Malliavin-Stein Method in Stochastic Geometry

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Abstract

In this thesis, abstract bounds for the normal approximation of Poisson functionals are computed by the Malliavin-Stein method and used to derive central limit theorems for problems from stochastic geometry. As a Poisson functional we denote a random variable depending on a Poisson point process. It is known from stochastic analysis that every square integrable Poisson functional has a representation as a (possibly infinite) sum of multiple Wiener-Itô integrals. This decomposition is called Wiener-Itô chaos expansion, and the integrands are denoted as kernels of the Wiener-Itô chaos expansion. An explicit formula for these kernels is known due to Last and Penrose.

Via their Wiener-Itô chaos expansions the so-called Malliavin operators are defined. By combining Malliavin calculus and Stein's method, a well-known technique to derive limit theorems in probability theory, bounds for the normal approximation of Poisson functionals in the Wasserstein distance and vectors of Poisson functionals in a similar distance were obtained by Peccati, Solé, Taqqu, and Utzet and Peccati and Zheng, respectively. An analogous bound for the univariate normal approximation in Kolmogorov distance is derived.

In order to evaluate these bounds, one has to compute the expectation of products of multiple Wiener-Itô integrals, which are complicated sums of deterministic integrals. Therefore, the bounds for the normal approximation of Poisson functionals reduce to sums of integrals depending on the kernels of the Wiener-Itô chaos expansion.

The strategy to derive central limit theorems for Poisson functionals is to compute the kernels of their Wiener-Itô chaos expansions, to put the kernels in the bounds for the normal approximation, and to show that the bounds vanish asymptotically.

By this approach, central limit theorems for some problems from stochastic geometry are derived. Univariate and multivariate central limit theorems for some functionals of the intersection process of Poisson k-flats and the number of vertices and the total edge length of a Gilbert graph are shown. These Poisson functionals are so-called Poisson U-statistics which have an easier structure since their Wiener-Itô chaos expansions are finite, i.e. their Wiener-Itô chaos expansions consist of finitely many multiple Wiener-Itô integrals. As examples for Poisson functionals with infinite Wiener-Itô chaos expansions, central limit theorems for the volume of the Poisson-Voronoi approximation of a convex set and the intrinsic volumes of Boolean models are proven.

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Contents

Chapter 1

Introduction

The underlying idea of this thesis is to apply a recently developed technique from stochastic analysis, the Malliavin-Stein method, to derive central limit theorems for problems from stochastic geometry.

Throughout this work, we are interested in random variables depending on a Poisson point process, which we denote as Poisson functionals. A Poisson point process η over a measurable space $(X, \mathcal{X}, \mu)$ is a random collection of points in X, where $\eta(A)$ stands for the number of points of η in $A \in \mathcal{X}$, such that

- $\eta(A)$ follows a Poisson distribution with mean $\mu(A)$ for all $A \in \mathcal{X}$;
- $\eta(A_1), \dots, \eta(A_n)$ are independent for disjoint $A_1, \dots, A_n \in \mathcal{X}$, $n \in \mathbb{N}$.

The measure μ is called the intensity measure of η.

The easiest example of a Poisson point process is a stationary Poisson point process in $\mathbb{R}^d$, where the intensity measure is a constant times the Lebesgue measure. This process plays an important role in stochastic geometry since it is the natural way to choose infinitely many random points uniformly in the whole $\mathbb{R}^d$. Such a point configuration is the starting point for many problems in stochastic geometry. Even if only a random configuration of finitely many points is required, it can be easier to consider a Poisson point process than a fixed number of independently and identically distributed points since a Poisson point process has the independence property for disjoint sets. For some problems in stochastic geometry we consider a Poisson point process in another state space than $\mathbb{R}^d$, for example the set of k-dimensional affine subspaces or the set of all compact convex sets.

Poisson point processes also occur in other branches of probability theory such as Lévy processes or queueing theory. Every Lévy process can be decomposed into a sum of a deterministic linear drift, a Brownian motion, and a part driven by a Poisson point process. Lévy processes play an important role in mathematical finance. In queueing theory, a standard assumption is that the interarrival times are identically and exponentially distributed. Then the arrival times form a stationary Poisson point process on the positive real half-axis.

It is known from stochastic analysis that every square integrable Poisson functional F has a representation

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n) \tag{1.1}$$

with square integrable symmetric functions $f_n \in L_s^2(\mu^n)$, $n \in \mathbb{N}$. Here, $I_n(f)$ denotes the n-th multiple Wiener-Itô integral which is defined for $f \in L_s^2(\mu^n)$. The multiple Wiener-Itô integrals have expectation zero and are orthogonal in the sense that

$$\mathbb{E} I_n(f)\, I_m(g) = \begin{cases} n!\, \langle f, g\rangle_{L^2(\mu^n)}, & n = m \\ 0, & n \neq m \end{cases}$$

for $f \in L_s^2(\mu^n)$ and $g \in L_s^2(\mu^m)$ with $n, m \in \mathbb{N}$. The representation (1.1) is called Wiener-Itô chaos expansion, and we denote the functions f_n as the kernels of the Wiener-Itô chaos expansion of F. The multiple Wiener-Itô integral and the Wiener-Itô chaos expansion go back to classical works by Wiener and Itô (see [30, 31, 93]) and are also known if the underlying stochastic process is a Gaussian process.

In the Poisson case, the kernels of the Wiener-Itô chaos expansion of a square integrable Poisson functional F are given by the formula

$$f_n(x_1, \dots, x_n) = \frac{1}{n!} \mathbb{E} D_{x_1,\dots,x_n} F \tag{1.2}$$

for $x_1, \dots, x_n \in X$, where the difference operator $D_x F$ is defined by

$$D_x F = F(\eta + \delta_x) - F(\eta) \tag{1.3}$$

for $x \in X$ (here we think of η as a measure and δ_x is the Dirac measure concentrated at the point $x \in X$), and the iterated difference operator is recursively given by

$$D_{x_1,\dots,x_n} F = D_{x_1} D_{x_2,\dots,x_n} F$$

for $x_1, \dots, x_n \in X$. The formula (1.2) was proven by Last and Penrose in [41]. By the orthogonality of the multiple Wiener-Itô integrals, we obtain the variance formula

$$\operatorname{Var} F = \sum_{n=1}^{\infty} n!\, \|f_n\|_n^2. \tag{1.4}$$

In the Gaussian and in the Poisson case, we can define via their Wiener-Itô chaos expansions the difference operator (that coincides with the pathwise definition in formula (1.3) in the Poisson case), the Skorohod integral, and the Ornstein-Uhlenbeck generator, which are called Malliavin operators. Such operators were first defined for the Gaussian case, where they have many applications (see the monographs [13, 59] by DiNunno, Øksendal, and Proske and Nualart and the references therein). The investigation of the properties of these operators and their applications is called Malliavin calculus.

A common problem in probability theory is to show that the distribution of a random variable is close to the distribution of a Gaussian random variable or another well-known distribution. The distance between two random variables Y and Z or, more precisely, their distributions can be measured by a probability distance like the Wasserstein distance

$$d_W(Y, Z) = \sup_{h \in \operatorname{Lip}(1)} |\mathbb{E} h(Y) - \mathbb{E} h(Z)|,$$

where Lip(1) is the set of all functions $h : \mathbb{R} \to \mathbb{R}$ with a Lipschitz constant less than or equal to one, or the Kolmogorov distance

$$d_K(Y, Z) = \sup_{t\in\mathbb{R}} |\mathbb{P}(Y \leq t) - \mathbb{P}(Z \leq t)|.$$

Convergence in these probability distances implies convergence in distribution so that we can derive central limit theorems by showing that these distances vanish asymptotically.

A powerful technique to obtain bounds for the Wasserstein or the Kolmogorov distance is Stein's method. From now on, we focus on the distance to a standard Gaussian random variable, which we denote by N. By combining Stein's method with Malliavin calculus, one can obtain upper bounds for the probability distances that involve Malliavin operators. Both techniques were combined first by Nourdin and Peccati for the Gaussian case in [58]. Since the key ingredients of this approach are Stein's method and Malliavin calculus, it is called Malliavin-Stein method.

The main result for the Poisson case is due to Peccati, Solé, Taqqu, and Utzet (see [65]). They proved for a square integrable Poisson functional F with expectation zero that

$$d_W(F, N) \leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}| + \int_X \mathbb{E}(D_zF)^2\, |D_zL^{-1}F|\, \mathrm{d}\mu(z). \tag{1.5}$$

Here, D is the difference operator and L^{-1} is the inverse Ornstein-Uhlenbeck generator. The expressions on the right-hand side are given by

$$D_zF = \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(z, \cdot)) \quad \text{and} \quad D_zL^{-1}F = -\sum_{n=1}^{\infty} I_{n-1}(f_n(z, \cdot))$$

for $z \in X$, where f_n, $n \in \mathbb{N}$, are the kernels of the Wiener-Itô chaos expansion of F.

For the Kolmogorov distance between F and N, a similar bound as formula (1.5) is derived in this work. For the multivariate normal approximation of vectors of Poisson functionals Peccati and Zheng computed a bound close to formula (1.5) for the d_3-distance in [67]. Again, convergence in the d_3-distance, that is defined by some test functions with bounded second and third derivatives, implies convergence in distribution.

It can be a difficult task to evaluate the bounds derived by the Malliavin-Stein method for a given Poisson functional since it requires to compute the expectation of complicated expressions involving the Malliavin operators. These operators are (possibly infinite) sums of multiple Wiener-Itô integrals with integrands for which we only have formula (1.2).

For simplicity, we only describe in this introduction how the bound (1.5) can be evaluated. For the Kolmogorov distance and the d_3-distance this can be done in a similar way. We can remove the absolute values in formula (1.5) by some elementary inequalities. For the first term in the bound (1.5) we obtain sums of products of the type

$$\mathbb{E} I_{n_1}(f_1)\, I_{n_2}(f_2)\, I_{n_3}(f_3)\, I_{n_4}(f_4),$$

where $f_1 \in L^2_s(\mu^{n_1})$, $f_2 \in L^2_s(\mu^{n_2})$, $f_3 \in L^2_s(\mu^{n_3})$, $f_4 \in L^2_s(\mu^{n_4})$, $n_1, n_2, n_3, n_4 \in \mathbb{N}$, are basically the kernels of the Wiener-Itô chaos expansion of F. Products of multiple

Wiener-Itô integrals were considered by Peccati and Taqqu, and Surgailis in [66, 92]. The expectation is given by a sum of deterministic integrals depending on $f_1, \dots, f_4$. We prove such a product formula that is appropriate for our slightly more general setting than in the previous works.

As a consequence of this product formula for multiple Wiener-Itô integrals, the bound for the normal approximation of a Poisson functional F in the Wasserstein distance in Equation (1.5) can be simplified to

$$\begin{aligned} d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq & \frac{1}{\operatorname{Var} F} \sum_{i,j=1}^{\infty} i \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, \mathrm{d}\mu^{|\sigma|}} \\ & + \frac{1}{\operatorname{Var} F} \sqrt{\int_X \mathbb{E}(D_z F)^4 \, \mathrm{d}\mu(z)}. \end{aligned} \tag{1.6}$$

On the right-hand side, we have a sum of deterministic integrals depending on the kernels of the Wiener-Itô chaos expansion of F and a special class of partitions and an integral depending on the fourth moment of the difference operator.

In case that the considered Poisson functional has an infinite Wiener-Itô chaos expansion, i.e. it is a sum of infinitely many multiple Wiener-Itô integrals with integrands that are not constantly zero, one has to ensure that the new bound converges. In order to avoid this problem, we investigate Poisson functionals with a finite Wiener-Itô chaos expansion first. In this case, the second term on the right-hand side of formula (1.6) can be bounded by integrals as they occur in the first term. A special class of Poisson functionals with finite Wiener-Itô chaos expansion are so called Poisson U-statistics that have the form

$$S = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k)$$

with $f \in L^1(\mu^k)$ and $k \in \mathbb{N}$. Here, $\eta^k_{\neq}$ stands for the set of all k-tuples of distinct points of η. For the approximation of Poisson functionals with an infinite Wiener-Itô chaos expansion we present two results. The first one is based on the approximation of the Poisson functional by a Poisson functional with a finite Wiener-Itô chaos expansion, and the second result requires that the kernels of the Wiener-Itô chaos expansion satisfy some special integrability conditions.

Our strategy to provide a bound for the normal approximation of a given square integrable Poisson functional F is the following.

- We compute the kernels f_n, $n \in \mathbb{N}$, of the Wiener-Itô chaos expansion of F by formula (1.2).

- Depending on the structure of the derived formulas for the kernels, we can give an explicit formula for the variance or only lower and upper bounds for the variance by Equation (1.4).

- We show that the integrals depending on the kernels of the Wiener-Itô chaos expansion and the integral of the fourth moment of the difference operator on the right-hand side of formula (1.6) are small.

In stochastic geometry, as in other branches of probability, it is often not possible to determine exact distributions of random variables. In such a situation, one can try to investigate their asymptotic behaviour. Possible asymptotic regimes are that the intensity of the underlying point process is increased (in this case we consider a family of Poisson point processes $(\eta_t)_{t\geq 1}$ with intensity measures $\mu_t = t\mu$) or that the observation window is increased. Often one is interested in random variables that are sums of dependent random variables. Here, one expects the limiting distribution to be Gaussian. Therefore, we denote such results as central limit theorems.

Although there are many central limit theorems in stochastic geometry that were derived by different techniques, most of them belong to one of the following four approaches:

- Baryshnikov, Penrose, Wade, and Yukich introduced the so-called stabilization technique and applied it to many examples in [5, 68, 70, 72, 73, 74, 75]. They investigate random variables that are sums over all points of a Poisson point process where every summand depends only on a point and its neighbourhood. Using the size of this neighbourhood, described by the radius of stabilization, and some additional assumptions, they prove abstract central limit theorems. It is assumed that the underlying point process is Poisson but some of the results still hold for binomial point processes.

- Central limit theorems can be also derived from mixing properties of random fields. Examples are the works [2, 21, 24] by Baddeley, Heinrich, and Molchanov. An advantage of this method is that it works for many different underlying point processes.

- Stein's method consists of several techniques that can be directly applied to problems from stochastic geometry. For example, Avram and Bertsimas, Bárány and Reitzner, and Schreiber use in [1, 4, 80] the dependency graph method (see the paper [3] by Baldi and Rinott), and Goldstein and Penrose apply a coupling approach in [18].

- A martingale central limit theorem by McLeish (see [50]) is applied by Kesten and Lee in [33] and by Penrose in [69]. The technique developed by Kesten and Lee is also used by Lee in [44, 45], Kesten and Zhang in [34], and Zhang in [94]. A more recent example for martingale methods are the works [81, 82, 83] by Schreiber and Thäle.

These different methods are sometimes closely related. For example, the proofs of the stabilization method rest upon different techniques from probability theory as Stein's method and the application of martingales.

In this work, we want to show that the Malliavin-Stein method is a further approach to derive central limit theorems. We compute the Wiener-Itô chaos expansion of Poisson functionals occurring in stochastic geometry and apply our abstract approximation results. In particular, we consider the following problems:

- **Intersection process of Poisson k-flats**: We observe a stationary Poisson k-flat process η_t of intensity $t > 0$ in a compact convex observation window

$W \subset \mathbb{R}^d$. For $\ell \in \mathbb{N}$ with $d - \ell(d-k) \geq 0$, ℓ distinct k-flats of η_t intersect in a $d-\ell(d-k)$-dimensional flat. Now we apply a functional, for example an intrinsic volume, to the intersection of each of the intersection flats with W and sum over all intersection flats. These Poisson functionals are Poisson U-statistics, and it is proven that they converge after standardization in distribution to a standard Gaussian random variable for increasing intensity. The number of intersection flats in W or their total $d - \ell(d-k)$-dimensional volume in W are examples for this class of Poisson functionals. We also derive multivariate limit theorems for different observation windows or different functionals that are applied to the intersection flats.

- **Poisson hyperplane tessellation**: A special case of a Poisson k-flat process is a Poisson hyperplane process. A stationary Poisson hyperplane process η_t induces a random tessellation in $\mathbb{R}^d$, the Poisson hyperplane tessellation. Now one is interested in the numbers of ℓ-dimensional faces of a Poisson hyperplane tessellation in a compact convex observation window W. Again, we derive univariate and multivariate central limit theorems for increasing intensity of the underlying Poisson hyperplane process.

- **Gilbert graph**: Let η_t be the restriction of a stationary Poisson point process to a compact convex set with interior points. The so-called Gilbert graph is constructed by taking the points of η_t as vertices and connecting two points by an edge if their distance is not greater than a threshold δ_t. Assuming that $\delta_t \to 0$ as $t \to \infty$, we prove central limit theorems for the number of vertices and the total edge length of the Gilbert graph.

- **Poisson-Voronoi approximation**: For a compact convex set K with interior points and a stationary Poisson point process η_t in $\mathbb{R}^d$ we can construct the following random approximation of K that is called Poisson-Voronoi approximation. We construct the Voronoi tessellation induced by η_t and take the union of all cells with nucleus in K as approximation of K. We prove that the standardization of the volume of the Poisson-Voronoi approximation converges in distribution to a standard Gaussian random variable as $t \to \infty$.

- **Boolean model**: Let η be a stationary Poisson point process on the space of all compact convex sets in $\mathbb{R}^d$. The union of all these sets is called Boolean model. We observe the Boolean model in a sequence of increasing compact convex observation windows $(W_n)_{n\in\mathbb{N}}$ and prove univariate and multivariate central limit theorems for the intrinsic volumes of the Boolean model within W_n as $n \to \infty$.

Closely related to this work are the papers [39, 40] by Lachièze-Rey and Peccati, where similar bounds for the normal approximation of Poisson functionals with finite Wiener-Itô chaos expansion are proven and applied to the Gilbert graph described above and some of its generalizations.

In the following, we consider only the normal approximation of Poisson functionals, but the idea of the Malliavin-Stein method, to combine Stein's method and Malliavin calculus, can be also used for the approximation of Poisson functionals by other distributions than a Gaussian distribution. This is done for the Poisson distribution by

Peccati in [64] and for a vector of Gaussian and Poisson random variables by Bourguin and Peccati in [6]. The result for the Poisson approximation was used by Schulte and Thäle in [87] to derive non-central limit theorems and Poisson point process convergence for several problems in stochastic geometry and in [88], together with results for the normal approximation, to study the so-called proximity problem of non-intersecting Poisson k-flats.

This work is mainly based on the following papers, partially jointly written with Daniel Hug, Günter Last, Mathew Penrose, Matthias Reitzner, and Christoph Thäle:

- *Hug, Last, and Schulte 2012*:
  Second order properties and central limit theorems for Boolean models. In preparation.

- *Last, Penrose, Schulte, and Thäle 2012*:
  Moments and central limit theorems for some multivariate Poisson functionals. Preprint.

- *Reitzner and Schulte 2011*:
  Central limit theorems for U-statistics of Poisson point processes. To appear in Annals of Probability.

- *Schulte 2012a*:
  A central limit theorem for the Poisson-Voronoi approximation. Published in Advances in Applied Mathematics.

- *Schulte 2012b*:
  Normal approximation of Poisson functionals in Kolmogorov distance. Preprint.

In order to increase the readability of the text, we only discuss at the end of every chapter which result belongs to which paper.

This thesis is organized in the following way. In Chapter 2, we fix some basic notation, present some tools from convex and integral geometry we need for our applications to stochastic geometry, and recall some facts from probability theory. Then, we define Poisson U-statistics and multiple Wiener-Itô integrals and derive product formulas for them in Chapter 3. The product formula for multiple Wiener-Itô integrals is our main tool to evaluate abstract bounds derived by the Malliavin-Stein method. In Chapter 4, we give a brief introduction to Wiener-Itô chaos expansions and Malliavin operators and compute both for Poisson U-statistics.

Chapter 5 starts with bounds for the normal approximation of Poisson functionals derived by the Malliavin-Stein method by Peccati, Solé, Taqqu, and Utzet and Peccati and Zheng, respectively. A similar bound is proven for the univariate normal approximation in the Kolmogorov distance. The abstract bounds are further evaluated for Poisson functionals satisfying some additional assumptions. These results are applied to Poisson U-statistics in Chapter 6.

In Chapter 7, central limit theorems for some Poisson U-statistics from stochastic geometry are proven. As examples for Poisson functionals with infinite Wiener-Itô chaos expansions, the volume of the Poisson-Voronoi approximation of a convex body and the intrinsic volumes of Boolean models are considered in Chapter 8 and Chapter 9, respectively.

Chapter 2

Preliminaries

After introducing some basic notation in the first section, we recall some facts from convex and integral geometry in the second section. The third section is devoted to moments and cumulants, Poisson point processes, and Stein's method.

2.1 Basic notation

Let $(X, \mathcal{X})$ be a measurable space with a σ-finite measure μ. For $n \in \mathbb{N}$ (in our notation is $0 \notin \mathbb{N}$) we denote by μ^n the product measure of μ on the space X^n equipped with the σ-algebra generated by $\mathcal{X}^n$. For the integral of a measurable function $f : X^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ with respect to μ^n we write

$$\int_{X^n} f \, \mathrm{d}\mu^n \quad \text{or} \quad \int_{X^n} f(x_1, \ldots, x_n) \, \mathrm{d}\mu(x_1, \ldots, x_n).$$

By $L^p(\mu^n)$, $p > 0$, we denote the set of all measurable functions $f : X^n \to \overline{\mathbb{R}}$ such that

$$\int_{X^n} |f|^p \, \mathrm{d}\mu^n < \infty.$$

We call a function $f : X^n \to \overline{\mathbb{R}}$ symmetric if it is invariant under permutations of its arguments for μ-almost all $(x_1, \ldots, x_n) \in X^n$. Let $L^p_s(\mu^n)$ stand for the subspace of all symmetric functions in $L^p(\mu^n)$.

The space $L^2(\mu^n)$ equipped with the inner product

$$\langle f, g \rangle_{L^2(\mu^n)} := \int_{X^n} f \, g \, \mathrm{d}\mu^n \quad \text{for } f, g \in L^2(\mu^n)$$

and the norm $\|f\|_n := \sqrt{\langle f, f \rangle_{L^2(\mu^n)}}$ for $f \in L^2(\mu^n)$ is complete, which implies that $L^2(\mu^n)$ is a Hilbert space. Its subspace $L^2_s(\mu^n)$ is a Hilbert space as well.

A function $f \in L^2(\mu^n)$ of the form

$$f(x) = \sum_{i=1}^{m} c_i \, \mathbb{1}(x \in Z_1^{(i)} \times \ldots \times Z_n^{(i)})$$

with $c_i \in \mathbb{R}$ and $Z_1^{(i)}, \dots, Z_n^{(i)} \in \mathcal{X}$ for $i = 1, \dots, m$ is called simple. By $\mathcal{E}(\mu^n)$ and $\mathcal{E}_s(\mu^n)$ we denote the sets of simple functions in $L^2(X^n)$ and $L_s^2(X^n)$, respectively. It is important to note that $\mathcal{E}(\mu^n)$ and $\mathcal{E}_s(\mu^n)$ are dense in $L^2(\mu^n)$ and $L_s^2(\mu^n)$.

The d-dimensional Lebesgue measure is denoted by λ_d, and in the one-dimensional case we write λ. We use the standard notation where $\mathrm{d}x$ stands for integration with respect to the (d-dimensional) Lebesgue measure.

For a finite set A let $|A|$ be the number of elements of A. A partition σ of a non-empty set A is a collection of non-empty sets $B_1, \dots, B_m \subset A$ such that $\bigcup_{\ell=1}^m B_\ell = A$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$. We call $B_1, \dots, B_m$ blocks of σ and write $B_\ell \in \sigma$ for $\ell = 1, \dots, m$. Moreover, $|\sigma|$ stands for the number of blocks of σ. By $\mathcal{P}(A)$ we denote the set of all partitions of A. For $m \in \mathbb{N}$ we write $[m]$ as an abbreviation for $\{1, \dots, m\}$.

2.2 Background material from convex and integral geometry

In this section, we introduce some notation and results from convex and integral geometry that are necessary for our applications to stochastic geometry in the Chapters 7, 8, and 9. For more details we refer to the monographs [78, 79] by Schneider and Weil. Our notation is similar as in [79].

Let $\mathcal{C}^d$ stand for the system of all compact subsets of $\mathbb{R}^d$. The system of all compact convex sets is denoted by $\mathcal{K}^d$, and we use the convention that $\emptyset \in \mathcal{K}^d$. We call a compact convex subset of $\mathbb{R}^d$ with interior points a convex body and denote by $\mathcal{K}_0^d$ the set of all convex bodies in $\mathbb{R}^d$. A compact set in $\mathbb{R}^d$ is called polyconvex if it is the union of finitely many compact convex sets. The system of all polyconvex sets in $\mathbb{R}^d$ forms the convex ring, which is denoted by $\mathcal{R}^d$.

We call a linear map on $\mathbb{R}^d$ a rotation if it preserves angles and orientation and denote the set of all rotations on $\mathbb{R}^d$ by SO_d. A rigid motion is a combination of a rotation and a translation on $\mathbb{R}^d$. Let G_d be the set of all rigid motions on $\mathbb{R}^d$.

Let $\mathrm{Vol}(A) = \lambda_d(A)$ stand for the volume of a measurable set $A \subset \mathbb{R}^d$ and let $\mathcal{H}^m$ be the m-dimensional Hausdorff measure.

In the following, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. We write $\mathrm{dist}(x, y) = \|x - y\|$ for the usual Euclidean distance of two points $x, y \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$ we define $\mathrm{dist}(x, A) = \inf_{y \in A} \mathrm{dist}(x, y)$. By $B^d(x, r)$ we denote the closed ball in $\mathbb{R}^d$ with centre x and radius $r > 0$, and $B^d = B^d(0, 1)$ is the unit ball in $\mathbb{R}^d$. Let κ_n be the volume of the n-dimensional unit ball in $\mathbb{R}^n$ and let $\kappa_0 = 1$. For a compact convex set $K \in \mathcal{K}^d$ the inradius $r(K)$ is the largest radius of a ball contained in K, and the circumradius $R(K)$ is the smallest radius of a ball containing K.

The Minkowski sum $A + B$ for $A, B \subset \mathbb{R}^d$ and the dilation cA for $A \subset \mathbb{R}^d$ and $c > 0$ are defined by

$$A + B = \{x + y : x \in A, y \in B\} \quad \text{and} \quad cA = \{cx : x \in A\}.$$

For $K \in \mathcal{K}^d$ and $\varepsilon > 0$ the Minkowski sum $K + \varepsilon B^d$ is the set of all points in $\mathbb{R}^d$ with a distance less than or equal to ε to K. Its volume is given by the following formula (see [79, Equation (14.5)]):

Proposition 2.1 *There are functions $V_j : \mathcal{K}^d \to \mathbb{R}$, $j = 0, \ldots, d$, such that*

$$\mathrm{Vol}(K + \varepsilon B^d) = \sum_{j=0}^{d} \kappa_{d-j} V_j(K)\, \varepsilon^{d-j} \tag{2.1}$$

for all $K \in \mathcal{K}^d$ and $\varepsilon > 0$.

Equation (2.1) tells us that $\mathrm{Vol}(K + \varepsilon B^d)$ is in a polynomial in ε. The coefficients $V_j(K)$, $j = 0, \ldots, d$, are called intrinsic volumes of K. The intrinsic volumes are additive, meaning that

$$V_j(K \cup L) = V_j(K) + V_j(L) - V_j(K \cap L) \tag{2.2}$$

for all $K, L \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$. Moreover, they are non-negative, rigid motion invariant, continuous with respect to the Hausdorff metric

$$\delta(K, L) = \max\{\max_{x \in K} \mathrm{dist}(x, L), \max_{y \in L} \mathrm{dist}(y, K)\} \quad \text{for } K, L \in \mathcal{C}^d,$$

and monotone under set inclusion, i.e. $V_j(K) \leq V_j(L)$ for all $K, L \in \mathcal{K}^d$ with $K \subset L$.

By using the additivity (2.2), one can extend the intrinsic volumes from the compact convex sets $\mathcal{K}^d$ to the convex ring $\mathcal{R}^d$. The extension prevents additivity and rigid motion invariance, but we lose continuity and the intrinsic volumes can become negative. Some of the intrinsic volumes have special geometric meanings. For example, $V_d(K)$ is the usual volume of K, $V_{d-1}(K) = \frac{1}{2} S(K)$, where $S(K)$ stands for the surface area of K, and $V_0(K)$ is the Euler characteristic $\chi(K)$, a well-known topological invariant.

The Steiner formula Proposition 2.1 can be generalized to arbitrary intrinsic volumes (see [79, Theorem 14.2.4]):

Proposition 2.2 *For $0 \leq m \leq d$, $\varepsilon > 0$, and $K \in \mathcal{K}^d$ we have*

$$V_m(K + \varepsilon B^d) = \sum_{j=0}^{m} \frac{\kappa_{d-j}}{\kappa_{d-m}} \binom{d-j}{d-m} V_j(K)\, \varepsilon^{m-j}.$$

For $k \in \{0 \ldots, d\}$ let the Grassmannian $G(d, k)$ be the set of all k-dimensional linear subspaces of $\mathbb{R}^d$ and let the affine Grassmannian $A(d, k)$ be the set of all k-dimensional affine subspaces of $\mathbb{R}^d$. The elements of $A(d, k)$ are also called k-flats. For a fixed $k \in \{0, \ldots, d\}$ and $W \in \mathcal{K}^d$ we denote by $[W]$ the set of all k-flats hitting W.

A Haar measure on $G(d, k)$ is a SO_d invariant measure, and a Haar measure on $A(d, k)$ is a G_d invariant measure. The Haar measures on $G(d, k)$ and $A(d, k)$ are unique up to a constant. For a detailed construction we refer to [79, Section 13.2]. By integration with respect to the Haar measure on $A(d, k)$ we omit the measure in our notation and just write $\mathrm{d}E$. Here, we use the Haar measure with the normalization such that the measure of all k-flats hitting the d-dimensional unit ball B^d is one. A useful tool for the integration over the set of all k-flats is Crofton's formula (see [79, Theorem 5.1.1]):

Proposition 2.3 *For $K \in \mathcal{K}^d$, $1 \le k \le d-1$, and $0 \le j \le k$ we have*

$$\int_{A(d,k)} V_j(K \cap E)\,\mathrm{d}E = \frac{\binom{k}{j}\kappa_k\,\kappa_{d-k+j}}{\binom{d}{k-j}\kappa_j\,\kappa_d} V_{d-k+j}(K).$$

For the computation of some integrals we apply the so-called coarea formula. If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $x \in \mathbb{R}^m$, we define the Jacobian $Jf(x)$ by

$$Jf(x) = \sqrt{\det\left(f'(x)\ f'(x)^T\right)},$$

where f' stands for the Jacobi matrix of f. Observe that a Lipschitz function is almost everywhere differentiable in such a way that its Jacobian is almost everywhere defined. Using this notation, we have (see Corollary 5.2.6 in the monograph [38] by Krantz and Parks, for example):

Proposition 2.4 *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function and $m \ge n$, then*

$$\int_B g(x)\,Jf(x)\,\mathrm{d}\lambda_m(x) = \int_{\mathbb{R}^n}\int_{B\cap f^{-1}(y)} g(z)\,\mathrm{d}\mathcal{H}^{m-n}(z)\,\mathrm{d}\lambda_n(y)$$

holds for each Lebesgue measurable $B \subset \mathbb{R}^m$ and each non-negative λ_m-measurable function $g : B \to \mathbb{R}$.

For $n = 1$ we have $Jf(x) = \|\nabla f(x)\|$. Note that $\|\cdot\|$ stands for the usual Euclidean norm, whereas $\|\cdot\|_n$ is the norm in $L^2_s(\mu^n)$ or $L^2_s(\lambda^n_d)$.

2.3 Background material from probability theory

2.3.1 Moments and cumulants

Throughout this monograph, let $(\Omega, \mathcal{F}, \mathbb{P})$ always be the underlying probability space and let $\mathbb{E}X$ stand for the expectation of a random variable (or random vector) X over $(\Omega, \mathcal{F}, \mathbb{P})$. By $L^p(\mathbb{P})$, $p > 0$, we denote the set of all random variables X such that

$$\mathbb{E}|X|^p = \int_\Omega |X|^p\,\mathrm{d}\mathbb{P} < \infty.$$

In order to describe the behaviour of random variables, we use their moments and cumulants. For a random variable X and $m \in \mathbb{N}$ we call $\mathbb{E}X^m$ the m-th moment and $\mathbb{E}(X - \mathbb{E}X)^m$ the m-th centred moment. Both moments exist if and only if $X \in L^m(\mathbb{P})$. Analogously, there are mixed moments $\mathbb{E}\prod_{\ell=1}^m X_\ell$ and mixed centred moments $\mathbb{E}\prod_{\ell=1}^m (X_\ell - \mathbb{E}X_\ell)$ for random variables $X_1, \dots, X_m$. The characteristic function $\varphi_{(X_1 \dots, X_m)} : \mathbb{R}^m \to \mathbb{C}$ of a vector of random variables $X_1, \dots, X_m$ is defined as

$$\varphi_{(X_1,\dots,X_m)}(z_1, \dots, z_m) = \mathbb{E}\exp(\boldsymbol{i}(z_1X_1 + \ldots + z_mX_m)),$$

where $\boldsymbol{i}$ stands for the imaginary unit. The joint cumulant $\gamma(X_1, \dots, X_m)$ of $X_1, \dots, X_m$ is defined by

$$\gamma(X_1, \dots, X_m) = (-\boldsymbol{i})^m \frac{\partial^m \log \varphi_{(X_1,\dots,X_m)}}{\partial z_1 \dots \partial z_m}(z_1, \dots, z_m)|_{z_1=\ldots=z_m=0}.$$

Note that the joint cumulant $\gamma(X_1, \dots, X_m)$ is multilinear in $X_1, \dots, X_m$. The m-th cumulant $\gamma_m(X)$ of a random variable X is given by $\gamma_m(X) = \gamma(X, \dots, X)$, where X occurs m times. The following lemma (see Proposition 3.2.1 in the monograph [66] by Peccati and Taqqu or Equation (40) in II §12 in the textbook [89] by Shiryaev) gives us a relation between the cumulants and the mixed moments of a random vector:

Lemma 2.5 *Let $X_1, \dots, X_m$ be random variables and define $X_B = (X_{i_1}, \dots, X_{i_\ell})$ for $B = \{i_1, \dots, i_\ell\} \subset [m]$. Then*

$$\mathbb{E} \prod_{\ell=1}^{m} X_\ell = \sum_{\pi \in \mathcal{P}([m])} \prod_{B \in \pi} \gamma(X_B).$$

We say that a random variable is uniquely determined by its moments if every random variable with the same moments has the same probability distribution. In this case, the moments of random variables can be used to prove convergence in distribution. The following criterion (see Section 8.12 in the textbook [7] by Breiman) is called method of moments:

Proposition 2.6 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with finite moments and let X be a random variable with finite moments.*

a) The random variable X is uniquely determined by its moments if

$$\limsup_{m\to\infty} \frac{|\mathbb{E}X^m|^{\frac{1}{m}}}{m} < \infty.$$

b) Let X be uniquely determined by its moments. Then $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X if and only if

$$\lim_{n\to\infty} \mathbb{E}X_n^m = \mathbb{E}X^m \text{ for all } m \in \mathbb{N}. \tag{2.3}$$

Because of Lemma 2.5, condition (2.3) is equivalent to

$$\lim_{n\to\infty} \gamma_m(X_n) = \gamma_m(X) \text{ for all } m \in \mathbb{N}.$$

This formulation is useful in situations where the cumulants are easier to handle than the moments. Then the criterion is called method of cumulants.

2.3.2 Poisson point processes

In the following, we introduce Poisson point processes as random measures and present some of their properties. The definition of a Poisson point process follows the approach in the textbook [32] by Kallenberg and does not require any topological assumptions on the underlying measurable space.

Let $(X, \mathcal{X})$ be a measurable space. By $N(X)$ we denote the set of all σ-finite integer valued measures on $(X, \mathcal{X})$. The set $N(X)$ is equipped with the smallest σ-algebra $\mathcal{N}(X)$ such that all maps $g_A : N(X) \to \overline{\mathbb{R}}, \eta \mapsto \eta(A)$ with $A \in \mathcal{X}$ are measurable. From now on, let $(\Omega, \mathcal{F}, \mathbb{P})$ always be the underlying probability space. We call a measurable map $\eta : (\Omega, \mathcal{F}) \to (N(X), \mathcal{N}(X))$ an integer valued random measure. Now we can define a Poisson point process in the following way:

Definition 2.7 *Let $(X, \mathcal{X})$ be a measurable space with a σ-finite measure μ. A σ-finite integer valued random measure η is called a Poisson point process with intensity measure μ if*

1. *$\eta(A_1), \ldots, \eta(A_n)$ are independent for disjoint sets $A_1, \ldots, A_n \in \mathcal{X}$ and $n \in \mathbb{N}$;*
2. *$\eta(A)$ is Poisson distributed with parameter $\mu(A)$ for each $A \in \mathcal{X}$.*

In the second condition, it can happen that $\mu(A) = \infty$. In this case, $\eta(A)$ is infinite almost surely. For a proof of the existence of a Poisson point process we refer to [32, Theorem 12.7].

In the following, we are interested in the behaviour of a Poisson functional F which is a random variable depending on the Poisson point process η. More precisely, F is a measurable map from $N(X)$ to $\overline{\mathbb{R}}$. As usual, we treat Poisson functionals as random variables depending on the image measure $\mathbb{P}_\eta$ of η instead of the original probability measure $\mathbb{P}$.

The following result allows us without loss of generality to assume that all singletons of X belong to the σ-algebra $\mathcal{X}$. Since I do not know of a reference in the literature, it is proven in the following.

Proposition 2.8 *Let η_1 be a Poisson point process with a σ-finite intensity measure μ_1 over a measurable space $(X_1, \mathcal{X}_1)$ and let $F_1 : N(X_1) \to \overline{\mathbb{R}}$ be measurable. Then there is a Poisson point process η_2 with a σ-finite intensity measure μ_2 over a measurable space $(X_2, \mathcal{X}_2)$ satisfying $\{x\} \in \mathcal{X}_2$ for all $x \in X_2$ and a measurable $F_2 : N(X_2) \to \overline{\mathbb{R}}$ such that $F_1(\eta_1)$ and $F_2(\eta_2)$ have the same distribution.*

Proof. Let the relation $\sim$ on X_1 be given by $x \sim y$ for $x, y \in X_1$ if there are no sets $A, B \in \mathcal{X}_1$ such that $x \in A$, $y \notin A$, $x \notin B$, and $y \in B$ and let $[x] = \{y \in X_1 : x \sim y\}$ for $x \in X_1$. Now we define sets

$$M_1 = \{x \in X_1 : \text{ there is a countable set } Y \subset X_1 \text{ such that } [x] \cup \bigcup_{y \in Y} [y] \in \mathcal{X}_1\}$$

and $M_2 = X_1 \setminus M_1$. Let $x \in M_1$ and let $Y \subset X_1$ be the countable set. Without loss of generality we can assume that $[x] \cap Y = \emptyset$. By the definition of $\sim$, there exists for each $y \in Y$ a set $B_y \in \mathcal{X}_1$ such that $x \notin B_y$ (and hence $z \notin B_y$ for all $z \in [x]$) and $y \in B_y$ (and hence $[y] \subset B_y$). Therefore, we have the representation

$$[x] = \left([x] \cup \bigcup_{y \in Y} [y] \right) \cap \bigcap_{y \in Y} B_y^C,$$

which means that $[x] \in \mathcal{X}_1$ for all $x \in M_1$. Let $\widetilde{\mathcal{X}_1}$ be the σ-algebra generated by $\mathcal{X}_1 \cup \{[x] : x \in M_2\}$. All elements $A \in \widetilde{\mathcal{X}_1}$ are of the form

$$A = (A_0 \cup A_1) \setminus A_2,$$

where $A_0 \in \mathcal{X}_1$ and $A_1 = \bigcup_{x \in I} [x]$ and $A_2 = \bigcup_{x \in J} [x]$ with countable subsets I and J of M_2. Hence, we can define a σ-finite measure $\tilde{\mu}_1$ as $\tilde{\mu}_1(A) = \mu(A_0)$. Now there exists a Poisson point process over $(X_1, \widetilde{\mathcal{X}_1})$ with intensity measure $\tilde{\mu}_1$. We define $\tilde{F}_1$: $N((X_1, \widetilde{\mathcal{X}_1})) \to \overline{\mathbb{R}}$ by restricting $\tilde{\eta}_1$ to a measure on $\mathcal{X}_1$ and applying F_1 to this new integer valued measure. Since $\mathcal{X}_1 \subset \widetilde{\mathcal{X}_1}$, the restriction of the measure and, hence, $\tilde{F}_1$ are measurable. The restriction of the Poisson point process $\tilde{\eta}_1$ to $\mathcal{X}_1$ has by definition the same intensity measure as η_1 so that the Poisson functionals $F_1(\eta_1)$ and $\tilde{F}_1(\tilde{\eta}_1)$ have the same distributions.

Let X_2 be the set of all equivalence classes $[x]$. Since $[x] \cap A = [x]$ or $[x] \cap A = \emptyset$ for all $A \in \widetilde{\mathcal{X}_1}$, $\widetilde{\mathcal{X}_1}$ and $\tilde{\mu}_1$ induce a σ-algebra $\mathcal{X}_2$ and a measure μ_2 on X_2. Now every realization of a Poisson point process η_2 with intensity measure μ_2 is equivalent to a realization of the Poisson point process $\tilde{\eta}_1$. Thus, we have a Poisson functional $F_2 = F_2(\eta_2)$ with the same distribution as $F_1(\eta_1)$. Since $\{y\} \in \mathcal{X}_2$ for all $y \in X_2$, this concludes the proof. $\square$

From now on we assume that the measurable space $(X, \mathcal{X})$ satisfies

$$\{x\} \in \mathcal{X} \quad \text{for all} \quad x \in X. \tag{2.4}$$

We call a point $x \in X$ an atom of a measure ν if $\nu(\{x\}) > 0$. A measure ν is diffuse if it has no atoms. Under assumption (2.4) the Poisson point process η has almost surely a representation as a sum of its atoms, namely

$$\eta = \sum_{i \in I} \delta_{x_i} \quad \text{with} \quad x_i \in X \quad \text{and a countable index set } I, \tag{2.5}$$

where δ_x stands for the Dirac measure concentrated at the point $x \in X$. In case that the intensity measure μ has atoms, it can happen that there are $i, j \in I$ with $i \neq j$ such that $x_i = x_j$, which means that $\eta(x_i) > 1$ and x_i is a multiple point of η. On the other hand, if μ is diffuse, it holds almost surely that $\eta(x) \leq 1$ for all $x \in X$ and there are no multiple points. Because of the representation (2.5), we can think of a Poisson point process as a random collection of countable points in X. If μ is diffuse, we can identify the point process η with its support. In the case that μ has atoms, this is wrong since we would lose the multiplicity of points.

In order to distinct multiple points $x_i = x_j$ for $i \neq j$, we give each of them a number as a mark, which allows us to consider η as a set. This is the background for the notation $x \in \eta$. Analogously, we define $\eta^k_{\neq}$ for $k \in \mathbb{N}$ as the set of all k-tuples of distinct points. Here, two points are distinct, if they have the same location but different marks. This notation is used for the so-called Slivnyak-Mecke formula, which plays an important role in the following.

Proposition 2.9 *For every $f \in L^1(\mathbb{P}_\eta \times \mu^k)$ it holds that*

$$\mathbb{E} \sum_{(x_1,\dots,x_k) \in \eta^k_{\neq}} f(\eta, x_1, \dots, x_k) = \int_{X^k} \mathbb{E} f(\eta + \sum_{i=1}^k \delta_{x_i}, x_1, \dots, x_k) \, \mathrm{d}\mu(x_1, \dots, x_k).$$

Proof. The proofs in [79, Theorem 3.2.5 and Corollary 3.2.3] can be extended to our slightly different setting. $\square$

We often need only a special case of the Slivnyak-Mecke formula, where the function f only depends on the points $x_1, \dots, x_k$ and not on the whole process η.

Corollary 2.10 *For every $f \in L^1(\mu^k)$ we have*

$$\mathbb{E} \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k) = \int_{X^k} f(x_1,\dots,x_k)\,\mathrm{d}\mu(x_1,\dots,x_k).$$

2.3.3 Stein's method

A common problem in probability theory is to decide if a family of random variables (or random vectors) $(X_t)_{t\geq 0}$ converges in distribution to a random variable (or a random vector) X. Since one is also interested in the question how fast this convergence takes place, one needs a measure for the distance between two random variables (or random vectors) Y and Z. This can be a distance $d_{\mathcal{H}}(\cdot,\cdot)$ given by

$$d_{\mathcal{H}}(Y,Z) = \sup_{h\in\mathcal{H}} |\mathbb{E}h(Y) - \mathbb{E}h(Z)|, \tag{2.6}$$

where $\mathcal{H}$ is a suitable set of test functions.

For two random variables Y and Z the choice $\mathcal{H} = \mathrm{Lip}(1)$, where $\mathrm{Lip}(1)$ stands for the set of all functions $h : \mathbb{R} \to \mathbb{R}$ with a Lipschitz constant less than or equal to one, leads to the Wasserstein distance

$$d_W(Y,Z) = \sup_{h\in\mathrm{Lip}(1)} |\mathbb{E}h(Y) - \mathbb{E}h(Z)|.$$

Taking for $\mathcal{H}$ the set of indicator functions of intervals $(-\infty, t]$, $t \in \mathbb{R}$, we obtain the Kolmogorov distance

$$d_K(Y,Z) = \sup_{t\in\mathbb{R}} |\mathbb{P}(Y \leq t) - \mathbb{P}(Z \leq t)|,$$

which is the supremum norm of the difference between the distribution functions of Y and Z. Wasserstein distance and Kolmogorov distance are both zero if and only if Y and Z follow the same distribution, whence they are pseudo metrics on the space of all random variables. Now the idea is that the random variables Y and Z have similar distributions if these distances are small. This is true since convergence in Wasserstein distance or in Kolmogorov distance implies convergence in distribution. The converse does not hold because there are examples of random variables converging in distribution but not in these distances.

