$ and $i \in F_s$, $j \in F_r$ we have $i > j$. Moreover, let $M_i = \cup_{j \leq i} F_j$. [**8**, Theorem 2.8 (5)] also implies that $\widehat{0} = M_0 < M_1 < \ldots < M_r = \widehat{1}$ is a maximal chain of modular elements of the supersolvable lattice $L(\mathcal{A})$. Note that we identify M_i with the element $\cap_{j\in M_i} H_j$ in $L(\mathcal{A})$.

Let $<$ be the reverse lexicographic order on E with $e_1 > e_2 > \ldots > e_n$. Recall that a broken circuit w.r.t. $<$ of $M(\mathcal{A})$ is the set received from a circuit by deleting the largest index in the circuit. We claim that if $\{x,z\}$ is a broken circuit with $x < z$ then x, z belong to the same F_i. Let $\{w,x,z\}$ be the circuit containing $\{x,z\}$, where

$x < z < w$ and $x \in F_i$. Note that if $z \in F_j$, then $j \leq i$. If $w \in F_k$ and $z \in F_j$ with $k \leq j < i$, then $H_z \cap H_w \supset M_j$. Since $\{w, x, z\}$ is a circuit, we have $H_x \supset H_z \cap H_w$. Thus $H_x \supset M_j$. This implies that $x \in F_t$ with $t \leq j$. This contradicts the fact that $x \in F_i$ and $x \notin \cup_{k \leq j} F_k$ since $i > j$. Thus $z \in F_i$.

Next we claim that the condition (ii) of Proposition 5.4.5 holds for J. By [**50**, Proposition 4.2, Theorem 4.1, 4.3], we have that $\text{in}_<(J)$ is quadratic. More precisely, $\text{in}_<(J)$ is generated by squarefree monomials corresponding to broken circuits of 2 indices of $M(\mathcal{A})$. Now for a set of indices $\{x, y, z\}$ with $x < y < z$, we have that if $e_x e_z \in \text{in}(J)$ then $\{x, z\}$ is a broken circuit of $M(\mathcal{A})$. By the above argument, there exists $1 \leq i \leq r$ such that $\{x, z\} \subset F_i$. By the assumption for F_i, we also have that $y \in F_i$ because $x < y < z$. Since $\{y, z\} \in F_i$, there exists $t \in F_j$ where $j < i$ and $t > y, z$ such that $\{t, y, z\}$ is a circuit of $\mathcal{A}$. Thus $\{y, z\}$ is also a broken circuit and $e_y e_z \in \text{in}_<(J)$. Hence the condition (ii) of Proposition 5.4.5 is fulfilled. So A has a Gröbner flag by Proposition 5.4.5, i.e., A is initially Koszul. $\square$

We illustrate the proof of the theorem above by the following example:

Example 6.4.6. Let $\mathcal{A}_3$ be the rank-three braid arrangement in $\mathbb{C}^4$ which is defined by the equation

$$Q = (x - y)(x - z)(y - z)(x - t)(y - t)(z - t).$$

It is well-known that $\mathcal{A}_3$ is a supersolvable arrangement; see, e.g., [**47**, Example 2.33]. From the matroid of $\mathcal{A}_3$ (see Figure 2), we get that the Orlik-Solomon ideal of $\mathcal{A}_3$ is

$$J = (\partial\, e_{125}, \partial\, e_{134}, \partial\, e_{236}, \partial\, e_{456}) \subset E = K\langle e_1, \ldots, e_6 \rangle.$$

The partition of [6] satisfying the condition in [**8**, Theorem 2.8 (5)] is: $(6|5, 4|3, 2, 1)$, i.e., $F_1 = \{6\}$, $F_2 = \{5, 4\}$, $F_3 = \{3, 2, 1\}$. The broken circuits of size 2 of $\mathcal{A}_3$ are $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}$. We see that two elements of every broken circuit are in one F_i. Moreover, we can check directly that

$$0 \subset \text{span}_K\{e_6\} \subset \text{span}_K\{e_6, e_5\} \subset \ldots \subset \text{span}_K\{e_6, \ldots, e_1\}$$

is a Gröbner flag of the Orlik-Solomon algebra $A = E/J$ of $\mathcal{A}_3$ since

$$0 :_A e_6 = (e_1), \quad (e_6) :_A e_5 = (e_6, e_5, e_4), \quad (e_6, e_5) :_A e_4 = (e_6, e_5, e_4)$$

and

$$(e_6, e_5, e_4) :_A e_3 = (e_6, \ldots, e_3) :_A e_2 = (e_6, \ldots, e_2) :_A e_1 = (e_6, \ldots, e_1).$$

Thus A is initially Koszul.

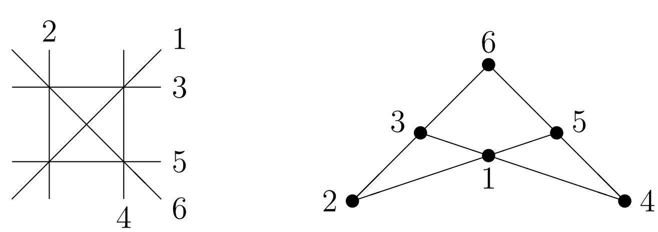

Figure 2: The braid arrangement and its matroid

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