The Wasserstein distance and the Kolmogorov distance are also defined for m-dimensional random vectors by taking functions $h : \mathbb{R}^m \to \mathbb{R}$ with a Lipschitz constant less than or equal to one or indicator functions of cartesian products $(-\infty, t_1] \times \ldots \times (-\infty, t_m]$ as test functions. But these distances are too strong for our purposes so that we use a distance d_3 instead. For two m-dimensional random vectors Y and Z the distance $d_3(Y,Z)$ is defined by

$$d_3(Y,Z) = \sup_{g\in\mathcal{H}_m} |\mathbb{E}g(Y) - \mathbb{E}g(Z)|,$$

where $\mathcal{H}_m$ is the set of all thrice continuously differentiable functions $g : \mathbb{R}^m \to \mathbb{R}$ such that

$$\max_{1\leq i_1\leq i_2\leq m} \sup_{x\in\mathbb{R}^m} \left|\frac{\partial^2 g}{\partial x_{i_1}\partial x_{i_2}}(x)\right| \leq 1 \quad \text{and} \quad \max_{1\leq i_1\leq i_2\leq i_3\leq m} \sup_{x\in\mathbb{R}^m} \left|\frac{\partial^3 g}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}(x)\right| \leq 1.$$

Again convergence in d_3 implies convergence in distribution.

A method to bound such distance and, hence, to prove limit theorems is Stein's method which goes back to Charles Stein (see [90, 91]). Although it can be used for many distributions, we focus on univariate normal approximation in the following. In this context, it was originally introduced by Stein. The underlying idea of this method is to find solutions g_h of the differential equation

$$g_h'(w) - wg_h(w) = h(w) - \mathbb{E}h(N) \tag{2.7}$$

for test functions $h \in \mathcal{H}$, where N is a standard Gaussian random variable. The differential equation (2.7) is called Stein's equation. Now we can replace w by a random variable Y, take the expectation, and put it in formula (2.6), which yields

$$d_{\mathcal{H}}(Y,N) = \sup_{h\in\mathcal{H}} |\mathbb{E}[g_h'(Y) - Y\, g_h(Y)]|.$$

Now there are several techniques to evaluate the right-hand side. In Chapter 5, this will be done by Malliavin calculus. For a detailed and more general introduction into Stein's method we refer to the works [8, 9, 91] by Chen, Goldstein, Shao, and Stein.

Later we will apply Stein's method for the normal approximation in Kolmogorov distance which rests upon the the following lemma (see Chapter II in [91]):

Lemma 2.11 *Let N be a standard Gaussian random variable and let $t \in \mathbb{R}$. Then $g_t : \mathbb{R} \to \mathbb{R}$ with*

$$g_t(w) = e^{\frac{w^2}{2}} \int_{-\infty}^{w} \left(\mathbb{1}_{(-\infty,t]}(s) - \mathbb{P}(N \leq t)\right) e^{-\frac{s^2}{2}}\, \mathrm{d}s \tag{2.8}$$

is a solution of the differential equation

$$g_t'(w) - wg_t(w) = \mathbb{1}_{(-\infty,t]}(w) - \mathbb{P}(N \leq t) \tag{2.9}$$

and satisfies

$$0 < g_t(w) \leq \frac{\sqrt{2\pi}}{4}, \quad |g_t'(w)| \leq 1, \quad \textit{and} \quad |wg_t(w)| \leq 1 \tag{2.10}$$

for any $w \in \mathbb{R}$.

The function g_t is infinitely differentiable on $\mathbb{R} \setminus \{t\}$, but it is not differentiable at t. We denote the left-sided and right-sided limits of the derivatives in t by $g_t^{(m)}(t-)$ and $g_t^{(m)}(t+)$, respectively. For the first derivative, a direct computation proves

$$g_t'(t+) = -1 + g_t'(t-), \tag{2.11}$$

and we define $g_t'(t) := g_t'(t-)$ so that Equation (2.9) holds for $w = t$. By replacing w by a random variable Z and taking the expectation in Equation (2.9), one obtains

$$\mathbb{E}[g_t'(Z) - Z\, g_t(Z)] = \mathbb{P}(Z \leq t) - \mathbb{P}(N \leq t)$$

and as a consequence of the definition of the Kolmogorov distance

$$d_K(Z, N) = \sup_{t \in \mathbb{R}} |\mathbb{E}[g_t'(Z) - Z g_t(Z)]|. \tag{2.12}$$

The identity (2.12) will be our starting point for the proof of Theorem 5.2 in Chapter 5. Note furthermore, that we obtain, by combining the inequalities (2.7) and (2.10), the upper bound

$$|g_t''(w)| \leq \frac{\sqrt{2\pi}}{4} + |w| \tag{2.13}$$

for $w \in \mathbb{R} \setminus \{t\}$.

Chapter 3

Poisson U-statistics, Wiener-Itô integrals, and the product formula

In this chapter, we introduce so-called Poisson U-statistics and derive formulas for their moments and cumulants. Then we use Poisson U-statistics to define multiple Wiener-Itô integrals and prove formulas for their moments and cumulants by using the formulas for Poisson U-statistics. In order to formulate our results, we need some special classes of partitions, which are introduced in the first section.

Let us recall that $(X, \mathcal{X})$ is a measurable space such that $\{x\} \in \mathcal{X}$ for all $x \in X$ and μ is a σ-finite measure on X. Moreover, η is a Poisson point process with state space X and intensity measure μ. We tacitly assume this setting from now on.

3.1 Partitions

Let A be an arbitrary non-empty finite set. Recall that $\mathcal{P}(A)$ stands for the set of all partitions of A. In our terminology, a partition consists of so-called blocks and $|\sigma|$ denotes the number of blocks of a partition $\sigma \in \mathcal{P}(A)$.

On $\mathcal{P}(A)$ we define a partial order $\leq$ in the following way. For two partitions $\sigma, \tau \in \mathcal{P}(A)$ we say that $\sigma \leq \tau$ if each block of σ is contained in a block of τ. It is easy to see that the partition $\widehat{1}$ which has only the single block A is the maximal partition. On the other hand, the partition $\widehat{0}$ which consists of blocks having only one element, so-called singletons, is the minimal partition. For two partitions $\sigma, \tau \in \mathcal{P}(A)$ let $\sigma \vee \tau$ be the minimal partition of $\mathcal{P}(A)$ such that $\sigma \leq \sigma \vee \tau$ and $\tau \leq \sigma \vee \tau$. Similarly, let $\sigma \wedge \tau$ be the maximal partition of $\mathcal{P}(A)$ such that $\sigma \wedge \tau \leq \sigma$ and $\sigma \wedge \tau \leq \tau$. A short proof yields that $\sigma \vee \tau$ and $\sigma \wedge \tau$ are uniquely defined.

Before we use some partitions of variables to define functions, we need to introduce a tensor product of functions. Let $f^{(\ell)} : X^{n_\ell} \to \overline{\mathbb{R}}$ with $n_\ell \in \mathbb{N}$ for $\ell = 1, \dots, m$. Then the function $\otimes_{\ell=1}^m f^{(\ell)} : X^{\sum_{\ell=1}^m n_\ell} \to \overline{\mathbb{R}}$ is given by

$$\otimes_{\ell=1}^m f^{(\ell)}(x_1^{(1)}, \dots, x_{n_m}^{(m)}) = \prod_{\ell=1}^m f^{(\ell)}(x_1^{(\ell)}, \dots, x_{n_\ell}^{(\ell)}).$$

Now we think of the variables of $\otimes_{\ell=1}^m f^{(\ell)}$ as combinatorial objects and introduce some partitions of them. We define the set

$$V(n_1,\dots,n_m)=\{x_1^{(1)},\dots,x_{n_1}^{(1)},x_1^{(2)},\dots,x_{n_2}^{(2)},\dots,x_1^{(m-1)},\dots,x_{n_{m-1}}^{(m-1)},x_1^{(m)},\dots,x_{n_m}^{(m)}\}.$$

For each $\sigma\in\mathcal{P}(V(n_1,\dots,n_m))$ we can define a partition $\sigma^*\in\mathcal{P}([m])$ as the minimal partition in $\mathcal{P}([m])$ such that $i,j\in[m]$ are in the same block of σ^* whenever two variables $x_u^{(i)},x_v^{(j)}\in V(n_1,\dots,n_m)$ are in the same block of σ.

Definition 3.1 *For $n_\ell\in\mathbb{N}$, $\ell=1,\dots,m$, let $\overline{\pi}\in\mathcal{P}(V(n_1,\dots,n_m))$ be given by the blocks $\{x_1^{(\ell)},\dots,x_{n_\ell}^{(\ell)}\}$, $\ell=1,\dots,m$, and put*

$$\begin{aligned}
\Pi(n_1,\dots,n_m) &= \Big\{\sigma\in\mathcal{P}(V(n_1,\dots,n_m)):\sigma\wedge\overline{\pi}=\widehat{0}\Big\}\\
\Pi_{\geq 2}(n_1,\dots,n_m) &= \Big\{\sigma\in\mathcal{P}(V(n_1,\dots,n_m)):\sigma\wedge\overline{\pi}=\widehat{0},\ |B|\geq 2\ \forall B\in\sigma\Big\}\\
\widetilde{\Pi}(n_1,\dots,n_m) &= \Big\{\sigma\in\mathcal{P}(V(n_1,\dots,n_m)):\sigma\wedge\overline{\pi}=\widehat{0},\ \sigma^*=\widehat{1}\Big\}\\
\widetilde{\Pi}_{\geq 2}(n_1,\dots,n_m) &= \Big\{\sigma\in\mathcal{P}(V(n_1,\dots,n_m)):\sigma\wedge\overline{\pi}=\widehat{0},\ |B|\geq 2\ \forall B\in\sigma,\ \sigma^*=\widehat{1}\Big\}.
\end{aligned}$$

In this definition, $\widehat{0}$ is the minimal partition in $\mathcal{P}(V(n_1,\dots,n_m))$, and $\widehat{1}$ is the maximal partition in $\mathcal{P}([m])$.

It is easy to see that $\widetilde{\Pi}_{\geq 2}(n_1,\dots,n_m)\subset\widetilde{\Pi}(n_1,\dots,n_m)\subset\Pi(n_1,\dots,n_m)$ and that $\widetilde{\Pi}_{\geq 2}(n_1,\dots,n_m)\subset\Pi_{\geq 2}(n_1,\dots,n_m)\subset\Pi(n_1,\dots,n_m)$. The condition $\sigma\wedge\overline{\pi}=\widehat{0}$ means that variables with the same upper index are in different blocks of σ. If $\sigma\in\Pi_{\geq 2}(n_1,\dots,n_m)$, each block of σ has at least two elements. The condition $\sigma^*=\widehat{1}$ implies that it is not possible to divide the blocks $B_1,\dots,B_{|\sigma|}$ of σ in two non-empty sets σ_1 and σ_2 such that σ_1 is a partition of the variables with upper index $\ell\in A$ and σ_2 is a partition of the variables with upper index $\ell\in[m]\setminus A$.

We call a block of a partition a singleton if it has exactly one element. For a partition $\sigma\in\Pi(n_1,\dots,n_m)$ we denote by $S(\sigma)$ the set of all singletons of σ and by $s(\sigma)$ the vector $(s_1,\dots,s_m)$ where s_ℓ, $\ell=1,\dots,m$, is the number of variables with upper index ℓ that are included in the partition σ as singletons.

Now we are able to combine the tensor product notation and the partitions of variables in the following way. For given functions $f^{(\ell)}:X^{n_\ell}\to\overline{\mathbb{R}}$ with $n_\ell\in\mathbb{N}$ for $\ell=1,\dots,m$ and a partition $\sigma\in\Pi(n_1,\dots,n_m)$ we construct a new function $\left(\otimes_{\ell=1}^m f^{(\ell)}\right)_\sigma:X^{|\sigma|}\to\overline{\mathbb{R}}$ by replacing all variables that belong to the same block of σ by a new common variable. In order to uniquely define this new function, we must order the new variables. This can be done by taking the order of occurrence of the new variables in the tensor product.

Our notation can be extended to the case $n_\ell=0$. Then we have no variables with upper index ℓ, and the function $f^{(\ell)}$ is only a constant.

The classes of partitions introduced here are very similar to the partitions in the monograph [66] by Peccati and Taqqu. A formal difference is that there sets of numbers are partitioned, and the variables are identified with numbers. In the paper [92] by Surgailis, the partitions $\Pi(n_1,\dots,n_m)$ are defined as a special class of graphs where the variables are the vertices, and two vertices are connected by an edge if both variables belong to the same block.

We illustrate the partitions and the tensor product notation with the following example:

Example 3.2 We have $V(2,1,2,1) = \Big\{x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_1^{(3)}, x_2^{(3)}, x_1^{(4)}\Big\}$. Now the partitions

$$\begin{aligned}
\sigma_1 &= \Big\{\{x_1^{(1)}, x_1^{(2)}\}, \{x_2^{(1)}\}, \{x_1^{(3)}, x_1^{(4)}\}, \{x_2^{(3)}\}\Big\} \\
\sigma_2 &= \Big\{\{x_1^{(1)}, x_1^{(3)}\}, \{x_2^{(1)}, x_2^{(3)}\}, \{x_1^{(2)}, x_1^{(4)}\}\Big\} \\
\sigma_3 &= \Big\{\{x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, x_1^{(4)}\}, \{x_2^{(1)}\}, \{x_2^{(3)}\}\Big\} \\
\sigma_4 &= \Big\{\{x_1^{(1)}, x_1^{(2)}\}, \{x_2^{(1)}, x_2^{(3)}\}, \{x_1^{(3)}, x_1^{(4)}\}\Big\}
\end{aligned}$$

belong to $\Pi(2,1,2,1)$ since variables with the same upper index are always in different blocks. We have $|\sigma_1| = 4$ and $|\sigma_2| = |\sigma_3| = |\sigma_4| = 3$ as well as

$$S(\sigma_1) = S(\sigma_3) = \Big\{\{x_2^{(1)}\}, \{x_2^{(3)}\}\Big\} \quad \text{and} \quad S(\sigma_2) = S(\sigma_4) = \emptyset,$$

whence $s(\sigma_1) = s(\sigma_3) = (1,0,1,0)$ and $s(\sigma_2) = s(\sigma_4) = (0,0,0,0)$. Thus, σ_2 and σ_4 are elements of $\Pi_{\geq 2}(2,1,2,1)$ but not σ_1 and σ_3. Since

$$\sigma_1 = \Big\{\{x_1^{(1)}, x_1^{(2)}\}, \{x_2^{(1)}\}\Big\} \cup \Big\{\{x_1^{(3)}, x_1^{(4)}\}, \{x_2^{(3)}\}\Big\}$$

and

$$\sigma_2 = \Big\{\{x_1^{(1)}, x_1^{(3)}\}, \{x_2^{(1)}, x_2^{(3)}\}\Big\} \cup \Big\{\{x_1^{(2)}, x_1^{(4)}\}\Big\},$$

we have $\sigma_1^* = \{\{1,2\},\{3,4\}\}$ and $\sigma_2^* = \{\{1,3\},\{2,4\}\}$ so that σ_1 and σ_2 do not belong to $\widetilde{\Pi}(2,1,2,1)$. For σ_3 there is no such decomposition since the first block contains variables with all possible upper indices, which implies $\sigma_3^* = \{\{1,2,3,4\}\}$. The partition σ_4 has blocks with variables having the upper indices $\{1,2\}$, $\{1,3\}$, and $\{3,4\}$ so that $\sigma_4^* = \{\{1,2,3,4\}\}$. Hence, we obtain $\sigma_3 \in \widetilde{\Pi}(2,1,2,1)$ and $\sigma_4 \in \widetilde{\Pi}_{\geq 2}(2,1,2,1)$.

For $f^{(1)} : X^2 \to \overline{\mathbb{R}}$, $f^{(2)} : X \to \overline{\mathbb{R}}$, $f^{(3)} : X^2 \to \overline{\mathbb{R}}$, and $f^{(4)} : X \to \overline{\mathbb{R}}$ we have

$$\begin{aligned}
&f^{(1)} \otimes f^{(2)} \otimes f^{(3)} \otimes f^{(4)}(x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_1^{(3)}, x_2^{(3)}, x_1^{(4)}) \\
&= f^{(1)}(x_1^{(1)}, x_2^{(1)})\, f^{(2)}(x_1^{(2)})\, f^{(3)}(x_1^{(3)}, x_2^{(3)})\, f^{(4)}(x_1^{(4)})
\end{aligned}$$

and

$$\begin{aligned}
\big(f^{(1)} \otimes f^{(2)} \otimes f^{(3)} \otimes f^{(4)}\big)_{\sigma_1}(y_1, y_2, y_3, y_4) &= f^{(1)}(y_1, y_2)\, f^{(2)}(y_1)\, f^{(3)}(y_3, y_4)\, f^{(4)}(y_3) \\
\big(f^{(1)} \otimes f^{(2)} \otimes f^{(3)} \otimes f^{(4)}\big)_{\sigma_2}(y_1, y_2, y_3) &= f^{(1)}(y_1, y_2)\, f^{(2)}(y_3)\, f^{(3)}(y_1, y_2)\, f^{(4)}(y_3) \\
\big(f^{(1)} \otimes f^{(2)} \otimes f^{(3)} \otimes f^{(4)}\big)_{\sigma_3}(y_1, y_2, y_3) &= f^{(1)}(y_1, y_2)\, f^{(2)}(y_1)\, f^{(3)}(y_1, y_3)\, f^{(4)}(y_1) \\
\big(f^{(1)} \otimes f^{(2)} \otimes f^{(3)} \otimes f^{(4)}\big)_{\sigma_4}(y_1, y_2, y_3) &= f^{(1)}(y_1, y_2)\, f^{(2)}(y_1)\, f^{(3)}(y_3, y_2)\, f^{(4)}(y_3).
\end{aligned}$$

3.2 Poisson U-statistics and their moments and cumulants

In this section, we discuss random variables that depend on η and have the following form:

Definition 3.3 *A Poisson functional S of the form*

$$S = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k)$$

with $k \in \mathbb{N}$ and $f \in L^1_s(\mu^k)$ is called a Poisson U-statistic of order k.

Recall that $\eta^k_{\neq}$ stands for the set of all k-tuples of distinct points of η. Since we sum over all permutations of a fixed k-tuple of distinct points, we can assume without loss of generality that f is symmetric.

We call the Poisson functionals in Definition 3.3 Poisson U-statistics due to their similarity to classical U-statistics. Let $X_1,\dots,X_m$ be a fixed number of independently and identically distributed random variables and let $\zeta^k_{m,\neq}$ be the set of all k-tuples of distinct random variables from $X_1,\dots,X_m$. Then a random variable

$$S = \sum_{(x_1,\dots,x_k)\in\zeta^k_{m,\neq}} f(x_1,\dots,x_k)$$

is called a (classical) U-statistic. The sum is often divided by the number of summands. But since m is fixed, this is a fixed number we can neglect. One can think of $X_1,\dots,X_m$ as a binomial point process ζ_m. Then the difference between a Poisson U-statistic and a classical U-statistic is whether the underlying point process is a Poisson point process or a binomial point process. In the Poisson case, the number of summands is random and can be infinite, whereas the number of summands of a classical U-statistic is fixed and finite. For more details on classical U-statistics and their applications in statistics we refer to the classical work [27] by Hoeffding and the monographs [37, 43] by Korolyuk and Borovskich, and Lee.

The fact that f is integrable allows us to apply the Slivnyak-Mecke formula from Corollary 2.10, which ensures that $S \in L^1(\mathbb{P}_\eta)$ and yields

$$\mathbb{E}S = \int_{X^k} f(x_1,\dots,x_k)\,\mathrm{d}\mu(x_1,\dots,x_k). \tag{3.1}$$

Using the notation from Section 3.1, we can state the following formulas for the moments and cumulants of Poisson U-statistics.

Theorem 3.4 *Let the Poisson U-statistics $S^{(1)},\dots,S^{(m)}$, $m \geq 2$, be given by*

$$S^{(\ell)} = \sum_{(x_1,\dots,x_{n_\ell})\in\eta^{n_\ell}_{\neq}} f^{(\ell)}(x_1,\dots,x_{n_\ell}) \quad \textit{with} \quad f^{(\ell)} \in L^1_s(\mu^{n_\ell}) \quad \textit{and} \quad n_\ell \in \mathbb{N}$$

for $\ell = 1,\dots,m$ and assume that

$$\int_{X^{|\sigma|}} |(\otimes_{\ell=1}^m f^{(\ell)})_\sigma|\,\mathrm{d}\mu^{|\sigma|} < \infty \quad \textit{for all} \quad \sigma \in \Pi(n_1,\dots,n_m). \tag{3.2}$$

Then we have

$$\mathbb{E}\prod_{\ell=1}^m S^{(\ell)} = \sum_{\sigma\in\Pi(n_1,\dots,n_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma\,\mathrm{d}\mu^{|\sigma|}, \tag{3.3}$$

$$\mathbb{E}\prod_{\ell=1}^{m}(S^{(\ell)}-\mathbb{E}S^{(\ell)}) = \sum_{\substack{\sigma\in\Pi(n_1,\dots,n_m),\\ s(\sigma)\leq(n_1-1,\dots,n_m-1)}} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}, \tag{3.4}$$

and

$$\gamma(S^{(1)},\dots,S^{(m)}) = \sum_{\sigma\in\widetilde{\Pi}(n_1,\dots,n_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}. \tag{3.5}$$

Proof. Because of $f^{(\ell)} \in L^1_s(\mu^{n_\ell})$, each of the sums $S^{(1)},\dots,S^{(m)}$ is almost surely absolutely convergent. Since two points we sum over in two different U-statistics can be either identical or distinct, we can rewrite the product of $S^{(1)},\dots,S^{(m)}$ as

$$\prod_{\ell=1}^{m} S^{(\ell)} = \sum_{\sigma\in\Pi(n_1,\dots,n_m)} \sum_{(y_1,\dots,y_{|\sigma|})\in\eta_{\neq}^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma(y_1,\dots,y_{|\sigma|})$$

almost surely. Now assumption (3.2) allows us to apply the Slivnyak-Mecke formula from Corollary 2.10, which yields formula (3.3).

The left-hand side in Equation (3.4) is the expectation of a sum of 2^m products, where the ℓ-th factor is either $S^{(\ell)}$ or $-\mathbb{E}S^{(\ell)}$, and each of them can be computed by formula (3.3). Now we count for every partition $\sigma\in\Pi(n_1,\dots,n_m)$ how often the integral $\int_{X^{|\sigma|}}(\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}$ occurs. We obtain the integral related to a partition $\sigma\in\Pi(n_1,\dots,n_m)$ such that $s_j(\sigma)=n_j$ for $j\in J\subset[m]$ and $s_j(\sigma)\leq n_j-1$ for $j\in[m]\setminus J$ if and only if the factors $j\in[m]\setminus J$ are $S^{(j)}$ and the remaining factors are $S^{(j)}$ or $-\mathbb{E}S^{(j)}$ for $j\in J$. For $J\neq\emptyset$ we have $2^{|J|-1}$ combinations with a plus sign and $2^{|J|-1}$ combinations with a minus sign that cancel out. For the partitions with $J=\emptyset$ we obtain the integrals on the right-hand side of formula (3.4).

We prove formula (3.5) by induction over m. The identity holds for $m=2$ since $\mathbb{E}(S^{(1)}-\mathbb{E}S^{(1)})(S^{(2)}-\mathbb{E}S^{(2)})=\gamma(S^{(1)},S^{(2)})$ and

$$\{\sigma\in\Pi(n_1,n_2): s(\sigma)\leq(n_1-1,n_2-1)\}=\widetilde{\Pi}(n_1,n_2).$$

For $m\geq 3$ we have by Lemma 2.5, formula (3.3), and the assumption of the induction

$$\begin{aligned}
\gamma(S^{(1)},\dots,S^{(m)}) &= \sum_{\sigma\in\Pi(n_1,\dots,n_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|} - \sum_{\substack{\pi\in\mathcal{P}([m]),\\ |\pi|>1}} \prod_{J\in\pi} \gamma(S^J) \\
&= \sum_{\sigma\in\Pi(n_1,\dots,n_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|} \\
&\quad - \sum_{\substack{\pi\in\mathcal{P}([m]),\\ |\pi|>1}} \prod_{J\in\pi} \sum_{\sigma_J\in\widetilde{\Pi}(n_J)} \int_{X^{|\sigma_J|}} (\otimes_{\ell\in J} f^{(\ell)})_{\sigma_J} \,\mathrm{d}\mu^{|\sigma_J|}
\end{aligned}$$

with $S^J=\{S^{(j)}: j\in J\}$ and $n_J=\{n_j: j\in J\}$. Since each partition $\sigma\in\Pi(n_1,\dots,n_m)$ defines a partition $\sigma^*\in\mathcal{P}([m])$ and partitions $\sigma_J\in\widetilde{\Pi}(n_J)$ for $J\in\sigma^*$ and vice versa, we can rewrite the previous equation as

$$\begin{aligned}\gamma(S^{(1)},\dots,S^{(m)}) &= \sum_{\sigma\in\Pi(n_1,\dots,n_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|} \\ &\quad - \sum_{\sigma\in\Pi(n_1,\dots,n_m),|\sigma^*|>1} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}.\end{aligned}$$

Now only partitions $\sigma \in \Pi(n_1,\dots,n_m)$ with $|\sigma^*| = 1$ remain on the right-hand side. By definition of $\widetilde{\Pi}(n_1,\dots,n_m)$, these are exactly the partitions on the right-hand side of formula (3.5). □

The previous result can be reformulated in the following way:

Corollary 3.5 *Let $f^{(\ell)} \in L^1_s(\mu^{n_\ell})$ with $n_\ell \in \mathbb{N}$ for $\ell = 1,\dots,m$ and*

$$S^{(\ell)} = \sum_{(x_1,\dots,x_{n_\ell})\in\eta^{n_\ell}_{\neq}} f^{(\ell)}(x_1,\dots,x_{n_\ell})$$

for $\ell = 1,\dots,m$. We define

$$\widehat{f}^{(\ell)}_i(x_1,\dots,x_i) = \int_{X^{n_\ell-i}} f^{(\ell)}(x_1,\dots,x_i,y_1,\dots,y_{n_\ell-i}) \,\mathrm{d}\mu(y_1,\dots,y_{n_\ell-i})$$

for $i = 0,\dots,n_\ell$ and $\ell = 1,\dots,m$ and assume that

$$\int_{X^{|\sigma|}} |(\otimes_{\ell=1}^m \widehat{f}^{(\ell)}_{i_\ell})_\sigma| \,\mathrm{d}\mu^{|\sigma|} < \infty \text{ for } \sigma \in \Pi(i_1,\dots,i_m) \text{ and } 0 \le i_\ell \le n_\ell, \ell = 1,\dots,m. \tag{3.6}$$

Then we have

$$\mathbb{E}\prod_{\ell=1}^m S^{(\ell)} = \sum_{0\le i_1\le n_1,\dots,0\le i_m\le n_m} \prod_{\ell=1}^m \binom{n_\ell}{i_\ell} \sum_{\sigma\in\Pi_{\ge2}(i_1,\dots,i_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m \widehat{f}^{(\ell)}_{i_\ell})_\sigma \,\mathrm{d}\mu^{|\sigma|}, \tag{3.7}$$

$$\mathbb{E}\prod_{\ell=1}^m (S^{(\ell)} - \mathbb{E}S^{(\ell)}) = \sum_{1\le i_1\le n_1,\dots,1\le i_m\le n_m} \prod_{\ell=1}^m \binom{n_\ell}{i_\ell} \sum_{\sigma\in\Pi_{\ge2}(i_1,\dots,i_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m \widehat{f}^{(\ell)}_{i_\ell})_\sigma \,\mathrm{d}\mu^{|\sigma|}, \tag{3.8}$$

and

$$\gamma(S^{(1)},\dots,S^{(m)}) = \sum_{1\le i_1\le n_1,\dots,1\le i_m\le n_m} \prod_{\ell=1}^m \binom{n_\ell}{i_\ell} \sum_{\sigma\in\widetilde{\Pi}_{\ge2}(i_1,\dots,i_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m \widehat{f}^{(\ell)}_{i_\ell})_\sigma \,\mathrm{d}\mu^{|\sigma|}. \tag{3.9}$$

Proof. For a partition $\sigma \in \Pi(n_1,\dots,n_m)$ with $s(\sigma) = (s_1,\dots,s_m)$, we can construct a reduced partition $\tau \in \Pi_{\ge2}(n_1 - s_1,\dots,n_m - s_m)$ by removing the singletons of σ and relabelling the remaining variables. Together with the definition of $\widehat{f}_i^{(\ell)}$ and Fubini's theorem, we obtain the identity

$$\int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|} = \int_{X^{|\tau|}} (\otimes_{\ell=1}^m \widehat{f}^{(\ell)}_{n_\ell - s_\ell})_\tau \,\mathrm{d}\mu^{|\tau|}.$$

There are $\prod_{\ell=1}^m \binom{n_\ell}{s_\ell}$ partitions $\sigma \in \Pi(n_1, \dots, n_m)$ with $s(\sigma) = (s_1, \dots, s_m)$ that have the same reduced partition $\tau \in \Pi_{\geq 2}(n_1 - s_1, \dots, n_m - s_m)$ since we have for every $f^{(\ell)}$, $\binom{n_\ell}{s_\ell}$ possibilities to choose the singletons. Hence, it follows that

$$\sum_{\substack{\sigma \in \Pi(n_1,\dots,n_m),\\ s(\sigma)=(s_1,\dots,s_m)}} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}$$
$$= \prod_{\ell=1}^m \binom{n_\ell}{s_\ell} \sum_{\sigma \in \Pi_{\geq 2}(n_1-s_1,\dots,n_m-s_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m \widehat{f}^{(\ell)}_{n_\ell - s_\ell})_\sigma \,\mathrm{d}\mu^{|\sigma|}.$$

Since assumption (3.6) for $i_1 = n_1, \dots, i_m = n_m$ implies that condition (3.2) is satisfied, the formulas (3.7), (3.8), and (3.9) are direct consequences of Theorem 3.4. □

3.3 Multiple Wiener-Itô integrals

The aim of this section is to introduce multiple Wiener-Itô integrals that play a crucial role in the sequel. We do this in a similar way as in the work [41] by Last and Penrose. We start by defining the multiple Wiener-Itô integral for integrable functions and investigate its properties before we use it to define the multiple Wiener-Itô integral for square integrable functions.

Definition 3.6 *For $n \in \mathbb{N}$ the n-th multiple Wiener-Itô integral $I_n(f)$ of a function $f \in L_s^1(\mu^n)$ is given by*

$$I_n(f) = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \sum_{(x_1,\dots,x_i) \in \eta^i_{\neq}} \int_{X^{n-i}} f(x_1, \dots, x_i, y_1, \dots, y_{n-i}) \,\mathrm{d}\mu(y_1, \dots, y_{n-i}).$$

This pathwise definition of a multiple Wiener-Itô integral is close to the definition in [41], where factorial moment measures are used. But because of the assumption that $\{x\} \in \mathcal{X}$ for all $x \in X$, integrals with respect to factorial moment measures reduce to sums, and $I_n(f)$ is a sum of Poisson U-statistics as considered in the previous section. Moreover, we only require that f is integrable, whereas it must be bounded and have a support with finite measure in [41].

Now it follows from formula (3.1) that

$$\mathbb{E} I_n(f) = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \int_{X^n} f(y_1, \dots, y_n) \,\mathrm{d}\mu(y_1, \dots, y_n) = 0.$$

Similarly, we can use the results for Poisson U-statistics to obtain a product formula for multiple Wiener-Itô integrals:

Theorem 3.7 *Let $f^{(\ell)} \in L_s^1(\mu^{n_\ell})$ with $n_\ell \in \mathbb{N}$ for $\ell = 1, \dots, m$ such that*

$$\int_{X^{|\sigma|}} |(\otimes_{\ell=1}^m f^{(\ell)})_\sigma| \, \mathrm{d}\mu^{|\sigma|} < \infty \quad \text{for all} \quad \sigma \in \Pi(n_1, \dots, n_m).$$

Then we have

$$\mathbb{E} \prod_{\ell=1}^m I_{n_\ell}(f^{(\ell)}) = \sum_{\sigma \in \Pi_{\geq 2}(n_1,\dots,n_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|} \tag{3.10}$$

and

$$\gamma\left(I_{n_1}(f^{(1)}), \dots, I_{n_m}(f^{(m)})\right) = \sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(n_1,\dots,n_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|}. \tag{3.11}$$

Proof. As in the previous section, we use the abbreviations

$$\widehat{f}_i^{(\ell)}(x_1, \dots, x_i) = \int_{X^{n_\ell - i}} f^{(\ell)}(x_1, \dots, x_i, y_1, \dots, y_{n_\ell - i}) \, \mathrm{d}\mu(y_1, \dots, y_{n_\ell - i})$$

for $i = 0, \dots, n_\ell$ and $\ell = 1, \dots, m$, which implies that

$$\int_{X^{i-j}} \widehat{f}_i^{(\ell)}(x_1, \dots, x_j, y_1, \dots, y_{i-j}) \, \mathrm{d}\mu(y_1, \dots, y_{i-j}) = \widehat{f}_j^{(\ell)}(x_1, \dots, x_j) \tag{3.12}$$

for $j \leq i$. Using this notation, we can write $I_{n_\ell}(f^{(\ell)})$ as

$$I_{n_\ell}(f^{(\ell)}) = \sum_{i=0}^{n_\ell} (-1)^{n_\ell - i} \binom{n_\ell}{i} \sum_{(x_1,\dots,x_i) \in \eta_{\neq}^i} \widehat{f}_i^{(\ell)}(x_1, \dots, x_i),$$

and it follows from Corollary 3.5 and formula (3.12) that

$$\begin{aligned} \mathbb{E} \prod_{\ell=1}^m I_{n_\ell}(f^{(\ell)}) &= \sum_{0 \leq i_1 \leq n_1, \dots, 0 \leq i_m \leq n_m} \mathbb{E} \prod_{\ell=1}^m (-1)^{n_\ell - i_\ell} \binom{n_\ell}{i_\ell} \sum_{(x_1,\dots,x_{i_\ell}) \in \eta_{\neq}^{i_\ell}} \widehat{f}_{i_\ell}^{(\ell)}(x_1, \dots, x_{i_\ell}) \\ &= \sum_{0 \leq j_1 \leq i_1 \leq n_1, \dots, 0 \leq j_m \leq i_m \leq n_m} \prod_{\ell=1}^m (-1)^{n_\ell - i_\ell} \binom{n_\ell}{i_\ell} \binom{i_\ell}{j_\ell} \sum_{\sigma \in \Pi_{\geq 2}(j_1,\dots,j_m)} \int_{X^{|\sigma|}} (\otimes_{\ell=1}^m \widehat{f}_{j_\ell}^{(\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|}. \end{aligned} \tag{3.13}$$

Now the integral belonging to a partition $\sigma \in \Pi_{\geq 2}(j_1, \dots, j_m)$ occurs

$$\sum_{j_1 \leq i_1 \leq n_1, \dots, j_m \leq i_m \leq n_m} \prod_{\ell=1}^m (-1)^{n_\ell - i_\ell} \binom{n_\ell}{i_\ell} \binom{i_\ell}{j_\ell}$$

times on the right-hand side. Straightforward computations yield

$$\sum_{j_1 \leq i_1 \leq n_1, \dots, j_m \leq i_m \leq n_m} \prod_{\ell=1}^m (-1)^{n_\ell - i_\ell} \binom{n_\ell}{i_\ell} \binom{i_\ell}{j_\ell} = \prod_{\ell=1}^m \sum_{i_\ell = j_\ell}^{n_\ell} (-1)^{n_\ell - i_\ell} \binom{n_\ell}{i_\ell} \binom{i_\ell}{j_\ell}$$

and

$$\sum_{i_\ell=j_\ell}^{n_\ell}(-1)^{n_\ell-i_\ell}\binom{n_\ell}{i_\ell}\binom{i_\ell}{j_\ell} = \frac{n_\ell!}{j_\ell!}\sum_{i_\ell=j_\ell}^{n_\ell}\frac{(-1)^{n_\ell-i_\ell}}{(n_\ell-i_\ell)!\,(i_\ell-j_\ell)!}$$
$$= \frac{n_\ell!}{j_\ell!}\sum_{j=0}^{n_\ell-j_\ell}\frac{(-1)^{n_\ell-j_\ell-j}}{(n_\ell-j_\ell-j)!\,j!} = \begin{cases}1, & n_\ell=j_\ell\\ 0, & n_\ell\neq j_\ell\end{cases}$$

so that formula (3.13) simplifies to formula (3.10).

By Corollary 3.5 and the multilinearity of the joint cumulants, we obtain

$$\gamma(I_{n_1}(f^{(1)}),\dots,I_{n_m}(f^{(m)}))$$
$$= \sum_{0\le j_1\le i_1\le n_1,\dots,0\le j_m\le i_m\le n_m}\prod_{\ell=1}^{m}(-1)^{n_\ell-i_\ell}\binom{n_\ell}{i_\ell}\binom{i_\ell}{j_\ell}\sum_{\sigma\in\widetilde{\Pi}_{\ge 2}(j_1,\dots,j_m)}\int_{X^{|\sigma|}}(\otimes_{\ell=1}^{m}\widehat{f}_{j_\ell}^{(\ell)})_\sigma\,\mathrm{d}\mu^{|\sigma|}.$$

Exactly as for the expectation above, one can show that only the integrals on the right-hand side in formula (3.11) remain. □

For the construction of the multiple Wiener-Itô integral of a function $f\in L^2_s(\mu^n)$ we use an approximation by simple symmetric functions (recall from Chapter 2 that we denote the class of these functions by $\mathcal{E}_s(\mu^n)$). This approach relies on the following consequence of Theorem 3.7 (see also [41, Equation (3.5)]):

Corollary 3.8 *For $f\in\mathcal{E}_s(\mu^n)$ and $g\in\mathcal{E}_s(\mu^m)$ with $n,m\in\mathbb{N}$ we have*

$$\mathbb{E}I_n(f)^2 = n!\,\|f\|_n^2 \tag{3.14}$$

and

$$\mathbb{E}I_n(f)I_m(g) = \begin{cases} n!\,\langle f,g\rangle_{L^2(\mu^n)}, & n=m\\ 0, & n\neq m\end{cases}.$$

For $n\in\mathbb{N}$ we define $\mathcal{I}_n^0=\{I_n(f): f\in\mathcal{E}_s(\mu^n)\}$. As consequence of formula (3.14), we have $\mathcal{I}_n^0\subset L^2(\mathbb{P}_\eta)$ and an isometry relation between $\mathcal{I}_n^0$ and $\mathcal{E}_s(\mu^n)$. Since $\mathcal{E}_s(\mu^n)$ is dense in $L^2_s(\mu^n)$, we can approximate a function $f\in L^2_s(\mu^n)$ by a sequence $(f_j)_{j\in\mathbb{N}}$ such that $f_j\in\mathcal{E}_s(\mu^n)$ and

$$\|f-f_j\|_n^2\to 0 \text{ as } j\to\infty,$$

which means that $(f_j)_{j\in\mathbb{N}}$ is a Cauchy-sequence in $L^2_s(\mu^n)$. Because of the isometry (3.14) the sequence $(I_n(f_j))_{j\in\mathbb{N}}$ in $\mathcal{I}_n^0$ is a Cauchy sequence in $L^2(\mathbb{P}_\eta)$ as well. Now the completeness of $L^2(\mathbb{P}_\eta)$ implies that $(I_n(f_j))_{j\in\mathbb{N}}$ has a limit in $L^2(\mathbb{P}_\eta)$. Since this limit is independent of the choice of the approximating sequence $(f_j)_{j\in\mathbb{N}}$, we can take it as definition of $I_n(f)$.

Definition 3.9 *For $n\in\mathbb{N}$ and $f\in L^2_s(\mu^n)$ let $(f_j)_{j\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}_s(\mu^n)$ such that $\lim_{j\to\infty}\|f-f_j\|_n=0$. Then we define the n-th multiple Wiener-Itô integral of f by*

$$I_n(f)=\lim_{j\to\infty} I_n(f_j),$$

where lim *stands for the limit in $L^2(\mathbb{P}_\eta)$.*

In the works [66, 92] by Peccati and Taqqu, and Surgailis, the n-th multiple Wiener-Itô integral is at first defined for simple functions from $\mathcal{E}_s(\mu^n)$ before the definition is extended to $L^2_s(\mu^n)$ as above. The authors of both works assume that the the measure μ is non-atomic. Since they exclude the diagonals in the integration, this assumption cannot be easily dispensed. Our Definition 3.6 of a multiple Wiener-Itô integral is derived as Theorem 4.1 in [92] for the setting considered there. In the paper [46] by Liebscher, the multiple Wiener-Itô integral is defined in terms of the so-called Charlier polynomials for some special functions, and the definition is extended to $L^2_s(\mu^n)$ by the isometry relation as above. The intensity measure is allowed to have atoms, but there are some weak assumptions on the underlying space and the intensity measure. The approach presented above requires less assumptions on the underlying space and the intensity measure of the Poisson point process and is, thus, more general.

From the properties of multiple Wiener-Itô integrals with integrands in $\mathcal{E}_s(\mu^n)$ and the construction of a multiple Wiener-Itô integral of a function in $L^2_s(\mu^n)$, it follows that (see [41, Chapter 3], for example):

Lemma 3.10 *Let $f \in L^2_s(\mu^n)$ and $g \in L^2_s(\mu^m)$ with $n, m \in \mathbb{N}$. Then*

1. $\mathbb{E} I_n(f) = 0$,
2. $\mathbb{E} I_n(f)^2 = n!\,\|f\|_n^2$,
3. $\mathbb{E} I_n(f)\, I_m(g) = \begin{cases} n!\,\langle f, g\rangle_{L^2(\mu^n)}, & n = m \\ 0, & n \neq m \end{cases}.$

At the begin of this section, we defined the multiple Wiener-Itô integral of an L^1-integrable symmetric function as a sum of Poisson U-statistics. In the following proposition, we show that every Poisson U-statistic can be written as a sum of multiple Wiener-Itô integrals. Thereby, we waive the assumption that the function f we sum over in the Poisson U-statistic is symmetric. In oder to give formulas for the integrands f_n of the multiple Wiener-Itô integrals, we need the symmetrization operator sym that is given by

$$(\text{sym } g)(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in \text{Per}(n)} g(x_{\pi(1)}, \dots, x_{\pi(n)})$$

for $g : X^n \to \overline{\mathbb{R}}$, where $\text{Per}(n)$ stands for the set of all permutations of $\{1, \dots, n\}$. For $f \in L^1(X^k)$ and $J \subset [k]$ with $|J| = k - n$ and $0 \leq n \leq k$ we denote by $\int_{X^{|J|}} f \, \mathrm{d}\mu_J$ the function from X^n to $\overline{\mathbb{R}}$ we obtain by integrating over all variables x_j of f with $j \in J$. We use the convention $I_0(c) = c$ for $c \in \mathbb{R}$ in the sequel.

Proposition 3.11 *For $f \in L^1(\mu^k)$ with $k \in \mathbb{N}$ the equation*

$$\sum_{(x_1,\dots,x_k) \in \eta^k_{\neq}} f(x_1, \dots, x_k) = \sum_{n=0}^{k} I_n(f_n)$$

holds almost surely with $f_0 = \int_{X^k} f \, \mathrm{d}\mu^k$ and

$$f_n(x_1, \dots, x_n) = \sum_{\substack{J \subset [k], \\ |J| = k-n}} \left(\text{sym} \int_{X^{|J|}} f \, \mathrm{d}\mu_J \right) (x_1, \dots, x_n)$$

for $n = 1, \dots, k$.

Proof. Note that the functions f_n are symmetric and that

$$\begin{aligned}
&\int_X f_n(x_1, \dots, x_n)\, \mathrm{d}\mu(x_n) \\
&= \sum_{\substack{J \subset [k], \\ |J| = k-n}} \frac{1}{n!} \int_X \sum_{\pi \in \mathrm{Per}(n)} \left(\int_{X^{|J|}} f\, \mathrm{d}\mu_J \right) (x_{\pi(1)}, \dots, x_{\pi(n)})\, \mathrm{d}\mu(x_n) \\
&= \sum_{\substack{J \subset [k], \\ |J| = k-n}} \frac{1}{n!} \sum_{j \in [k] \setminus J} \sum_{\pi \in \mathrm{Per}(n-1)} \left(\int_{X^{|J|+1}} f\, \mathrm{d}\mu_{J \cup \{j\}} \right) (x_{\pi(1)}, \dots, x_{\pi(n-1)}) \\
&= \sum_{\substack{J \subset [k], \\ |J| = k-n}} \frac{1}{n} \sum_{j \in [k] \setminus J} \left(\mathrm{sym} \int_{X^{|J|+1}} f\, \mathrm{d}\mu_{J \cup \{j\}} \right) (x_1, \dots, x_{n-1}) \\
&= \frac{k-n+1}{n} \sum_{\substack{J \subset [k], \\ |J| = k-(n-1)}} \left(\mathrm{sym} \int_{X^{|J|}} f\, \mathrm{d}\mu_J \right) (x_1, \dots, x_{n-1}) \\
&= \frac{k-n+1}{n} f_{n-1}(x_1, \dots, x_{n-1}).
\end{aligned}$$

Combining this with the definition of the multiple Wiener-Itô integral for L^1-functions yields

$$\begin{aligned}
I_n(f_n) &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{(x_1, \dots, x_j) \in \eta^j_{\neq}} \int_{X^{n-j}} f_n(x_1, \dots, x_j, y_1, \dots, y_{n-j})\, \mathrm{d}\mu(y_1, \dots, y_{n-j}) \\
&= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \prod_{\ell=j+1}^{n} \frac{k-\ell+1}{\ell} \sum_{(x_1, \dots, x_j) \in \eta^j_{\neq}} f_j(x_1, \dots, x_j).
\end{aligned}$$

Together with $\binom{n}{j} \prod_{\ell=j+1}^{n} \frac{k-\ell+1}{\ell} = \binom{n}{j} \frac{j!}{n!} \frac{(k-j)!}{(k-n)!} = \binom{k-j}{k-n}$, we obtain

$$\begin{aligned}
\sum_{n=0}^{k} I_n(f_n) &= \sum_{j=0}^{k} \sum_{n=0}^{k} (-1)^{n-j} \binom{k-j}{k-n} \sum_{(x_1, \dots, x_j) \in \eta^j_{\neq}} f_j(x_1, \dots, x_j) \\
&= \sum_{(x_1, \dots, x_k) \in \eta^k_{\neq}} f_k(x_1, \dots, x_k) = \sum_{(x_1, \dots, x_k) \in \eta^k_{\neq}} f(x_1, \dots, x_k),
\end{aligned}$$

which proves the claim. □

The next proposition allows us to write the product of two multiple Wiener-Itô integrals as a finite sum of multiple Wiener-Itô integrals in the L^1-sense.

Proposition 3.12 *Let $f \in L^1_s(\mu^n)$ and $g \in L^1_s(\mu^m)$ with $n, m \in \mathbb{N}$. We assume that $(f \otimes g)_\sigma \in L^1(\mu^{|\sigma|})$ for all $\sigma \in \Pi(n,m)$ and define*

$$h_\ell(x_1,\dots,x_\ell) = \sum_{\sigma\in\Pi(n,m)} \sum_{\substack{J\subset\sigma\setminus S(\sigma),\\ |J|=|\sigma|-\ell}} \left(\operatorname{sym}\int_{X^{|J|}} (f\otimes g)_\sigma \,\mathrm{d}\mu_J\right)(x_1,\dots,x_\ell)$$

for $\ell = |n-m|,\dots,n+m$. Then

$$I_n(f)\, I_m(g) = \sum_{\ell=|n-m|}^{n+m} I_\ell(h_\ell)$$

holds almost surely.

Proof. Using Definition 3.6 and the abbreviations

$$\widehat{f}_i(x_1,\dots,x_i) = \int_{X^{n-i}} f(x_1,\dots,x_i,y_1,\dots,y_{n-i})\,\mathrm{d}\mu(y_1,\dots,y_{n-i})$$

for $i = 0,\dots,n$ and

$$\widehat{g}_j(x_1,\dots,x_j) = \int_{X^{m-j}} g(x_1,\dots,x_j,y_1,\dots,y_{m-j})\,\mathrm{d}\mu(y_1,\dots,y_{m-j})$$

for $j = 0,\dots,m$, we obtain

$$\begin{aligned}
&I_n(f)\, I_m(g)\\
&= \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{n+m-(i+j)}\binom{n}{i}\binom{m}{j} \sum_{(x_1,\dots,x_i)\in\eta^i_{\neq}} \widehat{f}_i(x_1,\dots,x_i) \sum_{(x_1,\dots,x_j)\in\eta^j_{\neq}} \widehat{g}_j(x_1,\dots,x_j)\\
&= \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{n+m-(i+j)}\binom{n}{i}\binom{m}{j} \sum_{\sigma\in\Pi(i,j)} \sum_{(x_1,\dots,x_{|\sigma|})\in\eta^{|\sigma|}_{\neq}} (\widehat{f}_i\otimes\widehat{g}_j)_\sigma(x_1,\dots,x_{|\sigma|}).
\end{aligned}$$

Here, the Slivnyak-Mecke formula and the assumptions on f, g, and $(f\otimes g)_\sigma$ ensure that all sums are absolutely convergent almost surely. Since the right-hand side is a sum of Poisson U-statistics as considered in Proposition 3.11, we see that

$$I_n(f)\, I_m(g) = \sum_{\ell=0}^{n+m} I_\ell(\widetilde{h}_\ell)$$

almost surely, where the functions $\widetilde{h}_\ell : X^\ell \to \overline{\mathbb{R}}$ are given by

$$\widetilde{h}_\ell = \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{n+m-(i+j)}\binom{n}{i}\binom{m}{j} \sum_{\sigma\in\Pi(i,j)} \sum_{\substack{J\subset\sigma,\\ |J|=|\sigma|-\ell}} \operatorname{sym}\int_{X^{|J|}} (\widehat{f}_i\otimes\widehat{g}_j)_\sigma\,\mathrm{d}\mu_J$$

for $\ell = 0, \dots, n+m$. Now we can include the integration steps in the definitions of $\widehat{f}_i$ and $\widehat{g}_j$ in our partition. Together with the binomial coefficients, we obtain

$$\widetilde{h}_\ell = \sum_{\sigma\in\Pi(n,m)} \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{n+m-(i+j)} \sum_{\substack{J_1\subset S_1(\sigma),|J_1|=n-i,\\ J_2\subset S_2(\sigma),|J_2|=m-j,\\ J_3\subset\sigma\setminus(J_1\cup J_2),|J_3|=|\sigma|-n-m+i+j-\ell}} \operatorname{sym}\int_{X^{|\sigma|-\ell}} (f\otimes g)_\sigma \, \mathrm{d}\mu_{J_1\cup J_2\cup J_3},$$

where $S_1(\sigma)$ (resp. $S_2(\sigma)$) is the set of singletons of σ belonging to f (resp. g). For a fixed $J \subset \sigma$ all sets J_1, J_2, J_3 with $J_1 \cup J_2 \cup J_3 = J$ lead to the same functions on the right-hand side. In case that $J \cap S_1(\sigma) \neq \emptyset$ or $J \cap S_2(\sigma) \neq \emptyset$, all these combinations cancel out because of the alternating sign. Hence, we have

$$\widetilde{h}_\ell = \sum_{\sigma\in\Pi(n,m)} \sum_{\substack{J\subset\sigma\setminus S(\sigma),\\ |J|=|\sigma|-\ell}} \operatorname{sym}\int_{X^{|J|}} (f\otimes g)_\sigma \, \mathrm{d}\mu_J,$$

which is h_ℓ for $\ell = |n-m|, \dots, n+m$ by definition of h_ℓ. Since $|J| \leq |\sigma \setminus S(\sigma)| = |\sigma| - |S(\sigma)| \leq |\sigma| - |n-m|$, we see that $\widetilde{h}_\ell \equiv 0$ for $\ell < |n-m|$. □

Proposition 3.12 allows us to write the product of two multiple Wiener-Itô integrals of L^1-functions as a sum of multiple Wiener-Itô integrals of L^1-functions. In our next result, we replace L^1-functions by L^2-functions.

Proposition 3.13 *Let $f \in L_s^2(\mu^n)$ and $g \in L_s^2(\mu^m)$ with $n, m \in \mathbb{N}$ such that*

$$\int_{X^{|J|}} |(f\otimes g)_\sigma| \, \mathrm{d}\mu_J \in L^2(\mu^{|\sigma|-|J|}) \quad \text{for all} \quad J\subset\sigma\setminus S(\sigma) \quad \text{and} \quad \sigma\in\Pi(n,m) \tag{3.15}$$

and define h_ℓ, $\ell = |n-m|, \dots, n+m$, as in Proposition 3.12. Then

$$I_n(f)\, I_m(g) = \sum_{\ell=|n-m|}^{n+m} I_\ell(h_\ell) \tag{3.16}$$

holds almost surely.

Proof. Since μ is a σ-finite measure, there exists a sequence of measurable sets $(A_k)_{k\in\mathbb{N}}$ in X such that

$$A_k \subset A_{k+1} \quad \text{for} \quad k\in\mathbb{N}, \qquad \bigcup_{k\in\mathbb{N}} A_k = X, \qquad \text{and} \qquad \mu(A_k) < \infty \quad \text{for } k\in\mathbb{N}.$$

Now we define $f^{(k)} : X^n \to \overline{\mathbb{R}}$, $k \in \mathbb{N}$, and $g^{(k)} : X^m \to \overline{\mathbb{R}}$, $k \in \mathbb{N}$, by

$$f^{(k)}(x_1, \dots, x_n) = \mathbb{1}(x_1, \dots, x_n \in A_k)\; f(x_1, \dots, x_n)$$

and

$$g^{(k)}(x_1, \dots, x_m) = \mathbb{1}(x_1, \dots, x_m \in A_k)\; g(x_1, \dots, x_m).$$

The definitions of $f^{(k)}$ and $g^{(k)}$ and the assumption (3.15) imply that $f^{(k)} \in L^1_s(\mu^n)$ and $g^{(k)} \in L^1_s(\mu^m)$ and that $(f^{(k)} \otimes g^{(k)})_\sigma \in L^1(\mu^{|\sigma|})$ for all $\sigma \in \Pi(n,m)$. Hence, it follows from Proposition 3.12 that

$$I_n(f^{(k)})\, I_m(g^{(k)}) = \sum_{\ell=|n-m|}^{n+m} I_\ell(h_\ell^{(k)}) \tag{3.17}$$

holds almost surely with

$$h_\ell^{(k)}(x_1,\dots,x_\ell) = \sum_{\sigma\in\Pi(n,m)} \sum_{\substack{J\subset\sigma\setminus S(\sigma),\\ |J|=|\sigma|-\ell}} \left(\operatorname{sym} \int_{X^{|J|}} (f^{(k)} \otimes g^{(k)})_\sigma \,\mathrm{d}\mu_J\right)(x_1,\dots,x_\ell)$$

for $\ell = |n-m|,\dots,n+m$. Because of the assumption (3.15), we have for fixed $\sigma \in \Pi(n,m)$ and $J \subset \sigma \setminus S(\sigma)$ that

$$\left(\operatorname{sym} \int_{X^{|J|}} |(f \otimes g)_\sigma| \,\mathrm{d}\mu_J\right)(x_1,\dots,x_\ell) < \infty$$

for μ-almost all $(x_1,\dots,x_\ell) \in X^\ell$ and obtain, by the dominated convergence theorem,

$$\lim_{k\to\infty} \left(\operatorname{sym} \int_{X^{|J|}} (f^{(k)} \otimes g^{(k)})_\sigma \,\mathrm{d}\mu_J\right)(x_1,\dots,x_\ell) = \left(\operatorname{sym} \int_{X^{|J|}} (f \otimes g)_\sigma \,\mathrm{d}\mu_J\right)(x_1,\dots,x_\ell)$$

for μ-almost all $(x_1,\dots,x_\ell) \in X^\ell$. As a consequence, we have

$$\lim_{k\to\infty} h_\ell^{(k)}(x_1,\dots,x_\ell) = h_\ell(x_1,\dots,x_\ell)$$

for μ-almost all $(x_1,\dots,x_\ell) \in X^\ell$. Since $h_\ell \in L^2_s(\mu^\ell)$ and the definitions of $f^{(k)}$ and $g^{(k)}$ and assumption (3.15) guarantee that

$$|h_\ell^{(k)}| \le \sum_{\sigma\in\Pi(n,m)} \sum_{\substack{J\subset\sigma\setminus S(\sigma),\\ |J|=|\sigma|-\ell}} \left(\operatorname{sym} \int_{X^{|J|}} |(f \otimes g)_\sigma| \,\mathrm{d}\mu_J\right) \in L^2_s(\mu^\ell),$$

the dominated convergence theorem implies $\|h_\ell^{(k)} - h_\ell\|_\ell^2 \to 0$ as $k \to \infty$. It follows from the definition of the multiple Wiener-Itô integral of an L^2-function that

$$I_\ell(h_\ell^{(k)}) \to I_\ell(h_\ell) \quad \text{as} \quad k \to \infty \quad \text{almost surely.} \tag{3.18}$$

On the other hand, the Cauchy-Schwarz inequality and Lemma 3.10 yield that

$$\begin{aligned}
&\mathbb{E}|I_n(f)\, I_m(g) - I_n(f^{(k)})\, I_m(g^{(k)})| \\
&\le \mathbb{E}|(I_n(f) - I_n(f^{(k)}))I_m(g)| + \mathbb{E}|I_n(f^{(k)})(I_m(g) - I_m(g^{(k)}))| \\
&\le \sqrt{n!\,m!}\left(\|f - f^{(k)}\|_n \,\|g\|_m + \|f^{(k)}\|_n \,\|g - g^{(k)}\|_m\right).
\end{aligned}$$

Together with $f^{(k)} \to f$ in $L^2_s(\mu^n)$ and $g^{(k)} \to g$ in $L^2_s(\mu^m)$ as $k \to \infty$, we obtain

$$I_n(f^{(k)})\, I_m(g^{(k)}) \to I_n(f)\, I_m(g) \quad \text{as} \quad k \to \infty \quad \text{almost surely.} \tag{3.19}$$

Combining the limits in formula (3.18) and formula (3.19) with identity (3.17) concludes the proof of Equation (3.16). □

The previous proposition generalizes a result due to Surgailis (see [92, Theorem 3.1]), where it is required that the underlying measure space $(X, \mathcal{X}, \mu)$ satisfies additional assumptions, in particular it has to be non-atomic. Peccati and Taqqu treat product formulas for stochastic integrals with respect to completely random measures in their book [66]. Examples are the Gaussian case, where the integration is with respect to a Gaussian measure, and the Poisson case, that is considered here. Our Propositions 3.12 and 3.13 for simple functions as integrands and a non-atomic Poisson point process are stated as Proposition 6.5.1 in [66]. The authors use this formula to derive Corollary 7.4.1 which is our Theorem 3.7 under the restrictions mentioned before.

In a similar way, we use our Proposition 3.13 to compute the expectation of a product of four multiple Wiener-Itô integrals.

Corollary 3.14 *Let* $f_\ell \in L^2_s(\mu^{n_\ell})$ *with* $n_\ell \in \mathbb{N}$ *for* $\ell = 1, \dots, 4$ *and assume that*

$$\int_{X^{|J|}} |(f_1 \otimes f_2)_\sigma| \, \mathrm{d}\mu_J \in L^2(\mu^{|\sigma|-|J|}) \text{ for all } J \subset \sigma \setminus S(\sigma) \text{ and } \sigma \in \Pi(n_1, n_2)$$

and

$$\int_{X^{|J|}} |(f_3 \otimes f_4)_\sigma| \, \mathrm{d}\mu_J \in L^2(\mu^{|\sigma|-|J|}) \text{ for all } J \subset \sigma \setminus S(\sigma) \text{ and } \sigma \in \Pi(n_3, n_4).$$

Then

$$\mathbb{E} \prod_{\ell=1}^{4} I_{n_\ell}(f_\ell) = \sum_{\sigma \in \Pi_{\geq 2}(n_1,n_2,n_3,n_4)} \int_{X^{|\sigma|}} (f_1 \otimes f_2 \otimes f_3 \otimes f_4)_\sigma \, \mathrm{d}\mu^{|\sigma|}. \tag{3.20}$$

Proof. Proposition 3.13 implies that $I_{n_1}(f_1)\, I_{n_2}(f_2)$ and $I_{n_3}(f_3)\, I_{n_4}(f_4)$ can be written as finite sums of multiple Wiener-Itô integrals with square integrable integrands. Hence, it follows from the orthogonality of the multiple Wiener-Itô integrals (see Lemma 3.10) that

$$\begin{aligned}\mathbb{E} \prod_{\ell=1}^{4} I_{n_\ell}(f_\ell) \;=\; & \sum_{n=n_{min}}^{n_{max}} n! \sum_{\substack{\sigma_1 \in \Pi(n_1,n_2),\\ \sigma_2 \in \Pi(n_3,n_4)}} \sum_{\substack{J_1 \subset \sigma_1 \setminus S(\sigma_1), |J_1| = |\sigma_1| - n,\\ J_2 \subset \sigma_2 \setminus S(\sigma_2), |J_2| = |\sigma_2| - n}} \\ & \int_{X^n} \left(\operatorname{sym} \int_{X^{|J_1|}} (f_1 \otimes f_2)_{\sigma_1} \, \mathrm{d}\mu_{J_1} \right) \left(\operatorname{sym} \int_{X^{|J_2|}} (f_3 \otimes f_4)_{\sigma_2} \, \mathrm{d}\mu_{J_2} \right) \mathrm{d}\mu^n\end{aligned}$$

with $n_{min} = \max\{|n_1-n_2|, |n_3-n_4|\}$ and $n_{max} = \min\{n_1+n_2, n_3+n_4\}$. This expression can be further simplified to

$$\begin{aligned}\mathbb{E}\prod_{\ell=1}^{4} I_{n_\ell}(f_\ell) = \sum_{n=n_{min}}^{n_{max}} \sum_{\substack{\sigma_1\in\Pi(n_1,n_2),\\ \sigma_2\in\Pi(n_3,n_4)}} &\sum_{\substack{J_1\subset\sigma_1\setminus S(\sigma_1),|J_1|=|\sigma_1|-n,\\ J_2\subset\sigma_2\setminus S(\sigma_2),|J_2|=|\sigma_2|-n}} \\ &\int_{X^n}\left(\int_{X^{|J_1|}}(f_1\otimes f_2)_{\sigma_1}\,\mathrm{d}\mu_{J_1}\right)(x_1,\dots,x_n) \\ &\sum_{\pi\in\mathrm{Per}(n)}\left(\int_{X^{|J_2|}}(f_3\otimes f_4)_{\sigma_2}\,\mathrm{d}\mu_{J_2}\right)(x_{\pi(1)},\dots,x_{\pi(n)})\,\mathrm{d}\mu^n.\end{aligned} \tag{3.21}$$

Now observe that each partition $\sigma \in \Pi_{\geq 2}(n_1, n_2, n_3, n_4)$ has a decomposition into

- two partitions $\sigma_1 \in \Pi(n_1, n_2)$ and $\sigma_2 \in \Pi(n_3, n_4)$;
- a list that identifies blocks of σ_1 and σ_2 such that each block occurs at most once and each block that is a singleton occurs exactly once.

Using this decomposition, we see that the right-hand side of formula (3.21) equals the right-hand side of formula (3.20). □

We can derive Equation (3.20) by applying Theorem 3.7 as well. But in this case we have to assume that

$$\int_{X^{|\sigma|}} |(f_1\otimes f_2\otimes f_3\otimes f_4)_\sigma|\,\mathrm{d}\mu^{|\sigma|} < \infty \quad \text{for all} \quad \sigma \in \Pi(n_1, n_2, n_3, n_4).$$

For our applications in Chapter 5 the assumptions of Corollary 3.14 work better.

Notes: The definition of a Poisson U-statistic is taken from *Reitzner and Schulte 2011*. Theorem 3.4, Corollary 3.5, and Theorem 3.7 are from *Last, Penrose, Schulte, and Thäle 2012*. But in the present work they are presented in a reverse order since the formulas for Poisson U-statistics are used to prove the product formula for multiple Wiener-Itô integrals. In the mentioned paper, the product formula for multiple Wiener-Itô integrals is proven first and used to derive the formulas for moments and cumulants of Poisson U-statistics.

Chapter 4

Wiener-Itô chaos expansion and Malliavin calculus

In this chapter, we give an introduction to the Wiener-Itô chaos expansion and the Malliavin operators that will play an important role in the sequel. The Wiener-Itô chaos expansion is a representation of a square integrable Poisson functional as a possibly infinite sum of multiple Wiener-Itô integrals. Via this decomposition, one can define the so-called Malliavin operators. As an example, we compute the Wiener-Itô chaos expansion and the Malliavin operators of a Poisson U-statistic.

4.1 Wiener-Itô chaos expansion

We start with some notation we need to introduce the Wiener-Itô chaos expansion as in the paper [41] by Last and Penrose. For a Poisson functional F and $x \in X$ the difference operator $D_x F$ is given by

$$D_x F = F(\eta + \delta_x) - F(\eta), \tag{4.1}$$

where δ_x stands for the Dirac measure concentrated at the point $x \in X$. On the left-hand side of formula (4.1), we suppress the dependence on η as usual in our notation. Since on the right-hand side F is evaluated for two different random measures, namely $\eta + \delta_x$ and η, we mention them explicitly. By its definition, the difference operator describes the behaviour of F if we add the point x to the Poisson point process. For this reason, the difference operator is sometimes called add-one-cost operator.

For $x_1, \ldots, x_n \in X$ the n-th iterated difference operator $D_{x_1,\ldots,x_n} F$ is recursively defined by

$$D_{x_1,\ldots,x_n} F = D_{x_1} D_{x_2,\ldots,x_n} F. \tag{4.2}$$

Using the definition of $D_x F$, we obtain the explicit representation

$$D_{x_1,\ldots,x_n} F = \sum_{I \subset [n]} (-1)^{n-|I|} F(\eta + \sum_{i \in I} \delta_{x_i}). \tag{4.3}$$

It is easy to see that the n-th iterated difference operator is symmetric in $x_1, \ldots, x_n$.

Now we can introduce functions $f_n : X^n \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, by

$$f_n(x_1, \ldots, x_n) = \frac{1}{n!} \mathbb{E} D_{x_1,\ldots,x_n} F. \tag{4.4}$$

Note that the functions f_n, $n \in \mathbb{N}$, are not necessarily defined since it can happen that the expectation does not exist. The symmetry of the iterated difference operator implies that the functions f_n, $n \in \mathbb{N}$, are symmetric if they exist. Using these functions and multiple Wiener-Itô integrals, we have the following representation for square integrable Poisson functionals (see Theorem 1.3 in [41]):

Theorem 4.1 *Let $F \in L^2(\mathbb{P}_\eta)$ be a Poisson functional. Then $f_n \in L_s^2(\mu^n)$ for $n \in \mathbb{N}$ and*

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n), \tag{4.5}$$

where the right-hand side converges in $L^2(\mathbb{P}_\eta)$. Moreover, the functions f_n, $n \in \mathbb{N}$, are the μ^n-unique functions g_n, $n \in \mathbb{N}$, such that $F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(g_n)$ in $L^2(\mathbb{P}_\eta)$.

The right-hand side of Equation (4.5) is called the Wiener-Itô chaos expansion of F, and we denote the functions f_n, $n \in \mathbb{N}$, as kernels. Combining the L^2-convergence of the Wiener-Itô chaos expansion with the orthogonality of the multiple Wiener-Itô integrals (see Lemma 3.10), we obtain the following formulas for the variance and covariance of square integrable Poisson functionals (see Theorem 1.1 in [41]):

Theorem 4.2 *Let $F, G \in L^2(\mathbb{P}_\eta)$ be Poisson functionals with Wiener-Itô chaos expansions $F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n)$ and $G = \mathbb{E}G + \sum_{n=1}^{\infty} I_n(g_n)$. Then*

$$\operatorname{Var} F = \sum_{n=1}^{\infty} n! \, \|f_n\|_n^2$$

and

$$\operatorname{Cov}(F, G) = \sum_{n=1}^{\infty} n! \, \langle f_n, g_n \rangle_{L^2(\mu^n)}.$$

In other words, Theorem 4.1 tells us that there is an isomorphism between $L^2(\mathbb{P}_\eta)$ and the set of all sequences $(f_n)_{n \in \mathbb{N} \cup \{0\}}$ with $f_0 \in \mathbb{R}$ and $f_n \in L_s^2(\mu^n)$ for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} n! \, \|f_n\|_n^2 < \infty$. The set of all these sequences is denoted as Fock space and the isomorphism is called Fock space representation.

The representation of a square integrable random variable as a sum of multiple stochastic integrals goes back to classical works of Wiener and Itô. Continuing earlier work of Wiener in [93], Itô introduced the multiple Wiener-Itô integral with respect to a Gaussian random measure in [30] and proved that every square integrable random variable depending on a Gaussian random measure has a representation as a sum of multiple Wiener-Itô integrals. In his paper [31], Itô extended the multiple Wiener-Itô integral and this representation to a more general class of random measures that also includes the Poisson case discussed above.

More recent works concerning the Wiener-Itô chaos expansion of a square integrable Poisson functional are [28, 41, 46, 61] by Houdre and Perez-Abreu, Last and Penrose, Liebscher, and Nualart and Vives.

4.2 Malliavin operators

For the central limit theorems that occur in the next chapter we need the so-called Malliavin operators and some of their properties. These operators are maps between some sets of Poisson functionals and random functions depending on the Poisson point process η and are defined in terms of the Wiener-Itô chaos expansion. Such operators also exist if we have a Gaussian random measure instead of a Poisson point process. In this context, they were first introduced by Malliavin in [47] to investigate smoothness properties of solutions of partial differential equations. Since then the theory has developed in several directions and can be applied to many problems in probability theory. For an overview of the most important results and applications we refer to the monographs [13, 59] by Di Nunno, Øksendal, and F. Proske and Nualart. One can think of Malliavin calculus as a stochastic calculus of variations. Although the major part of Malliavin calculus is concerned with the Gaussian case, we focus on the Poisson case under the same assumptions as before in the sequel. For more details on the Mallivin operators in the Poisson setting we also refer to the works [41, 61, 65] by Last and Penrose, Nualart and Vivies, and Peccati, Solé, Taqqu, and Utzet.

In the following, let a Poisson functional $F \in L^2(\mathbb{P}_\eta)$ have the Wiener-Itô chaos expansion

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n) \quad \text{with} \quad f_n \in L_s^2(\mu^n), n \in \mathbb{N}.$$

By a random function $g : X \to \overline{\mathbb{R}}$ we mean a collection of random variables $(g(x))_{x\in X}$. We denote by $L^p(\mathbb{P}_\eta \times \mu)$, $p > 0$, the set of all random functions $g : X \to \overline{\mathbb{R}}$ with

$$\int_X \mathbb{E}|g(x)|^p \,\mathrm{d}\mu(x) < \infty.$$

If $g \in L^2(\mathbb{P}_\eta \times \mu)$, we have $g(x) \in L^2(\mathbb{P}_\eta)$ for μ-almost all $x \in X$. Hence, we have for μ-almost all $x \in X$ a Wiener-Itô chaos expansion

$$g(x) = g_0(x) + \sum_{n=1}^{\infty} I_n(g_n(x,\cdot)). \tag{4.6}$$

Here, $g_0 : X \to \overline{\mathbb{R}}$ and $g_n : X \times X^n \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, are deterministic functions such that $g_n(x,\cdot) \in L_s^2(\mu^n)$ for $n \in \mathbb{N}$ and μ-almost all $x \in X$. Together with $g \in L^2(\mathbb{P}_\eta \times \mu)$, we obtain

$$\int_X \mathbb{E}\, g(x)^2 \,\mathrm{d}\mu(x) = \sum_{n=0}^{\infty} n!\, \|g_n\|_{n+1}^2 < \infty.$$

In the following, let for $g \in L^2(\mathbb{P}_\eta \times \mu)$ the functions g_0 and g_n, $n \in \mathbb{N}$, be given by formula (4.6).

The difference operator defined in Equation (4.1) is a Malliavin operator and has the following representation (see [41, Theorem 3.3] or [61, Theorem 6.2]):

Lemma 4.3 *Let* $\operatorname{dom} D$ *be the set of all Poisson functionals* $F \in L^2(\mathbb{P}_\eta)$ *such that* $\sum_{n=1}^{\infty} n\, n!\, \|f_n\|_n^2 < \infty$. *If* $F \in \operatorname{dom} D$, *then*

$$D_x F = \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(x,\cdot))$$

holds almost surely for μ-almost all $x \in X$.

Thus, we can regard the difference operator as a map $D : \operatorname{dom} D \to L^2(\mathbb{P}_\eta \times \mu)$ with $DF : x \mapsto D_x F$. The second Malliavin operator is the Ornstein-Uhlenbeck generator:

Definition 4.4 *Let $\operatorname{dom} L$ be the set of all Poisson functionals $F \in L^2(\mathbb{P}_\eta)$ such that $\sum_{n=1}^{\infty} n^2\, n!\, ||f_n||_n^2 < \infty$. The Ornstein-Uhlenbeck generator is the map $L : \operatorname{dom} L \to L^2(\mathbb{P}_\eta)$ that is given by*

$$LF = -\sum_{n=1}^{\infty} n\, I_n(f_n).$$

For centred random variables $F \in L^2(\mathbb{P}_\eta)$, i.e. $\mathbb{E}F = 0$, the inverse Ornstein-Uhlenbeck generator is given by

$$L^{-1}F = -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n). \tag{4.7}$$

The last Malliavin operator we need is the Skorohod integral:

Definition 4.5 *Let $\operatorname{dom} \delta$ be the set of all $g \in L^2(\mathbb{P}_\eta \times \mu)$ such that $\sum_{n=0}^{\infty} (n+1)!\, \|g_n\|_{n+1}^2 < \infty$. The Skorohod integral is the map $\delta : \operatorname{dom} \delta \to L^2(\mathbb{P}_\eta)$ that is defined by*

$$\delta(g) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{g}_n),$$

where $\tilde{g}_n$ is the symmetrization

$$\tilde{g}_n(x_1, \dots, x_{n+1}) = \frac{1}{(n+1)!} \sum_{\pi \in \mathrm{Per}(n+1)} g_n(x_{\pi(1)}, \dots, x_{\pi(n+1)})$$

over all permutations of the $n+1$ variables.

The following lemma summarizes how the operators from Malliavin calculus are related (see [65, Lemma 2.11] and [41, Proposition 3.4] or [61, Proposition 3.4]):

Lemma 4.6 *a) For every $F \in \operatorname{dom} L$ we have $F \in \operatorname{dom} D$, $DF \in \operatorname{dom} \delta$, and*

$$\delta DF = -LF. \tag{4.8}$$

b) Let $F \in \operatorname{dom} D$ and $g \in \operatorname{dom} \delta$. Then

$$\mathbb{E}\langle DF, g\rangle_{L^2(\mu)} = \mathbb{E}[F\delta(g)]. \tag{4.9}$$

Equation (4.9) is often called the integration by parts formula and is at the heart of Malliavin calculus. Because of this identity, one can see the difference operator and the Skorohod integral as dual operators.

In part b) of Lemma 4.6, it is required that $F \in \operatorname{dom} D$. The next lemma allows us to drop this assumption if some other conditions are satisfied. In this case, DF is given by formula (4.1).

Lemma 4.7 *Let $F \in L^2(\mathbb{P}_\eta)$, $t \in \mathbb{R}$ and let $g \in \operatorname{dom} \delta$ have the form*

$$g(z) = \sum_{n=0}^{k} I_n(g_n(z, \cdot))$$

for μ-almost all $z \in X$ with a fixed $k \in \mathbb{N}$. Moreover, assume that $D_z \mathbb{1}(F > t)\, g(z) \geq 0$ a.s. for μ-almost all $z \in X$. Then

$$\mathbb{E}\langle D\mathbb{1}(F > t), g\rangle_{L^2(\mu)} = \mathbb{E}[\mathbb{1}(F > t)\, \delta(g)].$$

Proof. Obviously, we have $\mathbb{1}(F > t) \in L^2(\mathbb{P}_\eta)$, whence it has the Wiener-Itô chaos expansion

$$\mathbb{1}(F > t) = \sum_{n=0}^{\infty} I_n(h_n).$$

For fixed $z \in X$ we have $D_z \mathbb{1}(F > t) \in L^2(\mathbb{P}_\eta)$ since it is bounded. For the kernels of the Wiener-Itô chaos expansions of $\mathbb{1}(F > t)$ and $D_z \mathbb{1}(F > t)$ Theorem 4.1 yields that

$$\begin{aligned}
&\frac{1}{n!} \mathbb{E} D_{x_1,\dots,x_n} D_z \mathbb{1}(F > t) \\
&= \frac{1}{n!} \mathbb{E}\left[\sum_{I \subset \{1,\dots,n\}} (-1)^{n+|I|} \Big(\mathbb{1}(F(\eta + \delta_z + \sum_{i \in I} \delta_{x_i}) > t) - \mathbb{1}(F(\eta + \sum_{i \in I} \delta_{x_i}) > t) \Big) \right] \\
&= (n+1)\, h_{n+1}(z, x_1, \dots, x_n).
\end{aligned}$$

Hence, we obtain the representation

$$D_z \mathbb{1}(F > t) = \sum_{n=1}^{\infty} n\, I_{n-1}(h_n(z, \cdot))$$

for μ-almost all $z \in X$. Now Fubini's Theorem and Theorem 4.2 imply that

$$\begin{aligned}
&\mathbb{E}\langle D\mathbb{1}(F > t), g\rangle_{L^2(\mu)} \\
&= \int_X \mathbb{E}[D_z \mathbb{1}(F > t)\, g(z)]\, \mathrm{d}\mu(z) \\
&= \int_X \mathbb{E}\left[\sum_{n=1}^{\infty} n\, I_{n-1}(h_n(z, \cdot)) \sum_{n=0}^{k} I_n(g_n(z, \cdot)) \right] \mathrm{d}\mu(z) \\
&= \int_X \sum_{n=1}^{k+1} n! \int_{X^{n-1}} h_n(z, x_1, \dots, x_{n-1})\, g_{n-1}(z, x_1, \dots, x_{n-1})\, \mathrm{d}\mu(x_1, \dots, x_{n-1})\, \mathrm{d}\mu(z).
\end{aligned} \tag{4.10}$$

On the other hand, we have

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}(F > t)\, \delta(g)\right] &= \mathbb{E}\left[\sum_{n=0}^{\infty} I_n(h_n) \sum_{n=0}^{k} I_{n+1}(\tilde{g}_n) \right] \\
&= \sum_{n=1}^{k+1} n! \int_{X^n} h_n(x_1, \dots, x_n)\, \tilde{g}_{n-1}(x_1, \dots, x_n)\, \mathrm{d}\mu(x_1, \dots, x_n) \qquad (4.11) \\
&= \sum_{n=1}^{k+1} n! \int_{X^n} h_n(x_1, \dots, x_n)\, g_{n-1}(x_1, \dots, x_n)\, \mathrm{d}\mu(x_1, \dots, x_n),
\end{aligned}$$

where we use that h_n is symmetric. Comparing the right-hand sides of formula (4.10) and formula (4.11) concludes the proof. $\square$

Similarly as the difference operator, the Skorohod integral also has a pathwise representation (see [41, Theorem 3.5]).

Lemma 4.8 *Let $g \in L^1(\mathbb{P}_\eta \times \mu) \cap \operatorname{dom} \delta$. Then*

$$\delta(g) = \sum_{x \in \eta} g(\eta - \delta_x, x) - \int_X g(\eta, x) \, \mathrm{d}\mu(x)$$

holds almost surely.

For some computations in Chapter 5 and Chapter 6 we need the following inequality:

Lemma 4.9 *Let $f \in L^2(\mu^{k+1})$, $k \in \mathbb{N} \cup \{0\}$, be symmetric in its last k arguments. For $h(x) = I_k(f(x, \cdot))$ we have*

$$\mathbb{E}\left[\delta(h)^2\right] \leq (k+1)\,\mathbb{E} \int_X I_k(f(x, \cdot))^2 \, \mathrm{d}\mu(x).$$

Proof. By the definition of δ, we obtain $\delta(h) = I_{k+1}(\tilde{f})$ with the symmetrization

$$\tilde{f}(x_1, \ldots, x_{k+1}) = \frac{1}{(k+1)!} \sum_{\pi \in \operatorname{Per}(k+1)} f(x_{\pi(1)}, \ldots, x_{\pi(k+1)})$$

as above. From the triangle inequality, it follows that $\|\tilde{f}\|_{k+1}^2 \leq \|f\|_{k+1}^2$. Combining this with Fubini's theorem, we find

$$\mathbb{E}\left[\delta(h)^2\right] = (k+1)!\, \|\tilde{f}\|_{k+1}^2 \leq (k+1)!\, \|f\|_{k+1}^2 = (k+1)\,\mathbb{E} \int_X I_k(f(x, \cdot))^2 \, \mathrm{d}\mu(x),$$

which completes the proof. $\square$

4.3 Wiener-Itô chaos expansion of Poisson U-statistics

In this section, we consider a Poisson U-statistic

$$S = \sum_{(x_1, \ldots, x_k) \in \eta_{\neq}^k} f(x_1, \ldots, x_k)$$

with $f \in L_s^1(\mu^k)$ and assume that $S \in L^2(\mathbb{P}_\eta)$. This assumption ensures that the Wiener-Itô chaos expansion of S exists. We compute its kernels, which also yields a formula for the variance of S. Moreover, we apply the Malliavin operators to S. Recall from Section 3.2 that

$$\mathbb{E} \sum_{(x_1, \ldots, x_k) \in \eta_{\neq}^k} f(x_1, \ldots, x_k) = \int_{X^k} f(x_1, \ldots, x_k) \, \mathrm{d}\mu(x_1, \ldots, x_k).$$

We start with the difference operator of the Poisson U-statistic S.

Lemma 4.10 *Let $S \in L^2(\mathbb{P}_\eta)$ be a Poisson U-statistic of order k. Then the difference operator applied to S yields*

$$D_y S = k \sum_{(x_1,\dots,x_{k-1})\in\eta^{k-1}_{\neq}} f(y, x_1, \dots, x_{k-1})$$

for $y \in X$.

Proof. By the definition of the difference operator D_y and the symmetry of f, we obtain for a Poisson U-statistic S

$$\begin{aligned} D_y S &= \sum_{(x_1,\dots,x_k)\in(\eta\cup\{y\})^k_{\neq}} f(x_1,\dots,x_k) - \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k) \\ &= \sum_{(x_1,\dots,x_{k-1})\in\eta^{k-1}_{\neq}} \left(f(y,x_1,\dots,x_{k-1}) + \dots + f(x_1,\dots,x_{k-1},y)\right) \\ &= k \sum_{(x_1,\dots,x_{k-1})\in\eta^{k-1}_{\neq}} f(y,x_1,\dots,x_{k-1}), \end{aligned}$$

which completes the proof. □

In Proposition 3.11, a representation of a Poisson U-statistic as a finite sum of multiple Wiener-Itô integrals is given. At the first glance, it looks like the Wiener-Itô chaos expansion of a Poisson U-statistic. But the Wiener-Itô chaos expansion is defined for square integrable Poisson functionals and has square integrable kernels, whereas we only assume in Proposition 3.11 that the Poisson U-statistic is in $L^1(\mathbb{P}_\eta)$ and that the integrands of the multiple Wiener-Itô integrals are L^1-functions. The following result computes the Wiener-Itô chaos expansion as given in Theorem 4.1 for a square integrable Poisson U-statistic and shows that it coincides with the representation in Proposition 3.11.

Theorem 4.11 *Let S be a Poisson U-statistic of order k and define $f_n : X^n \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, by*

$$f_n(y_1,\dots,y_n) = \begin{cases} \binom{k}{n} \int_{X^{k-n}} f(y_1,\dots,y_n,x_1,\dots,x_{k-n}) \, \mathrm{d}\mu(x_1,\dots,x_{k-n}), & n \le k \\ 0, & n > k \end{cases}. \tag{4.12}$$

Then $S \in L^2(\mathbb{P}_\eta)$ holds if and only if $f_n \in L^2_s(\mu^n)$ for $n \in \mathbb{N}$. In this case, the functions f_n, $n \in \mathbb{N}$, are the kernels of the Wiener-Itô chaos expansion of S, and S has the variance

$$\begin{aligned} &\operatorname{Var} S \\ &= \sum_{n=1}^{k} n! \binom{k}{n}^2 \int_{X^n} \left(\int_{X^{k-n}} f(y_1,\dots,y_n,x_1,\dots,x_{k-n}) \, \mathrm{d}\mu(x_1,\dots,x_{k-n}) \right)^2 \mathrm{d}\mu(y_1,\dots,y_n). \end{aligned}$$

Proof. At first we assume that $S \in L^2(\mathbb{P}_\eta)$ and use Theorem 4.1 to compute the Wiener-Itô chaos expansion of S. In Lemma 4.10, the difference operator of a Poisson U-statistic was computed. Proceeding by induction, we get

$$D_{y_1,\dots,y_n} S = \frac{k!}{(k-n)!} \sum_{(x_1,\dots,x_{k-n}) \in \eta_{\neq}^{k-n}} f(y_1,\dots,y_n,x_1,\dots,x_{k-n})$$

for $n \leq k$. Hence, $D_{y_1,\dots,y_k} S = k! f(y_1,\dots,y_k)$ only depends on $y_1,\dots,y_k$ and is independent of the Poisson point process η. This yields

$$D_{y_1,\dots,y_{k+1}} S = 0 \quad \text{and} \quad D_{y_1,\dots,y_n} S = 0 \quad \text{for} \quad n > k.$$

We just proved

$$D_{y_1,\dots,y_n} S = \begin{cases} \frac{k!}{(k-n)!} \sum_{(x_1,\dots,x_{k-n}) \in \eta_{\neq}^{k-n}} f(y_1,\dots,y_n,x_1,\dots,x_{k-n}), & n \leq k \\ 0, & \text{otherwise} \end{cases}.$$

By Corollary 2.10, we obtain

$$\begin{aligned} f_n(y_1,\dots,y_n) &= \frac{1}{n!} \mathbb{E} D_{y_1,\dots,y_n} S \\ &= \frac{1}{n!} \frac{k!}{(k-n)!} \mathbb{E} \sum_{(x_1,\dots,x_{k-n}) \in \eta_{\neq}^{k-n}} f(y_1,\dots,y_n,x_1,\dots,x_{k-n}) \\ &= \binom{k}{n} \int_{X^{k-n}} f(y_1,\dots,y_n,x_1,\dots,x_{k-n}) \, \mathrm{d}\mu(x_1,\dots,x_{k-n}) \end{aligned}$$

for $n \leq k$ and $f_n \equiv 0$ for $n > k$. Now Theorem 4.1 implies that $f_n \in L_s^2(\mu^n)$. The formula for the variance follows from Theorem 4.2.

On the other hand, Proposition 3.11 tells us that every Poisson U-statistic in $L^1(\mathbb{P}_\eta)$ has a representation as a finite sum of multiple Wiener-Itô integrals of L^1-functions regardless of whether it is in $L^2(\mathbb{P}_\eta)$ or not. Since the function f is symmetric, the formula for the integrands in Proposition 3.11 coincides with formula (4.12). If these functions are square integrable, Proposition 3.11 gives us a representation of S as a finite sum of multiple Wiener-Itô integrals in the L^2-sense so that $S \in L^2(\mathbb{P}_\eta)$. □

For the special case $k = 2$ the formulas for the kernels are already implicit in the paper [56] by Molchanov and Zuyev, where ideas closely related to Malliavin calculus are used.

Note that $S \in L^2(\mathbb{P}_\eta)$ implies $f_n \in L_s^2(\mu^n)$ for $n \in \mathbb{N}$, and thus that for all $n = 1,\dots,k$

$$\int_{X^n} \Big(\int_{X^{k-n}} f(y_1,\dots,y_n,x_1,\dots,x_{k-n}) \, \mathrm{d}\mu(x_1,\dots,x_{k-n}) \Big)^2 \mathrm{d}\mu(y_1,\dots,y_n) < \infty.$$

In particular, we have $f \in L_s^2(\mu^k)$.

By Theorem 4.11, Poisson U-statistics only have a finite number of non-vanishing kernels. The following theorem characterizes a Poisson U-statistic by this property. We call a Wiener-Itô chaos expansion finite if only a finite number of kernels do not vanish, and its order is the highest order of a multiple Wiener-Itô integral with a non-vanishing integrand.

Corollary 4.12 *Assume that $F \in L^2(\mathbb{P}_\eta)$.*

a) If F is a Poisson U-statistic, then F has a finite Wiener-Itô chaos expansion of order k with kernels $f_n \in L^1_s(\mu^n) \cap L^2_s(\mu^n)$, $n = 1, \dots, k$.

b) If F has a finite Wiener-Itô chaos expansion of order k with kernels $f_n \in L^1_s(\mu^n) \cap L^2_s(\mu^n)$, $n = 1, \dots, k$, then F is a (finite) sum of Poisson U-statistics of the orders 1 to k and a constant.

Proof. The fact that a Poisson U-statistic $F \in L^2(\mathbb{P}_\eta)$ has a finite Wiener-Itô chaos expansion of order k with $f_n \in L^1_s(\mu^n)$ for $n = 1, \dots, k$ follows from Theorem 4.11 and $f \in L^1_s(\mu^k)$. The L^2-integrability of the kernels is a direct consequence of Theorem 4.1.

For the second part of the proof let $F \in L^2(\mathbb{P}_\eta)$ have a finite Wiener-Itô chaos expansion of order k, i.e.

$$F = \mathbb{E}F + \sum_{n=1}^{k} I_n(f_n)$$

with kernels $f_n \in L^1_s(\mu^n) \cap L^2_s(\mu^n)$, $n = 1, \dots, k$, and $k \in \mathbb{N}$. Hence, $I_n(f_n)$ is given by Definition 3.6, which tells us that it is a finite sum of Poisson U-statistics of the orders 1 to n and a constant. Now we can write F as a sum of Poisson U-statistics of the orders 1 to k and a constant. □

There exist random variables in $L^2(\mathbb{P}_\eta)$ with finite Wiener-Itô chaos expansions which are not sums of Poisson U-statistics. This is possible if a kernel f_n is in $L^2_s(\mu^n) \backslash L^1_s(\mu^n)$.

Example 4.13 Define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \frac{1}{x}\, \mathbb{1}(x > 1)$$

which is in $L^2(\mathbb{R}) \backslash L^1(\mathbb{R})$. Now we define the random variable $F = I_1(g)$. F is in $L^2(\mathbb{P}_\eta)$ and has a finite Wiener-Itô chaos expansion. But the formal representation

$$I_1(g) = \sum_{x \in \eta} g(x) - \int_{\mathbb{R}} g(x)\, \mathrm{d}x$$

we used in the proof of Corollary 4.12 fails because the integral does not exist.

There also exist Poisson U-statistics $S \in L^1(\mathbb{P}_\eta)$ with $f \in L^1_s(\mu^k) \cap L^2_s(\mu^k)$ which are not in $L^2(\mathbb{P}_\eta)$.

Example 4.14 We construct $f \in L^1_s(\mathbb{R}^2) \cap L^2_s(\mathbb{R}^2)$ with $\|f_1\|_1 = \infty$ by putting

$$f(x_1, x_2) = \mathbb{1}(0 \le x_1 \sqrt{|x_2|} \le 1)\, \mathbb{1}(0 \le x_2 \sqrt{|x_1|} \le 1)$$

and define

$$S = \sum_{(x_1, x_2) \in \eta^2_{\neq}} f(x_1, x_2).$$

In this case the first kernel

$$f_1(y) = \mathbb{E}[2\sum_{x\in\eta} f(y,x)] = 2\int_{\mathbb{R}} f(y,x)\,\mathrm{d}x = 2\,\mathbb{1}(y\geq 0)\min\left\{\frac{1}{y^2},\frac{1}{\sqrt{y}}\right\}$$

is not in $L^2_s(\mathbb{R})$ so that S has no Wiener-Itô chaos expansion and cannot be in $L^2(\mathbb{P}_\eta)$.

By Corollary 4.12 b), a functional $F\in L^2(\mathbb{P}_\eta)$ with a finite Wiener-Itô chaos expansion of order k and kernels $f_n\in L^1_s(\mu^n)\cap L^2_s(\mu^n)$, $n=1,\dots,k$, is a (finite) sum of Poisson U-statistics and a constant. Our next example shows that neither the single Poisson U-statistics are in $L^2(\mathbb{P}_\eta)$ nor the functions we sum over are necessarily in $L^2_s(\mu^n)$.

Example 4.15 Set $F=I_2(f)$ with f as in Example 4.14. Then

$$F=\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y - 2\sum_{x\in\eta}\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y + \sum_{(x_1,x_2)\in\eta^2_{\neq}} f(x_1,x_2)$$

so that F is a sum of Poisson U-statistics. Regarding the second Poisson U-statistic

$$S_2=\sum_{x\in\eta}\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y,$$

we know from the previous example that $x\mapsto\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y$ is not in $L^2(\mathbb{R})$. This is in contrast to the remark after the proof of Theorem 4.11 that for a Poisson U-statistic $S\in L^2(\mathbb{P}_\eta)$ of order k we always have $f\in L^2_s(\mu^k)$, whence $S_2\notin L^2(\mathbb{P}_\eta)$.

The knowledge of the Wiener-Itô chaos expansion of a Poisson U-statistic enables us to give pathwise representations for the Ornstein-Uhlenbeck generator and the inverse Ornstein-Uhlenbeck generator of a Poisson U-statistic.

Lemma 4.16 *Let $S\in L^2(\mathbb{P}_\eta)$ be a Poisson U-statistic of order k. Then*

$$LS=-kS+k\int_X \sum_{(x_1,\dots,x_{k-1})\in\eta^{k-1}_{\neq}} f(x_1,\dots,x_{k-1},y)\,\mathrm{d}\mu(y)$$

and

$$\begin{aligned}-L^{-1}(S-\mathbb{E}S) &= \sum_{m=1}^{k}\frac{1}{m}\sum_{(x_1,\dots,x_m)\in\eta^m_{\neq}}\int_{X^{k-m}} f(x_1,\dots,x_m,y_1,\dots,y_{k-m})\,\mathrm{d}\mu(y_1,\dots,y_{k-m})\\ &\quad -\sum_{m=1}^{k}\frac{1}{m}\int_{X^k} f(y_1,\dots,y_k)\,\mathrm{d}\mu(y_1,\dots,y_k) \qquad (4.13)\end{aligned}$$

hold almost surely.

Proof. Since S has a finite Wiener-Itô chaos expansion, we have $S \in \operatorname{dom} L$. The representation of the Ornstein-Uhlenbeck generator is a consequence of Lemma 4.6 a) and Lemma 4.8.

For the proof of the second identity we define $\widehat{f_n} : X^n \to \overline{\mathbb{R}}$ by $\widehat{f_n}(x_1, \dots, x_n) = \binom{k}{n}^{-1} f_n(x_1, \dots, x_n)$ for $n = 1, \dots, k$. Using this notation and formula (4.12) for the kernels of a Poisson U-statistic, we obtain the Wiener-Itô chaos expansion

$$\begin{aligned} &\sum_{(x_1,\dots,x_m)\in\eta^m_{\neq}} \int_{X^{k-m}} f(x_1, \dots, x_m, y_1, \dots, y_{k-m}) \, \mathrm{d}\mu(y_1, \dots, y_{k-m}) \\ &= \int_{X^k} f(y_1, \dots, y_k) \, \mathrm{d}\mu(dy_1 \dots, y_k) + \sum_{n=1}^{m} \binom{m}{n} I_n(\widehat{f_n}) \end{aligned}$$

for $m = 1, \dots, k$. Combining this with an identity for binomial coefficients, we see that the right-hand side in Equation (4.13) equals

$$\begin{aligned} \sum_{m=1}^{k} \frac{1}{m} \sum_{n=1}^{m} \binom{m}{n} I_n(\widehat{f_n}) &= \sum_{m=1}^{k} \sum_{n=1}^{m} \frac{1}{m} \binom{m}{n} I_n(\widehat{f_n}) = \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{1}{m} \binom{m}{n} I_n(\widehat{f_n}) \\ &= \sum_{n=1}^{k} \frac{1}{n} \binom{k}{n} I_n(\widehat{f_n}) = \sum_{n=1}^{k} \frac{1}{n} \, I_n(f_n), \end{aligned}$$

which is the Wiener-Itô chaos expansion of $-L^{-1}(S - \mathbb{E}S)$ by definition. □

Notes: Lemma 4.9 and the second part of Lemma 4.16 are in *Schulte 2012b*. Lemma 4.10, Theorem 4.11, Corollary 4.12, the Examples 4.13, 4.14, and 4.15, and the first part of Lemma 4.16 are from *Reitzner and Schulte 2011*.

Chapter 5

Normal approximation of Poisson functionals

The aim of this chapter is to develop bounds for the normal approximation of Poisson functionals and vectors of Poisson functionals. These bounds can be used to derive univariate and multivariate central limit theorems. Throughout this chapter let $F \in L^2(\mathbb{P}_\eta)$ be a square integrable Poisson functional with the Wiener-Itô chaos expansion

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n) \quad \text{with} \quad f_n \in L_s^2(\mu^n),\ n \in \mathbb{N}.$$

Moreover, we consider a random vector $\mathbf{F} = (F^{(1)}, \dots, F^{(m)})$ of Poisson functionals $F^{(\ell)} \in L^2(\mathbb{P}_\eta)$ having Wiener-Itô chaos expansions

$$F^{(\ell)} = \mathbb{E}F^{(\ell)} + \sum_{n=1}^{\infty} I_n(f_n^{(\ell)}) \quad \text{with} \quad f_n^{(\ell)} \in L_s^2(\mu^n),\ n \in \mathbb{N},$$

for $\ell = 1, \dots, m$. We denote the covariance matrix of $\mathbf{F}$ by Σ and its elements by σ_{uv} for $u, v = 1, \dots, m$.

For the formulation of our central limit theorems we consider a family of Poisson point processes $(\eta_t)_{t \geq 1}$ with σ-finite intensity measures $(\mu_t)_{t\geq 1}$ and families of square integrable Poisson functionals $(F_t)_{t\geq 1}$ and $(F_t^{(\ell)})_{t\geq 1}$, $\ell = 1, \dots, m$, depending on the point processes $(\eta_t)_{t\geq 1}$. We put $\mathbf{F}_t = (F_t^{(1)}, \dots, F_t^{(m)})$. For $F_t \in L^2(\mathbb{P}_{\eta_t})$ and $F_t^{(\ell)} \in L^2(\mathbb{P}_{\eta_t})$, $\ell = 1. \dots, m$, we have the Wiener-Itô chaos expansions

$$F_t = \mathbb{E}F_t + \sum_{n=1}^{\infty} I_{n,t}(f_{n,t})$$

with $f_{n,t} \in L_s^2(\mu_t^n)$, $n \in \mathbb{N}$, and

$$F_t^{(\ell)} = \mathbb{E}F_t^{(\ell)} + \sum_{n=1}^{\infty} I_{n,t}(f_{n,t}^{(\ell)})$$

with $f_{n,t}^{(\ell)} \in L_s^2(\mu_t^n)$, $n \in \mathbb{N}$. Here, $I_{n,t}(\cdot)$ stands for the n-th multiple Wiener-Itô integral with respect to the Poisson point process η_t. Moreover, we always assume that the variances of $(F_t)_{t\geq 1}$ and the covariances of $(\mathbf{F}_t)_{t\geq 1}$ are uniformly bounded, i.e.

$$\sup_{t\geq 1} \operatorname{Var} F_t < \infty \quad \text{and} \quad \max_{1\leq u,v\leq m} \sup_{t\geq 1} |\operatorname{Cov}(F_t^{(u)}, F_t^{(v)})| < \infty.$$

These assumptions are no restrictions since they are satisfied if we rescale each Poisson functional by the square root of its variance, for example.

In the following, we write $\|\cdot\|_{n,t}$ for the norm in $L^2(\mu_t^n)$.

5.1 Malliavin-Stein method for Poisson functionals

In this section, we present three abstract results for the normal approximation of Poisson functionals, which can be derived by a combination of Malliavin calculus and Stein's method and are the base for our theorems in the next sections. We start with a result for the normal approximation of a square integrable Poisson functional in the Wasserstein distance due to Peccati, Solé, Taqqu, and Utzet (see [65, Theorem 3.1]):

Theorem 5.1 *Let $F \in L^2(\mathbb{P}_\eta)$ be a Poisson functional with $F \in \operatorname{dom} D$ and $\mathbb{E}F = 0$ and let N be a standard Gaussian random variable. Then*

$$d_W(F, N) \leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}| + \int_X \mathbb{E}(D_z F)^2 \, |D_z L^{-1} F| \, \mathrm{d}\mu(z). \tag{5.1}$$

Recall that the Wasserstein distance used in the previous theorem is defined as

$$d_W(Y, Z) = \sup_{h\in \mathrm{Lip}(1)} |\mathbb{E}h(Y) - \mathbb{E}h(Z)|$$

for two random variables Y and Z. But in some situations one is more interested in the Kolmogorov distance given by

$$d_K(Y, Z) = \sup_{x\in\mathbb{R}} |\mathbb{P}(Y \leq x) - \mathbb{P}(Z \leq x)|,$$

which is the supremum-norm of the difference between the distribution functions of Y and Z. In the situation that one of the random variables is a standard Gaussian random variable N, it is known (see [9, Theorem 3.1], for example) that

$$d_K(Y, N) \leq 2\sqrt{d_W(Y, N)}. \tag{5.2}$$

Hence, Theorem 5.1 also gives bound for the Kolmogorov distance between a Poisson functional and a standard Gaussian random variable. For a family of Poisson functionals $(F_t)_{t\geq 1}$ that converges in distribution to a standard Gaussian random variable a combination of Theorem 5.1 and the inequality (5.2) yields a weaker rate of convergence for the Kolmogorov distance than for the Wasserstein distance due to the square root in formula (5.2). But in many situation (e.g. the classical central limit theorem for i.i.d. random variables), one has the same rate of convergence for both distances.

By a similar technique as in the proof of Theorem 5.1 in [65], we obtain the following bound for the normal approximation of Poisson functionals in Kolmogorov distance. Later, we present examples where this theorem yields the same rate of convergence as Theorem 5.1.

Theorem 5.2 *Let $F \in L^2(\mathbb{P}_\eta)$ with $\mathbb{E}F = 0$ and $F \in \operatorname{dom} D$ and let N be a standard Gaussian random variable. Then*

$$\begin{aligned} d_K(F,N) &\le \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}| + 2\,\mathbb{E}\langle (DF)^2, |DL^{-1}F|\rangle_{L^2(\mu)} \\ &\quad + 2\,\mathbb{E}\langle (DF)^2, |F\ DL^{-1}F|\rangle_{L^2(\mu)} + 2\,\mathbb{E}\langle (DF)^2, |DF\,DL^{-1}F|\rangle_{L^2(\mu)} \\ &\quad + \sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}\,(F>t)\,, DF\,|DL^{-1}F|\rangle_{L^2(\mu)} \\ &\le \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}| + 2c(F)\sqrt{\mathbb{E}\langle (DF)^2, (DL^{-1}F)^2\rangle_{L^2(\mu)}} \\ &\quad + \sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}\,(F>t)\,, DF\,|DL^{-1}F|\rangle_{L^2(\mu)} \end{aligned} \tag{5.3}$$

with

$$c(F) = \sqrt{\mathbb{E}\langle (DF)^2, (DF)^2\rangle_{L^2(\mu)}} + \left(\mathbb{E}\langle DF, DF\rangle^2_{L^2(\mu)}\right)^{\frac{1}{4}} \left(\left(\mathbb{E}F^4\right)^{\frac{1}{4}} + 1\right).$$

Proof. Let $g_t : \mathbb{R} \to \mathbb{R}$, $t \in \mathbb{R}$, be defined as in Equation (2.8). Now formula (2.12) tells us that

$$d_K(F,N) = \sup_{t\in\mathbb{R}} |\mathbb{E}[g_t'(F) - F g_t(F)]|. \tag{5.4}$$

Thus, we obtain a bound for the Kolmogorov distance by estimating the right-hand side of Equation (5.4).

Using identity (4.8) and the integration by parts formula (4.9), we obtain

$$\begin{aligned} \mathbb{E}[F\,g_t(F)] &= \mathbb{E}[LL^{-1}F\,g_t(F)] = \mathbb{E}[\delta(-DL^{-1}F)\,g_t(F)] \\ &= \mathbb{E}\langle -DL^{-1}F, Dg_t(F)\rangle_{L^2(\mu)}. \end{aligned} \tag{5.5}$$

In order to compute $D_z g_t(F)$ for a fixed $z \in X$, we consider the following cases:

1. $F, F + D_zF \le t$ or $F, F + D_zF > t$;
2. $F \le t < F + D_zF$;
3. $F + D_zF \le t < F$.

For $F, F + D_zF \le t$ or $F, F + D_zF > t$ it follows by Taylor expansion that

$$\begin{aligned} D_z g_t(F) &= g_t(F + D_zF) - g_t(F) = g_t'(F)D_zF + \frac{1}{2}g_t''(\tilde{F})(D_zF)^2 \\ &=: g_t'(F)D_zF + r_1(F,z,t), \end{aligned}$$

where $\tilde{F}$ is between F and $F + D_zF$.

Note that g_t is not differentiable at $t \in \mathbb{R}$ and that we defined $g_t'(t)$ as the left-sided limit of g_t' in t in Chapter 2. For $F \le t < F + D_zF$, we obtain by Taylor expansion and Equation (2.11)

$$\begin{aligned}
D_z g_t(F) &= g_t(F+D_zF) - g_t(F) = g_t(F+D_zF) - g_t(t) + g_t(t) - g_t(F)\\
&= g_t'(t+)(F+D_zF-t) + \frac{1}{2}g_t''(\tilde{F}_1)(F+D_zF-t)^2\\
&\quad + g_t'(F)(t-F) + \frac{1}{2}g_t''(\tilde{F}_2)(t-F)^2\\
&= g_t'(F)D_zF + (g_t'(t-) - 1 - g_t'(F))(F+D_zF-t)\\
&\quad + \frac{1}{2}g_t''(\tilde{F}_1)(F+D_zF-t)^2 + \frac{1}{2}g_t''(\tilde{F}_2)(t-F)^2\\
&= g_t'(F)D_zF - (F+D_zF-t) + g_t''(\tilde{F}_0)(t-F)(F+D_zF-t)\\
&\quad + \frac{1}{2}g_t''(\tilde{F}_1)(F+D_zF-t)^2 + \frac{1}{2}g_t''(\tilde{F}_2)(t-F)^2\\
&=: g_t'(F)D_zF - (F+D_zF-t) + r_2(F,z,t)
\end{aligned}$$

with $\tilde{F}_0, \tilde{F}_1, \tilde{F}_2 \in (F, F+D_zF)$. For $F + D_zF \leq t < F$, we have analogously

$$\begin{aligned}
D_z g_t(F) &= g_t(F+D_zF) - g_t(F) = g_t(F+D_zF) - g_t(t) + g_t(t) - g_t(F)\\
&= g_t'(t-)(F+D_zF-t) + \frac{1}{2}g_t''(\tilde{F}_1)(F+D_zF-t)^2\\
&\quad + g_t'(F)(t-F) + \frac{1}{2}g_t''(\tilde{F}_2)(t-F)^2\\
&= g_t'(F)D_zF + (g_t'(t+) + 1 - g_t'(F))(F+D_zF-t)\\
&\quad + \frac{1}{2}g_t''(\tilde{F}_1)(F+D_zF-t)^2 + \frac{1}{2}g_t''(\tilde{F}_2)(t-F)^2\\
&= g_t'(F)D_zF + (F+D_zF-t) + g_t''(\tilde{F}_0)(t-F)(F+D_zF-t)\\
&\quad + \frac{1}{2}g_t''(\tilde{F}_1)(F+D_zF-t)^2 + \frac{1}{2}g_t''(\tilde{F}_2)(t-F)^2\\
&=: g_t'(F)D_zF + (F+D_zF-t) + r_2(F,z,t)
\end{aligned}$$

with $\tilde{F}_0, \tilde{F}_1, \tilde{F}_2 \in (F+D_zF, F)$. Thus, $D_z g_t(F)$ has a representation

$$D_z g_t(F) = g_t'(F)D_zF + R_{F,t}(z), \tag{5.6}$$

where $R_{F,t} : X \to \mathbb{R}$ is given by

$$\begin{aligned}
&R_{F,t}(z)\\
&= (\mathbb{1}(F, F+D_zF \leq t) + \mathbb{1}(F, F+D_zF > t))\, r_1(F,z,t)\\
&\quad + (\mathbb{1}(F \leq t < F+D_zF) + \mathbb{1}(F+D_zF \leq t < F))\,(r_2(F,z,t) - |F+D_zF-t|).
\end{aligned}$$

Combining the Equations (5.5) and (5.6) yields

$$\mathbb{E}\left[g_t'(F) - Fg_t(F)\right] = \mathbb{E}\left[g_t'(F) - \langle g_t'(F)DF + R_{F,t}, -DL^{-1}F\rangle_{L^2(\mu)}\right],$$

and the triangle inequality and $|g_t'(F)| \leq 1$ (see Lemma 2.11) lead to

$$\begin{aligned}
|\mathbb{E}\left[g_t'(F) - Fg_t(F)\right]| &\leq |\mathbb{E}\left[g_t'(F)\left(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right)\right]|\\
&\quad + |\mathbb{E}\langle R_{F,t}, DL^{-1}F\rangle_{L^2(\mu)}|\\
&\leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}| + \mathbb{E}\langle |R_{F,t}|, |DL^{-1}F|\rangle_{L^2(\mu)}.
\end{aligned} \tag{5.7}$$

In $r_2(F,z,t)$, we assume that t is between F and $F+D_zF$, so that

$$|F+D_zF-t|\le |D_zF| \ \text{ and } \ |F-t|\le |D_zF|.$$

The inequality (2.13) allows us to bound all second derivatives in $R_{F,t}(z)$ by

$$|g_t''(\tilde{F}_i)|\le \frac{\sqrt{2\pi}}{4}+|F|+|D_zF|.$$

Now it is easy to see that

$$\begin{aligned}
&|R_{F,t}(z)|\\
&\le \left(\mathbb{1}\left(F,F+D_zF\le t\right)+\mathbb{1}\left(F,F+D_zF>t\right)\right)\frac{1}{2}\left(\frac{\sqrt{2\pi}}{4}+|F|+|D_zF|\right)(D_zF)^2\\
&\quad +\left(\mathbb{1}\left(F\le t<F+D_zF\right)+\mathbb{1}\left(F+D_zF\le t<F\right)\right)|D_zF|\\
&\quad +\left(\mathbb{1}\left(F\le t<F+D_zF\right)+\mathbb{1}\left(F+D_zF\le t<F\right)\right)2\left(\frac{\sqrt{2\pi}}{4}+|F|+|D_zF|\right)(D_zF)^2\\
&\le 2\left(\frac{\sqrt{2\pi}}{4}+|F|+|D_zF|\right)(D_zF)^2\\
&\quad +\left(\mathbb{1}\left(F\le t<F+D_zF\right)+\mathbb{1}\left(F+D_zF\le t<F\right)\right)|D_zF|,
\end{aligned}$$

where the last summand can be rewritten as

$$\left(\mathbb{1}\left(F\le t<F+D_zF\right)+\mathbb{1}\left(F+D_zF\le t<F\right)\right)|D_zF|=D_z\mathbb{1}\left(F>t\right)D_zF.$$

Hence, it follows directly that

$$\begin{aligned}
\mathbb{E}\langle |R_{F,t}|,|DL^{-1}F|\rangle_{L^2(\mu)} \ &\le\ 2\mathbb{E}\langle (DF)^2,|DL^{-1}F|\rangle_{L^2(\mu)}+2\mathbb{E}\langle (DF)^2,|F\ DL^{-1}F|\rangle_{L^2(\mu)}\\
&\quad +2\mathbb{E}\langle (DF)^2,|DF\ DL^{-1}F|\rangle_{L^2(\mu)}\\
&\quad +\mathbb{E}\langle D\mathbb{1}\left(F>t\right)DF,|DL^{-1}F|\rangle_{L^2(\mu)}.
\end{aligned}$$

Combining this with the formulas (5.7) and (5.4) concludes the proof of the first inequality in formula (5.3). The second bound in formula (5.3) is a direct consequence of the Cauchy-Schwarz inequality. □

In the proof of Theorem 5.1 in [65], the right-hand side of the Equation (5.5) is evaluated for twice differentiable functions $f:\mathbb{R}\to\mathbb{R}$ with $\sup_{x\in\mathbb{R}}|f'(x)|\le 1$ and $\sup_{x\in\mathbb{R}}|f''(x)|\le 2$ instead of the functions g_t as defined in formula (2.8) because for the Wasserstein distance one regards solutions of Stein's equation (2.7) for $h\in \mathrm{Lip}(1)$, which must have these properties. For such a function f it holds that

$$D_zf(F)=f'(F)D_zF+\tilde{r}(F)$$

with $|\tilde{r}(F)|\le (D_zF)^2$. Since this representation is easier than the representation we obtain for $D_zg_t(F)$, the bound for the Wasserstein distance in Theorem 5.1 is shorter and easier to evaluate than the bound for the Kolmogorov distance in Theorem 5.2.

Since our univariate Theorems 5.1 and 5.2 are for the distance to a standard Gaussian random variable, we apply it to the standardization $(F-\mathbb{E}F)/\sqrt{\operatorname{Var} F}$ of a Poisson functional $F \in L^2(\mathbb{P}_\eta)$. But in some situations, it is more convenient to use a different rescaling. For example, we can use an approximation of the variance if the exact variance is unknown. For such situations the following corollary allows us to use a different rescaling:

Corollary 5.3 *For a Poisson functional $F \in L^2(\mathbb{P}_\eta)$ with $F \in \operatorname{dom} D$ we have*

$$\mathbb{E}\left|1-\frac{1}{V}\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right| \\ \leq \left|1-\frac{\operatorname{Var} F}{V}\right| + \frac{\operatorname{Var} F}{V}\,\mathbb{E}\left|1-\frac{1}{\operatorname{Var} F}\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right|$$

for $V > 0$.

Proof. A straightforward computation shows that

$$\begin{aligned}&\mathbb{E}\left|1-\frac{1}{V}\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right| \\ &= \frac{\operatorname{Var} F}{V}\,\mathbb{E}\left|\frac{V}{\operatorname{Var} F}-\frac{1}{\operatorname{Var} F}\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right| \\ &\leq \frac{\operatorname{Var} F}{V}\left|1-\frac{V}{\operatorname{Var} F}\right| + \frac{\operatorname{Var} F}{V}\,\mathbb{E}\left|1-\frac{1}{\operatorname{Var} F}\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right|,\end{aligned}$$

which completes the proof. □

As a first application of Theorem 5.1 and Theorem 5.2, we bound the Wasserstein distance and the Kolmogorov distance between a Poisson random variable and a Gaussian random variable and see that we obtain the optimal rate of convergence for both distances. In Example 3.5 in [65], the bound (5.1) in Theorem 5.1 is used to compute the Wasserstein distance and the known optimal rate of convergence $t^{-\frac{1}{2}}$ is obtained.

Example 5.4 Let Y be a Poisson distributed random variable with $\mathbb{E}Y = t > 0$. It has the same distribution as $F_t = |\eta_t| = \sum_{x\in\eta_t} 1$, where η_t is a Poisson point process on $[0,1]$ with t times the restriction of the Lebesgue measure as intensity measure μ_t. The representation

$$I_{1,t}(f) = \sum_{x\in\eta_t} f(x) - \int_X f(x)\,\mathrm{d}\mu_t(x)$$

for a Wiener-Itô integral of a function $f \in L^1(\mu_t) \cap L^2(\mu_t)$ and the fact that

$$F_t = t\int_0^1 1\ \mathrm{d}x + \sum_{x\in\eta_t} 1 - t\int_0^1 1\,\mathrm{d}x$$

imply that F_t has the Wiener-Itô chaos expansion $F_t = \mathbb{E}F_t + I_{1,t}(f_{1,t}) = t + I_{1,t}(1)$. Hence, the standardized random variable

$$G_t = \frac{F_t - \mathbb{E}F_t}{\sqrt{\operatorname{Var} F_t}} = \frac{F_t - t}{\sqrt{t}}$$

has the chaos expansion $G_t = I_{1,t}(1)/\sqrt{t}$ and $D_z G_t = -D_z L^{-1} G_t = 1/\sqrt{t}$ for $z \in [0,1]$. It is easy to see that

$$\mathbb{E}|1 - \langle DG_t, -DL^{-1}G_t\rangle_{L^2(\mu_t)}| = |1 - \frac{1}{t}\langle 1,1\rangle_{L^2(\mu_t)}| = |1 - \frac{t}{t}| = 0,$$

and we obtain

$$\mathbb{E}\langle (DG_t)^2, (DL^{-1}G_t)^2\rangle_{L^2(\mu_t)} = \mathbb{E}\langle (DG_t)^2, (DG_t)^2\rangle_{L^2(\mu_t)} = \frac{1}{t},$$

$\mathbb{E}\langle (DG_t)^2, |DL^{-1}G_t|\rangle_{L^2(\mu_t)} = 1/\sqrt{t}$, $\mathbb{E}\langle DG_t, DG_t\rangle^2_{L^2(\mu_t)} = 1$, and $\mathbb{E}G_t^4 = 3 + 1/t$ by analogous computations. Since $D_z \mathbb{1}(G_t > s)\, D_z G_t\, |D_z L^{-1} G_t| \geq 0$ for $z \in [0,1]$ and $s \in \mathbb{R}$ and $D_z G_t\, |D_z L^{-1} G_t| = 1/t$ for $z \in [0,1]$, it follows from Lemma 4.7 and the Cauchy-Schwarz inequality that

$$\begin{aligned}\sup_{s\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}(G_t > s), DG_t\, |DL^{-1}G_t|\rangle_{L^2(\mu)} &= \sup_{s\in\mathbb{R}} \mathbb{E}\left[\mathbb{1}(G_t > s)\, \delta(DG_t\, |DL^{-1}G_t|)\right] \\ &\leq \mathbb{E}\left[\delta(DG_t\, |DL^{-1}G_t|)^2\right]^{\frac{1}{2}} \\ &= \frac{1}{t}\,\mathbb{E}[\delta(1)^2]^{\frac{1}{2}} = \frac{1}{t}\,\mathbb{E}[I_{1,t}(1)^2]^{\frac{1}{2}} = \frac{1}{\sqrt{t}}.\end{aligned}$$

Now Theorem 5.1 and Theorem 5.2 yield

$$d_W\left(\frac{Y-t}{\sqrt{t}}, N\right) \leq \frac{1}{\sqrt{t}}$$

and

$$d_K\left(\frac{Y-t}{\sqrt{t}}, N\right) \leq 2\left(\frac{1}{\sqrt{t}} + \left(3 + \frac{1}{t}\right)^{\frac{1}{4}} + 1\right)\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t}} \leq \frac{8}{\sqrt{t}}$$

for $t \geq 1$, which are the classical Berry-Esseen inequalities for the Wasserstein and the Kolmogorov distance with the optimal rates of convergence (up to constants).

For the Wasserstein distance the optimality of the rate follows from taking

$$h_t(x) = \min\{|x - (\lfloor\sqrt{t}x + t\rfloor - t)/\sqrt{t}|, |x - (\lceil\sqrt{t}x + t\rceil - t)/\sqrt{t}|\}$$

as test function. Since the distribution function of Y has a jump at $\lfloor t\rfloor$ of size $t^{\lfloor t\rfloor}e^{-t}/\lfloor t\rfloor!$, which is greater than a constant times $t^{-\frac{1}{2}}$ for $t \to \infty$, the rate for the Kolmogorov distance cannot be better than $t^{-\frac{1}{2}}$.

For the normal approximation of vectors of Poisson functionals we have the following result due to Peccati and Zheng (see [67, Theorem 4.2]).

Theorem 5.5 *Let $\boldsymbol{F} = (F^{(1)}, \dots, F^{(m)})$ with Poisson functionals $F^{(\ell)} \in L^2(\mathbb{P}_\eta)$ satisfying $\mathbb{E}F^{(\ell)} = 0$ and $F^{(\ell)} \in \operatorname{dom} D$ for $\ell = 1, \dots, m$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector with a positive semidefinite covariance matrix Σ. Then*

$$\begin{aligned}d_3(\boldsymbol{F}, \boldsymbol{N}(\Sigma)) \leq &\frac{1}{2}\sum_{u,v=1}^m \mathbb{E}|\sigma_{uv} - \langle DF^{(u)}, -DL^{-1}F^{(v)}\rangle_{L^2(\mu)}| \\ &+ \frac{1}{4}\int_X \mathbb{E}\left(\sum_{\ell=1}^m |D_z F^{(\ell)}|\right)^2 \sum_{\ell=1}^m |D_z L^{-1} F^{(\ell)}|\, \mathrm{d}\mu(z).\end{aligned}$$

This result is proven by a combination of Malliavin calculus and an interpolation technique. The authors of [67] start with the expression $|\mathbb{E}h(\mathbf{F})-\mathbb{E}h(\mathbf{N}(\Sigma))|$, where h is a thrice continuously differentiable function with bounded second and third derivatives as in the definition of the d_3-distance, and bound it by a direct computation using the boundedness of the derivatives of h and Malliavin calculus. They also derive a similar bound for the d_2-distance that is defined via slightly different test functions. It is shown by Malliavin calculus and multivariate Stein's method in a similar way as in the univariate case.

The bound for the d_3-distance has the advantage that the covariance matrix needs to be only positive semidefinite, whereas it has to be positive definite for the d_2-distance. In some of the examples from stochastic geometry in Chapter 7, the asymptotic covariance matrix is singular. For this reason, we only use the d_3-distance in the following. But all our abstract bounds for the normal approximation of vectors of Poisson functionals can be derived for the d_2-distance as well.

It would be desirable to have bounds for the multivariate Wasserstein distance (or even Kolmogorov distance), but the multivariate Stein's equation involves the Hessian, and for the interpolation method one also needs derivatives of higher order. Therefore, the test functions must have a higher regularity than in the univariate case.

In order to evaluate the right-hand sides of the bounds in Theorem 5.1, Theorem 5.2, and Theorem 5.5, we need to compute the expectations of products of Malliavin operators which is done in the following. We present four results that belong to different types of Poisson functionals and are later applied to examples from stochastic geometry in the Chapters 7, 8, and 9.

5.2 Normal approximation of asymptotic first order Poisson functionals

The underlying idea of our first result is that the bounds in Theorem 5.1 and in Theorem 5.5 are easier to evaluate if the Poisson functionals are first order Wiener-Itô integrals because then DF and $DL^{-1}F$ are deterministic functions. Hence, we approximate a Poisson functional by the first Wiener-Itô integral of its chaos expansion and approximate this integral by a standard Gaussian random variable. For this approach we need to control the error we obtain by ignoring the Wiener-Itô integrals of higher order.

Theorem 5.6 *a) Let $F \in L^2(\mathbb{P}_\eta)$ and let N be a standard Gaussian random variable. Then*

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq 2\sqrt{1-\frac{\|f_1\|_1^2}{\operatorname{Var} F}} + \frac{1}{(\operatorname{Var} F)^{3/2}} \int_X |f_1(z)|^3 \, \mathrm{d}\mu(z). \tag{5.8}$$

b) Let $\boldsymbol{F} = (F^{(1)}, \ldots, F^{(m)})$ be a vector of Poisson functionals $F^{(\ell)} \in L^2(\mathbb{P}_\eta)$ for $\ell = 1, \ldots, m$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector *with a positive semidefinite covariance matrix* Σ. *Then*

$$d_3(\boldsymbol{F}-\mathbb{E}\boldsymbol{F},\boldsymbol{N}(\Sigma)) \;\leq\; \frac{1}{2}\sum_{u,v=1}^{m}|\sigma_{uv}-\langle f_1^{(u)},f_1^{(v)}\rangle_{L^2(\mu)}|+\frac{m^2}{4}\sum_{\ell=1}^{m}\int_X |f_1^{(\ell)}(z)|^3\,\mathrm{d}\mu(z)$$
$$+\sqrt{2}m\sqrt{\sum_{\ell=1}^{m}\operatorname{Var}F^{(\ell)}}\sqrt{\sum_{\ell=1}^{m}\sum_{n=2}^{\infty}n!\,\|f_n^{(\ell)}\|_n^2}. \tag{5.9}$$

If F and $F^{(1)},\dots,F^{(m)}$ are first order Wiener-Itô integrals, Theorem 5.6 coincides with [65, Corollary 3.4] and [67, Corollary 4.3] that are direct consequences of the results denoted as Theorem 5.1 and Theorem 5.5 in this work. An example for the multivariate normal approximation of first order Wiener-Itô integrals is the paper [14] by Durastanti, Marinucci, and Peccati, where wavelet coefficients, occurring for example in astrophysics and cosmology, are investigated.

Theorem 5.6 is helpful to formulate central limit theorems for Poisson functionals F_t and vectors $\mathbf{F}_t=(F_t^{(1)},\dots,F_t^{(m)})$ of Poisson functionals that are asymptotically dominated by the first Wiener-Itô integral in their chaos expansion in the sense that

$$\lim_{t\to\infty}\operatorname{Var}F_t-\langle f_{1,t},f_{1,t}\rangle_{L^2(\mu_t)}=0$$

and

$$\lim_{t\to\infty}\operatorname{Cov}(F_t^{(u)},F_t^{(v)})-\langle f_{1,t}^{(u)},f_{1,t}^{(v)}\rangle_{L^2(\mu_t)}=0$$

for $u,v=1,\dots,m$. For a family of such Poisson functionals $(F_t)_{t\geq 1}$ with $\liminf_{t\to\infty}\operatorname{Var}F_t>0$ the first expression in formula (5.8) vanishes for $t\to\infty$. Since $\operatorname{Var}F_t\geq\|f_{1,t}\|_{1,t}^2$, the second summand is bounded by

$$\frac{1}{\|f_{1,t}\|_{1,t}^3}\int_X |f_{1,t}(z)|^3\,\mathrm{d}\mu(z)$$

so that we only have to deal with an expression depending on $f_{1,t}$. The assumption $\liminf_{t\to\infty}\operatorname{Var}F_t>0$ is necessary to prevent that $\lim_{t\to\infty}\operatorname{Var}F_t=0$.

In the multivariate case, the last expression on the right-hand side of formula (5.9) vanishes for $t\to\infty$, and only expressions involving $f_{1,t}^{(1)},\dots,f_{1,t}^{(m)}$ remain.

We prepare for the proof of Theorem 5.6 by the following Lemma:

Lemma 5.7 *a) For square integrable random variables* Y,Z *we have*

$$d_W(Y,Z)\leq\sqrt{\mathbb{E}(Y-Z)^2}.$$

b) Let Y *and* Z *be* m*-dimensional random vectors with* $\mathbb{E}Y=\mathbb{E}Z$ *and Euclidean norms* $||Y||$ *and* $||Z||$ *satisfying* $\mathbb{E}||Y||^2<\infty$ *and* $\mathbb{E}||Z||^2<\infty$. *Then*

$$d_3(Y,Z)\leq m\sqrt{\mathbb{E}||Y||^2+\mathbb{E}||Z||^2}\sqrt{\mathbb{E}||Y-Z||^2}.$$

Proof. It follows from the definition of the Wasserstein distance and the the Cauchy-Schwarz inequality that

$$d_W(Y,Z) = \sup_{h\in \mathrm{Lip}(1)} |\mathbb{E}h(Y) - \mathbb{E}h(Z)| \leq \mathbb{E}|Y-Z| \leq \sqrt{\mathbb{E}(Y-Z)^2},$$

which concludes the proof of part a).

Recall from Chapter 2 that $\mathcal{H}_m$ are the test functions for the d_3-distance. For $h \in \mathcal{H}_m$ and $Y=(Y_1,\dots,Y_m), Z=(Z_1,\dots,Z_m)$, we obtain by the mean value theorem

$$|\mathbb{E}h(Y) - \mathbb{E}h(Z)| = |\mathbb{E}[h'(W)(Y-Z)] - \mathbb{E}[h'(0)(Y-Z)]|\,,$$

where $W = Z + U(Y-Z)$ for some random variable U in $[0,1]$ and where we use that all components of $Y-Z$ have expectation zero. Applying the mean value theorem again as well as the Cauchy-Schwarz inequality yields

$$\begin{aligned}
|\mathbb{E}h(Y) - \mathbb{E}h(Z)| &= \left|\mathbb{E}\sum_{i=1}^m \left(\frac{\partial h(W)}{\partial u_i} - \frac{\partial h(0)}{\partial u_i}\right)(Y_i - Z_i)\right| \\
&= \left|\mathbb{E}\sum_{i=1}^m\sum_{j=1}^m \frac{\partial^2 h(\tilde{W}^{(i)})}{\partial u_j \partial u_i} W_j (Y_i - Z_i)\right| \\
&\leq \sqrt{\mathbb{E}\sum_{i=1}^m \left(\sum_{j=1}^m \frac{\partial^2 h(\tilde{W}^{(i)})}{\partial u_j \partial u_i} W_j\right)^2}\sqrt{\mathbb{E}||Y-Z||^2}
\end{aligned}$$

with random vectors $\tilde{W}^{(i)} = U_i W$ and random variables $U_i \in [0,1]$, $i=1,\dots,m$. Because of $h \in \mathcal{H}_m$ and the Cauchy-Schwarz inequality, it follows that

$$\mathbb{E}\sum_{i=1}^m \left(\sum_{j=1}^m \frac{\partial^2 h}{\partial u_j \partial u_i}(\tilde{W}^{(i)}) W_j\right)^2 \leq m^2\,\mathbb{E}||W||^2 \leq m^2\left(\mathbb{E}||Y||^2 + \mathbb{E}||Z||^2\right),$$

which completes the argument. □

Proof of Theorem 5.6: By the triangle inequality for the Wasserstein distance, we obtain

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathrm{Var}\,F}}, N\right) \leq d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathrm{Var}\,F}}, \frac{I_1(f_1)}{\sqrt{\mathrm{Var}\,F}}\right) + d_W\left(\frac{I_1(f_1)}{\sqrt{\mathrm{Var}\,F}}, N\right).$$

Now Lemma 5.7 a) and Theorem 4.2 imply that

$$\begin{aligned}
d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathrm{Var}\,F}}, \frac{I_1(f_1)}{\sqrt{\mathrm{Var}\,F}}\right) &\leq \sqrt{\frac{\mathbb{E}(F-\mathbb{E}F - I_1(f_1))^2}{\mathrm{Var}\,F}} \\
&= \sqrt{\frac{\sum_{n=2}^\infty n!\,\|f_n\|_n^2}{\mathrm{Var}\,F}} = \sqrt{1 - \frac{\|f_1\|_1^2}{\mathrm{Var}\,F}},
\end{aligned}$$

and it follows from Theorem 5.1 that

$$d_W\left(\frac{I_1(f_1)}{\sqrt{\operatorname{Var} F}}, N\right) \leq \left|1 - \frac{\langle f_1, f_1\rangle_{L^2(\mu)}}{\operatorname{Var} F}\right| + \frac{1}{(\operatorname{Var} F)^{3/2}} \int_X |f_1(z)|^3 \, \mathrm{d}\mu(z).$$

Combining these inequalities with $1 - \|f_1\|_1^2 / \operatorname{Var} F \leq 1$ concludes the proof of the bound (5.8).

The triangle inequality for the d_3-distance implies that

$$d_3(\mathbf{F} - \mathbb{E}\mathbf{F}, \mathbf{N}(\Sigma)) \leq d_3(\mathbf{F} - \mathbb{E}\mathbf{F}, \mathbf{G}) + d_3(\mathbf{G}, \mathbf{N}(\Sigma))$$

with $\mathbf{G} = (I_1(f_1^{(1)}), \dots, I_1(f_1^{(m)}))$. It follows from Lemma 5.7 b) and Theorem 4.2 that

$$\begin{aligned}
&d_3(\mathbf{F} - \mathbb{E}\mathbf{F}, \mathbf{G}) \\
&\leq m \sqrt{\sum_{\ell=1}^m E I_1(f_1^{(\ell)})^2 + \mathbb{E}(F^{(\ell)} - \mathbb{E}F^{(\ell)})^2} \sqrt{\sum_{\ell=1}^m \mathbb{E}\left(F^{(\ell)} - \mathbb{E}F^{(\ell)} - I_1(f_1^{(\ell)})\right)^2} \\
&\leq m \sqrt{2 \sum_{\ell=1}^m \operatorname{Var} F^{(\ell)}} \sqrt{\sum_{\ell=1}^m \sum_{n=2}^\infty n! \, \|f_n^{(\ell)}\|_n^2}.
\end{aligned}$$

We deduce from Theorem 5.5 and Jensen's inequality that

$$\begin{aligned}
d_3(\mathbf{G}, \mathbf{N}(\Sigma)) &\leq \frac{1}{2} \sum_{u,v=1}^m |\sigma_{uv} - \langle f_1^{(u)}, f_1^{(v)}\rangle_{L^2(\mu)}| + \frac{1}{4} \int_X \left(\sum_{\ell=1}^m |f_1^{(\ell)}(z)|\right)^2 \sum_{\ell=1}^m |f_1^{(\ell)}(z)| \, \mathrm{d}\mu(z) \\
&\leq \frac{1}{2} \sum_{u,v=1}^m |\sigma_{uv} - \langle f_1^{(u)}, f_1^{(v)}\rangle_{L^2(\mu)}| + \frac{m^2}{4} \sum_{\ell=1}^m \int_X |f_1^{(\ell)}(z)|^3 \, \mathrm{d}\mu(z).
\end{aligned}$$

This bound is also formulated in [67, Corollary 4.3]. Combining the inequalities above concludes the proof of the bound (5.9). □

If we rescale in Equation (5.8) with the square root of a constant $V > 0$ instead of the square root of $\operatorname{Var} F$, we obtain by the same arguments

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{V}}, N\right) \leq \sqrt{\frac{\operatorname{Var} F - \|f_1\|_1^2}{V}} + \left|1 - \frac{\|f_1\|_1^2}{V}\right| + \frac{1}{V^{3/2}} \int_X |f_1(z)|^3 \, \mathrm{d}\mu(z). \quad (5.10)$$

5.3 Normal approximation of Poisson functionals with finite Wiener-Itô chaos expansion

The aim of this section is to derive bounds for the normal approximation of Poisson functionals and vectors of Poisson functionals that have finite Wiener-Itô chaos expansions. This is done by evaluating the bounds in the Theorems 5.1, 5.2, and 5.5 and using some elementary inequalities and the product formula for multiple Wiener-Itô integrals. We begin with a proposition that is used in the proofs of the main results of this and the following section.

Proposition 5.8 *a) Let $F \in L^2(\mathbb{P}_\eta)$ be a Poisson functional with $F \in \operatorname{dom} D$ and let N be a standard Gaussian random variable. Then*

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq \sum_{i,j=1}^{\infty} i\, \frac{\sqrt{R_{ij}}}{\operatorname{Var} F} + \frac{1}{\operatorname{Var} F}\left(\int_X \mathbb{E}(D_z F)^4 \,\mathrm{d}\mu(z)\right)^{\frac{1}{2}}$$

with

$$\begin{aligned} R_{ij} &= \mathbb{E}\left(\int_X I_{i-1}(f_i(z,\cdot))\, I_{j-1}(f_j(z,\cdot))\,\mathrm{d}\mu(z)\right)^2 \\ &\quad -\left(\mathbb{E}\int_X I_{i-1}(f_i(z,\cdot))\, I_{j-1}(f_j(z,\cdot))\,\mathrm{d}\mu(z)\right)^2 \end{aligned}$$

for $i, j \in \mathbb{N}$.

b) Let $\boldsymbol{F} = (F^{(1)}, \ldots, F^{(m)})$ with Poisson functionals $F^{(\ell)} \in L^2(\mathbb{P}_\eta)$, $\ell = 1, \ldots, m$, satisfying $F^{(\ell)} \in \operatorname{dom} D$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector with a positive semidefinite covariance matrix Σ. Then

$$\begin{aligned} d_3\left(\boldsymbol{F} - \mathbb{E}\boldsymbol{F}, \boldsymbol{N}(\Sigma)\right) &\leq \frac{1}{2}\sum_{u,v=1}^{m} |\sigma_{uv} - \operatorname{Cov}(F^{(u)}, F^{(v)})| + \frac{1}{2}\sum_{u,v=1}^{m}\sum_{i,j=1}^{\infty} i\sqrt{R_{ij}^{(u,v)}} \\ &\quad + \frac{m}{4}\sum_{u,v=1}^{m}\sqrt{\operatorname{Var} F^{(u)}}\left(\int_X \mathbb{E}(D_z F^{(v)})^4\,\mathrm{d}\mu(z)\right)^{\frac{1}{2}} \end{aligned}$$

with

$$\begin{aligned} R_{ij}^{(u,v)} &= \mathbb{E}\left(\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{j-1}(f_j^{(v)}(z,\cdot))\,\mathrm{d}\mu(z)\right)^2 \\ &\quad -\left(\mathbb{E}\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{j-1}(f_j^{(v)}(z,\cdot))\,\mathrm{d}\mu(z)\right)^2 \end{aligned}$$

for $i, j \in \mathbb{N}$ and $u, v = 1, \ldots, m$.

Proof. We start with the univariate case. Combining Lemma 4.3 and Equation (4.7) with the representation $\operatorname{Var} F = \sum_{n=1}^{\infty} n!\, \|f_n\|_n^2$, we obtain

$$\begin{aligned} &\mathbb{E}\left|1 - \frac{1}{\operatorname{Var} F}\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right| \\ &= \frac{1}{\operatorname{Var} F}\,\mathbb{E}\left|\sum_{n=1}^{\infty} n!\,\|f_n\|_n^2 - \int_X \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(z,\cdot)) \sum_{n=1}^{\infty} I_{n-1}(f_n(z,\cdot))\,\mathrm{d}\mu(z)\right| \\ &\leq \sum_{i=1}^{\infty} i\ \mathbb{E}\left|\int_X I_{i-1}(f_i(z,\cdot))\, I_{i-1}(f_i(z,\cdot))\,\mathrm{d}\mu(z) - (i-1)!\,\|f_i\|_i^2\right| \\ &\quad + \sum_{i,j=1, i\neq j}^{\infty} i\ \mathbb{E}\left|\int_X I_{i-1}(f_i(z,\cdot))\, I_{j-1}(f_j(z,\cdot))\,\mathrm{d}\mu(z)\right|. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$\mathbb{E}\left|\int_X I_{i-1}(f_i(z,\cdot))\, I_{i-1}(f_i(z,\cdot))\,\mathrm{d}\mu(z) - (i-1)!\,\|f_i\|_i^2\right| \leq \sqrt{R_{ii}}$$

for $i \in \mathbb{N}$ and

$$\mathbb{E}\left|\int_X I_{i-1}(f_i(z,\cdot))\, I_{j-1}(f_j(z,\cdot))\,\mathrm{d}\mu(z)\right| \leq \sqrt{R_{ij}}$$

for $i, j \in \mathbb{N}$ with $i \neq j$. Here, we used the fact that

$$\mathbb{E}\int_X I_{i-1}(f_i(z,\cdot))\, I_{j-1}(f_j(z,\cdot))\,\mathrm{d}\mu(z) = \begin{cases} (i-1)!\,\|f_i\|_i^2, & i = j \\ 0, & i \neq j \end{cases},$$

which is a consequence of Fubini's theorem and the orthogonality of the multiple Wiener-Itô integrals (see Lemma 3.10). Combining the Cauchy-Schwarz inequality with

$$\int_X \mathbb{E}(D_z L^{-1}F)^2\,\mathrm{d}\mu(z) = \sum_{n=1}^{\infty}(n-1)!\,\|f_n\|_n^2 \leq \operatorname{Var} F,$$

we see that

$$\begin{aligned}
&\frac{1}{(\operatorname{Var} F)^{\frac{3}{2}}}\int_X \mathbb{E}(D_z F)^2\,|D_z L^{-1}F|\,\mathrm{d}\mu(z)\\
&\leq \frac{1}{(\operatorname{Var} F)^{\frac{3}{2}}}\left(\int_X \mathbb{E}(D_z F)^4\,\mathrm{d}\mu(z)\right)^{\frac{1}{2}}\left(\int_X \mathbb{E}(D_z L^{-1}F)^2\,\mathrm{d}\mu(z)\right)^{\frac{1}{2}}\\
&\leq \frac{1}{\operatorname{Var} F}\left(\int_X \mathbb{E}(D_z F)^4\,\mathrm{d}\mu(z)\right)^{\frac{1}{2}}.
\end{aligned}$$

Now part a) is a direct consequence of Theorem 5.1. In the multivariate case, we have

$$\begin{aligned}
&\mathbb{E}\left|\sigma_{uv} - \langle DF^{(u)}, -DL^{-1}F^{(v)}\rangle_{L^2(\mu)}\right|\\
&\leq |\sigma_{uv} - \operatorname{Cov}(F^{(u)}, F^{(v)})| + \mathbb{E}\left|\operatorname{Cov}(F^{(u)}, F^{(v)}) - \langle DF^{(u)}, -DL^{-1}F^{(v)}\rangle_{L^2(\mu)}\right|.
\end{aligned}$$

An analogous computation as in the univariate case yields

$$\mathbb{E}\left|\operatorname{Cov}(F^{(u)}, F^{(v)}) - \langle DF^{(u)}, -DL^{-1}F^{(v)}\rangle_{L^2(\mu)}\right| \leq \sum_{i,j=1}^{\infty} i\sqrt{R_{ij}^{(u,v)}}.$$

By the Cauchy-Schwarz inequality and the same arguments as in the univariate case, we obtain

$$\begin{aligned}
&\int_X \mathbb{E}\left(\sum_{\ell=1}^{m}|D_z F^{(\ell)}|\right)^2 \sum_{\ell=1}^{m}|D_z L^{-1}F^{(\ell)}|\,\mathrm{d}\mu(z)\\
&\leq m\sum_{u,v=1}^{m}\mathbb{E}\int_X |D_z F^{(u)}|^2\,|D_z L^{-1}F^{(v)}|\,\mathrm{d}\mu(z)\\
&\leq m\sum_{u,v=1}\sqrt{\operatorname{Var} F^{(u)}}\left(\int_X \mathbb{E}(D_z F^{(v)})^4\,\mathrm{d}\mu(z)\right)^{\frac{1}{2}},
\end{aligned}$$

and Theorem 5.5 concludes the proof. □

In order to further evaluate R_{ij} and $R_{ij}^{(u,v)}$, we need the following classes of partitions from $\widetilde{\Pi}_{\geq 2}(i,i,j,j)$:

Definition 5.9 *For $i,j \in \mathbb{N}$ let $\Pi_{\geq 2}^{(1)}(i,i,j,j)$ (resp. $\widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)$) be the set of all partitions $\sigma \in \Pi_{\geq 2}(i,i,j,j)$ (resp. $\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)$) such that $\{x_1^{(1)}, x_1^{(3)}\}$ and $\{x_1^{(2)}, x_1^{(4)}\}$ are blocks of σ.*

Using this notation, we can compute the expectations in R_{ij} and $R_{ij}^{(u,v)}$.

Lemma 5.10 *Let $i,j \in \mathbb{N}$ and $1 \leq u,v \leq m$.*

a) If

$$\int_{X^{|\sigma|}} |(f_\ell \otimes f_\ell \otimes f_\ell \otimes f_\ell)_\sigma| \,\mathrm{d}\mu^{|\sigma|} < \infty \quad \text{for all} \quad \sigma \in \Pi_{\geq 2}^{(1)}(\ell,\ell,\ell,\ell) \tag{5.11}$$

for $\ell \in \{i,j\}$, then

$$R_{ij} = \sum_{\sigma \in \widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)} \int_{X^{|\sigma|}} (f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma \,\mathrm{d}\mu^{|\sigma|}.$$

b) We assume that

$$\int_{X^{|\sigma|}} |(f_i^{(u)} \otimes f_i^{(u)} \otimes f_i^{(u)} \otimes f_i^{(u)})_\sigma| \,\mathrm{d}\mu^{|\sigma|} < \infty \quad \text{for all} \quad \sigma \in \Pi_{\geq 2}^{(1)}(i,i,i,i) \tag{5.12}$$

and that

$$\int_{X^{|\sigma|}} |(f_j^{(v)} \otimes f_j^{(v)} \otimes f_j^{(v)} \otimes f_j^{(v)})_\sigma| \,\mathrm{d}\mu^{|\sigma|} < \infty \quad \text{for all} \quad \sigma \in \Pi_{\geq 2}^{(1)}(j,j,j,j). \tag{5.13}$$

Then

$$R_{ij}^{(u,v)} = \sum_{\sigma \in \widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)} \int_{X^{|\sigma|}} (f_i^{(u)} \otimes f_i^{(u)} \otimes f_j^{(v)} \otimes f_j^{(v)})_\sigma \,\mathrm{d}\mu^{|\sigma|}.$$

Proof. It is sufficient to prove part b) since part a) is the special case $m = 1$. We start with $i = j$ and $u = v$ and obtain, by Fubini's theorem,

$$\begin{aligned}
&\mathbb{E}\left(\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{i-1}(f_i^{(u)}(z,\cdot)) \,\mathrm{d}\mu(z)\right)^2 \\
&= \int_{X^2} \mathbb{E} I_{i-1}(f_i^{(u)}(s,\cdot))\, I_{i-1}(f_i^{(u)}(t,\cdot))\, I_{i-1}(f_i^{(u)}(s,\cdot))\, I_{i-1}(f_i^{(u)}(t,\cdot)) \,\mathrm{d}\mu(s,t).
\end{aligned} \tag{5.14}$$

In the general case, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\mathbb{E}\left(\int_X |I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{j-1}(f_j^{(v)}(z,\cdot))| \,\mathrm{d}\mu(z)\right)^2 \\
&\le \left(\mathbb{E}\left(\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{i-1}(f_i^{(u)}(z,\cdot)) \,\mathrm{d}\mu(z)\right)^2\right)^{\frac{1}{2}} \\
&\quad \left(\mathbb{E}\left(\int_X I_{j-1}(f_j^{(v)}(z,\cdot))\, I_{j-1}(f_j^{(v)}(z,\cdot)) \,\mathrm{d}\mu(z)\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Under the assumption that the right-hand side in formula (5.14) is finite (this follows from our computations in the following), we can apply Fubini's theorem and obtain

$$\begin{aligned}
&\mathbb{E}\left(\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{j-1}(f_j^{(v)}(z,\cdot)) \,\mathrm{d}\mu(z)\right)^2 \\
&= \int_{X^2} \mathbb{E} I_{i-1}(f_i^{(u)}(s,\cdot))\, I_{i-1}(f_i^{(u)}(t,\cdot))\, I_{j-1}(f_j^{(v)}(s,\cdot))\, I_{j-1}(f_j^{(v)}(t,\cdot)) \,\mathrm{d}\mu(s,t).
\end{aligned} \tag{5.15}$$

For each $\tau \in \Pi(i-1,i-1)$ and $J \subset \tau \setminus S(\tau)$, we can construct a partition $\widetilde{\sigma} \in \Pi_{\ge 2}(i-1,i-1,i-1,i-1)$ by taking two copies of τ and merging all pairs of blocks that do not belong to J. By adding the blocks for the variables s and t, we obtain a partition $\sigma \in \Pi^{(1)}_{\ge 2}(i,i,i,i)$. Thus, we have

$$\begin{aligned}
&\int_{X^2}\int_{X^{|\tau|-|J|}} \left(\int_{X^{|J|}} |(f_i^{(u)}(s,\cdot)\otimes f_i^{(u)}(t,\cdot))_\tau| \,\mathrm{d}\mu_J\right)^2 \mathrm{d}\mu^{|\tau|-|J|}\,\mathrm{d}\mu(s,t) \\
&= \int_{X^2}\int_{X^{|\widetilde{\sigma}|}} |(f_i^{(u)}(s,\cdot)\otimes f_i^{(u)}(t,\cdot)\otimes f_i^{(u)}(s,\cdot)\otimes f_i^{(u)}(t,\cdot))_{\widetilde{\sigma}}| \,\mathrm{d}\mu^{|\widetilde{\sigma}|}\,\mathrm{d}\mu(s,t) \\
&= \int_{X^{|\sigma|}} |(f_i^{(u)}\otimes f_i^{(u)}\otimes f_i^{(u)}\otimes f_i^{(u)})_\sigma| \,\mathrm{d}\mu^{|\sigma|}.
\end{aligned} \tag{5.16}$$

By the assumptions (5.12) and (5.13) (respectively (5.11) in the univariate case), the right-hand side of formula (5.16) is finite so that

$$\int_{X^{|J|}} |(f_i^{(u)}(s,\cdot)\otimes f_i^{(u)}(t,\cdot))_\tau| \,\mathrm{d}\mu_J \in L^2_s(\mu^{|\tau|-|J|})$$

for μ-almost all $(s,t) \in X^2$, and the same holds if we replace i and u by j and v. This allows us to apply Corollary 3.14 to the right-hand side of formula (5.15), which yields

$$\begin{aligned}
&\mathbb{E}\left(\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{j-1}(f_j^{(v)}(z,\cdot)) \,\mathrm{d}\mu(z)\right)^2 \\
&= \sum_{\widetilde{\sigma}\in\Pi_{\ge 2}(i-1,i-1,j-1,j-1)} \int_{X^2}\int_{X^{|\widetilde{\sigma}|}} (f_i^{(u)}(s,\cdot)\otimes f_i^{(u)}(t,\cdot)\otimes f_j^{(v)}(s,\cdot)\otimes f_j^{(v)}(t,\cdot))_{\widetilde{\sigma}} \,\mathrm{d}\mu^{|\widetilde{\sigma}|}\,\mathrm{d}\mu(s,t) \\
&= \sum_{\sigma\in\Pi^{(1)}_{\ge 2}(i,i,j,j)} \int_{X^{|\sigma|}} (f_i^{(u)}\otimes f_i^{(u)}\otimes f_j^{(v)}\otimes f_j^{(v)})_\sigma \,\mathrm{d}\mu^{|\sigma|}.
\end{aligned}$$

Now we discuss the cases $i=j$ and $i\neq j$ separately. By the definitions of the partitions, a partition σ is in $\Pi_{\geq 2}^{(1)}(i,i,i,i)\setminus\widetilde{\Pi}_{\geq 2}^{(1)}(i,i,i,i)$ if and only if every block of σ contains either two variables belonging to the first and the third function or to the second and fourth function of the tensor product. For each of these partitions the integral in the sum above equals $\langle f_i^{(u)}, f_i^{(v)}\rangle_{L^2(\mu^i)}^2$. Since there are $((i-1)!)^2$ such partitions, the sum over these partitions cancels out with

$$-\left(\mathbb{E}\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{i-1}(f_i^{(v)}(z,\cdot))\,\mathrm{d}\mu(z)\right)^2=-\left((i-1)!\,\langle f_i^{(u)}, f_i^{(v)}\rangle_{L^2(\mu^i)}\right)^2.$$

For $i\neq j$ we have $\Pi_{\geq 2}^{(1)}(i,i,j,j)=\widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)$ since there are no partitions where all blocks have two elements that belong either to the first and third or to the second and fourth function of the tensor product. Moreover, Fubini's theorem and the orthogonality of multiple Wiener-Itô integrals (see Lemma 3.10) imply that

$$\left(\mathbb{E}\int_X I_{i-1}(f_i^{(u)}(z,\cdot))\, I_{j-1}(f_j^{(v)}(z,\cdot))\,\mathrm{d}\mu(z)\right)^2=0.$$

Altogether, we obtain

$$R_{ij}^{(u,v)}=\sum_{\sigma\in\widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)}\int_{X^{|\sigma|}}(f_i^{(u)}\otimes f_i^{(u)}\otimes f_j^{(v)}\otimes f_j^{(v)})_\sigma\,\mathrm{d}\mu^{|\sigma|},$$

which completes the proof. □

As pointed out in the previous proof, $\Pi_{\geq 2}^{(1)}(\ell,\ell,\ell,\ell)\setminus\widetilde{\Pi}_{\geq 2}^{(1)}(\ell,\ell,\ell,\ell)$ is the set of all partitions such that each block has size two and consists either of variables from the first and third function or from the second and fourth function. In this case the integral in assumption (5.11) equals $\|f_\ell\|_\ell^4$. Hence, we can replace $\sigma\in\Pi_{\geq 2}^{(1)}(\ell,\ell,\ell,\ell)$ by $\sigma\in\widetilde{\Pi}_{\geq 2}^{(1)}(\ell,\ell,\ell,\ell)$ in assumption (5.11) if $f_\ell\in L_s^2(\mu^\ell)$. The same holds for the assumptions (5.12) and (5.13).

Proposition 5.8 and Lemma 5.10 allow us to prove bounds for the normal approximation of Poisson functionals that are applied in the next chapters. We begin with the situation that the Poisson functionals have finite Wiener-Itô chaos expansions.

Theorem 5.11 *a) Let $F\in L^2(\mathbb{P}_\eta)$ be a Poisson functional with a finite Wiener-Itô chaos expansion of order $k\in\mathbb{N}$ such that*

$$\int_{X^{|\sigma|}}|(f_n\otimes f_n\otimes f_n\otimes f_n)_\sigma|\,\mathrm{d}\mu^{|\sigma|}<\infty\quad\textit{for all}\quad\sigma\in\widetilde{\Pi}_{\geq 2}(n,n,n,n)$$

for $n=1,\dots,k$ and let N be a standard Gaussian random variable. Then

$$\begin{aligned} & d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \\ & \leq \frac{1}{\operatorname{Var} F}\sum_{i,j=1}^{k} i\sqrt{\sum_{\sigma\in\widetilde{\Pi}^{(1)}_{\geq 2}(i,i,j,j)}\int_{X^{|\sigma|}}(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma \,\mathrm{d}\mu^{|\sigma|}} \\ & \quad + \frac{1}{\operatorname{Var} F}\sqrt{\int_X \mathbb{E}(D_zF)^4\,\mathrm{d}\mu(z)} \\ & \leq \frac{2k^{\frac{7}{2}}}{\operatorname{Var} F}\sum_{1\leq i\leq j\leq k}\sqrt{\sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)}\int_{X^{|\sigma|}}|(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma|\,\mathrm{d}\mu^{|\sigma|}}. \end{aligned} \tag{5.17}$$

b) Let $\boldsymbol{F}=(F^{(1)},\dots,F^{(m)})$ with Poisson functionals $F^{(\ell)}\in L^2(\mathbb{P}_\eta)$, $\ell=1,\dots,m$, that have finite Wiener-Itô chaos expansions of order $k_\ell\in\mathbb{N}$ and satisfy

$$\int_{X^{|\sigma|}}|(f_n^{(\ell)}\otimes f_n^{(\ell)}\otimes f_n^{(\ell)}\otimes f_n^{(\ell)})_\sigma|\,\mathrm{d}\mu^{|\sigma|}<\infty \quad \text{for all} \quad \sigma\in\widetilde{\Pi}_{\geq 2}(n,n,n,n)$$

for $n=1,\dots,k_\ell$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector with a positive semidefinite covariance matrix Σ. Then

$$d_3(\boldsymbol{F}-\mathbb{E}\boldsymbol{F},\boldsymbol{N}(\Sigma))\leq \frac{1}{2}\sum_{u,v=1}^{m}|\sigma_{uv}-\operatorname{Cov}(F^{(u)},F^{(v)})|+R(\boldsymbol{F})$$

with

$$\begin{aligned} R(\boldsymbol{F}) &= \frac{1}{2}\sum_{u,v=1}^{m}\sum_{i=1}^{k_u}\sum_{j=1}^{k_v} i\sqrt{\sum_{\sigma\in\widetilde{\Pi}^{(1)}_{\geq 2}(i,i,j,j)}\int_{X^{|\sigma|}}(f_i^{(u)}\otimes f_i^{(u)}\otimes f_j^{(v)}\otimes f_j^{(v)})_\sigma\,\mathrm{d}\mu^{|\sigma|}} \\ & \quad +\frac{m}{4}\sum_{u,v=1}^{m}\sqrt{\operatorname{Var} F^{(u)}}\left(\int_X \mathbb{E}(D_zF^{(v)})^4\,\mathrm{d}\mu(z)\right)^{\frac{1}{2}} \\ &\leq \frac{m}{2}\left(\sum_{\ell=1}^{m}\sqrt{\operatorname{Var} F^{(\ell)}}+1\right) \\ & \quad \sum_{u,v=1}^{m}\sum_{i=1}^{k_u}\sum_{j=1}^{k_v}k_u^{\frac{7}{2}}\sqrt{\sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)}\int_{X^{|\sigma|}}|(f_i^{(u)}\otimes f_i^{(u)}\otimes f_j^{(v)}\otimes f_j^{(v)})_\sigma|\,\mathrm{d}\mu^{|\sigma|}}. \end{aligned}$$

Proof. It follows from Theorem 4.2 that $\operatorname{Var} F=\sum_{n=1}^{k} n!\,\|f_n\|_n^2<\infty$. Now it is easy to see that

$$\sum_{n=1}^{k} n\, n!\,\|f_n\|_n^2<\infty$$

so that $F\in\operatorname{dom} D$. The same argument holds for $F^{(\ell)}$. Hence, we can apply Proposition 5.8. Combining this with Lemma 5.10 for the computation of R_{ij} and $R_{ij}^{(u,v)}$ yields the first bounds in part a) and part b).

It follows from the representation of the difference operator in Lemma 4.3 and Jensen's inequality that

$$\begin{aligned}\int_X \mathbb{E}(D_zF)^4 \,\mathrm{d}\mu(z) &= \int_X \mathbb{E}\left(\sum_{n=1}^{k} n\, I_{n-1}(f_n(z,\cdot))\right)^4 \mathrm{d}\mu(z)\\ &\leq k^3 \sum_{n=1}^{k} n^4 \int_X \mathbb{E}I_{n-1}(f_n(z,\cdot))^4 \,\mathrm{d}\mu(z).\end{aligned} \tag{5.18}$$

For fixed $\tau \in \Pi(n-1,n-1)$ and $J \subset \tau \setminus S(\tau)$ we can construct a partition $\widetilde{\sigma} \in \Pi_{\geq 2}(n-1,n-1,n-1,n-1)$ by taking two copies of τ and merging all copies of blocks that are not in J so that

$$\begin{aligned}&\int_{X^{|\tau|-|J|}} \left(\int_{X^{|J|}} |(f_n(z,\cdot)\otimes f_n(z,\cdot))_\tau| \,\mathrm{d}\mu_J\right)^2 \mathrm{d}\mu^{|\tau|-|J|}\\ &= \int_{X^{|\widetilde{\sigma}|}} |(f_n(z,\cdot)\otimes f_n(z,\cdot)\otimes f_n(z,\cdot)\otimes f_n(z,\cdot))_{\widetilde{\sigma}}| \,\mathrm{d}\mu^{|\widetilde{\sigma}|}.\end{aligned} \tag{5.19}$$

If we integrate over z, we can add the variable z to $\widetilde{\sigma}$ and obtain a partition $\sigma \in \widetilde{\Pi}_{\geq 2}(n,n,n,n)$. Due to the assumptions of part a), this integral is finite so that the right-hand side of formula (5.19) is finite for μ-almost all $z \in X$. This allows us to apply Corollary 3.14 on the right-hand side of formula (5.18), which yields

$$\begin{aligned}&\int_X \mathbb{E}I_{n-1}(f_n(z,\cdot))^4 \,\mathrm{d}\mu(z)\\ &= \int_X \sum_{\widetilde{\sigma}\in\Pi_{\geq 2}(n-1,n-1,n-1,n-1)} (f_n(z,\cdot)\otimes f_n(z,\cdot)\otimes f_n(z,\cdot)\otimes f_n(z,\cdot))_{\widetilde{\sigma}} \,\mathrm{d}\mu(z).\end{aligned}$$

Including the integration with respect to z into the notation, we obtain the bound

$$\begin{aligned}\int_X \mathbb{E}(D_zF)^4 \,\mathrm{d}\mu(z) &\leq k^3 \sum_{n=1}^{k} n^4 \sum_{\substack{\sigma\in\widetilde{\Pi}_{\geq 2}(n,n,n,n)\\ \{x_1^{(1)},x_1^{(2)},x_1^{(3)},x_1^{(4)}\}\in\sigma}} \int_{X^{|\sigma|}} (f_n\otimes f_n\otimes f_n\otimes f_n)_\sigma \,\mathrm{d}\mu^{|\sigma|}\\ &\leq k^3 \sum_{n=1}^{k} n^4 \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(n,n,n,n)} \int_{X^{|\sigma|}} |(f_n\otimes f_n\otimes f_n\otimes f_n)_\sigma| \,\mathrm{d}\mu^{|\sigma|}.\end{aligned} \tag{5.20}$$

In the multivariate case, we can use the same upper bound. Combining the first inequalities in part a) and part b) with the bound (5.20) leads to the second bounds in the univariate and in the multivariate case. □

Note that a similar bound as the second inequality in formula (5.17) even holds without absolute values inside the integrals, i.e.

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var}F}},N\right) \leq \frac{k}{\operatorname{Var}F}\sum_{i,j=1}^{k}\sqrt{\sum_{\sigma\in\widetilde{\Pi}^{(1)}_{\geq 2}(i,i,j,j)}\int_{X^{|\sigma|}}(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma \,\mathrm{d}\mu^{|\sigma|}} + \frac{k^{\frac{7}{2}}}{\operatorname{Var}F}\sum_{i=1}^{k}\sqrt{\sum_{\substack{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,i,i)\\ \{x_1^{(1)},x_1^{(2)},x_1^{(3)},x_1^{(4)}\}\in\sigma}}\int_{X^{|\sigma|}}(f_i\otimes f_i\otimes f_i\otimes f_i)_\sigma \,\mathrm{d}\mu^{|\sigma|}}. \tag{5.21}$$

This follows from the inequality (5.20). For the same reason, one has a similar bound without absolute values for $R(\boldsymbol{F})$ in part b) of Theorem 5.11. For the applications that are considered in the present work the bounds with the absolute values are sufficient. Moreover, it can happen that only upper bounds for the absolute values of the kernels of the Wiener-Itô chaos expansion of F are available and no exact formulas that can be used to evaluate the integrals in formula (5.21).

Lachièze-Rey and Peccati derive a similar bound as in formula (5.17) and in formula (5.21) in [39, Theorem 3.5]. In their formulation, the bound depends on L^2-norms of the contractions $f_i \star_r^\ell f_j$. The contraction $f_i \star_r^\ell f_j : X^{i+j-r-\ell} \to \overline{\mathbb{R}}$ for $0 \leq \ell \leq r$ and $1 \leq r \leq \min\{i,j\}$ is defined by replacing r variables of f_i and f_j by new common variables and integrating over ℓ of the new variables. Then we have

$$\|f_i \star_r^\ell f_j\|_{i+j-r-\ell}^2 = \int_{X^{|\sigma|}} (f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma \,\mathrm{d}\mu^{|\sigma|}$$

with a partition $\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)$ if not $\ell=r=i=j$ (such contractions do not occur in [39, Theorem 3.5]). On the other hand, the Cauchy-Schwarz inequality implies that every integral

$$\int_{X^{|\sigma|}} (f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma \,\mathrm{d}\mu^{|\sigma|}$$

with a partition $\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)$ can be bounded by the product of two L^2-norms of contraction operators. This means that the bounds in formula (5.21) and in [39] are equivalent up to constants.

In [39, Proposition 3.9], it is shown that

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var}F}},N\right) \leq c_{W,k}\sqrt{\frac{\mathbb{E}(F-\mathbb{E}F)^4}{(\operatorname{Var}F)^2}-3} \tag{5.22}$$

with a constant $c_{W,k}$ only depending on k if the kernels f_n, $n=1,\dots,k$, are non-negative. This holds since the same integrals as in formula (5.17) also occur in the fourth centred moment that can be computed by Corollary 3.14. In case of negative kernels, this argument fails because some of the integrals in the centred fourth moment could be negative and cancel out with positive integrals.

The bound (5.22) is helpful since it implies that for a family of Poisson functionals $(F_t)_{t\geq 1}$ with Wiener-Itô chaos expansions of order k and non-negative kernels the standardizations $(F_t - \mathbb{E}F_t)/\sqrt{\operatorname{Var} F_t}$ converge in distribution to a standard Gaussian random variable if

$$\frac{\mathbb{E}(F_t - \mathbb{E}F_t)^4}{(\operatorname{Var} F_t)^2} \to 3 \quad \text{as} \quad t \to \infty.$$

For a similar fourth moment criterion for multiple Wiener-Itô integrals with respect to a Gaussian measure we refer to the work [60] by Nualart and Peccati.

Combining Corollary 5.3 with the proof of the previous theorem, we see that

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{V}}, N\right) \leq \left|1 - \frac{\operatorname{Var} F}{V}\right| + \left(1 + \sqrt{\frac{\operatorname{Var} F}{V}}\right) \frac{\operatorname{Var} F}{V} B_W \tag{5.23}$$

for $V > 0$, where B_W is the right-hand side of formula (5.17). In case that $V = \operatorname{Var} F$, the right-hand side reduces to $2B_W$. The constant 2 is due to an upper estimate we used to simplify the expression.

Our next theorem is the counterpart of Theorem 5.11 a) for the Kolmogorov distance:

Theorem 5.12 *Let $F \in L^2(\mathbb{P}_\eta)$ be a Poisson functional with a finite Wiener-Itô chaos expansion of order k such that $f_n \in L^1_s(\mu^n)$ for $n = 1, \dots, k$ and*

$$\int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_i \otimes f_i)_\sigma| \, \mathrm{d}\mu^{|\sigma|} < \infty$$

for all $\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,i,i)$ and $i = 1, \dots, k$ and let N be a standard Gaussian random variable. Then

$$\begin{aligned} & d_K\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \\ & \leq \frac{17k^5}{\operatorname{Var} F} \sum_{i,j=1}^{k} \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, \mathrm{d}\mu^{|\sigma|}} \\ & \quad + \frac{1}{\operatorname{Var} F} \sup_{t \in \mathbb{R}} \mathbb{E}\langle D\mathbb{1}(F > t), DF \, |DL^{-1}(F - \mathbb{E}F)| \rangle_{L^2(\mu)} \\ & \leq \frac{2^k \sqrt{2k}\, k^2 + 17k^5}{\operatorname{Var} F} \sum_{i,j,\ell=1}^{k} \sqrt{\sum_{\sigma \in \widetilde{\Pi}(i,i,j,\ell)} \int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_\ell)_\sigma| \, \mathrm{d}\mu^{|\sigma|}}. \end{aligned} \tag{5.24}$$

Proof. Throughout this proof, we put

$$A = \left(\sum_{i,j=1}^{k} \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, \mathrm{d}\mu^{|\sigma|}}\right)^2.$$

Without loss of generality we can assume that $\mathbb{E}F = 0$. By the same reasoning as in the proofs of Proposition 5.8 and Lemma 5.10, we obtain

$$\begin{aligned} & \mathbb{E}|1 - \frac{1}{\operatorname{Var} F} \langle DF, -DL^{-1}F \rangle_{L^2(\mu)}| \\ & \leq \frac{k}{\operatorname{Var} F} \sum_{i,j=1}^{k} \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, \mathrm{d}\mu^{|\sigma|}} = \frac{k\sqrt{A}}{\operatorname{Var} F}. \end{aligned} \tag{5.25}$$

It follows from Fubini's theorem, Jensen's inequality, and the product formula in Corollary 3.14 that

$$
\begin{aligned}
&\mathbb{E}\langle (DF)^2, (DL^{-1}F)^2\rangle_{L^2(\mu)} \\
&= \int \mathbb{E}\left(\sum_{i=1}^{k} i\ I_{i-1}(f_i(z,\cdot))\right)^2 \left(\sum_{j=1}^{k} I_{j-1}(f_j(z,\cdot))\right)^2 \mathrm{d}\mu(z) \\
&\leq k^4 \int_X \mathbb{E}\sum_{i=1}^{k} I_{i-1}(f_i(z,\cdot))^2 \sum_{j=1}^{k} I_{j-1}(f_j(z,\cdot))^2 \,\mathrm{d}\mu(z) \\
&\leq k^4 \sum_{i,j=1}^{k} \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma| \,\mathrm{d}\mu^{|\sigma|} \leq k^4 A,
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
\mathbb{E}\langle (DF)^2, (DF)^2\rangle_{L^2(\mu)} &= \int \mathbb{E}\left(\sum_{i=1}^{k} i\ I_{i-1}(f_i(z,\cdot))\right)^2 \left(\sum_{j=1}^{k} j\ I_{j-1}(f_j(z,\cdot))\right)^2 \mathrm{d}\mu(z) \\
&\leq k^6 \int_X \mathbb{E}\sum_{i=1}^{k} I_{i-1}(f_i(z,\cdot))^2 \sum_{j=1}^{k} I_{j-1}(f_j(z,\cdot))^2 \,\mathrm{d}\mu(z) \\
&\leq k^6 \sum_{i,j=1}^{k} \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma| \,\mathrm{d}\mu^{|\sigma|} \leq k^6 A,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\langle DF, DF\rangle_{L^2(\mu)}^2 \\
&= \mathbb{E}\int_{X^2} \left(\sum_{i=1}^{k} i\ I_{i-1}(f_i(s,\cdot))\right)^2 \left(\sum_{j=1}^{k} j\ I_{j-1}(f_j(t,\cdot))\right)^2 \mathrm{d}\mu(s,t) \\
&\leq k^2 \int_{X^2} \mathbb{E}\sum_{i=1}^{k} i^2\ I_{i-1}(f_i(s,\cdot))^2 \sum_{j=1}^{k} j^2\ I_{j-1}(f_j(t,\cdot))^2 \,\mathrm{d}\mu(s,t) \\
&\leq k^2 \sum_{i,j=1}^{k} i^2 j^2 \\
&\qquad \sum_{\sigma\in\Pi_{\geq 2}(i-1,i-1,j-1,j-1)} \int_{X^2}\int_{X^{|\sigma|}} |(f_i(s,\cdot)\otimes f_i(s,\cdot)\otimes f_j(t,\cdot)\otimes f_j(t,\cdot))_\sigma| \,\mathrm{d}\mu^{|\sigma|}\,\mathrm{d}\mu(s,t) \\
&\leq k^6 \sum_{i,j=1}^{k} \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma| \,\mathrm{d}\mu^{|\sigma|} + k^4 \sum_{i,j=1}^{k} i!\,\|f_i\|_i^2\ j!\,\|f_j\|_j^2 \\
&\leq k^6 A + k^4 (\operatorname{Var} F)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}F^4 &= \mathbb{E}\left(\sum_{i=1}^k I_i(f_i)\right)^2 \left(\sum_{j=1}^k I_j(f_j)\right)^2 \leq k^2\, \mathbb{E}\sum_{i=1}^k I_i(f_i)^2 \sum_{j=1}^k I_j(f_j)^2 \\
&\leq k^2 \sum_{i,j=1}^k \sum_{\sigma\in\Pi_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma| \,\mathrm{d}\mu^{|\sigma|} \\
&\leq k^2 \sum_{i,j=1}^k \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma| \,\mathrm{d}\mu^{|\sigma|} + 3k^2 \sum_{i,j=1}^k i!\,\|f_i\|_i^2\; j!\,\|f_j\|_j^2 \\
&\leq k^2 A + 3k^2 (\operatorname{Var} F)^2.
\end{aligned}
$$

Combining the last three inequalities, we see that

$$
\begin{aligned}
&2\,\frac{\sqrt{\mathbb{E}\langle (DF)^2,(DF)^2\rangle_{L^2(\mu)}}}{\operatorname{Var}F} + 2\,\frac{\left(\mathbb{E}\langle DF,DF\rangle_{L^2(\mu)}^2\right)^{\frac14}}{\sqrt{\operatorname{Var}F}} \left(\frac{(\mathbb{E}F^4)^{\frac14}}{\sqrt{\operatorname{Var}F}}+1\right) \\
&\leq 2\,\frac{k^3\sqrt{A}}{\operatorname{Var}F} + 2\left(\frac{k^{\frac32}A^{\frac14}}{\sqrt{\operatorname{Var}F}}+k\right)\left(\frac{\sqrt{k}A^{\frac14}}{\sqrt{\operatorname{Var}F}}+3^{\frac14}\sqrt{k}+1\right) \leq 16k^3
\end{aligned}
$$

for $\sqrt{A}/\operatorname{Var}F \leq 1$. Together with the inequalities (5.25) and (5.26) and Theorem 5.2, we obtain

$$
\begin{aligned}
&d_K\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var}F}},N\right) \\
&\leq \frac{k\sqrt{A}}{\operatorname{Var}F} + 16k^3\,\frac{k^2\sqrt{A}}{\operatorname{Var}F} \\
&\quad + \frac{1}{\operatorname{Var}F}\sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}\left((F-\mathbb{E}F)/\sqrt{\operatorname{Var}F}>t\right), DF\,|DL^{-1}F|\rangle_{L^2(\mu)} \\
&\leq \frac{17k^5\sqrt{A}}{\operatorname{Var}F} + \frac{1}{\operatorname{Var}F}\sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}\,(F>t)\,, DF\,|DL^{-1}F|\rangle_{L^2(\mu)}
\end{aligned}
\tag{5.27}
$$

for $\sqrt{A}/\operatorname{Var}F \leq 1$. Otherwise, the right-hand side is still an upper bound since the Kolmogorov distance is by definition at most 1.

For the second inequality we evaluate the expression with the supremum in formula (5.27). Here, we can assume that all integrals occurring in the second bound in formula (5.24) are finite since the the inequality is obviously true otherwise. Because of the assumption $f_n \in L_s^1(\mu^n)$ for $n=1,\dots,k$, Corollary 4.12 implies that

$$
F = \sum_{\ell=1}^k \sum_{(x_1,\dots,x_\ell)\in\eta^\ell_{\neq}} g_\ell(x_1,\dots,x_\ell) - \int_{X^\ell} g_\ell(y_1,\dots,y_\ell)\,\mathrm{d}\mu(y_1,\dots,y_\ell)
$$

with

$$
g_\ell(x_1,\dots,x_\ell) = \sum_{n=\ell}^k (-1)^{n-\ell}\binom{n}{\ell}\int_{X^{n-\ell}} f_n(x_1,\dots,x_\ell,y_1,\dots,y_{n-\ell})\,\mathrm{d}\mu(y_1,\dots,y_{n-\ell})
$$

for $\ell = 1, \dots, k$. Defining

$$F^+ = \sum_{\ell=1}^{k} \sum_{(x_1,\dots,x_\ell)\in\eta^\ell_{\neq}} g_\ell^+(x_1,\dots,x_\ell) \quad \text{and} \quad F^- = \sum_{\ell=1}^{k} \sum_{(x_1,\dots,x_\ell)\in\eta^\ell_{\neq}} g_\ell^-(x_1,\dots,x_\ell)$$

with $g_\ell^+ = \max\{g_\ell, 0\}$ and $g_\ell^- = \max\{-g_\ell, 0\}$ for $\ell = 1, \dots, k$, we can rewrite F as

$$F = (F^+ - \mathbb{E}F^+) - (F^- - \mathbb{E}F^-).$$

and put

$$\overline{F} = F^+ + F^- = \sum_{\ell=1}^{k} \sum_{(x_1,\dots,x_\ell)\in\eta^\ell_{\neq}} |g_\ell(x_1,\dots,x_\ell)|.$$

As a consequence of Lemma 4.10, we know that $D_zV \geq 0$ for a U-statistic V where we sum over a non-negative function. Combining this with $g_\ell^+, g_\ell^- \geq 0$ for $\ell = 1, \dots, k$ and Lemma 4.16, we see that

$$-D_zL^{-1}\left(F^+ - \mathbb{E}F^+\right) \geq 0 \quad \text{and} \quad -D_zL^{-1}\left(F^- - \mathbb{E}F^-\right) \geq 0.$$

Moreover, it holds that $D_z\mathbb{1}(F > t)\, D_zF \geq 0$. Proposition 3.13 implies that the product $D_zF\ D_zL^{-1}\left(\overline{F} - \mathbb{E}\overline{F}\right)$ has a finite Wiener-Itô chaos expansion with an order less than or equal to $2k-2$. Together with Lemma 4.7, the Cauchy-Schwarz inequality, and Lemma 4.9, we obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}(F > t), DF\,|DL^{-1}F|\rangle_{L^2(\mu)} \\
&= \sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}(F > t), DF\,|DL^{-1}\left(F^+ - \mathbb{E}F^+ - F^- + \mathbb{E}F^-\right)|\rangle_{L^2(\mu)} \\
&\leq \sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}(F > t), DF\left(-DL^{-1}(F^+ - \mathbb{E}F^+) - DL^{-1}(F^- - \mathbb{E}F^-)\right)\rangle_{L^2(\mu)} \\
&= \sup_{t\in\mathbb{R}} \mathbb{E}[\mathbb{1}(F > t)\,\delta\left(-DF\,DL^{-1}(\overline{F} - \mathbb{E}\overline{F})\right)] \\
&\leq \mathbb{E}\left[\delta\left(DF\ DL^{-1}\left(\overline{F} - \mathbb{E}\overline{F}\right)\right)^2\right]^{\frac{1}{2}} \\
&\leq \sqrt{(2k-1)\mathbb{E}\langle (DF)^2, (DL^{-1}\left(\overline{F} - \mathbb{E}\overline{F}\right))^2\rangle_{L^2(\mu)}}.
\end{aligned}$$

Now $\overline{F} - \mathbb{E}\overline{F}$ has the Wiener-Itô chaos expansion

$$\overline{F} - \mathbb{E}\overline{F} = \sum_{m=1}^{k} I_m(h_m)$$

with non-negative kernels h_m, $m = 1, \dots, k$, satisfying

$$\begin{aligned}
&h_m(x_1,\dots,x_m) \\
&= \sum_{\ell=m}^{k} \binom{\ell}{m} \int_{X^{\ell-m}} |g_\ell(x_1,\dots,x_m,y_1,\dots,y_{\ell-m})|\,\mathrm{d}\mu(y_1,\dots,y_{\ell-m}) \\
&\leq \sum_{\ell=m}^{k}\sum_{n=\ell}^{k} \binom{\ell}{m}\binom{n}{\ell} \int_{X^{n-m}} |f_n(x_1,\dots,x_m,y_1,\dots,y_{n-m})|\,\mathrm{d}\mu(y_1,\dots,y_{n-m}) \\
&\leq 2^{k-m} \sum_{n=m}^{k} \binom{n}{m} \int_{X^{n-m}} |f_n(x_1,\dots,x_m,y_1,\dots,y_{n-m})|\,\mathrm{d}\mu(y_1,\dots,y_{n-m}).
\end{aligned} \tag{5.28}$$

Applying Fubini's theorem, Jensen's inequality, and Corollary 3.14 again yields

$$\begin{aligned}
&\mathbb{E}\langle (DF)^2, (DL^{-1}\left(\overline{F}-\mathbb{E}\overline{F}\right))^2\rangle_{L^2(\mu)}\\
&\leq \int_X \mathbb{E}\left(\sum_{i=1}^k i\ I_{i-1}(f_i(z,\cdot))\right)^2\left(\sum_{j=1}^k I_{j-1}(h_j(z,\cdot))\right)^2 \mathrm{d}\mu(z)\\
&\leq k^4 \int_X \mathbb{E}\sum_{i=1}^k I_{i-1}(f_i(z,\cdot))^2 \sum_{j=1}^k I_{j-1}(h_j(z,\cdot))^2 \,\mathrm{d}\mu(z)\\
&\leq k^4 \sum_{i,j=1}^k \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes h_j\otimes h_j)_\sigma| \,\mathrm{d}\mu^{|\sigma|}.
\end{aligned}$$

Together with the special structure of the upper bound (5.28), we obtain

$$\mathbb{E}\langle (DF)^2, (DL^{-1}\left(\overline{F}-\mathbb{E}\overline{F}\right))^2\rangle_{L^2(\mu)} \leq k^4 4^k \sum_{i,j,\ell=1}^k \sum_{\sigma\in\widetilde{\Pi}(i,i,j,\ell)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes f_j\otimes f_\ell)_\sigma| \,\mathrm{d}\mu^{|\sigma|},$$

so that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}\left(F>t\right), DF\,|DL^{-1}F|\rangle_{L^2(\mu)}\\
&\leq 2^k\sqrt{2k}\,k^2 \sum_{i,j,\ell=1}^k \sqrt{\sum_{\sigma\in\widetilde{\Pi}(i,i,j,\ell)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes f_j\otimes f_\ell)_\sigma| \,\mathrm{d}\mu^{|\sigma|}}.
\end{aligned} \tag{5.29}$$

Combining the inequalities (5.27) and (5.29) concludes the proof of Theorem 5.12. □

Similarly as in the Equations (5.10) and (5.23), we have

$$d_K\left(\frac{F-\mathbb{E}F}{\sqrt{V}}, N\right) \leq \left|1-\frac{\operatorname{Var}F}{V}\right| + \left(\frac{\operatorname{Var}F}{V}+\frac{(\operatorname{Var}F)^2}{V^2}\right) B_K \tag{5.30}$$

for $V>0$, where B_K is the right-hand side of formula (5.24). This is a direct consequence of Corollary 5.3 and the proof of Theorem 5.12.

5.4 Normal approximation of Poisson functionals with infinite Wiener-Itô chaos expansion

In this section, we turn our attention to Poisson functionals having infinite Wiener-Itô chaos expansions. The problem of dealing with such Poisson functionals is that an application of Proposition 5.8 leads to infinite sums as upper bounds for the Wasserstein distance and d_3-distance, respectively. Therefore, we need to make sure that the resulting expressions converge. This is even harder since every R_{ij} and $R_{ij}^{(u,v)}$ itself is a sum of integrals. The first result of this section, Theorem 5.13, is very general, but delivers weaker central limit theorems than the more specific Theorem 5.15 as the applications in Chapter 8 and Chapter 9 will show.

Since we have a bound for the normal approximation of Poisson functionals with finite Wiener-Itô chaos expansion, it is a natural approach to replace the original Poisson functional by a Poisson functional with a finite Wiener-Itô chaos expansion which we can approximate and to control the distance between both Poisson functionals. This is a generalization of Theorem 5.6 where a Poisson functional is approximated by the first order Wiener-Itô integral of its chaos expansion.

Theorem 5.13 *a) Let $F \in L^2(\mathbb{P}_\eta)$ be a Poisson functional such that*

$$\int_{X^{|\sigma|}} |(f_n \otimes f_n \otimes f_n \otimes f_n)_\sigma| \, \mathrm{d}\mu^{|\sigma|} < \infty \quad \text{for all } \sigma \in \widetilde{\Pi}_{\geq 2}(n,n,n,n) \text{ and } n \in \mathbb{N}$$

and let N be a standard Gaussian random variable. Then

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq 2 \, \frac{\sqrt{\sum_{n=k+1}^{\infty} n! \, \|f_n\|_n^2}}{\sqrt{\operatorname{Var} F}} + \frac{2k^{\frac{7}{2}}}{\operatorname{Var} F} \sum_{1 \leq i \leq j \leq k} \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, \mathrm{d}\mu^{|\sigma|}}$$

for all $k \in \mathbb{N}$.

b) Let $\boldsymbol{F} = (F^{(1)}, \ldots, F^{(m)})$ with Poisson functionals $F^{(\ell)} \in L^2(\mathbb{P}_\eta)$, $\ell = 1, \ldots, m$, such that

$$\int_{X^{|\sigma|}} |(f_n^{(\ell)} \otimes f_n^{(\ell)} \otimes f_n^{(\ell)} \otimes f_n^{(\ell)})_\sigma| \, \mathrm{d}\mu^{|\sigma|} < \infty \quad \text{for all } \sigma \in \widetilde{\Pi}_{\geq 2}(n,n,n,n) \text{ and } n \in \mathbb{N}$$

and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector with a positive semidefinite covariance matrix Σ. Then

$$\begin{aligned} & d_3(\boldsymbol{F} - \mathbb{E}\boldsymbol{F}, \boldsymbol{N}(\Sigma)) \\ & \leq \frac{1}{2} \sum_{u,v=1}^{m} |\sigma_{u,v} - \sum_{n=1}^{k} n! \, \langle f_n^{(u)}, f_n^{(v)} \rangle_{L^2(\mu^n)}| \\ & \quad + m \sqrt{2 \sum_{\ell=1}^{m} \operatorname{Var} F^{(\ell)}} \sqrt{\sum_{\ell=1}^{m} \sum_{n=k+1}^{\infty} n! \, \|f_n^{(\ell)}\|_n^2} \\ & \quad + \frac{m}{2} k^{\frac{7}{2}} \left(\sum_{\ell=1}^{m} \sqrt{\operatorname{Var} F^{(\ell)}} + 1 \right) \\ & \qquad \sum_{u,v=1}^{m} \sum_{i,j=1}^{k} \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i^{(u)} \otimes f_i^{(u)} \otimes f_j^{(v)} \otimes f_j^{(v)})_\sigma| \, \mathrm{d}\mu^{|\sigma|}} \end{aligned}$$

for all $k \in \mathbb{N}$.

Proof. We fix $k \in \mathbb{N}$ and define the truncated Poisson functional $F_k = \sum_{n=1}^k I_n(f_n)$. The triangle inequality for the Wasserstein distance implies that

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, \frac{F_k}{\sqrt{\operatorname{Var} F}}\right) + d_W\left(\frac{F_k}{\sqrt{\operatorname{Var} F}}, N\right).$$

It follows from Lemma 5.7 a) and Theorem 4.2 that

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, \frac{F_k}{\sqrt{\operatorname{Var} F}}\right) \leq \frac{\sqrt{\mathbb{E}(F-\mathbb{E}F-F_k)^2}}{\sqrt{\operatorname{Var} F}} = \frac{\sqrt{\sum_{n=k+1}^\infty n!\, \|f_n\|_n^2}}{\sqrt{\operatorname{Var} F}}.$$

Theorem 5.1 and the triangle inequality imply that

$$\begin{aligned} d_W\left(\frac{F_k}{\sqrt{\operatorname{Var} F}}, N\right) &\leq \mathbb{E}\left|1-\frac{1}{\operatorname{Var} F}\langle DF_k, -DL^{-1}F_k\rangle_{L^2(\mu)}\right| \\ &\quad + \frac{1}{(\operatorname{Var} F)^{\frac{3}{2}}}\int_X \mathbb{E}(D_zF_k)^2\,|D_zL^{-1}F_k|\,\mathrm{d}\mu(z) \\ &\leq \frac{\operatorname{Var} F - \operatorname{Var} F_k}{\operatorname{Var} F} + \frac{1}{\operatorname{Var} F}\,\mathbb{E}|\operatorname{Var} F_k - \langle DF_k, -DL^{-1}F_k\rangle_{L^2(\mu)}| \\ &\quad + \frac{1}{(\operatorname{Var} F)^{\frac{3}{2}}}\int_X \mathbb{E}(D_zF_k)^2\,|D_zL^{-1}F_k|\,\mathrm{d}\mu(z). \end{aligned}$$

For the first summand we have the upper bound

$$\frac{\operatorname{Var} F - \operatorname{Var} F_k}{\operatorname{Var} F} = \frac{\sum_{n=k+1}^\infty n!\,\|f_n\|_n^2}{\operatorname{Var} F} \leq \frac{\sqrt{\sum_{n=k+1}^\infty n!\,\|f_n\|_n^2}}{\sqrt{\operatorname{Var} F}}.$$

The remaining terms can be bounded in the same way as in Proposition 5.8 and Theorem 5.11. Combining all these estimates, we obtain the bound in part a).

For the multivariate setting we define $\mathbf{F}_k = (F_k^{(1)}, \dots, F_k^{(m)})$ with $F_k^{(\ell)} = \sum_{n=1}^k I_n(f_n)$ for $\ell = 1, \dots, m$ and obtain, by the triangle inequality for d_3,

$$d_3(\mathbf{F}-\mathbb{E}\mathbf{F}, \mathbf{N}(\Sigma)) \leq d_3(\mathbf{F}-\mathbb{E}\mathbf{F}, \mathbf{F}_k) + d_3(\mathbf{F}_k, \mathbf{N}(\Sigma)).$$

For the first expression Lemma 5.7 b) and Theorem 4.2 yield the bound

$$\begin{aligned} d_3(\mathbf{F}-\mathbb{E}\mathbf{F}, \mathbf{F}_k) &\leq m\sqrt{\sum_{\ell=1}^m \operatorname{Var} F^{(\ell)} + \operatorname{Var} F_k^{(\ell)}}\sqrt{\sum_{\ell=1}^m \operatorname{Var} F^{(\ell)} - \operatorname{Var} F_k^{(\ell)}} \\ &\leq m\sqrt{2\sum_{\ell=1}^m \operatorname{Var} F^{(\ell)}}\sqrt{\sum_{\ell=1}^m \sum_{n=k+1}^\infty n!\,\|f_n^{(\ell)}\|_n^2}. \end{aligned}$$

Combining this estimate with the bound we obtain from Theorem 5.11 for $d_3(\mathbf{F}_k, \mathbf{N}(\Sigma))$ concludes the proof. □

Using the previous theorem, we can state the following central limit theorems:

Corollary 5.14 *a) Let $F_t \in L^2(\mathbb{P}_{\eta_t})$ for $t \geq 1$ and let N be a standard Gaussian random variable. If there are constants $(c_n)_{n\in\mathbb{N}}$ and $t_0 \geq 1$ such that*

$$\frac{n!\,\|f_{n,t}\|_{n,t}^2}{\operatorname{Var} F_t} \leq c_n \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty \tag{5.31}$$

and if

$$\lim_{t\to\infty} \frac{1}{(\operatorname{Var} F_t)^2} \int_{X^{|\sigma|}} |(f_{i,t} \otimes f_{i,t} \otimes f_{j,t} \otimes f_{j,t})_\sigma| \,\mathrm{d}\mu_t^{|\sigma|} = 0 \tag{5.32}$$

for $\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)$ and $i,j \in \mathbb{N}$, then $(F_t - \mathbb{E}F_t)/\sqrt{\operatorname{Var} F_t} \to N$ in distribution as $t \to \infty$.

b) Let $\boldsymbol{F}_t = (F_t^{(1)}, \dots, F_t^{(m)})$ with $F_t^{(\ell)} \in L^2(\mathbb{P}_{\eta_t})$ for $\ell = 1, \dots, m$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector with a covariance matrix Σ given by

$$\sigma_{uv} = \lim_{t\to\infty} \operatorname{Cov}(F_t^{(u)}, F_t^{(v)})$$

for $u, v = 1, \dots, m$. Assume that there are constants $(c_n^{(\ell)})_{n\in\mathbb{N}}$, $\ell = 1, \dots, m$, and $t_0 \geq 1$ such that

$$\frac{n!\,\|f_{n,t}^{(\ell)}\|_{n,t}^2}{\operatorname{Var} F_t^{(\ell)}} \leq c_n^{(\ell)} \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n^{(\ell)} < \infty \tag{5.33}$$

for $\ell = 1, \dots, m$ and that

$$\lim_{t\to\infty} \int_{X^{|\sigma|}} |(f_{i,t}^{(u)} \otimes f_{i,t}^{(u)} \otimes f_{j,t}^{(v)} \otimes f_{j,t}^{(v)})_\sigma| \,\mathrm{d}\mu_t^{|\sigma|} = 0 \quad \text{for} \quad \sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j) \tag{5.34}$$

for $i, j \in \mathbb{N}$ and $u, v \in \{1, \dots, m\}$. Then $\boldsymbol{F}_t \to \boldsymbol{N}(\Sigma)$ in distribution as $t \to \infty$.

Proof. We start with part a). For an arbitrary $\varepsilon > 0$ the assumption (5.31) ensures that there exists an $n_0 = n_0(\varepsilon)$ with

$$2\,\frac{\sqrt{\sum_{n=n_0+1}^{\infty} n!\,\|f_{n,t}\|_{n,t}^2}}{\sqrt{\operatorname{Var} F_t}} < \frac{\varepsilon}{2}$$

for $t \geq t_0$. It follows from the assumption (5.32) that we can choose a constant $\tilde{t} \geq 1$ such that

$$\frac{2\,n_0^{\frac{7}{2}}}{\operatorname{Var} F_t} \sum_{1 \leq i \leq j \leq n_0} \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_{i,t} \otimes f_{i,t} \otimes f_{j,t} \otimes f_{j,t})_\sigma| \,\mathrm{d}\mu_t^{|\sigma|}} \leq \frac{\varepsilon}{2}$$

for all $t \geq \tilde{t}$. Together with Theorem 5.13 a), we obtain that

$$d_W\left(\frac{F_t - \mathbb{E}F_t}{\sqrt{\operatorname{Var} F_t}}, N\right) \leq \varepsilon$$

for $t \geq \max\{t_0, \tilde{t}\}$, which concludes the proof of the univariate case. The multivariate version can be proven by combining the assumptions (5.33) and (5.34) with Theorem 5.13 b) in a similar way. □

In case that the kernels of the Wiener-Itô chaos expansion of a Poisson functional satisfy some technical integrability conditions, we can apply the technique we used for Poisson functionals with a finite Wiener-Itô chaos expansion directly to a Poisson functional with an infinite Wiener-Itô chaos expansion:

Theorem 5.15 *a) Let $F \in L^2(\mathbb{P}_\eta)$ be a Poisson functional satisfying $F \in \operatorname{dom} D$ and let N be a standard Gaussian random variable. Moreover, assume that there are constants $a > 0$ and $b \geq 1$ such that*

$$\int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, \mathrm{d}\mu^{|\sigma|} \leq \frac{a \; b^{i+j}}{(i! \; j!)^2} \tag{5.35}$$

for all $\sigma \in \widetilde{\Pi}^{(1)}_{\geq 2}(i,i,j,j)$ and $i,j \in \mathbb{N}$. Then

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq \frac{c_b \sqrt{a}}{\operatorname{Var} F} + \frac{1}{\operatorname{Var} F} \left(\int_X \mathbb{E}(D_z F)^4 \, \mathrm{d}\mu(z)\right)^{\frac{1}{2}}$$

with a constant $c_b > 0$ only depending on b.

b) Let $\boldsymbol{F} = (F^{(1)}, \dots, F^{(m)})$ be a vector of Poisson functionals $F^{(\ell)} \in L^2(\mathbb{P}_\eta)$ with $F^{(\ell)} \in \operatorname{dom} D$ for $\ell = 1, \dots, m$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector with a positive semidefinite covariance matrix Σ. If there are constants $a > 0$ and $b \geq 1$ such that

$$\int_{X^{|\sigma|}} |(f_i^{(u)} \otimes f_i^{(u)} \otimes f_j^{(v)} \otimes f_j^{(v)})_\sigma| \, \mathrm{d}\mu^{|\sigma|} \leq \frac{a \; b^{i+j}}{(i! \; j!)^2} \tag{5.36}$$

for all $\sigma \in \widetilde{\Pi}^{(1)}_{\geq 2}(i,i,j,j)$, $i,j \in \mathbb{N}$, and $1 \leq u,v \leq m$, then

$$\begin{aligned} d_3(\boldsymbol{F} - \mathbb{E}\boldsymbol{F}, \boldsymbol{N}(\Sigma)) \;\; &\leq \;\; \frac{1}{2} \sum_{u,v=1}^m |\sigma_{uv} - \operatorname{Cov}(F^{(u)}, F^{(v)})| + \frac{m^2 c_b \sqrt{a}}{2} \\ &\quad + \frac{m}{4} \sum_{u,v=1}^m \left(\int \mathbb{E}(D_z F^{(u)})^4\right)^{\frac{1}{2}} \sqrt{\operatorname{Var} F^{(v)}} \end{aligned}$$

with the same constant $c_b > 0$ as in a).

Proof. Combing Lemma 5.10 and the assumptions (5.35) and (5.36) yields

$$R_{ij} = \sum_{\sigma \in \widetilde{\Pi}^{(1)}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} (f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma \, \mathrm{d}\mu^{|\sigma|} \leq |\widetilde{\Pi}^{(1)}_{\geq 2}(i,i,j,j)| \, \frac{a \; b^{i+j}}{(i! \; j!)^2}$$

for $i, j \in \mathbb{N}$ and

$$R_{ij}^{(u,v)} = \sum_{\sigma \in \widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)} \int_{X^{|\sigma|}} (f_i^{(u)} \otimes f_i^{(u)} \otimes f_j^{(v)} \otimes f_j^{(v)})_\sigma \, \mathrm{d}\mu^{|\sigma|} \leq |\widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)| \, \frac{a \; b^{i+j}}{(i! \; j!)^2}$$

for $i, j \in \mathbb{N}$ and $1 \leq u, v \leq m$. Note that $|\widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j| \leq |\Pi_{\geq 2}(i,i,j,j)|$ since $\widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j) \subset \Pi_{\geq 2}(i,i,j,j)$. For a partition $\sigma \in \Pi_{\geq 2}(i,i,j,j)$ we denote by k_{pq} with $1 \leq p < q \leq 4$ the number of blocks including only variables of the p-th and q-th function of the tensor product. Moreover, let k_{pqr} with $1 \leq p < q < r \leq 4$ be the number of blocks consisting of variables of the p-th, q-th, and r-th function of the tensor product and let k_{1234} be the number of blocks with four variables. For given $k_{12}, \dots, k_{1234}$ there are at most

$$\frac{(i! \; j!)^2}{k_{12}! \dots k_{1234}!}$$

partitions in $\Pi_{\geq 2}(i,i,j,j)$.

Since there are eleven different numbers of blocks $k_{12}, \dots, k_{1234}$ and each of them must be between 0 and i or j, we have at most $\max\{i+1, j+1\}^{11}$ possible choices for $k_{12}, \dots, , k_{1234}$. The seven numbers of $k_{12}, \dots, k_{1234}$ that have an index $\ell \in \{1,2,3,4\}$ must sum up to i for $\ell \in \{1,2\}$ and j for $\ell \in \{3,4\}$, respectively. This implies that $\max\{k_{12}, \dots, k_{1234}\} \geq \lceil \max\{i,j\}/7 \rceil$. Altogether, we obtain

$$|\Pi_{\geq 2}(i,i,j,j)| \leq \frac{\max\{i+1, j+1\}^{11} (i! \; j!)^2}{\lceil \max\{i,j\}/7 \rceil !} \leq \frac{2^{11} \max\{i,j\}^{11} (i! \; j!)^2}{\lceil \max\{i,j\}/7 \rceil !}$$

so that

$$R_{ij} \leq \frac{2^{11} \max\{i,j\}^{11} \, b^{i+j}}{\lceil \max\{i,j\}/7 \rceil !} \, a \quad \text{and} \quad R_{ij}^{(u,v)} \leq \frac{2^{11} \max\{i,j\}^{11} \, b^{i+j}}{\lceil \max\{i,j\}/7 \rceil !} \, a.$$

A short computation shows that

$$\begin{aligned} \sum_{i,j=1}^{\infty} i \sqrt{\frac{2^{11} \max\{i,j\}^{11} \, b^{i+j}}{\lceil \max\{i,j\}/7 \rceil !}} &\leq 2^{\frac{11}{2}} \sum_{i,j=1}^{\infty} \max\{i,j\} \sqrt{\frac{\max\{i,j\}^{11}}{\lceil \max\{i,j\}/7 \rceil !}} \, b^{\max\{i,j\}} \\ &\leq 2^{\frac{13}{2}} \sum_{k=1}^{\infty} k^2 \sqrt{\frac{k^{11}}{\lceil k/7 \rceil !}} \, b^k, \end{aligned}$$

which proves that the series converges. We take the sum on the right-hand side as definition of c_b. Now Theorem 5.15 is a direct consequence of Proposition 5.8. □

Notes: Theorem 5.2 and the example for the normal approximation of a Poisson distributed random variable in Section 5.1 are taken from *Schulte 2012b*.

The truncation argument in the proof of Theorem 5.6 and Lemma 5.7 are from *Last, Penrose, Schulte, and Thäle 2012*. But the setting of Section 5.2 is slightly more general since we consider general Poisson functionals instead of a special class of Poisson U-statistics.

Bounds derived by the Malliavin-Stein method for Poisson functionals with finite Wiener-Itô chaos expansion as in Section 5.3 are evaluated in *Reitzner and Schulte 2011*. The univariate part of Proposition 5.8 is very similar. Due to technical reason we use a different version of the product formula for multiple Wiener-Itô integrals. In Lemma 5.10, we carry out an idea that is briefly mentioned in *Schulte 2012a*. A special version of Theorem 5.11 for Poisson U-statistics is given in *Reitzner and Schulte 2011*, whereas the multivariate part is new. Theorem 5.12 is a generalization of a result from *Schulte 2012b*, where it is assumed that the Poisson functional is a Poisson U-statistic.

The univariate part of Theorem 5.13 and Corollary 5.14 are contained in *Schulte 2012a*. The multivariate result can be derived in a similar way. Theorem 5.15 is derived in *Hug, Last, and Schulte 2012*.

Chapter 6

Limit theorems for Poisson U-statistics

In this chapter, we use the results of the previous chapter to investigate the asymptotic behaviour of Poisson U-statistics. We derive abstract bounds for the normal approximation of Poisson U-statistics in the first section that are applied to two special classes of Poisson U-statistics, so-called geometric Poisson U-statistics and local Poisson U-statistics, in the second and third section.

6.1 Normal approximation of Poisson U-statistics

In the sequel, we consider a Poisson U-statistic of the form

$$S = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k)$$

with $f \in L^1_s(\mu^k)$. We assume henceforth that S is absolutely convergent.

Definition 6.1 *A Poisson U-statistic S is absolutely convergent if the Poisson U-statistic*

$$\overline{S} = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} |f(x_1,\dots,x_k)|$$

is in $L^2(\mathbb{P}_\eta)$.

To motivate the definition of an absolutely convergent Poisson U-statistic, we provide an example of a Poisson U-statistic that is in $L^2(\mathbb{P}_\eta)$ but not absolutely convergent.

Example 6.2 Similarly as in Example 4.14, we consider a stationary Poisson point process η on $\mathbb{R}$ with intensity one and set

$$f(x_1,x_2) = \mathbb{1}(0\leq |x_1|\sqrt{|x_2|}\leq 1)\,\mathbb{1}(0\leq |x_2|\sqrt{|x_1|}\leq 1)\,\left(2\,\mathbb{1}(x_1x_2\geq 0)-1\right)$$

and

$$S = \sum_{(x_1,x_2)\in\eta^2_{\neq}} f(x_1,x_2) \quad \text{and} \quad \overline{S} = \sum_{(x_1,x_2)\in\eta^2_{\neq}} |f(x_1,x_2)|.$$

Now it is easy to verify for the kernels of the Wiener-Itô chaos expansion of S that $f_1(x) = 0$ and $f_2(x_1, x_2) = f(x_1, x_2)$ so that $S \in L^2(\mathbb{P}_\eta)$. But the first kernel of the Wiener-Itô chaos expansion of $\overline{S}$ is not in $L^2(\mathbb{R})$ so that $\overline{S} \notin L^2(\mathbb{P}_\eta)$.

Note that S absolutely convergent implies that $S \in L^2(\mathbb{P}_\eta)$. Obviously, every $S \in L^2(\mathbb{P}_\eta)$ with $f \geq 0$ is absolutely convergent.

In Chapter 3 and in Chapter 4, it was shown that a Poisson U-statistic $S \in L^2(\mathbb{P}_\eta)$ of order k has the finite Wiener-Itô chaos expansion

$$S = \mathbb{E}S + \sum_{n=1}^{k} I_n(f_n)$$

with

$$\mathbb{E}S = \int_{X^k} f(y_1, \ldots, y_k) \, \mathrm{d}\mu(y_1, \ldots, y_k)$$

and

$$f_n(x_1, \ldots, x_n) = \binom{k}{n} \int_{X^{k-n}} f(x_1, \ldots, x_n, y_1, \ldots, y_{k-n}) \, \mathrm{d}\mu(y_1, \ldots, y_{k-n})$$

for $n = 1, \ldots, k$ and that the variance of S is given by

$$\begin{aligned} \operatorname{Var} S &= \sum_{n=1}^{k} n! \, \|f_n\|_n^2 \\ &= \sum_{n=1}^{k} n! \binom{k}{n}^2 \int_{X^n} \left(\int_{X^{k-n}} f(x_1, \ldots, x_n, y_1, \ldots, y_{k-n}) \, \mathrm{d}\mu(y_1, \ldots, y_{k-n}) \right)^2 \mathrm{d}\mu(x_1, \ldots, x_n). \end{aligned}$$

In our multivariate results, we investigate a vector $\mathbf{S} = (S^{(1)}, \ldots, S^{(m)})$ of Poisson U-statistics $S^{(\ell)} \in L^2(\mathbb{P}_\eta)$ given by

$$S^{(\ell)} = \sum_{(x_1, \ldots, x_{k_\ell}) \in \eta_{\neq}^{k_\ell}} f^{(\ell)}(x_1, \ldots, x_{k_\ell})$$

with $f^{(\ell)} \in L_s^1(\mu^{k_\ell})$ and $k_\ell \in \mathbb{N}$ for $\ell = 1, \ldots, m$. Analogously to the univariate setting, we have

$$\mathbb{E}S^{(\ell)} = \int_{X^{k_\ell}} f^{(\ell)}(y_1, \ldots, y_{k_\ell}) \, \mathrm{d}\mu(y_1, \ldots, y_{k_\ell}),$$

and $S^{(\ell)}$ has a finite Wiener-Itô chaos expansion of order k_ℓ with kernels

$$f_n^{(\ell)}(x_1, \ldots, x_n) = \binom{k_\ell}{n} \int_{X^{k_\ell - n}} f^{(\ell)}(x_1, \ldots, x_n, y_1, \ldots, y_{k_\ell - n}) \, \mathrm{d}\mu(y_1, \ldots, y_{k_\ell - n})$$

for $n = 1, \ldots, k_\ell$. The covariance matrix of $\mathbf{S}$ is given by

$$\operatorname{Cov}(S^{(u)}, S^{(v)}) = \sum_{n=1}^{\min\{k_u, k_v\}} n! \, \langle f_n^{(u)}, f_n^{(v)} \rangle_{L^2(\mu^n)}$$

for $u,v=1,\dots,m$.

In order to neatly formulate our results, we use the following notation. For a function $h:X^k\to\overline{\mathbb{R}}$ and $1\le i,j\le k$ we define

$$M_{ij}(h)=\sum_{\substack{\sigma\in\widetilde{\Pi}(k,k,k,k)\\ s(\sigma)=(k-i,k-i,k-j,k-j)}}\int_{X^{|\sigma|}}|(h\otimes h\otimes h\otimes h)_\sigma|\,\mathrm{d}\mu^{|\sigma|},$$

and for $h_1:X^{k_1}\to\overline{\mathbb{R}}$ and $h_2:X^{k_2}\to\overline{\mathbb{R}}$ and $1\le i\le k_1$ and $1\le j\le k_2$ we put

$$M_{ij}(h_1,h_2)=\sum_{\substack{\sigma\in\widetilde{\Pi}(k_1,k_1,k_2,k_2)\\ s(\sigma)=(k_1-i,k_1-i,k_2-j,k_2-j)}}\int_{X^{|\sigma|}}|(h_1\otimes h_1\otimes h_2\otimes h_2)_\sigma|\,\mathrm{d}\mu^{|\sigma|}.$$

By Theorem 4.11, the functions $\overline{f}_n:X^n\to\overline{\mathbb{R}}$ given by

$$\overline{f}_n(x_1,\dots,x_n)=\binom{k}{n}\int_{X^{k-n}}|f(x_1,\dots,x_n,y_1,\dots,y_{k-n})|\,\mathrm{d}\mu(y_1,\dots,y_{k-n})$$

for $n=1,\dots,k$ are the kernels of the Wiener-Itô chaos expansion of the Poisson U-statistic

$$\overline{S}=\sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}}|f(x_1,\dots,x_k)|$$

we used for the definition of an absolutely convergent Poisson U-statistic. It follows from Fubini's theorem that

$$M_{ij}(f)=\sum_{\sigma\in\widetilde{\Pi}_{\ge2}(i,i,j,j)}\int_{X^{|\sigma|}}(\overline{f}_i\otimes\overline{f}_i\otimes\overline{f}_j\otimes\overline{f}_j)_\sigma\,\mathrm{d}\mu^{|\sigma|}.$$

Since $|f_n|\le\overline{f}_n$ for $n=1,\dots,k$, it is easy to see that

$$M_{ij}(f)\ge\sum_{\sigma\in\widetilde{\Pi}_{\ge2}(i,i,j,j)}\int_{X^{|\sigma|}}|(f_i\otimes f_i\otimes f_j\otimes f_j)_\sigma|\,\mathrm{d}\mu^{|\sigma|},\tag{6.1}$$

where equality holds if and only if $f\ge0$. In the multivariate case, we have

$$M_{ij}(f^{(u)},f^{(v)})=\sum_{\sigma\in\widetilde{\Pi}_{\ge2}(i,i,j,j)}\int_{X^{|\sigma|}}(\overline{f}_i^{(u)}\otimes\overline{f}_i^{(u)}\otimes\overline{f}_j^{(v)}\otimes\overline{f}_j^{(v)})_\sigma\,\mathrm{d}\mu^{|\sigma|},$$

where $\overline{f}_i^{(u)}$, $i=1,\dots,k_u$, and $\overline{f}_j^{(v)}$, $j=1,\dots,k_v$, are the kernels of the Wiener-Itô chaos expansions of the Poisson U-statistics

$$\overline{S}^{(u)}=\sum_{(x_1,\dots,x_{k_u})\in\eta^{k_u}_{\neq}}|f^{(u)}(x_1,\dots,x_{k_u})|\quad\text{and}\quad\overline{S}^{(v)}=\sum_{(x_1,\dots,x_{k_v})\in\eta^{k_v}_{\neq}}|f^{(v)}(x_1,\dots,x_{k_v})|.$$

Analogously to inequality (6.1), we have

$$M_{ij}(f^{(u)},f^{(v)})\ge\sum_{\sigma\in\widetilde{\Pi}_{\ge2}(i,i,j,j)}\int_{X^{|\sigma|}}|(f_i^{(u)}\otimes f_i^{(u)}\otimes f_j^{(v)}\otimes f_j^{(v)})_\sigma|\,\mathrm{d}\mu^{|\sigma|}\tag{6.2}$$

with equality if and only if $f^{(u)}\ge0$ and $f^{(v)}\ge0$. Combining the inequalities (6.1) and (6.2) with Theorem 5.11 yields:

Theorem 6.3 *a) For an absolutely convergent Poisson U-statistic S of order k and a standard Gaussian random variable N we have*

$$d_W\left(\frac{S-\mathbb{E}S}{\sqrt{\operatorname{Var} S}}, N\right) \leq 2k^{\frac{7}{2}} \sum_{1\leq i\leq j\leq k} \frac{\sqrt{M_{ij}(f)}}{\operatorname{Var} S}.$$

b) Let $\boldsymbol{S}=(S^{(1)},\dots,S^{(m)})$ be a vector of absolutely convergent Poisson U-statistics of the orders $k_1,\dots,k_m$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional centred Gaussian random vector with a positive semidefinite covariance matrix Σ. Then

$$\begin{aligned} d_3(\boldsymbol{S}-\mathbb{E}\boldsymbol{S}, \boldsymbol{N}(\Sigma)) \leq &\frac{1}{2}\sum_{u,v=1}^{m} |\sigma_{uv} - \operatorname{Cov}(S^{(u)}, S^{(v)})| \\ &+ \frac{m}{2}\left(\sum_{\ell=1}^{m}\sqrt{\operatorname{Var} S^{(\ell)}}+1\right)\sum_{u,v=1}^{m}\sum_{i=1}^{k_u}\sum_{j=1}^{k_v} k_u^{\frac{7}{2}}\sqrt{M_{ij}(f^{(u)}, f^{(v)})}. \end{aligned}$$

We can also derive a bound similar to the first part of Theorem 6.3 for the Kolmogorov distance:

Theorem 6.4 *For an absolutely convergent Poisson U-statistic S of order k and a standard Gaussian random variable N we have*

$$d_K\left(\frac{S-\mathbb{E}S}{\sqrt{\operatorname{Var} S}}, N\right) \leq 19k^5 \sum_{i,j=1}^{k} \frac{\sqrt{M_{ij}(f)}}{\operatorname{Var} S}. \tag{6.3}$$

Proof. Using Theorem 5.12 and the inequality (6.1), we obtain

$$\begin{aligned} d_K\left(\frac{S-\mathbb{E}S}{\sqrt{\operatorname{Var} S}}, N\right) \leq &17k^5 \sum_{i,j=1}^{k} \frac{\sqrt{M_{ij}(f)}}{\operatorname{Var} S} \\ &+ \frac{1}{\operatorname{Var} S}\sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}(S>t), DS\,|DL^{-1}(S-\mathbb{E}S)|\rangle_{L^2(\mu)}. \end{aligned} \tag{6.4}$$

Now we bound the second part in a similar way as in the proof of Theorem 5.12 but make use of the fact that S is an absolutely convergent Poisson U-statistic. We define

$$S^+ = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f^+(x_1,\dots,x_k) \quad \text{and} \quad S^- = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f^-(x_1,\dots,x_k)$$

with $f^+=\max\{f,0\}$ and $f^-=\max\{-f,0\}$ and put

$$\overline{S} = S^+ + S^- = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} |f(x_1,\dots,x_k)|.$$

By the same reasoning as in the proof of Theorem 5.12, we obtain

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}(S>t), DS\,|DL^{-1}(S-\mathbb{E}S)|\rangle_{L^2(\mu)} \\ &\leq \sup_{t\in\mathbb{R}} \mathbb{E}\langle D\mathbb{1}(S>t), DS\left(-DL^{-1}(\overline{S}-\mathbb{E}\overline{S})\right)\rangle_{L^2(\mu)} \\ &\leq \sqrt{(2k-1)\mathbb{E}\langle (DS)^2, (DL^{-1}(\overline{S}-\mathbb{E}\overline{S}))^2\rangle_{L^2(\mu)}}. \end{aligned} \tag{6.5}$$

Evaluating the right-hand side by Corollary 3.14 in a similar way as in formula (5.26) yields

$$\begin{aligned}
&\mathbb{E}\langle (DS)^2, (DL^{-1}(\overline{S}-\mathbb{E}\overline{S}))^2\rangle_{L^2(\mu)} \\
&\leq k^4 \sum_{i,j=1}^{k} \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(i,i,j,j)} \int_{X^{|\sigma|}} |(f_i\otimes f_i\otimes \overline{f}_j\otimes \overline{f}_j)_\sigma| \,\mathrm{d}\mu^{|\sigma|} \leq k^4 \sum_{i,j=1}^{k} M_{ij}(f).
\end{aligned} \tag{6.6}$$

Here, $\overline{f}_j$ stands for the j-th kernel of the Wiener-Itô chaos expansion of $\overline{S}$. Combining the inequalities (6.5) and (6.6) with formula (6.4) concludes the proof of the bound (6.3). □

In Theorem 6.3 and Theorem 6.4, we tacitly assume that the expressions on the right-hand sides are finite. If the function f we sum over in the Poisson U-statistic S is non-negative, we have the following fourth moment criterion (see also the discussion next to Theorem 5.11):

Corollary 6.5 *For a Poisson U-statistic*

$$S = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k)$$

of order k with $f\in L^1_s(\mu^k)$ and $f\geq 0$ and a standard Gaussian random variable N we have

$$d_W\left(\frac{S-\mathbb{E}S}{\sqrt{\operatorname{Var} S}}, N\right) \leq 2k^{\frac{11}{2}} \sqrt{\frac{\mathbb{E}(S-\mathbb{E}S)^4}{(\operatorname{Var} S)^2} - 3}$$

and

$$d_K\left(\frac{S-\mathbb{E}S}{\sqrt{\operatorname{Var} S}}, N\right) \leq 19k^{7} \sqrt{\frac{\mathbb{E}(S-\mathbb{E}S)^4}{(\operatorname{Var} S)^2} - 3}.$$

Proof. Because of $f\geq 0$, the equation

$$S^4 = \sum_{\sigma\in\Pi(k,k,k,k)} \sum_{(x_1,\dots,x_{|\sigma|})\in\eta^{|\sigma|}_{\neq}} (f\otimes f\otimes f\otimes f)_\sigma(x_1,\dots,x_{|\sigma|})$$

holds almost surely. Now the Slivnyak-Mecke formula implies that $\mathbb{E}S^4<\infty$ if and only if

$$\int_{X^{|\sigma|}} (f\otimes f\otimes f\otimes f)_\sigma \,\mathrm{d}\mu^{|\sigma|} < \infty$$

for all $\sigma\in\Pi(k,k,k,k)$. For $\mathbb{E}S^4=\infty$ the assertions are obviously true. Otherwise, it follows from Theorem 3.4 and Lemma 2.5 that

$$\begin{aligned}
M_{ij}(f) &= \sum_{\substack{\sigma\in\widetilde{\Pi}(k,k,k,k)\\ s(\sigma)=(k-i,k-i,k-j,k-j)}} \int_{X^{|\sigma|}} (f\otimes f\otimes f\otimes f)_\sigma \,\mathrm{d}\mu^{|\sigma|} \\
&\leq \sum_{\sigma\in\widetilde{\Pi}(k,k,k,k)} \int_{X^{|\sigma|}} (f\otimes f\otimes f\otimes f)_\sigma \,\mathrm{d}\mu^{|\sigma|} \\
&\leq \gamma_4(S-\mathbb{E}S) = \mathbb{E}(S-\mathbb{E}S)^4 - 3(\operatorname{Var} F)^2
\end{aligned}$$

so that the bounds for the Wasserstein distance and the Kolmogorov distance are direct consequences of Theorem 6.3 and Theorem 6.4. □

6.2 Limit theorems for geometric Poisson U-statistics

In this section, we consider a family of Poisson point processes $(\eta_t)_{t\geq 1}$ with intensity measures $(\mu_t)_{t\geq 1}$ that are given by $\mu_t = t\mu$ with a fixed σ-finite measure μ and a special class of Poisson U-statistics $(S_t)_{t\geq 1}$, namely so-called geometric Poisson U-statistics.

Definition 6.6 *A family of Poisson U-statistics $(S_t)_{t\geq 1}$ of the form*

$$S_t = \sum_{(x_1,\ldots,x_k)\in\eta_{t,\neq}^k} f_t(x_1,\ldots,x_k)$$

is called geometric if $f_t \in L_s^1(\mu_t^k)$ satisfies

$$f_t(x_1,\ldots,x_k) = g(t)\,\tilde{f}(x_1,\ldots,x_k)$$

with $g : \mathbb{R} \to (0,\infty)$ and $\tilde{f} : X^k \to \overline{\mathbb{R}}$ not depending on t.

If $g \equiv 1$, we can write

$$S_t = S(\eta_t) = \sum_{(x_1,\ldots,x_k)\in\eta_{t,\neq}^k} \tilde{f}(x_1,\ldots,x_k).$$

Then the value of S for a given realization of η_t only depends on the geometry of the points of η_t and not on the intensity parameter t. For this reason, we use the term geometric. We allow an intensity related scaling factor $g(t)$ in the definition above since we only consider standardized random variables where the scaling factor $g(t)$ is cancelled out.

Theorem 6.7 *Let N be a standard Gaussian random variable and let $(S_t)_{t\geq 1}$ be a family of absolutely convergent geometric Poisson U-statistics of order k with*

$$\tilde{V} := k^2 \int_X \left(\int_{X^{k-1}} \tilde{f}(x,y_1,\ldots,y_{k-1})\,\mathrm{d}\mu(y_1,\ldots,y_{k-1})\right)^2 \mathrm{d}\mu(x) > 0. \qquad (6.7)$$

a) We have

$$\lim_{t\to\infty} \frac{\operatorname{Var} S_t}{g(t)^2\, t^{2k-1}} = \tilde{V}. \qquad (6.8)$$

b) If

$$\int_X \left|\int_{X^{k-1}} \tilde{f}(x,y_1,\ldots,y_{k-1})\,\mathrm{d}\mu(y_1,\ldots,y_{k-1})\right|^3 \mathrm{d}\mu(x) < \infty,$$

there is a constant $C_{W,\tilde{f}} > 0$ such that

$$d_W\left(\frac{S_t - \mathbb{E}S_t}{\sqrt{\operatorname{Var} S_t}}, N\right) \leq C_{W,\tilde{f}}\; t^{-\frac{1}{2}} \qquad (6.9)$$

for $t \geq 1$.

c) If $M_{ij}(\tilde{f}) < \infty$ for $i,j=1,\dots,k$, then there is a constant $C_{K,\tilde{f}}$ such that

$$d_K\left(\frac{S_t-\mathbb{E}S_t}{\sqrt{\operatorname{Var} S_t}}, N\right) \leq C_{K,\tilde{f}}\; t^{-\frac{1}{2}} \tag{6.10}$$

for $t \geq 1$.

Proof. It follows from Theorem 4.11 and the form of f_t and μ_t that the Wiener-Itô chaos expansion of S_t has the kernels

$$\begin{aligned} f_{n,t}(x_1,\dots,x_n) &= \binom{k}{n}\int_{X^{k-n}} f_t(x_1,\dots,x_n,y_1,\dots,y_{k-n})\,\mathrm{d}\mu_t(y_1,\dots,y_{k-n}) \\ &= g(t)\,t^{k-n}\underbrace{\binom{k}{n}\int_{X^{k-n}} \tilde{f}(x_1,\dots,x_n,y_1,\dots,y_{k-n})\,\mathrm{d}\mu(y_1,\dots,y_{k-n})}_{=:\tilde{f}_n(x_1,\dots,x_n)} \end{aligned}$$

for $n=1,\dots,k$. Since $f_{n,t}\in L_s^2(\mu_t^n)$, we have $\tilde{f}_n\in L_s^2(\mu^n)$. As before, we denote by $\|\cdot\|_n$ and $\|\cdot\|_{n,t}$ the L^2-norms in $L_s^2(\mu^n)$ and $L_s^2(\mu_t^n)$, respectively. Now Theorem 4.2 yields

$$\operatorname{Var} S_t = g(t)^2\sum_{n=1}^k n!\,\|\tilde{f}_n\|_n^2\; t^{2k-n}.$$

This sum is a polynomial of degree $2k-1$ in t with the leading coefficient $\|\tilde{f}_1\|_1^2=\tilde{V}>0$, which proves formula (6.8).

Theorem 5.6 a) implies that

$$d_W\left(\frac{S_t-\mathbb{E}S_t}{\sqrt{\operatorname{Var} S_t}}, N\right) \leq 2\sqrt{1-\frac{\|f_{1,t}\|_{1,t}^2}{\operatorname{Var} S_t}} + \frac{1}{(\operatorname{Var} S_t)^{\frac{3}{2}}}\int_X |f_{1,t}(z)|^3\,\mathrm{d}\mu_t(z).$$

Obviously, we have $\operatorname{Var} S_t \geq \|f_{1,t}\|_{1,t}^2 = g(t)^2\,\tilde{V}\,t^{2k-1}$. Combining this with

$$1-\frac{\|f_{1,t}\|_{1,t}^2}{\operatorname{Var} S_t} = \frac{\sum_{n=2}^k n!\,\|f_{n,t}\|_{n,t}^2}{\operatorname{Var} S_t} \leq \frac{g(t)^2\sum_{n=2}^k n!\,\|\tilde{f}_n\|_n^2\,t^{2k-n}}{g(t)^2\,\tilde{V}\,t^{2k-1}} \leq \frac{\sum_{n=2}^k n!\,\|\tilde{f}_n\|_n^2}{\tilde{V}}\;t^{-1}$$

for $t\geq 1$ and

$$\begin{aligned} &\frac{1}{(\operatorname{Var} S_t)^{\frac{3}{2}}}\int_X |f_{1,t}(z)|^3\,\mathrm{d}\mu_t(z) \\ &\leq \frac{1}{g(t)^3\,\tilde{V}^{\frac{3}{2}}\,t^{3k-\frac{3}{2}}}\,g(t)^3\,t^{3k-2}\int_X |\tilde{f}_1(z)|^3\,\mathrm{d}\mu(z) \\ &= \frac{k^3}{\tilde{V}^{\frac{3}{2}}}\int_X\left|\int_{X^{k-1}} \tilde{f}(x,y_1,\dots,y_{k-1})\,\mathrm{d}\mu(y_1,\dots,y_{k-1})\right|^3\mathrm{d}\mu(x)\;t^{-\frac{1}{2}} \end{aligned}$$

for $t\geq 1$ proves the bound (6.9).

We know from Theorem 6.4 that

$$d_K\left(\frac{S_t-\mathbb{E}S_t}{\sqrt{\operatorname{Var} S_t}}, N\right) \leq 19k^5\sum_{i,j=1}^k \frac{\sqrt{M_{ij}(f_t)}}{\operatorname{Var} S_t}.$$

Here, we integrate in $M_{ij}(\cdot)$ with respect to the measure μ_t. In contrast, we write $\tilde{M}_{ij}(\cdot)$ if we integrate with respect to μ. Since $|\sigma| \leq 2i + 2j - 3$ for $\sigma \in \widetilde{\Pi}(i, i, j, j)$, we obtain

$$\begin{aligned} M_{ij}(f_t) &= g(t)^4 \sum_{\substack{\sigma \in \widetilde{\Pi}(k,k,k,k) \\ s(\sigma) = (k-i,k-i,k-j,k-j)}} \int_{X^{|\sigma|}} |(\tilde{f} \otimes \tilde{f} \otimes \tilde{f} \otimes \tilde{f})_\sigma| \, \mathrm{d}\mu_t^{|\sigma|} \, t^{2(k-i)+2(k-j)} \\ &= g(t)^4 \tilde{M}_{ij}(\tilde{f}) \, t^{2(k-i)+2(k-j)+|\sigma|} \leq g(t)^4 \tilde{M}_{ij}(\tilde{f}) \, t^{4k-3} \end{aligned}$$

for $t \geq 1$. Together with $\operatorname{Var} S_t \geq g(t)^2 \, \tilde{V} \, t^{2k-1}$, this yields the bound (6.10). □

In case that $\mu(X) < \infty$ (and $f_t = \tilde{f}$), one can also consider a binomial point process ζ_m of m independently distributed points with respect to the probability measure $\mu(\cdot)/\mu(X)$ and a classical U-statistic

$$T_m = \sum_{(x_1,\ldots,x_k) \in \zeta_{m,\neq}^k} \tilde{f}(x_1, \ldots, x_k).$$

Recall from Section 3.2 that $\zeta_{m,\neq}^k$ stands for the set of all k-tuples of distinct points from ζ_m. In the classical works [17, 19, 36] by Grams and Serfling, Korolyuk and Borovskich, and Friedrich, bounds for the normal approximation of T_m in Kolmogorov distance were derived. Under some moment assumptions on $\tilde{f}$, these bounds are of order $m^{-\frac{1}{2}}$, which is very similar to $t^{-\frac{1}{2}}$ in formula (6.10). For a more recent proof we refer to the work [10] by Chen and Shao.

Classical U-statistics have a so-called Hoeffding decomposition (see the classical references [27, 37, 43] by Hoeffding, Korolyuk and Borovskich, and Lee, for example)

$$T_m = \mathbb{E} T_m + \sum_{n=1}^{k} \sum_{(x_1,\ldots,x_n) \in \zeta_{m,\neq}^n} \binom{m-n}{k-n} h_n(x_1, \ldots, x_n)$$

with functions $h_n \in L_s^1(\mu^n)$, $n = 1, \ldots, k$, such that

$$\int_X h_n(x_1, \ldots, x_{n-1}, y) \, \mathrm{d}\mu(y) = 0$$

for μ-almost all $x_1, \ldots, x_{n-1} \in X$. For explicit formulas for h_n we refer to [27, 37, 43]. If $\mu(X) < \infty$, the Poisson U-statistic S_t equals almost surely the Poissonized classical U-statistic $T_{\eta_t(X)}$ so that one can compute the Hoeffding decomposition of S_t (this was done by Lachièze-Rey and Peccati in [40] for the case $\|\tilde{f}_1\|_1^2 = 0$). Then the summands in the Hoeffding decomposition are Poisson U-statistics, that can be written as sums of Poisson U-statistics due to the definition of h_n. Comparing the n-th summand of the Hoeffding decomposition with the n-th multiple Wiener-Itô integral in the Wiener-Itô chaos expansion of S_t, that is itself a sum of Poisson U-statistics, one sees that both have a similar structure, but are distinct. Indeed, in the Hoeffding decomposition the Poisson U-statistics are multiplied by factors depending on the given realization of the Poisson point process, which is not the case if we write $I_{n,t}(f_{n,t})$ as a sum of Poisson U-statistics.

In [40, Theorem 7.3], the Hoeffding decomposition and a result due to Dynkin and Mandelbaum (see [15]) are used to prove that a geometric Poisson U-statistic converges after a suitable rescaling to the multiple Wiener-Itô integral with respect to a Gaussian measure with control μ of the first kernel $\tilde{f}_q$ with $\|\tilde{f}_q\|_q^2 \neq 0$ if $q > 1$. For $q = 1$ the argument still holds, and the Wiener-Itô integral is a Gaussian random variable so that we have convergence in distribution of the standardized Poisson U-statistic to a standard Gaussian random variable as in Theorem 6.7. But in contrast to the Malliavin-Stein method this approach does not deliver a rate of convergence.

The assumption (6.7) in Theorem 6.7 cannot be easily dispensed as the following example shows:

Example 6.8 Let η_t be the restriction of a stationary Poisson point process with intensity $t > 0$ on $[-1, 1]$, i.e. the intensity measure is $\mu_t(\cdot) = t\,\lambda(\cdot \cap [-1, 1])$. We define a Poisson U-statistic

$$S_t = \sum_{(x_1,x_2)\in\eta_{t,\neq}^2} f(x_1, x_2) \quad \text{with} \quad f(x_1, x_2) = \operatorname{sgn}(x_1)\operatorname{sgn}(x_2) = \begin{cases} 1, & x_1x_2 \geq 0 \\ -1, & x_1x_2 < 0 \end{cases},$$

where $\operatorname{sgn}(\cdot)$ stands for the sign of a real number. Since

$$\mathbb{E}S_t = t^2 \int_{-1}^{1}\int_{-1}^{1} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 = t^2 \int_{-1}^{1}\int_{-1}^{1} \operatorname{sgn}(y_1)\operatorname{sgn}(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 = 0$$

and

$$f_{1,t}(x) = 2t \int_{-1}^{1} f(x, y)\,\mathrm{d}y = 2t \int_{-1}^{1} \operatorname{sgn}(y)\,\mathrm{d}y\ \operatorname{sgn}(x) = 0$$

for all $x \in [-1, 1]$, S_t has the Wiener-Itô chaos expansion $S_t = I_{2,t}(f)$ and the variance

$$\operatorname{Var} S_t = 2t^2 \int_{-1}^{1}\int_{-1}^{1} f(y_1, y_2)^2\,\mathrm{d}y_1\,\mathrm{d}y_2 = 2t^2 \int_{-1}^{1}\int_{-1}^{1} 1\,\mathrm{d}y_1\,\mathrm{d}y_2 = 8t^2.$$

For the ℓ-th cumulant $\gamma_\ell(S_t)$ of S_t we obtain, by Theorem 3.7,

$$\gamma_\ell(S_t) = \sum_{\sigma\in\widetilde{\Pi}_{\geq 2}(2,\dots,2)} \int_{X^{|\sigma|}} (\otimes_{j=1}^{\ell} f)_\sigma\,\mathrm{d}\mu_t^{|\sigma|}.$$

Because of $\mu_t = t\mu$ and $|\sigma| \leq \ell$ for $\sigma \in \widetilde{\Pi}_{\geq 2}(2, \dots, 2)$, the ℓ-th cumulant is a polynomial in t, and its degree is at most ℓ. There are $2^{\ell-1}\,(\ell-1)!$ partitions $\sigma \in \widetilde{\Pi}_{\geq 2}(2, \dots, 2)$ with $|\sigma| = \ell$. Indeed, we obtain a cycle by connecting two functions having two variables in the same block of such a partition. There are $(\ell - 1)!$ possible cycles and for each cycle we can switch the two variables of a function, which gives us $2^{\ell-1}$ possible combinations. For such a partition $\sigma \in \widetilde{\Pi}_{\geq 2}(2, \dots, 2)$ with $|\sigma| = \ell$ we have

$$\int_{X^{|\sigma|}} (\otimes_{j=1}^{\ell} f)_\sigma\,\mathrm{d}\lambda^{|\sigma|} = \int_{-1}^{1} \dots \int_{-1}^{1} \prod_{j=1}^{|\sigma|} \operatorname{sgn}(y_j)^2\,\mathrm{d}y_1 \dots \mathrm{d}y_{|\sigma|} = 2^\ell$$

so that

$$\lim_{t\to\infty} \frac{\gamma_\ell(S_t)}{t^\ell} = 2^{\ell-1}\,(\ell-1)!\,2^\ell = 2^{2\ell-1}\,(\ell-1)!$$

for $\ell \in \mathbb{N}$. As a consequence, we have

$$\lim_{t\to\infty} \gamma_\ell(\sqrt{2}S_t/\sqrt{\operatorname{Var} S_t}) = \frac{2^{\frac{\ell}{2}}\,2^{2\ell-1}\,(\ell-1)!}{8^{\frac{\ell}{2}}} = 2^{\ell-1}\,(\ell-1)!$$

for $\ell \in \mathbb{N}$. Since these are the moments of a centred chi-square distribution with one degree of freedom, the method of moments or cumulants (see Proposition 2.6) yields that $\sqrt{2}S_t/\sqrt{\operatorname{Var} S_t}$ converges in distribution to such a random variable as $t \to \infty$.

The limiting distribution can be also computed by applying Theorem 7.3 from [40] that was discussed above.

Instead of a family of geometric Poisson U-statistics we can consider a family of random vectors $(\mathbf{S}_t)_{t\geq 1}$ of geometric Poisson U-statistics. More precisely, let $\mathbf{S}_t = (S_t^{(1)},\dots,S_t^{(m)})$ be such that

$$S_t^{(\ell)} = g^{(\ell)}(t) \sum_{(x_1,\dots,x_{k_\ell})\in\eta^{k_\ell}_{t,\neq}} \tilde{f}^{(\ell)}(x_1,\dots,x_{k_\ell})$$

with $g^{(\ell)} : \mathbb{R} \to (0,\infty)$ and $\tilde{f}^{(\ell)} \in L^1_s(\mu^{k_\ell})$ for $\ell = 1,\dots,m$. Now we are able to formulate the multivariate counterpart of Theorem 6.7:

Theorem 6.9 *Let $(\boldsymbol{S}_t)_{t\geq 1}$ be a family of vectors of absolutely convergent geometric Poisson U-statistics $(S_t^{(\ell)})_{t\geq 1}$ of order k_ℓ such that*

$$\int_X \left| \int_{X^{k-1}} \tilde{f}^{(\ell)}(x,y_1,\dots,y_{k-1})\,\mathrm{d}\mu(y_1,\dots,y_{k-1}) \right|^3 \mathrm{d}\mu(x) < \infty$$

for $\ell = 1,\dots,m$ and let $\boldsymbol{N}(\Sigma)$ be an m-dimensional Gaussian random vector with a covariance matrix Σ given by

$$\sigma_{uv} = \langle \tilde{f}_1^{(u)}, \tilde{f}_1^{(v)} \rangle_{L^2(\mu)} = k_u k_v \int_X \int_{X^{k_u-1}} \tilde{f}^{(u)}(x,y_1,\dots,y_{k_u-1})\,\mathrm{d}\mu(y_1,\dots,y_{k_u-1}) \int_{X^{k_v-1}} \tilde{f}^{(v)}(x,y_1,\dots,y_{k_v-1})\,\mathrm{d}\mu(y_1,\dots,y_{k_v-1})\,\mathrm{d}\mu(x)$$

for $u,v = 1,\dots,m$. Then there is a constant $C_{\tilde{f}^{(1)},\dots,\tilde{f}^{(m)}} > 0$ depending on $\tilde{f}^{(1)},\dots,\tilde{f}^{(m)}$ such that

$$d_3\left(\left(\frac{S_t^{(1)} - \mathbb{E}S_t^{(1)}}{g^{(1)}(t)\,t^{k_1-\frac{1}{2}}},\dots,\frac{S_t^{(m)} - \mathbb{E}S_t^{(m)}}{g^{(m)}(t)\,t^{k_m-\frac{1}{2}}} \right), \boldsymbol{N}(\Sigma) \right) \leq C_{\tilde{f}^{(1)},\dots,\tilde{f}^{(m)}}\, t^{-\frac{1}{2}}$$

for $t \geq 1$. For $m = 1$ we can replace the d_3-metric by the Wasserstein distance and, if $M_{ij}(\tilde{f}^{(1)}) < \infty$ for $i,j = 1,\dots,k_1$, by the Kolmogorov distance.

Proof. Without loss of generality we can assume that $g^{(\ell)} \equiv 1$ for $\ell = 1, \dots, m$. It follows from Theorem 5.6 b) that

$$\begin{aligned}
& d_3\left(\left(\frac{S_t^{(1)} - \mathbb{E}S_t^{(1)}}{g^{(1)}(t)\, t^{k_1-\frac{1}{2}}}, \dots, \frac{S_t^{(m)} - \mathbb{E}S_t^{(m)}}{g^{(m)}(t)\, t^{k_m-\frac{1}{2}}}\right), \mathbf{N}(\Sigma)\right) \\
& \leq \frac{1}{2}\sum_{u,v=1}^{m} |\langle \tilde{f}_1^{(u)}, \tilde{f}_1^{(v)} \rangle_{L^2(\mu)} - t^{-k_u-k_v+1} \langle f_{1,t}^{(u)}, f_{1,t}^{(v)} \rangle_{L^2(\mu_t)}| \\
& \quad + \frac{m^2}{4} \sum_{\ell=1}^{m} t^{-3k_\ell+\frac{3}{2}} \int_X |f_{1,t}^{(\ell)}(z)|^3 \, \mathrm{d}\mu_t(z) \\
& \quad + \sqrt{2}m \sqrt{\sum_{\ell=1}^{m} t^{-2k_\ell+1} \operatorname{Var} S_t^{(\ell)}} \sqrt{\sum_{\ell=1}^{m} t^{-2k_\ell+1} \sum_{n=2}^{k_\ell} n! \, \|f_{n,t}^{(\ell)}\|_{n,t}^2}.
\end{aligned}$$

Now the summands on the right-hand side simplify to

$$\begin{aligned}
& \langle \tilde{f}_1^{(u)}, \tilde{f}_1^{(v)} \rangle_{L^2(\mu)} - t^{-k_u-k_v+1} \langle f_{1,t}^{(u)}, f_{1,t}^{(v)} \rangle_{L^2(\mu_t)} \\
& = \langle \tilde{f}_1^{(u)}, \tilde{f}_1^{(v)} \rangle_{L^2(\mu)} - t^{-k_u-k_v+1+k_u-1+k_v-1+1} \, \langle \tilde{f}_1^{(u)}, \tilde{f}_1^{(v)} \rangle_{L^2(\mu)} = 0,
\end{aligned}$$

$$\begin{aligned}
t^{-3k_\ell+\frac{3}{2}} \int_X |f_{1,t}^{(\ell)}(z)|^3 \, \mathrm{d}\mu_t(z) &= t^{-3k_\ell+\frac{3}{2}+3k_\ell-3+1} \int_X |\tilde{f}_1^{(\ell)}(z)|^3 \, \mathrm{d}\mu(z) \\
&\leq \int_X |\tilde{f}_1^{(\ell)}(z)|^3 \, \mathrm{d}\mu(z) \, t^{-\frac{1}{2}},
\end{aligned}$$

$$t^{-2k_\ell+1} \operatorname{Var} S_t^{(\ell)} = t^{-2k_\ell+1} \sum_{n=1}^{k_\ell} n! \, \|\tilde{f}_n\|_n^2 \, t^{2k_\ell-n} \leq \sum_{n=1}^{k_\ell} n! \, \|\tilde{f}_n\|_n^2,$$

and

$$t^{-2k_\ell+1} \sum_{n=2}^{k_\ell} n! \, \|f_{n,t}^{(\ell)}\|_{n,t}^2 = t^{-2k_\ell+1} \sum_{n=2}^{k_\ell} n! \, \|\tilde{f}_n^{(\ell)}\|_n^2 \, t^{n+2(k_\ell-n)} \leq \sum_{n=2}^{k_\ell} n! \, \|\tilde{f}_n^{(\ell)}\|_n^2 \, t^{-1}$$

for $t \geq 1$, which proves the assertion. The special case for $m = 1$ follows from Equation (5.10) for the Wasserstein distance and from Equation (5.30) and the proofs of the Theorems 6.4 and 6.7 for the Kolmogorov distance. $\square$

6.3 Local Poisson U-statistics

Throughout this section, we assume that the space X is equipped with a metric and that $\mathcal{X}$ is the Borel σ-algebra generated by this metric. Let $B(z, r)$ be a ball with centre z and radius $r > 0$ in X. For points $x_1, \dots, x_k \in X$ we denote by $\operatorname{diam}(\{x_1, \dots, x_k\})$ the diameter of the points with respect to the given metric. We are interested in the behaviour of Poisson U-statistics that only depend on k-tuples of distinct points of η with diameter less than or equal to a given threshold.

Definition 6.10 *A Poisson U-statistic S of the form*

$$S = \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k)$$

with $f \in L^1_s(\mu^k)$ and a constant $\delta > 0$ such that $f(x_1,\dots,x_k) = 0$ for all $x_1,\dots,x_k \in X$ with $\operatorname{diam}(\{x_1,\dots,x_k\}) > \delta$ is called local.

If we consider a vector $\mathbf{S} = (S^{(1)},\dots,S^{(m)})$ of local Poisson U-statistics $S^{(\ell)}$ of order k_ℓ, $\ell = 1,\dots,m$, we denote the functions we sum over by $f^{(\ell)}$ and the thresholds for the diameter by δ_ℓ.

Theorem 6.11 *a) Let S be an absolutely convergent local Poisson U-statistic of order k and let N be a standard Gaussian random variable. Then there are constants $c_W > 0$ and $c_K > 0$ only depending on k such that*

$$d_W\left(\frac{S-\mathbb{E}S}{\sqrt{\operatorname{Var} S}}, N\right) \leq c_W\Big(1 + \Big(\sup_{z\in X} \mu(B(z,4\delta))\Big)^{\frac{3k-3}{2}}\Big)\frac{\|f^2\|_k}{\operatorname{Var} S}$$

and

$$d_K\left(\frac{S-\mathbb{E}S}{\sqrt{\operatorname{Var} S}}, N\right) \leq c_K\Big(1 + \Big(\sup_{z\in X} \mu(B(z,4\delta))\Big)^{\frac{3k-3}{2}}\Big)\frac{\|f^2\|_k}{\operatorname{Var} S}.$$

b) Let $\boldsymbol{S} = (S^{(1)},\dots,S^{(m)})$ be a vector of absolutely convergent local Poisson U-statistics of the orders $k_1,\dots,k_m$ and let $\delta = \max\limits_{\ell=1,\dots,m} \delta_\ell$. By $\boldsymbol{N}(\Sigma)$ we denote an m-dimensional centred Gaussian random vector with a positive semidefinite covariance matrix Σ. Then there is a constant $\tilde{c} > 0$ depending on $k_1,\dots,k_m$ such that

$$\begin{aligned}
& d_3\left(\boldsymbol{S}-\mathbb{E}\boldsymbol{S}, \boldsymbol{N}(\Sigma)\right) \\
& \leq \frac{1}{2}\sum_{u,v=1}^m |\sigma_{uv} - \operatorname{Cov}(S^{(u)},S^{(v)})| \\
& \quad + \tilde{c}\left(\sum_{\ell=1}^m \sqrt{\operatorname{Var} S^{(\ell)}} + 1\right)\sum_{u,v=1}^m \Big(1 + \Big(\sup_{z\in X}\mu(B(z,4\delta))\Big)^{\frac{3}{4}(k_u+k_v)-\frac{3}{2}}\Big) \\
& \qquad \left(\|(f^{(u)})^2\|_{k_u} + \|(f^{(v)})^2\|_{k_v}\right).
\end{aligned}$$

Proof. For $\sigma \in \widetilde{\Pi}(k_u,k_u,k_v,k_v)$ the function $(f^{(u)}\otimes f^{(u)}\otimes f^{(v)}\otimes f^{(v)})_\sigma$ vanishes if two arguments have a distance larger than 4δ. Hence, we obtain by Hölder's inequality

$$\begin{aligned}
M_{ij}(f^{(u)},f^{(v)}) &= \sum_{\substack{\sigma\in\widetilde{\Pi}(k_u,k_u,k_v,k_v)\\ s(\sigma)=(k_u-i,k_u-i,k_v-j,k_v-j)}} \int_{X^{|\sigma|}} |(f^{(u)}\otimes f^{(u)}\otimes f^{(v)}\otimes f^{(v)})_\sigma|\,\mathrm{d}\mu^{|\sigma|} \\
&= \sum_{\substack{\sigma\in\widetilde{\Pi}(k_u,k_u,k_v,k_v)\\ s(\sigma)=(k_u-i,k_u-i,k_v-j,k_v-j)}} \int_{X^{|\sigma|}} \mathbb{1}(\operatorname{diam}(\{y_1,\dots,y_{|\sigma|}\} \leq 4\delta) \\
&\qquad |(f^{(u)}\otimes f^{(u)}\otimes f^{(v)}\otimes f^{(v)})_\sigma(y_1,\dots,y_{|\sigma|})|\,\mathrm{d}\mu(y_1,\dots,y_{|\sigma|}) \\
&\leq \sum_{\substack{\sigma\in\widetilde{\Pi}(k_u,k_u,k_v,k_v)\\ s(\sigma)=(k_u-i,k_u-i,k_v-j,k_v-j)}} \Big(\sup_{z\in X}\mu(B(z,4\delta))\Big)^{|\sigma|-(k_u+k_v)/2} \|(f^{(u)})^2\|_{k_u}\|(f^{(v)})^2\|_{k_v}.
\end{aligned}$$

For $\sigma \in \widetilde{\Pi}(k_u, k_u, k_v, k_v)$ we have $|\sigma| \leq 2(k_u + k_v) - 3$ so that

$$\begin{aligned}
&M_{ij}(f^{(u)}, f^{(v)}) \\
&\leq |\widetilde{\Pi}(k_u, k_u, k_v, k_v)| \Big(1 + \big(\sup_{z \in X} \mu(B(z, 4\delta))\big)^{\frac{3}{2}(k_u+k_v)-3}\Big) \|(f^{(u)})^2\|_{k_u} \|(f^{(v)})^2\|_{k_v} \\
&\leq \frac{1}{2} |\widetilde{\Pi}(k_u, k_u, k_v, k_v)| \Big(1 + \big(\sup_{z \in X} \mu(B(z, 4\delta))\big)^{\frac{3}{2}(k_u+k_v)-3}\Big) \left(\|(f^{(u)})^2\|_{k_u}^2 + \|(f^{(v)})^2\|_{k_v}^2\right)
\end{aligned}$$

for $i = 1, \ldots, k_u$, $j = 1, \ldots, k_v$, and $u, v = 1, \ldots, m$. Since $M_{ij}(f) = M_{ij}(f, f)$, we also obtain

$$M_{ij}(f) \leq |\widetilde{\Pi}(k, k, k, k)| \Big(1 + \big(\sup_{z \in X} \mu(B(z, 4\delta))\big)^{3k-3}\Big) \|f^2\|_k^2$$

for $i, j = 1, \ldots, k$. Now Theorem 6.11 is a direct consequence of Theorem 6.3 and Theorem 6.4. □

Notes: The example for a not absolutely convergent Poisson U-statistic and the univariate part of Theorem 6.3 are from *Reitzner and Schulte 2011*. The bound for the Kolmogorov distance in Theorem 6.4 is proven in *Schulte 2012b*.

Geometric Poisson U-statistics are considered in *Reitzner and Schulte 2011*. The formula for the asymptotic variance of a geometric Poisson U-statistic and the counterexample motivating the condition $\tilde{V} > 0$ are also taken from there. The bound for the Wasserstein distance in Theorem 6.7 and the multivariate version, Theorem 6.9, are due to *Last, Penrose, Schulte, and Thäle 2012*. The bound for the Kolmogorov distance in Theorem 6.7 is derived in *Schulte 2012b*.

The definition of a local Poisson U-statistic and the bound for the Wasserstein distance in Theorem 6.11 are from *Reitzner and Schulte 2011*. The bound for the Kolmogorov distance and the multivariate result are new.

Chapter 7

Poisson U-statistics in stochastic geometry

The aim of this chapter is to apply the results of the previous chapter to problems from stochastic geometry. We investigate some functionals of the intersection process of Poisson k-flats. For $k = d - 1$ we have a Poisson hyperplane process that induces a random tessellation in $\mathbb{R}^d$. We are interested in the combinatorial structure of this tessellation in an observation window. Finally, we consider the number of edges and the total edge length of a random graph, the so-called Gilbert graph.

As a further example, one could take the number of k-simplices induced by a Poisson point process on the d-dimensional torus. Decreusefond, Ferraz, Randriambololona, and Vergne investigate this problem in [11], using the Malliavin-Stein method. But the considered Poisson functionals are geometric Poisson U-statistics so that central limit theorems follow directly from our results in Section 6.2.

7.1 Intersection process of Poisson k-flats

Throughout this section, η_t is a stationary Poisson k-flat process in $\mathbb{R}^d$ with $k \in \{1, \dots, d-1\}$. This is a Poisson point process on the space $A(d,k)$ of all k-dimensional affine subspaces of $\mathbb{R}^d$ whose distribution is invariant under translations. The intensity measure μ_t of η_t is of the form

$$\mu_t(\cdot) = t \int_{G(d,k)} \int_{E^\perp} \mathbb{I}\,(E + x \in \cdot)\, \mathrm{d}\mathcal{H}^{d-k}(x)\, \mathrm{d}\mathbb{Q}(E),$$

with $t > 0$ and a probability measure $\mathbb{Q}$ on $G(d,k)$, the space of all k-dimensional linear subspaces of $\mathbb{R}^d$. If $\mathbb{Q}$ is the uniform distribution (i.e. $\mathbb{Q}$ is the Haar measure on $G(d,k)$ with the right normalization), then η_t is isotropic, which means that its distribution is invariant under rotations. In the following, we use the convention $\mu = \mu_1$.

The intersection process of order $1 \leq m \leq \lfloor d/(d-k) \rfloor$ is given as the set of all intersections $E_1 \cap \ldots \cap E_m$ of m distinct flats $E_1, \dots, E_m$ of η_t. It is assumed that $m \leq \lfloor d/(d-k) \rfloor$ since, otherwise, all intersections would be empty almost surely. We are interested in the behaviour of Poisson functionals $F_t^{(\ell)}$, $\ell = 1, \dots, n$, of the form

$$F_t^{(\ell)} = \frac{1}{m_\ell!} \sum_{(E_1,\dots,E_{m_\ell}) \in \eta_{t,\neq}^{m_\ell}} \psi_\ell(E_1 \cap \dots \cap E_{m_\ell} \cap W_\ell) \tag{7.1}$$

with $1 \le m_\ell \le \lfloor d/(d-k) \rfloor$, $W_\ell \in \mathcal{K}^d$, and measurable functionals $\psi_\ell : \mathcal{K}^d \to \overline{\mathbb{R}}$. Additionally, we assume that $\psi_\ell(\emptyset) = 0$ and that there are constants $c_{\psi_\ell, W_\ell} > 0$ such that

$$|\psi_\ell(E_1 \cap \dots \cap E_{m_\ell} \cap W_\ell)| \le c_{\psi_\ell, W_\ell} \tag{7.2}$$

for μ-almost all $(E_1, \dots, E_{m_\ell}) \in A(d,k)^{m_\ell}$. Then the following multivariate central limit theorem holds:

Theorem 7.1 *Let* $\boldsymbol{F}_t = \left(t^{-(m_1-1/2)}(F_t^{(1)} - \mathbb{E}F_t^{(1)}), \dots, t^{-(m_n-1/2)}(F_t^{(n)} - \mathbb{E}F_t^{(n)})\right)$ *with* $F_t^{(\ell)}$, $\ell = 1, \dots, n$, *as in formula* (7.1), *suppose that condition* (7.2) *is satisfied, and let* $\boldsymbol{N}(\Sigma)$ *be an* n*-dimensional centred Gaussian random vector with covariance matrix* $\Sigma = (\sigma_{uv})_{u,v=1,\dots,n}$ *given by*

$$\sigma_{uv} = \frac{1}{(m_u-1)!(m_v-1)!} \int\limits_{[W_u]\cap[W_v]} \int\limits_{[W_u]^{m_u-1}} \psi_u(E_1 \cap E_2 \cap \dots \cap E_{m_u} \cap W_u)\, \mathrm{d}E_2 \dots \mathrm{d}E_{m_u}$$
$$\int\limits_{[W_v]^{m_v-1}} \psi_v(E_1 \cap E_2' \cap \dots \cap E_{m_v}' \cap W_v)\, \mathrm{d}E_2' \dots \mathrm{d}E_{m_v}'\, \mathrm{d}E_1$$

for $u, v = 1, \dots, n$. *Then there is a constant* $c > 0$ *depending on* $m_1, \dots, m_n$, $W_1, \dots,$ W_n, *and* $\psi_1, \dots, \psi_n$ *such that*

$$d_3(\boldsymbol{F}_t, \boldsymbol{N}(\Sigma)) \le c\, t^{-\frac{1}{2}}$$

for $t \ge 1$. *For* $n = 1$ *the* d_3*-distance can be replaced by the Wasserstein distance or the Kolmogorov distance.*

Proof. The Poisson functionals $(F_t^{(\ell)})_{t\ge1}$, $\ell = 1, \dots, n$, are geometric Poisson U-statistics. Because of $\mu([W_\ell]) < \infty$ and the assumption (7.2), all integrals involving $\psi_\ell(E_1 \cap \dots \cap E_{m_\ell} \cap W_\ell)$ are finite. Hence, the assertion is a direct consequence of Theorem 6.9. □

In Theorem 7.1, we consider fixed observation windows and increase the intensity of the underlying Poisson point process η_t. An alternative regime is to keep the intensity fixed and to increase the observation windows. More precisely, we work with families of homotetic observation windows $(\varrho W_\ell)_{\varrho \ge 1}$ with $W_\ell \in \mathcal{K}^d$ for $\ell = 1, \dots, n$. For this setting we have to assume that the functionals ψ_ℓ are α_ℓ-homogeneous, i.e.

$$\psi_\ell(\varrho A) = \varrho^{\alpha_\ell} \psi_\ell(A)$$

for all $\varrho > 0$ and $A \in \mathcal{K}^d$. We consider a Poisson point process η_{t_0} of fixed intensity $t_0 > 0$ and Poisson functionals

$$F_\varrho^{(\ell)} = \frac{1}{m_\ell!} \sum_{(E_1,\dots,E_{m_\ell}) \in \eta_{t_0,\neq}^{m_\ell}} \psi_\ell(E_1 \cap \dots \cap E_{m_\ell} \cap \varrho W_\ell) \tag{7.3}$$

with $1 \leq m_\ell \leq \lfloor d/(d-k) \rfloor$, $W_\ell \in \mathcal{K}^d$, and $\psi_\ell : \mathcal{K}^d \to \overline{\mathbb{R}}$ satisfying condition (7.2) for $\ell = 1, \ldots, n$. Now we can keep the intensity parameter $t_0 > 0$ fixed and formulate a central limit theorem for increasing observation windows.

Corollary 7.2 *Let*

$$\boldsymbol{F}_\varrho = \left(\varrho^{-\alpha_1-(m_1-1/2)(d-k)}(F_\varrho^{(1)} - \mathbb{E}F_\varrho^{(1)}), \ldots, \varrho^{-\alpha_n-(m_n-1/2)(d-k)}(F_\varrho^{(n)} - \mathbb{E}F_\varrho^{(n)})\right)$$

with $F_\varrho^{(\ell)}$ as in formula (7.3) and let $\boldsymbol{N}(\Sigma_{t_0})$ be an n-dimensional centred Gaussian random vector with covariance matrix $\Sigma_{t_0} = t_0^{2d-1}\Sigma$, where Σ is the same matrix as in Theorem 7.1. Then there is a constant $c > 0$ depending on $m_1, \ldots, m_n$, $W_1, \ldots, W_n$, $\psi_1, \ldots, \psi_n$, and t_0 such that

$$d_3(\boldsymbol{F}_r, \boldsymbol{N}(\Sigma_{t_0})) \leq c\, \varrho^{-\frac{1}{2}(d-k)}$$

for $\varrho \geq 1$. For $n = 1$ the d_3-distance can be replaced by the Wasserstein distance or the Kolmogorov distance.

Proof. Observe that the Poisson point processes $\varrho^{-1}\eta_{t_0}$ and $\eta_{\varrho^{d-k}t_0}$ have the same distribution. Since

$$\begin{aligned} F_\varrho^{(\ell)} &= \varrho^{\alpha_\ell}\frac{1}{m_\ell!}\sum_{(E_1,\ldots,E_{m_\ell})\in\eta_{t_0,\neq}^{m_\ell}} \psi_\ell(\varrho^{-1}E_1 \cap \ldots \cap \varrho^{-1}E_{m_\ell} \cap W_\ell) \\ &= \varrho^{\alpha_\ell}\frac{1}{m_\ell!}\sum_{(E_1,\ldots,E_{m_\ell})\in\eta_{\varrho^{d-k}t_0,\neq}^{m_\ell}} \psi_\ell(E_1 \cap \ldots \cap E_{m_\ell} \cap W_\ell) = \varrho^{\alpha_\ell}F_{\varrho^{d-k}t_0}^{(\ell)}, \end{aligned}$$

where equality means equality in distribution, the corollary is a direct consequence of Theorem 7.1. □

We say that $E_1, \ldots, E_m \in A(d,k)$ are in general position if $E_1 \cap \ldots \cap E_m$ is a $d-m(d-k)$-dimensional affine subspace of $\mathbb{R}^d$. For the rest of this section, we assume that for every $2 \leq m \leq \lfloor d/(d-k) \rfloor$ all combinations of m distinct flats of η_t are in general position almost surely. This assumption is always satisfied if the directional distribution $\mathbb{Q}$ is absolutely continuous with respect to the invariant measure on $G(d,k)$ (see [79, Lemma 13.2.1]).

Now Theorem 7.1 and Corollary 7.2 can be applied in the following situations, where they give us (multivariate) central limit theorems for increasing intensity or increasing observation windows:

Example 7.3 *Let $W_1, \ldots, W_n \in \mathcal{K}^d$, $\psi_1 = \ldots = \psi_n = \chi$, and $m_1 = \ldots = m_n = m \in \{1, \ldots, \lfloor d/(d-k) \rfloor\}$. Then we count the numbers of $d-m(d-k)$-dimensional intersection flats of η_t (resp. η_{t_0}) that hit the windows $W_1, \ldots, W_n$ (resp. $\varrho W_1, \ldots, \varrho W_n$).*

Example 7.4 *Choose $W_1, \ldots, W_n \in \mathcal{K}_0^d$ and $m_1 = \ldots = m_n = m \in \{1, \ldots, \lfloor d/(d-k) \rfloor\}$ as in the previous example and let $\psi_1 = \ldots = \psi_n = V_{d-m(d-k)}$ (which is homogeneous of degree $d-m(d-k)$). For each window W_ℓ (resp. ϱW_ℓ) we compute the sum of the $d-m(d-k)$-dimensional volumes of all $d-m(d-k)$-dimensional intersection flats of η_t (resp. η_{t_0}) in W_ℓ (resp. ϱW_ℓ).*

Example 7.5 *Let* $n = \lfloor d/(d-k) \rfloor$. *For* $W_1 = \ldots = W_n = W \in \mathcal{K}^d$, $\psi_1 = \ldots = \psi_n = \chi$, *and* $m_\ell = \ell$ *for* $\ell = 1, \ldots, n$ *we count the numbers of* $d - \ell(d-k)$*-dimensional intersections flats of* η_t *(resp.* η_{t_0}*) hitting* W *(resp.* ϱW*).*

Example 7.6 *For* $n = \lfloor d/(d-k) \rfloor$, $W_1 = \ldots = W_n = W \in \mathcal{K}_0^d$, *and* $\psi_\ell = V_{d-\ell(d-k)}$ *and* $m_\ell = \ell$ *for* $\ell = 1, \ldots, n$ *we sum over the* $d - \ell(d-k)$*-dimensional volumes of the* $d - \ell(d-k)$*-dimensional intersection flats of* η_t *(resp.* η_t*) in* W *(resp.* ϱW*). Here,* ψ_ℓ *is homogeneous of degree* $d - \ell(d-k)$*.*

The Examples 7.5 and 7.6 for the special case $k = d-1$ are considered by Heinrich and Heinrich, Schmidt, and Schmidt in [23, 25]. The proofs in both works make use of the Hoeffding decomposition of U-statistics. In [25], it is additionally assumed that W is the unit ball. The number of intersection points of a Poisson line process within increasing circles is investigated by Paroux in [63].

Example 7.7 *Fix* $1 \leq m \leq \lfloor d/(d-k) \rfloor$ *and let* $m_1 = \ldots = m_{d-m(d-k)+1} = m$, $W_1 = \ldots = W_{d-m(d-k)+1} = W \in \mathcal{K}_0^d$, *and* $\psi_\ell = V_{\ell-1}$ *for* $\ell = 1, \ldots, d - m(d-k) + 1$*. Now sum over the intrinsic volumes of the* $d - m(d-k)$*-dimensional intersection flats of* η_t *(resp.* η_{t_0}*) in* W *(resp.* ϱW*). Here,* ψ_ℓ *is homogeneous of degree* $\ell - 1$*.*

Whenever the functional ψ_ℓ is the n-dimensional volume of an n-dimensional intersection flat, we assume that W_ℓ is full dimensional since the Poisson U-statistic is zero almost surely, otherwise.

Due to the general structure of Theorem 7.1 and Corollary 7.2, we can also consider more complicated situations with different observation windows W_ℓ, functionals ψ_ℓ, and orders m_ℓ. Moreover, the functionals ψ_ℓ do not need to be additive.

7.2 Poisson hyperplane tessellations

For this section we fix $k = d-1$ so that η_t is a Poisson hyperplane process. Moreover, we assume that η_t is isotropic, which means that the probability measure $\mathbb{Q}$ is the Haar measure on $G(d, d-1)$ with the right normalization. This restriction allows us to derive explicit formulas for the asymptotic covariances. But central limit theorems still hold under the weaker assumption that the hyperplanes of η_t are in general position almost surely, i.e. the intersection of $1 \leq m \leq d$ distinct hyperplanes of η_t is a $(d-m)$-flat almost surely. This is always satisfied if the directional distribution $\mathbb{Q}$ is not concentrated on a great subsphere (see [79, Theorem 10.3.2]).

The Poisson hyperplane process η_t decomposes $\mathbb{R}^d$ into a system of polytopes with disjoint interiors. We call this system the Poisson hyperplane tessellation generated by η_t and denote the single polytopes as cells. As the ℓ-dimensional faces of a Poisson hyperplane tessellation we regard the system of all ℓ-dimensional faces of the cells. For a window $W \in \mathcal{K}_0^d$ let $N_t^{(\ell)}(W)$ be the number of ℓ-dimensional faces of the Poisson hyperplane tessellation induced by η_t that hit W. We are interested in the behaviour of $N_t^{(\ell)}(W)$ for increasing intensity t.

It is due to Miles (see [53, Theorem A]) that

$$N_t^{(\ell)}(W) = \sum_{j=d-\ell}^{d} \binom{j}{d-\ell} \Phi_t^{(j)}(W) \tag{7.4}$$

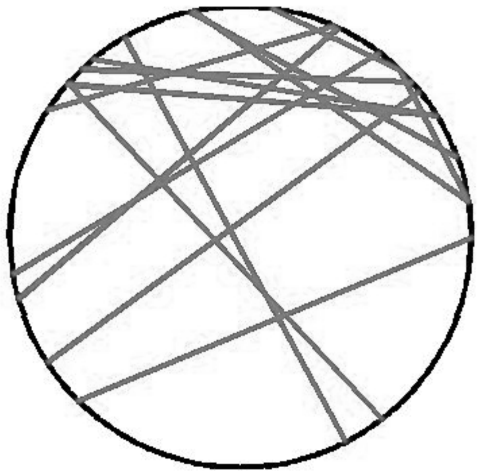

Figure 7.1: Poisson line tessellation within a circle

with $\Phi_t^{(0)}(W)=1$ and

$$\Phi_t^{(j)}(W)=\frac{1}{j!}\sum_{(H_1,\dots,H_j)\in\eta^j_{t,\neq}}\chi(H_1\cap\ldots\cap H_j\cap W)$$

for $j=1,\dots,d$. Here, $\Phi_t^{(j)}(W)$ is the number of $(d-j)$-dimensional intersection flats of η_t hitting W. Therefore, $N_t^{(\ell)}(W)$ is a sum of geometric Poisson U-statistics. This fact allows us to determine the asymptotic covariance structure of the numbers of ℓ-dimensional faces.

Proposition 7.8 *Define*

$$\sigma_{uv}=\binom{d}{u}\binom{d}{v}\kappa_{d-1}^2\left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{2(d-1)}\int_{[W]}V_{d-1}(H\cap W)^2\,\mathrm{d}H \tag{7.5}$$

for $u,v=0,\dots,d$. There are constants c_{uv}, $u,v\in\{0,\dots,d\}$, such that

$$\left|\frac{\operatorname{Cov}(N_t^{(u)}(W),N_t^{(v)}(W))}{t^{2d-1}}-\sigma_{uv}\right|\le c_{uv}\,t^{-\frac{1}{2}}$$

for $t\geq 1$.

Proof. It follows directly from Equation (7.4) that

$$\operatorname{Cov}(N_t^{(u)}(W),N_t^{(v)}(W))=\sum_{i=d-u}^{d}\sum_{j=d-v}^{d}\binom{i}{d-u}\binom{j}{d-v}\operatorname{Cov}(\Phi_t^{(i)}(W),\Phi_t^{(j)}(W)).$$

Formula (3.4) in Theorem 3.4 yields that $\operatorname{Cov}(\Phi_t^{(i)}(W),\Phi_t^{(j)}(W))$ is a polynomial of degree $i+j-1$ in t for $i,j\geq 1$. For $i=0$ or $j=0$ we have $\operatorname{Cov}(\Phi_t^{(i)}(W),\Phi_t^{(j)}(W))=0$.

Hence, $\operatorname{Cov}(N_t^{(u)}(W), N_t^{(v)}(W))$ is a polynomial of degree $2d-1$ in t. It has (up to the binomial coefficients) the same leading coefficient as $\operatorname{Cov}(\Phi_t^{(d)}(W), \Phi_t^{(d)}(W))$, which is

$$\begin{aligned} & d^2 \int_{[W]} \left(\frac{1}{d!} \int_{[W]^{d-1}} \chi(H \cap H_1 \cap \ldots \cap H_{d-1} \cap W) \, \mathrm{d}H_1 \ldots \mathrm{d}H_{d-1} \right)^2 \mathrm{d}H \\ & = \kappa_{d-1}^2 \left(\frac{\kappa_{d-1}}{d\, \kappa_d} \right)^{2(d-1)} \int_{[W]} V_{d-1}(H \cap W)^2 \, \mathrm{d}H. \end{aligned}$$

In the last step, we applied $(d-1)$-times Crofton's formula (see Proposition 2.3). □

This implies that the asymptotic covariance matrix of the random vector

$$t^{-d+\frac{1}{2}} (N_t^{(0)}(W), \ldots, N_t^{(d)}(W))$$

has rank 1, and a similar computation shows that

$$\lim_{t \to \infty} \frac{\mathbb{E} N_t^{(\ell)}}{t^d} = \kappa_d \binom{d}{\ell} \left(\frac{\kappa_{d-1}}{d\, \kappa_d} \right)^d \operatorname{Vol}(W) \tag{7.6}$$

for $\ell = 0, \ldots, d$. This is caused by the fact that for every $N_t^{(\ell)}(W)$ the expression $\binom{d}{d-\ell} \Phi_t^{(d)}(W)$ is the asymptotically leading term. Equation (7.6), which gives us the intensity of the ℓ-faces, was first obtained by Mecke in [51]. For further results about intensities of Poisson hyperplane tessellations we refer to the monograph [79] by Schneider and Weil and the references therein. The knowledge of the covariance structure allows us to formulate the following central limit theorem:

Theorem 7.9 *a) Let N be a standard Gaussian random variable. For $\ell \in \{0, \ldots, d\}$ there are constants $c_W^{(\ell)} > 0$ and $c_K^{(\ell)} > 0$ such that*

$$d_W \left(\frac{N_t^{(\ell)}(W) - \mathbb{E} N_t^{(\ell)}(W)}{\sqrt{\operatorname{Var} N_t^{(\ell)}(W)}}, N \right) \leq c_W^{(\ell)} \, t^{-\frac{1}{2}}$$

and

$$d_K \left(\frac{N_t^{(\ell)}(W) - \mathbb{E} N_t^{(\ell)}(W)}{\sqrt{\operatorname{Var} N_t^{(\ell)}(W)}}, N \right) \leq c_K^{(\ell)} \, t^{-\frac{1}{2}}$$

for $t \geq 1$.

b) Let $\boldsymbol{N}(\Sigma)$ be a $d+1$-dimensional centred Gaussian random vector with a covariance matrix $\Sigma = (\sigma_{uv})_{u,v=0,\ldots,d}$ given by formula (7.5). Then there is a constant $\tilde{c} > 0$ such that

$$d_3 \left(t^{-d+\frac{1}{2}} (N_t^{(0)}(W) - \mathbb{E} N_t^{(0)}(W), \ldots, N_t^{(d)}(W) - \mathbb{E} N_t^{(d)}(W)), \boldsymbol{N}(\Sigma) \right) \leq \tilde{c}\, t^{-\frac{1}{2}}$$

for $t \geq 1$.

Proof. It follows from Theorem 4.11 that the n-th kernel of the Wiener-Itô chaos expansion of $\Phi_t^{(\ell)}(W)$ is a polynomial of degree $\ell - n$ in t, if $n \leq \ell$. Otherwise the kernel is zero. By the linearity of the multiple Wiener-Itô integral, we obtain that $N_t^{(\ell)}(W)$ has a Wiener-Itô chaos expansion of order d, where the n-th kernel $f_{n,t}^{(\ell)}$, $n \leq d$, is a polynomial of degree $d - n$ in t.

Together with $|\sigma| \leq 2i + j + k - 3$ for $\sigma \in \widetilde{\Pi}(i, i, j, k)$, we see that

$$\int_{[W]^{|\sigma|}} |(f_{i,t} \otimes f_{i,t} \otimes f_{j,t} \otimes f_{k,t})_\sigma| \, \mathrm{d}\mu_t^{|\sigma|}$$

with $\sigma \in \widetilde{\Pi}(i, i, j, k)$ is a polynomial in t with a degree of at most

$$|\sigma| + 2(d - i) + d - j + d - k \leq 2i + j + k - 3 + 2(d - i) + d - j + d - k = 4d - 3.$$

Since $\operatorname{Var} N_t^{(\ell)}(W)$ is a polynomial of degree $2d-1$ in t, the right-hand sides in Theorem 5.11 a) and in Theorem 5.12 are of order $t^{-\frac{1}{2}}$ or less which proves the univariate bounds. By the same argument and Proposition 7.8, the right-hand side in Theorem 5.11 b) is of order $t^{-\frac{1}{2}}$ or less, which implies the multivariate bound. □

For the same reason as in the previous section, we can also consider the regime where we keep the intensity fixed and increase the observation window.

7.3 Gilbert graph

The aim of this section is to investigate a random graph, the so-called Gilbert graph or random geometric graph, that is constructed in the following way. Let η_t be the restriction of a stationary Poisson point process in $\mathbb{R}^d$ with intensity $t > 0$ to an observation window $W \in \mathcal{K}_0^d$. We take the points of η_t as vertices and connect two vertices by an edge if and only if their Euclidean distance is not greater than a given threshold δ. We denote this graph by $G(\eta_t, \delta)$.

The monograph [69] by Penrose is an exhaustive reference for these random graphs, and for further developments we refer to the works [12, 20, 49, 57] by Devroye, György, Han, Lugosi, Makowski, McDiarmid, Müller, and Udina. In their recent works [6, 39, 40] concerning the Malliavin-Stein method, Bourguin, Lachiéze-Rey, and Peccati consider the Gilbert graph as an example and derive similar results as we do in the following.

We allow that the threshold for connecting two vertices by an edge depends on the intensity parameter $t > 0$ and denote it by δ_t. We are interested in the behaviour of $G(\eta_t, \delta_t)$ for $t \to \infty$. In particular, we consider the number of edges of $G(\eta_t, \delta_t)$

$$S_t = \frac{1}{2} \sum_{(x_1, x_2) \in \eta_{t,\neq}^2} \mathbb{1}(\operatorname{dist}(x_1, x_2) \leq \delta_t)$$

and the total edge length of $G(\eta_t, \delta_t)$

$$L_t = \frac{1}{2} \sum_{(x_1, x_2) \in \eta_{t,\neq}^2} \mathbb{1}(\operatorname{dist}(x_1, x_2) \leq \delta_t) \operatorname{dist}(x_1, x_2).$$

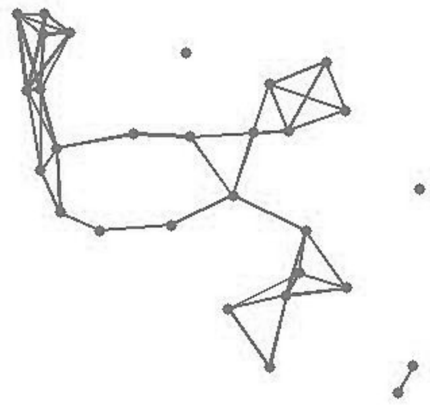

Figure 7.2: Gilbert graph in the plane

Note that S_t and L_t are Poisson U-statistics. By the Slivnyak-Mecke formula, we obtain

$$\mathbb{E}S_t = \frac{t^2}{2}\int_W\int_W \mathbb{1}(\operatorname{dist}(y_1,y_2)\leq \delta_t)\,\mathrm{d}y_1\,\mathrm{d}y_2$$

and

$$\mathbb{E}L_t = \frac{t^2}{2}\int_W\int_W \mathbb{1}(\operatorname{dist}(y_1,y_2)\leq \delta_t)\operatorname{dist}(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

For $\delta_t=\delta\in\mathbb{R}$ we have geometric Poisson U-statistics and can apply the results from Section 6.2. Hence, we focus on the situation that $\delta_t\to 0$ as $t\to\infty$ from now on. Moreover, we assume that δ_t is chosen in such a way that the limit of $t\,\delta_t^d$ for $t\to\infty$ exists (or is infinite).

Proposition 7.10 *a) We have*

$$\mathbb{E}S_t = \frac{t^2}{2}\big(\operatorname{Vol}(W)\,\kappa_d\,\delta_t^d + O(\delta_t^{d+1})\big) \text{ and } \mathbb{E}L_t = \frac{t^2}{2}\big(\operatorname{Vol}(W)\,\frac{d\,\kappa_d}{d+1}\,\delta_t^{d+1} + O(\delta_t^{d+2})\big)$$

for $\delta_t\to 0$.

b) For $\tilde{S}_t = S_t/\sqrt{\max\{t^3\delta_t^{2d}, t^2\delta_t^d\}}$ and $\tilde{L}_t = L_t/\sqrt{\max\{t^3\delta_t^{2d+2}, t^2\delta_t^{d+2}\}}$ we have

$$\lim_{t\to\infty}\begin{pmatrix}\operatorname{Var}\tilde{S}_t & \operatorname{Cov}(\tilde{S}_t,\tilde{L}_t)\\ \operatorname{Cov}(\tilde{S}_t,\tilde{L}_t) & \operatorname{Var}\tilde{L}_t\end{pmatrix} = \begin{cases}\Sigma_1, & \lim\limits_{t\to\infty} t\,\delta_t^d=\infty\\ \Sigma_1+\frac{1}{c}\,\Sigma_2, & \lim\limits_{t\to\infty} t\,\delta_t^d = c\in(1,\infty)\\ c\,\Sigma_1+\Sigma_2, & \lim\limits_{t\to\infty} t\,\delta_t^d=c\in(0,1]\\ \Sigma_2, & \lim\limits_{t\to\infty} t\,\delta_t^d = 0\end{cases} \tag{7.7}$$

with

$$\Sigma_1 = \operatorname{Vol}(W)\begin{pmatrix}\kappa_d^2 & d\,\kappa_d^2/(d+1)\\ d\,\kappa_d^2/(d+1) & (d\,\kappa_d/(d+1))^2\end{pmatrix}$$

and

$$\Sigma_2 = \frac{\mathrm{Vol}(W)}{2} \begin{pmatrix} \kappa_d & d\,\kappa_d/(d+1) \\ d\,\kappa_d/(d+1) & d\,\kappa_d/(d+2) \end{pmatrix}.$$

Proof. It follows from Theorem 4.11 that $S_t = \mathbb{E}S_t + I_{1,t}(g_{1,t}) + I_{2,t}(g_{2,t})$ with

$$g_{1,t}(x_1) = t \int_W \mathbb{1}(\mathrm{dist}(x_1,y) \le \delta_t)\,\mathrm{d}y \quad \text{and} \quad g_{2,t}(x_1,x_2) = \frac{1}{2}\,\mathbb{1}(\mathrm{dist}(x_1,x_2) \le \delta_t)$$

and $L_t = \mathbb{E}L_t + I_{1,t}(h_{1,t}) + I_{2,t}(h_{2,t})$ with

$$h_{1,t}(x_1) = t \int_W \mathbb{1}(\mathrm{dist}(x_1,y) \le \delta_t)\,\mathrm{dist}(x_1,y)\,\mathrm{d}y$$

and

$$h_{2,t}(x_1,x_2) = \frac{1}{2}\,\mathbb{1}(\mathrm{dist}(x_1,x_2) \le \delta_t)\,\mathrm{dist}(x_1,x_2).$$

Let $W_{-\delta_t} = W \setminus \{x \in W : \mathrm{dist}(x,\partial W) \le \delta_t\}$. A short computation with polar coordinates proves that

$$g_{1,t}(x_1) = t\,\kappa_d\,\delta_t^d \quad \text{and} \quad h_{1,t}(x_1) = t\,\frac{d\,\kappa_d}{d+1}\,\delta_t^{d+1}$$

for $x_1 \in W_{-\delta_t}$. On the remaining part $W \setminus W_{-\delta_t}$, whose volume is of order δ_t for $\delta_t \to 0$, the absolute values of $g_{1,t}$ and $h_{1,t}$ are smaller so that

$$\begin{aligned}
\mathbb{E}S_t &= \frac{t}{2}\int_W g_1(x_1)\,\mathrm{d}x_1 = \frac{t^2}{2}\big(\mathrm{Vol}(W)\,\kappa_d\,\delta_t^d + O(\delta_t^{d+1})\big) \\
\mathbb{E}L_t &= \frac{t}{2}\int_W h_1(x_1)\,\mathrm{d}x_1 = \frac{t^2}{2}\Big(\mathrm{Vol}(W)\,\frac{d\,\kappa_d}{d+1}\,\delta_t^{d+1} + O(\delta_t^{d+2})\Big)
\end{aligned}$$

and

$$\begin{aligned}
\|g_{1,t}\|_{1,t}^2 &= t^3\big(\mathrm{Vol}(W)\,\kappa_d^2\,\delta_t^{2d} + O(\delta_t^{2d+1})\big) \\
\|h_{1,t}\|_{1,t}^2 &= t^3\Big(\mathrm{Vol}(W)\left(\frac{d\,\kappa_d}{d+1}\right)^2 \delta_t^{2d+2} + O(\delta_t^{2d+3})\Big) \\
\langle g_{1,t}, h_{1,t}\rangle_{L^2(t\lambda_d)} &= t^3\Big(\mathrm{Vol}(W)\,\frac{d\,\kappa_d^2}{d+1}\,\delta_t^{2d+1} + O(\delta_t^{2d+2})\Big).
\end{aligned}$$

Using the fact that the inner integral is the same for all $x_2 \in W_{-\delta_t}$ and polar coordinates, we obtain

$$\begin{aligned}
\|g_{2,t}\|_{2,t}^2 &= \frac{t^2}{4}\int_{W^2} \mathbb{1}(\mathrm{dist}(x_1,x_2) \le \delta_t)^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= t^2\Big(\mathrm{Vol}(W)\,\frac{\kappa_d}{4}\,\delta_t^d + O(\delta_t^{d+1})\Big) \\
\|h_{2,t}\|_{2,t}^2 &= \frac{t^2}{4}\int_{W^2} \mathbb{1}(\mathrm{dist}(x_1,x_2) \le \delta_t)^2\,\mathrm{dist}(x_1,x_2)^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= t^2\Big(\mathrm{Vol}(W)\,\frac{d\,\kappa_d}{4\,(d+2)}\,\delta_t^{d+2} + O(\delta_t^{d+3})\Big) \\
\langle g_{2,t}, h_{2,t}\rangle_{L^2((t\lambda_d)^2)} &= \frac{t^2}{4}\int_{W^2} \mathbb{1}(\mathrm{dist}(x_1,x_2) \le \delta_t)^2\,\mathrm{dist}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= t^2\Big(\mathrm{Vol}(W)\,\frac{d\,\kappa_d}{4\,(d+1)}\,\delta_d^{d+1} + O(\delta_t^{d+2})\Big).
\end{aligned}$$

Now Theorem 4.2 concludes the proof. □

Observe that Σ_1 has rank 1, whereas Σ_2 has full rank. Hence, the asymptotic covariance matrix is regular if $\lim_{t\to\infty} t\,\delta_t^d < \infty$. Knowing the covariance structure, we can state the following central limit theorem.

Theorem 7.11 *a) Let N be a standard Gaussian random variable. Then there are constants $c_{S,W}, c_{S,K}, c_{L,W}, c_{L,K} > 0$ such that*

$$\begin{aligned} d_W\left(\frac{S_t - \mathbb{E}S_t}{\sqrt{\operatorname{Var} S_t}}, N\right) &\leq c_{S,W}\, t^{-\frac{1}{2}} \max\{1, (t\,\delta_t^d)^{-\frac{1}{2}}\} \\ d_K\left(\frac{S_t - \mathbb{E}S_t}{\sqrt{\operatorname{Var} S_t}}, N\right) &\leq c_{S,K}\, t^{-\frac{1}{2}} \max\{1, (t\,\delta_t^d)^{-\frac{1}{2}}\} \\ d_W\left(\frac{L_t - \mathbb{E}L_t}{\sqrt{\operatorname{Var} L_t}}, N\right) &\leq c_{L,W}\, t^{-\frac{1}{2}} \max\{1, (t\,\delta_t^d)^{-\frac{1}{2}}\} \\ d_K\left(\frac{L_t - \mathbb{E}L_t}{\sqrt{\operatorname{Var} L_t}}, N\right) &\leq c_{L,K}\, t^{-\frac{1}{2}} \max\{1, (t\,\delta_t^d)^{-\frac{1}{2}}\} \end{aligned}$$

for $t \geq 1$.

b) Let $\boldsymbol{N}(\Sigma)$ be a two-dimensional centred Gaussian random vector with the covariance matrix Σ given by formula (7.7) and let $\tilde{S}_t$ and $\tilde{L}_t$ be as in Proposition 7.10. Then there are constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ such that

$$d_3\left((\tilde{S}_t - \mathbb{E}\tilde{S}_t, \tilde{L}_t - \mathbb{E}\tilde{L}_t), \boldsymbol{N}(\Sigma)\right) \leq \tilde{c}_1\,\delta_t + \tilde{c}_2\,\tilde{R}_t + \tilde{c}_3\, t^{-\frac{1}{2}} \max\{1, (t\,\delta_t^d)^{-\frac{1}{2}}\}$$

for $t \geq 1$ with

$$\tilde{R}_t = \begin{cases} t^{-1}\delta_t^{-d}, & \lim_{t\to\infty} t\,\delta_t^d = \infty \\ 0, & \lim_{t\to\infty} t\,\delta_t^d = c \in (0,\infty) \\ t\,\delta_t^d, & \lim_{t\to\infty} t\,\delta_t^d = 0 \end{cases}.$$

Proof. By the same arguments as in the proof of Proposition 7.10, we obtain

$$\frac{t^2}{16}\int_{W^2} \mathbb{1}(\operatorname{dist}(x_1,x_2) \leq \delta_t)^4 \,\mathrm{d}x_1\,\mathrm{d}x_2 = t^2 \operatorname{Vol}(W)\left(\frac{\kappa_d}{16}\,\delta_t^d + O(\delta_t^{d+1})\right)$$

and

$$\begin{aligned} &\frac{t^2}{16}\int_{W^2} \mathbb{1}(\operatorname{dist}(x_1,x_2) \leq \delta_t)^4 \operatorname{dist}(x_1,x_2)^4 \,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= t^2 \operatorname{Vol}(W)\left(\frac{d\,\kappa_d}{16\,(d+4)}\,\delta_t^{d+4} + O(\delta_t^{d+5})\right). \end{aligned}$$

By applying Theorem 6.11 a) to the standardizations of S_t and L_t, we obtain bounds of the orders

$$\left(1 + (t\,\delta_t^d)^{\frac{3}{2}}\right)\frac{t\,\delta_t^{\frac{1}{2}d}}{\max\{t^3\,\delta_t^{2d}, t^2\,\delta_t^d\}} = \left(1 + (t\,\delta_t^d)^{\frac{3}{2}}\right) t^{-\frac{1}{2}} \min\{(t\,\delta_t^d)^{-\frac{3}{2}}, (t\,\delta_t^d)^{-\frac{1}{2}}\}$$

and

$$\Big(1+(t\,\delta_t^d)^{\frac{3}{2}}\Big)\,\frac{t\,\delta_t^{\frac{1}{2}d+2}}{\max\{t^3\,\delta_t^{2d+2},t^2\delta_t^{d+2}\}}=\Big(1+(t\,\delta_t^d)^{\frac{3}{2}}\Big)\,t^{-\frac{1}{2}}\min\{(t\,\delta_t^d)^{-\frac{3}{2}},(t\,\delta_t^d)^{-\frac{1}{2}}\}.$$

Considering the cases $t\,\delta_t^d<1$ and $t\,\delta_t^d\geq 1$, one sees that both expressions behave like $t^{-\frac{1}{2}}\max\{1,(t\,\delta_t^d)^{-\frac{1}{2}}\}$, which concludes the proof of part a).

From the proof of Proposition 7.10, we know that there are constants $\tilde{c}_1,\tilde{c}_2>0$ such that

$$|\sigma_{11}-\operatorname{Var}\tilde{S}_t|+2\,|\sigma_{12}-\operatorname{Cov}(\tilde{S}_t,\tilde{L}_t)|+|\sigma_{22}-\operatorname{Var}\tilde{L}_t|\leq\tilde{c}_1\,\delta_t+\tilde{c}_2\,\tilde{R}_t$$

for $t\geq 1$. Now part b) is a direct consequences of Theorem 6.11 b). □

Note that Theorem 7.11 implies convergence in distribution if $t^2\delta_t^d\to\infty$ as $t\to\infty$. This means that central limit theorems for S_t and L_t hold if $\mathbb{E}S_t\to\infty$ as $t\to\infty$. The univariate central limit theorem for S_t with the same rate of convergence for the Wasserstein distance is derived in [39, Example 4.15]. It can also be deduced (without a rate of convergence) from Theorem 3.9 in [69], where subgraphs in $G(\eta_t,\delta_t)$ are counted. Regarding single edges as subgraphs, we can apply this result to S_t.

Notes: Section 7.1 is close to Section 6 in *Last, Penrose, Schulte, and Thäle 2012*, where a single functional was evaluated for different windows. In our setting, we can have different functionals for different observation windows. The bound for the Kolmogorov distance is new.

The results in Section 7.2 are not contained in any of the underlying works.

The Gilbert graph is considered as an example in *Reitzner and Schulte 2011*. The univariate results for the Wasserstein distance in Section 7.3 are a slight modification of the setting there. The extensions to the Kolmogorov distance and the multivariate central limit theorem are new.

Chapter 8

A central limit theorem for the Poisson-Voronoi approximation

In this chapter, we use the Malliavin-Stein method to derive a central limit theorem for the volume of the Poisson-Voronoi approximation. We present the problem and the main results in the first section. The second and the third section contain the proofs.

8.1 Introduction and results

Let $K \subset \mathbb{R}^d, d \geq 2$, be a compact convex set with interior points and let η_t be a Poisson point process in $\mathbb{R}^d$ with intensity measure $\mu_t = t\lambda_d$ with $t > 0$ and the d-dimensional Lebesgue measure λ_d. For every point $x \in \eta_t$ we define the Voronoi cell of x by

$$V_x = \left\{ z \in \mathbb{R}^d : ||x - z|| \leq ||y - z|| \text{ for all } y \in \eta_t \right\}$$

and call x the nucleus of V_x. We have $\operatorname{int}(V_x) \cap \operatorname{int}(V_y) = \emptyset$ for $x \neq y \in \eta_t$ and $\bigcup_{x \in \eta_t} V_x = \mathbb{R}^d$ so that the collection $(V_x)_{x \in \eta_t}$ of random polytopes constitutes a random tessellation of $\mathbb{R}^d$, the so-called Poisson-Voronoi tessellation, which is one of the standard models in stochastic geometry, and we refer to the monograph [79] by Schneider and Weil and the references therein for further details.

For our set K we define the Poisson-Voronoi approximation $\mathrm{A}_t(K)$ as

$$\mathrm{A}_t(K) = \bigcup_{x \in \eta_t \cap K} V_x,$$

which is a random approximation of K. It is possible to interpret the Poisson-Voronoi approximation in the following way. One wants to reconstruct an unknown convex body $K \in \mathcal{K}_0^d$, but the only information available is a kind of oracle which says for every point of a realization of the Poisson point process if it belongs to K. Now one approximates the unknown set K by taking the union of the Voronoi cells with nuclei in K.

In the sequel, we are interested in the volume of the Poisson-Voronoi approximation

$$\mathrm{PV}_t(K) = \mathrm{Vol}(\mathrm{A}_t(K)).$$

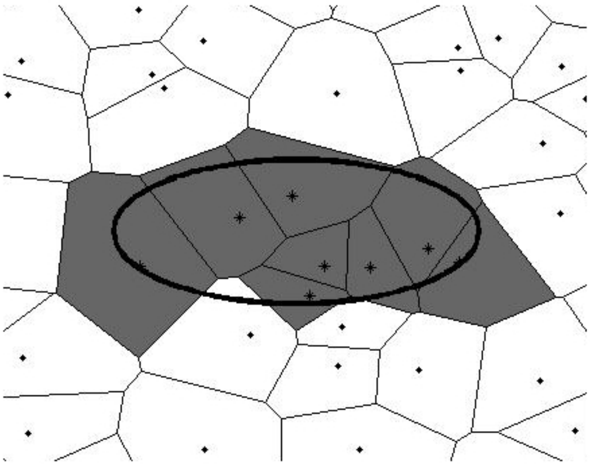

Figure 8.1: Poisson-Voronoi approximation of an ellipse

A short computation yields $\mathbb{E}\,\mathrm{PV}_t(K) = \mathrm{Vol}(K)$, which means that $\mathrm{PV}_t(K)$ is an unbiased estimator for the volume of K. Under weaker assumptions than convexity on the approximated set K, it is shown by Einmahl and Khmaladze, Khmaladze and Toronjadze, and Penrose in [16, 35, 71] that $\mathrm{PV}_t(K) \to \mathrm{Vol}(K)$ and $\mathrm{Vol}(\mathrm{A}_t(K)\Delta K) \to 0$ as $t \to \infty$, where $\mathrm{A}_t(K)\Delta K$ stands for the symmetric difference of $\mathrm{A}_t(K)$ and K. In [26], Heveling and Reitzner derive upper bounds for the asymptotic behaviour of $\mathrm{Var}\,\mathrm{PV}_t(K)$ and $\mathrm{Var}\,\mathrm{Vol}(\mathrm{A}_t(K)\Delta K)$ and large deviation inequalities for $\mathrm{PV}_t(K)$ and $\mathrm{Vol}(\mathrm{A}_t(K)\Delta K)$ for the same setting as in the present work. In [77], Reitzner, Spodarev, and Zaporozhets consider a more general class of approximated sets, namely sets of finite perimeter, and compute bounds for all non-centred moments

Our main result is that $\mathrm{PV}_t(K)$ behaves asymptotically like a Gaussian random variable if the intensity of the Poisson point process goes to infinity.

Theorem 8.1 *Let N be a standard Gaussian random variable. Then*

$$\frac{\mathrm{PV}_t(K) - \mathrm{Vol}(K)}{\sqrt{\mathrm{Var}\,\mathrm{PV}_t(K)}} \to N \quad \textit{in distribution as} \quad t \to \infty.$$

As pointed out in [26], the Poisson-Voronoi approximation has applications in non-parametric statistics, image analysis, and quantization problems. In this context, Theorem 8.1 can be helpful since it allows treating $\mathrm{PV}_t(K)$ as a Gaussian random variable if the intensity of the Poisson point process is sufficiently high.

For the proof of Theorem 8.1 we compute the Wiener-Itô chaos expansion of $\mathrm{PV}_t(K)$ and apply Corollary 5.14. In order to check the assumption (5.31) in this corollary, we have to prove some kind of uniform convergence for $n!\,\|f_{n,t}\|_{n,t}^2/\,\mathrm{Var}\,\mathrm{PV}_t(K)$, where $f_{n,t}$ is the n-th kernel of the Wiener-Itô chaos expansion of $\mathrm{PV}_t(K)$. Combining this property with the identity

$$\mathrm{Var}\,\mathrm{PV}_t(K) = \sum_{n=1}^{\infty} n!\,\|f_{n,t}\|_{n,t}^2,$$

we obtain as a byproduct lower and upper bounds for the variance of $\mathrm{PV}_t(K)$. Recall that $r(K)$ stands for the inradius of K, $V_i(K)$, $i = 0, \dots, d$, are the intrinsic volumes of K, and κ_j is the volume of the unit ball in $\mathbb{R}^j$.

Theorem 8.2 *There are explicit constants $\underline{C}, \overline{C} > 0$ depending only on the dimension d such that*

$$\underline{C}\, \kappa_1 V_{d-1}(K)\, t^{-1-\frac{1}{d}} \leq \operatorname{Var} \mathrm{PV}_t(K) \leq \overline{C} \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K)\, t^{-2+\frac{i}{d}} \tag{8.1}$$

for $t \geq (2/r(K))^d$.

Both bounds in formula (8.1) are of order $t^{-1-\frac{1}{d}}$ so that $\operatorname{Var} \mathrm{PV}_t(K)$ has order $t^{-1-\frac{1}{d}}$ as well. The asymptotically leading coefficients are constants times $V_{d-1}(K)$, which is proportional to the surface area $S(K)$ of K.

The upper bound in formula (8.1) is also contained in [26], where it is proven by a combination of the theory of valuations and the Poincaré inequality. The Poincaré inequality is related to the Wiener-Itô chaos expansion as well (for more details we refer to the work [41] by Last and Penrose). The lower bound is new as far as we know.

Although the construction of the Poisson-Voronoi approximation does not depend on the convexity of K and can also be done for more general classes of sets, we formulate our main results only for convex sets, in order to simplify the proofs. At the end of this chapter, we give two alternative conditions for the approximated set that allow us to weaken the convexity assumption.

8.2 Proof of Theorem 8.2

Because of Theorem 1 in [26], we know that $\mathrm{PV}_t(K) \in L^2(\mathbb{P}_{\eta_t})$ so that Theorem 4.1 implies the existence of a Wiener-Itô chaos expansion. In the following, we compute the kernels of this decomposition and use Theorem 4.2 to prove our bounds for the variance of $\mathrm{PV}_t(K)$ in Theorem 8.2.

By Equation (4.4), we can derive the following formula for the kernels of the Wiener-Itô chaos expansion of $\mathrm{PV}_t(K)$:

Lemma 8.3 *Let $x_1, \dots, x_n \in \mathbb{R}^d$. For $y \in \mathbb{R}^d$ we define $\overline{x}(y) := \arg \max_{x=x_1,\dots,x_n} \operatorname{dist}(y, x)$ and $z(y, \eta_t) := \arg\min_{z \in \eta_t} \operatorname{dist}(y, z)$. Then*

$$\begin{aligned} &f_{n,t}(x_1, \dots, x_n) \\ &= \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} \mathbb{1}\left(\overline{x}(y) \notin K\right)\ \mathbb{P}(z(y,\eta_t) \notin K^C \cup B^d(y, ||y - \overline{x}(y)||))\, \mathrm{d}y \\ &\quad - \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} \mathbb{1}\left(\overline{x}(y) \in K\right)\ \mathbb{P}(z(y,\eta_t) \notin K \cup B^d(y, ||y - \overline{x}(y)||))\, \mathrm{d}y. \end{aligned} \tag{8.2}$$

Proof. Since $z(y, \eta_t)$ is the nucleus of the Voronoi cell a point $y \in \mathbb{R}^d$ belongs to, it is easily seen that

$$\mathrm{PV}_t(K) = \operatorname{Vol}(\{y \in \mathbb{R}^d : z(y, \eta_t) \in K\}) = \int_{\mathbb{R}^d} \mathbb{1}\left(z(y, \eta_t) \in K\right)\, \mathrm{d}y.$$

Combining this with formula (4.3), we obtain

$$\begin{aligned} D_{x_1,\dots,x_n} \operatorname{PV}_t(K) &= \sum_{I\subset\{1,\dots,n\}} (-1)^{n+|I|} \operatorname{PV}_t(K)(\eta_t + \sum_{i\in I} \delta_{x_i}) \\ &= \int_{\mathbb{R}^d} \sum_{I\subset\{1,\dots,n\}} (-1)^{n+|I|} \mathbb{1}\left(z(y,\eta_t \cup \{x_i : i \in I\}) \in K\right) \mathrm{d}y. \end{aligned}$$

Now we consider the sum of the indicator functions on the right-hand side for a fixed $y \in \mathbb{R}^d$. Let i_{max} be the index of the x_i that maximizes $\overline{x}(y)$. For $I \subset \{1,\dots,n\}\backslash\{i_{max}\}$ with $I \neq \emptyset$, it holds that $z(y,\eta_t \cup \{x_i : i \in I\}) = z(y,\eta_t \cup \{x_i : i \in I \cup \{i_{max}\}\})$ and the summands for I and $I \cup \{i_{max}\}$ on the right-hand side cancel out because of the different signs. Hence, we obtain

$$D_{x_1,\dots,x_n} \operatorname{PV}_t(K) = \int_{\mathbb{R}^d} (-1)^n \left(\mathbb{1}\left(z(y,\eta_t) \in K\right) - \mathbb{1}\left(z(y,\eta_t \cup \{\overline{x}(y)\}) \in K\right)\right) \mathrm{d}y.$$

Now it is easy to see that

$$\begin{aligned} &\mathbb{1}\left(z(y,\eta_t) \in K\right) - \mathbb{1}\left(z(y,\eta_t \cup \{\overline{x}(y)\}) \in K\right) \\ &= \begin{cases} 1, & \operatorname{dist}(y,\overline{x}(y)) \le \operatorname{dist}(y,z(y,\eta_t)),\ z(y,\eta_t) \in K,\ \overline{x}(y) \notin K \\ -1, & \operatorname{dist}(y,\overline{x}(y)) \le \operatorname{dist}(y,z(y,\eta_t)),\ z(y,\eta_t) \notin K,\ \overline{x}(y) \in K \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Combining this with the definition of the kernels in formula (4.4), we obtain

$$\begin{aligned} f_{n,t}(x_1,\dots,x_n) &= \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} \mathbb{1}\left(\overline{x}(y) \notin K\right)\ \mathbb{P}(z(y,\eta_t) \notin K^C \cup B^d(y,||y-\overline{x}(y)||))\,\mathrm{d}y \\ &\quad - \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} \mathbb{1}\left(\overline{x}(y) \in K\right)\ \mathbb{P}(z(y,\eta_t) \notin K \cup B^d(y,||y-\overline{x}(y)||))\,\mathrm{d}y, \end{aligned}$$

which completes the proof. □

For $f_{1,t}$ we have the representation

$$f_{1,t}(x) = \begin{cases} \mathbb{E}\operatorname{Vol}(\{y \in \mathbb{R}^d : \operatorname{dist}(y,x) \le \operatorname{dist}(y,\eta_t \cap K^C) \le \operatorname{dist}(y,\eta_t \cap K)\}), & x \in K \\ -\mathbb{E}\operatorname{Vol}(\{y \in \mathbb{R}^d : \operatorname{dist}(y,x) \le \operatorname{dist}(y,\eta_t \cap K) \le \operatorname{dist}(y,\eta_t \cap K^C)\}), & x \in K^C \end{cases},$$

which means that $|f_{1,t}(x)|$ is the expectation of the volume of the points that change between $\mathrm{A}_t(K)$ and $\mathrm{A}_t(K)^C$ if the point x is added to the Poisson point process.

Our next goal is to compute upper bounds for $||f_{n,t}||_{n,t}^2$ so that we obtain by Theorem 4.2 an upper bound for the variance of $\operatorname{PV}_t(K)$ and can check condition (5.31) in Corollary 5.14. In formula (8.2), the distance between a point $y \in \mathbb{R}^d$ and $\overline{x}(y)$ plays an important role. In order to handle this quantity in the following, we define functions $h_n : (\mathbb{R}^d)^n \to \mathbb{R} \times \mathbb{R}^d$ by

$$h_n(x_1,\dots,x_n) = (\min_{y\in\mathbb{R}^d} \max_{i=1,\dots,n} \operatorname{dist}(y,x_i), \arg\min_{y\in\mathbb{R}^d} \max_{i=1,\dots,n} \operatorname{dist}(y,x_i)).$$

From a geometrical point of view, h_n gives the radius and the centre of the smallest ball that contains all points $x_1,\dots,x_n$.

The function h_n allows us to give the following upper bound for $f_{n,t}$:

Lemma 8.4 *Let $x_1,\dots,x_n\in\mathbb{R}^d$ and let $r=h_n^{(1)}(x_1,\dots,x_n)=\min\limits_{y\in\mathbb{R}^d}\max\limits_{i=1,\dots,n}\operatorname{dist}(y,x_i)$. Then*

$$|f_{n,t}(x_1,\dots,x_n)|\le\frac{1}{(n-1)!\,t}\exp(-t\,\kappa_d\,r^d).$$

Proof. As a consequence of Lemma 8.3, one has

$$|f_{n,t}(x_1,\dots,x_n)|\le\frac{1}{n!}\int_{\mathbb{R}^d}\mathbb{P}(z(y,\eta_t)\notin B^d(y,||y-\overline{x}(y)||))\,\mathrm{d}y.$$

By the definition of r, we know that the sets $\mathbb{R}^d\setminus\operatorname{int}(B^d(x_i,r))$, $i=1,\dots,n$, cover $\mathbb{R}^d$. Combining this with the previous inequality and using polar coordinates, we have

$$\begin{aligned}
|f_{n,t}(x_1,\dots,x_n)| &\le \frac{1}{n!}\sum_{i=1}^{n}\int_{\mathbb{R}^d\setminus B^d(x_i,r)}\mathbb{P}(z(y,\eta_t)\notin B^d(y,||x_i-y||))\,\mathrm{d}y\\
&= \frac{1}{n!}\sum_{i=1}^{n}\int_{\mathbb{R}^d\setminus B^d(x_i,r)}\exp(-t\kappa_d\|x_i-y\|^d)\,\mathrm{d}y\\
&= \frac{1}{n!}\sum_{i=1}^{n}\kappa_d d\int_r^\infty\exp(-t\kappa_d r^d)\,r^{d-1}\,\mathrm{d}r\\
&= \frac{1}{(n-1)!\,t}\exp(-t\kappa_d r^d),
\end{aligned}$$

which concludes the proof. □

By definition, $f_{n,t}(x_1,\dots,x_n)$ measures the effect on $\mathrm{PV}_t(K)$ of inserting points. Lemma 8.4 reflects the fact that this effect is small if the distances between the points are large. Similarly one expects that $f_{n,t}(x_1,\dots,x_n)$ is small if all points are close together but are far away from the boundary of K. This effect is described in the following lemma:

Lemma 8.5 *Let $x_1,\dots,x_n\in\mathbb{R}^d$ and $(r,\overline{y})=h_n(x_1,\dots,x_n)$. If $\delta=\operatorname{dist}(\overline{y},\partial K)>8r$, then*

$$|f_{n,t}(x_1,\dots,x_n)|\le\frac{2}{n!\,t}\exp(-t\kappa_d\delta^d/8^d). \tag{8.3}$$

Proof. Since $\operatorname{dist}(\overline{y},\partial K)>8r$, all $x_1,\dots,x_n$ are either in K or K^C. Let $\tilde{x}=\frac{1}{2}(\overline{y}+\operatorname{proj}_{\partial K}(\overline{y}))$, where $\operatorname{proj}_{\partial K}(\overline{y})$ stands for the metric projection of $\overline{y}$ on the boundary of K. If $\overline{y}\in K$, it can happen that the metric projection on ∂K is not unique. In this case, it does not matter which of the points is taken. Then, we have

$$\frac{\delta}{4}\le\operatorname{dist}(\tilde{x},y)\le\frac{3}{4}\delta\le\operatorname{dist}(y,\partial K)\ \text{ for all }\ y\in B^d(x_1,\delta/8)$$

and, by formula (8.2), it follows that

$$\begin{aligned}|f_{n,t}(x_1,\dots,x_n)| &\leq \frac{1}{n!}\int_{\mathbb{R}^d\setminus B^d(x_1,\delta/8)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,||y-x_1||))\,\mathrm{d}y\\ &\quad +\frac{1}{n!}\int_{B^d(x_1,\delta/8)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,\operatorname{dist}(y,\partial K)))\,\mathrm{d}y\\ &\leq \frac{1}{n!}\int_{\mathbb{R}^d\setminus B^d(x_1,\delta/8)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,||y-x_1||))\,\mathrm{d}y\\ &\quad +\frac{1}{n!}\int_{\mathbb{R}\setminus B^d(\tilde{x},\delta/8)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,||\tilde{x}-y||))\,\mathrm{d}y.\end{aligned}$$

A straightforward computation as in Lemma 8.4 yields the inequality (8.3). □

Combining Lemma 8.4 and Lemma 8.5 leads to the bound

$$|f_{n,t}(x_1,\dots,x_n)|\leq \overline{f}_{n,t}(h_n(x_1,\dots,x_n)),$$

where $\overline{f}_{n,t}:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is given by

$$\overline{f}_{n,t}(r,y)=\begin{cases}\frac{1}{(n-1)!\,t}\exp(-t\kappa_d r^d), & \operatorname{dist}(y,\partial K)\leq 8r\\ \frac{2}{n!\,t}\exp(-t\kappa_d \operatorname{dist}(y,\partial K)^d/8^d), & \operatorname{dist}(y,\partial K)> 8r\end{cases}. \tag{8.4}$$

By the coarea formula (see Proposition 2.4), we obtain for $n\geq 2$

$$\begin{aligned}||f_{n,t}||_{n,t}^2 &\leq t^n\int_{(\mathbb{R}^d)^n}\overline{f}_{n,t}(h_n(x_1,\dots,x_n))^2\,\mathrm{d}x_1\dots\,\mathrm{d}x_n \qquad (8.5)\\ &= t^n\int_0^\infty\int_{\mathbb{R}^d}\int_{h_n^{-1}(r,y)}\overline{f}_{n,t}(r,y)^2 Jh_n(x_1,\dots,x_n)^{-1}\,\mathrm{d}\mathcal{H}^{nd-d-1}((x_1,\dots,x_n))\,\mathrm{d}y\,\mathrm{d}r\\ &= t^n\int_0^\infty\int_{\mathbb{R}^d}\overline{f}_{n,t}(r,y)^2\int_{h_n^{-1}(r,y)} Jh_n(x_1,\dots,x_n)^{-1}\,\mathrm{d}\mathcal{H}^{nd-d-1}((x_1,\dots,x_n))\,\mathrm{d}y\,\mathrm{d}r.\end{aligned}$$

It is easy to see that $h_n(ax_1+v,\dots,ax_n+v)=ah_n(x_1,\dots,x_n)+(0,v)$ for all $a>0$ and $v\in\mathbb{R}^d$ and a short computation shows $h_n'(ax_1+v,\dots,ax_n+v)=h_n'(x_1,\dots,x_n)$, which implies

$$Jh_n(ax_1+v,\dots,ax_n+v)=Jh_n(x_1,\dots,x_n)$$

for all $a>0$ and $v\in\mathbb{R}^d$ and

$$\begin{aligned}&\int_{h_n^{-1}(r,y)} Jh_n(x_1,\dots,x_n)^{-1}\,\mathrm{d}\mathcal{H}^{nd-d-1}((x_1,\dots,x_n))\\ &= r^{nd-d-1}\int_{h_n^{-1}(1,0)} Jh_n(x_1,\dots,x_n)^{-1}\,\mathrm{d}\mathcal{H}^{nd-d-1}((x_1,\dots,x_n)).\end{aligned}$$

Hence, formula (8.5) simplifies to

$$||f_{n,t}||_{n,t}^2\leq C_n t^n\int_0^\infty\int_{\mathbb{R}^d}\overline{f}_{n,t}(r,y)^2(\kappa_d r^d)^{n-1-1/d}\,\mathrm{d}y\,\mathrm{d}r \tag{8.6}$$

for $n\geq 2$ with constants

$$C_n=\kappa_d^{-n+1+1/d}\int_{h_n^{-1}(1,0)} Jh_n(x_1,\dots,x_n)^{-1}\,\mathrm{d}\mathcal{H}^{nd-d-1}((x_1,\dots,x_n)).$$

Lemma 8.6 *The constants C_n, $n \geq 2$, are finite and there is a constant $\tilde{c}_d > 0$ only depending on the dimension d such that*

$$C_n \leq \tilde{c}_d \binom{n}{d+1} (n-1)$$

for $n \geq d+1$.

Proof. A straightforward computation yields

$$\begin{aligned} C_n &= \kappa_d^{-n+1+1/d} \int_{h_n^{-1}(1,0)} Jh_n(x_1,\dots,x_n)^{-1} \, \mathrm{d}\mathcal{H}^{nd-d-1}((x_1,\dots,x_n)) \\ &= (n-1)d\kappa_d^{-n+1/d} \int_0^1 \int_{B^d(0,1)} \int_{h_n^{-1}(r,y)} Jh_n(x_1,\dots,x_n)^{-1} \, \mathrm{d}\mathcal{H}^{nd-d-1}((x_1,\dots,x_n)) \, \mathrm{d}y \, \mathrm{d}r \\ &= (n-1)d\kappa_d^{-n+1/d} \int_{(\mathbb{R}^d)^n} \mathbb{1}\left(h_n(x_1,\dots,x_n) \in [0,1] \times B^d(0,1)\right) \mathrm{d}x_1 \dots \mathrm{d}x_n < \infty. \end{aligned}$$

For almost all $(x_1,\dots,x_n) \in (\mathbb{R}^d)^n$ at most $d+1$ points are on the boundary of the minimal ball that contains all points, and we assume that these are $x_1,\dots,x_{d+1}$. Since the centre of the minimal ball is in $B^d(0,1)$ and the radius is in $[0,1]$, these points must be in $B^d(0,2)$. The remaining points are in a ball with radius 1 around a centre given by the first $d+1$ points. These considerations lead to the bound

$$C_n \leq (n-1)d\kappa_d^{-n+1/d} \binom{n}{d+1} (\kappa_d 2^d)^{d+1} \kappa_d^{n-d-1} = d\kappa_d^{1/d} 2^{d^2+d} \binom{n}{d+1} (n-1)$$

for $n \geq d+1$. □

Our main tool for the computation of the right-hand side of formula (8.6) is the following inequality:

Lemma 8.7 *There are constants $c_{1,d}, c_{2,d} > 0$ only depending on the dimension d such that*

$$\int_{\mathbb{R}^d} \overline{f}_{n,t}(r,y)^2 \, \mathrm{d}y \leq \exp(-2t\kappa_d r^d) \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K) \left(\frac{c_{1,d}(\kappa_d r^d)^{1-\frac{i}{d}}}{((n-1)!)^2 \, t^2} + \frac{c_{2,d}(\kappa_d r^d)^{-\frac{i}{d}}}{(n!)^2 \, t^3} \right)$$

for all $r > 0$ and $n \geq 2$.

Proof. Let $(\partial K)_s = \{y \in \mathbb{R}^d : \mathrm{dist}(y, \partial K) \leq s\}$. By the definition of $\overline{f}_{n,t}$ in formula (8.4), we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \overline{f}_{n,t}(r,y)^2 \, \mathrm{d}y &= \int_{(\partial K)_{8r}} \frac{1}{((n-1)!)^2 \, t^2} \exp(-2t\kappa_d r^d) \, \mathrm{d}y \\ &\quad + \int_{\mathbb{R}^d \setminus (\partial K)_{8r}} \frac{4}{(n!)^2 \, t^2} \exp(-2t\kappa_d \mathrm{dist}(y,\partial K)^d / 8^d) \, \mathrm{d}y. \end{aligned} \tag{8.7}$$

It follows from $\mathrm{Vol}((\partial K)_{8r} \cap K) \leq \mathrm{Vol}((\partial K)_{8r} \cap K^C)$ and the Steiner formula (see Proposition 2.1) that

$$\begin{aligned}
&\int_{(\partial K)_{8r}} \frac{1}{((n-1)!)^2\, t^2} \exp(-2t\kappa_d r^d)\, \mathrm{d}y \\
&\leq 2\, \frac{1}{((n-1)!)^2\, t^2} \exp(-2t\kappa_d r^d) \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K)(8r)^{d-i} \\
&\leq \frac{c_{1,d}}{((n-1)!)^2\, t^2} \exp(-2t\kappa_d r^d) \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K)(\kappa_d r^d)^{1-\frac{i}{d}}
\end{aligned}$$

with a constant $c_{1,d} > 0$. To the second expression in Equation (8.7) we apply the coarea formula (see Proposition 2.4) with the Lipschitz function $u : \mathbb{R}^d \to \mathbb{R}, x \mapsto \mathrm{dist}(x, \partial K)$, which yields

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus (\partial K)_{8r}} \frac{4}{(n!)^2\, t^2} \exp(-2t\kappa_d \mathrm{dist}(y, \partial K)^d/8^d)\, \mathrm{d}y \\
&= \int_{8r}^{\infty} \int_{u^{-1}(\delta)} \frac{4}{(n!)^2\, t^2} \exp(-2t\kappa_d \delta^d/8^d) ||\nabla u(y)||^{-1}\, \mathrm{d}\mathcal{H}^{d-1}(y)\, \mathrm{d}\delta.
\end{aligned}$$

It is easy to see that $|\nabla_{v(x)} u(x)| = ||v(x)||$, where $\nabla_{v(x)} u(x)$ is the directional derivative in direction $v(x) = x - \mathrm{proj}_K(x)$. Hence, we have $||v(x)|| = |\nabla_{v(x)} u(x)| \leq ||\nabla u(x)||\ ||v(x)||$ and $||\nabla u(x)|| \geq 1$. By the Steiner formula for intrinsic volumes (see Proposition 2.2), we know that

$$\begin{aligned}
\mathcal{H}^{d-1}(\{z \in \mathbb{R}^d : \mathrm{dist}(z, \partial K) = \delta\}) &\leq 2\mathcal{H}^{d-1}(\{z \in K^C : \mathrm{dist}(z, \partial K) = \delta\}) \\
&= 2 \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} V_i(K) \delta^{d-1-i}
\end{aligned}$$

and altogether we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus (\partial K)_{8r}} \frac{4}{(n!)^2\, t^2} \exp(-2t\kappa_d \mathrm{dist}(y, \partial K)^d/8^d)\, \mathrm{d}y \\
&\leq \int_{8r}^{\infty} \frac{8}{(n!)^2\, t^2} \exp(-2t\kappa_d \delta^d/8^d) \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} V_i(K) \delta^{d-1-i}\, \mathrm{d}\delta \\
&\leq \frac{8}{(n!)^2\, t^2} \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} V_i(K)(8r)^{-i} \int_{8r}^{\infty} \exp(-2t\kappa_d \delta^d/8^d) \delta^{d-1}\, \mathrm{d}\delta \\
&= 4\, \frac{8^d}{\kappa_d d (n!)^2\, t^3} \exp(-2t\kappa_d r^d) \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} V_i(K)(8r)^{-i} \\
&\leq \frac{c_{2,d}}{(n!)^2\, t^3} \exp(-2t\kappa_d r^d) \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K)(\kappa_d r^d)^{-\frac{i}{d}}
\end{aligned}$$

with a constant $c_{2,d} > 0$. □

By $h_1(x_1) = (0, x_1)$, Lemma 8.5, and the coarea formula (see Proposition 2.4) with the same function u as in the previous proof, it follows that

$$\begin{aligned} ||f_{1,t}||_{1,t}^2 &\leq t \int_{\mathbb{R}^d} \frac{4}{t^2} \exp(-2t\kappa_d \mathrm{dist}(y, \partial K)^d / 8^d) \, \mathrm{d}y \\ &= \frac{4}{t} \int_0^\infty \int_{u^{-1}(r)} \exp(-2t\kappa_d r^d / 8^d) ||\nabla u(y)||^{-1} \, \mathrm{d}\mathcal{H}^{d-1}(y) \, \mathrm{d}r \\ &\leq \frac{8}{t} \int_0^\infty \exp(-2t\kappa_d r^d / 8^d) \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} V_i(K) r^{d-1-i} \, \mathrm{d}r. \end{aligned} \tag{8.8}$$

Combining inequality (8.6) and Lemma 8.7, we have

$$\begin{aligned} ||f_{n,t}||_{n,t}^2 &\leq C_n \int_0^\infty \exp(-2t\kappa_d r^d) \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K) \frac{c_{1,d} t^{-2+\frac{i+1}{d}}}{((n-1)!)^2} (t\kappa_d r^d)^{n-\frac{i+1}{d}} \, \mathrm{d}r \\ &\quad + C_n \int_0^\infty \exp(-2t\kappa_d r^d) \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K) \frac{c_{2,d} t^{-2+\frac{i+1}{d}}}{(n!)^2} (t\kappa_d r^d)^{n-1-\frac{i+1}{d}} \, \mathrm{d}r. \end{aligned}$$

for $n \geq 2$. Comparing this with formula (8.8), we see that the first summand on the right-hand side is an upper bound in the case $n = 1$ if the constant C_1 is chosen appropriately. For $n \geq 2$ substitution and the definition of the Gamma function lead to

$$\begin{aligned} ||f_{n,t}||_{n,t}^2 &\leq C_n \frac{c_{1,d}}{d\,((n-1)!)^2} \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K) \frac{t^{-2+\frac{i+1}{d}}}{2^{n-\frac{i+1}{d}}} \frac{1}{(2t\kappa_d)^{\frac{1}{d}}} \int_0^\infty \exp(-y)\, y^{n-1-\frac{i}{d}} \, \mathrm{d}y \\ &\quad + C_n \frac{c_{2,d}}{d(n!)^2} \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K) \frac{t^{-2+\frac{i+1}{d}}}{2^{n-1-\frac{i+1}{d}}} \frac{1}{(2t\kappa_d)^{\frac{1}{d}}} \int_0^\infty \exp(-y)\, y^{n-2-\frac{i}{d}} \, \mathrm{d}y \\ &\leq C_n \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K) \left(\frac{\tilde{c}_{1,d}\, \Gamma(n - \frac{i}{d})}{((n-1)!)^2\, 2^n} + \frac{\tilde{c}_{2,d}\, \Gamma(n - 1 - \frac{i}{d})}{(n!)^2\, 2^n} \right) t^{-2+\frac{i}{d}} \end{aligned}$$

with constants $\tilde{c}_{1,d}, \tilde{c}_{2,d} > 0$, and it is easy to see that

$$n! ||f_{n,t}||_{n,t}^2 \leq \frac{1}{2^n} C_n \left(\tilde{c}_{1,d} n + \tilde{c}_{2,d} \frac{1}{n(n-1)} \right) \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K) t^{-2+\frac{i}{d}}. \tag{8.9}$$

For $n = 1$ we obtain the inequality (8.9) with $\tilde{c}_{1,d}$ instead of the expression in brackets. By Lemma 8.6, we know that C_n is bounded by a polynomial of order $d + 2$ in n and it follows directly that the series

$$\sum_{n=2}^{\infty} \frac{1}{2^n} C_n \left(\tilde{c}_{1,d} n + \tilde{c}_{2,d} \frac{1}{n(n-1)} \right)$$

converges, which proves the upper bound in Theorem 8.2 and that the condition (5.31) in Corollary 5.14 is satisfied for the volume of the Poisson-Voronoi approximation.

In order to conclude the proof of Theorem 8.2, it remains to construct a lower bound. Because of Theorem 4.2 it is sufficient to give a lower bound for $||f_{1,t}||_{1,t}^2$.

Lemma 8.8 *There is a constant* $\underline{C}$ *only depending on the dimension* d *such that*

$$||f_{1,t}||_{1,t}^2 \geq \underline{C}\,\kappa_1 V_{d-1}(K)\, t^{-1-\frac{1}{d}} \tag{8.10}$$

for $t \geq (2/r(K))^d$*, where* $r(K)$ *is the inradius of* K*.*

Proof. Recall that $B^d(y,\delta) \subset \mathbb{R}^d$ stands for a ball with centre y and radius $\delta > 0$. We consider the set

$$M_\varepsilon = \left\{ x \in K^C : \operatorname{dist}(x,K) \leq \varepsilon, \operatorname{Vol}((B^d(x,2\varepsilon) \setminus B^d(x,\varepsilon)) \cap K) \geq \frac{\kappa_d \varepsilon^d}{2^d} \right\}$$

for $\varepsilon \leq r(K)/2$. By a result due to Schütt and Werner (see [84, Lemma 4]), it is known that

$$\mathcal{H}^{d-1}\left(\{x \in \partial K : \tilde{r}(x) \geq \varepsilon\}\right) \geq \left(1 - \frac{\varepsilon}{r(K)}\right)^{d-1} \kappa_1 V_{d-1}(K),$$

where $\tilde{r}(x)$ is the radius of the largest ball that is contained in K and contains x. It is easy to see that $x \in K^C$ with $\operatorname{dist}(x,K) \leq \varepsilon$ is in M_ε if $\tilde{r}(\operatorname{proj}_K(x)) \geq \varepsilon$. As a consequence, we have

$$\begin{aligned} \operatorname{Vol}(M_\varepsilon) &\geq \mathcal{H}^{d-1}\left(\{x \in \partial K : \tilde{r}(x) \geq \varepsilon\}\right)\varepsilon \\ &\geq \left(1 - \frac{\varepsilon}{r(K)}\right)^{d-1} \kappa_1 V_{d-1}(K)\varepsilon \geq \frac{1}{2^d}\kappa_1 V_{d-1}(K)\,\varepsilon. \end{aligned} \tag{8.11}$$

For $x \in M_\varepsilon$ it holds that

$$|f_{1,t}(x)| \geq \frac{\kappa_d \varepsilon^d}{2^d} \exp(-\left(4^d - 2^{-d}\right) t\kappa_d \varepsilon^d)\left(1 - \exp(-2^{-d}\, t\kappa_d\varepsilon^d)\right). \tag{8.12}$$

To see the inequality (8.12), the underlying idea is that for every $x \in M_\varepsilon$ there is, by definition of M_ε, a set $U \subset (B^d(x,2\varepsilon) \setminus B^d(x,\varepsilon)) \cap K$ with $\lambda_d(U) = 2^{-d}\kappa_d\varepsilon^d$. Then

$$\mathbb{P}(\eta_t(B^d(x,4\varepsilon) \setminus U) = 0, \eta_t(U) \geq 1) = \exp(-(4^d - 2^{-d})t\kappa_d\varepsilon^d)\left(1 - \exp(-2^{-d}\, t\kappa_d\varepsilon^d)\right),$$

and for this event the effect of adding x to the point process is larger than $\frac{\kappa_d\varepsilon^d}{2^d}$. Combining the estimates (8.11) and (8.12), we obtain

$$\begin{aligned} ||f_{1,t}||_{1,t}^2 &= t\int_{\mathbb{R}^d} f_{1,t}(x)^2\,\mathrm{d}x \geq t\int_{M_\varepsilon} f_{1,t}(x)^2\,\mathrm{d}x \\ &\geq t\,\frac{1}{2^d}\kappa_1 V_{d-1}(K)\,\varepsilon\,\frac{\kappa_d^2\varepsilon^{2d}}{4^d}\exp(-2(4^d - 2^{-d})t\kappa_d\varepsilon^d)\left(1 - \exp(2^{-d}\, t\kappa_d\varepsilon^d)\right)^2. \end{aligned}$$

Now the choice $\varepsilon = t^{-\frac{1}{d}}$ leads to

$$||f_{1,t}||_{1,t}^2 \geq \frac{\kappa_d^2}{8^d}\exp(-2(4^d - 2^{-d})\kappa_d)\left(1 - \exp(-2^{-d}\,\kappa_d)\right)^2 \kappa_1 V_{d-1}(K)\, t^{-1-\frac{1}{d}},$$

which concludes the proof. □

An inequality as formula (8.10) cannot hold for all $t > 0$ as the following consideration shows: We fix a convex body $K \in \mathcal{K}_0^d$ with $0 \in K$ and a compact window $W \supset K$ and set $K_r = rK = \{rx : x \in K\}$ for $r > 0$. We define the random variable $\widetilde{\mathrm{PV}}_t(W)$ as

$$\widetilde{\mathrm{PV}}_t(W) = \mathrm{Vol}\left(\{y \in \mathbb{R}^d : \mathrm{dist}(y, W) \leq ||y - x|| \ \forall x \in \eta_t \cap W^C\}\right).$$

A short computation proves $\mathbb{E}\, \widetilde{\mathrm{PV}}_t(W)^2 < \infty$. Then, it holds that

$$\begin{aligned} \mathrm{Var}\, \mathrm{PV}_t(K_r) &= \mathbb{E}\, \mathrm{PV}_t(K_r)^2 - \mathrm{Vol}(K_r)^2 \leq \mathbb{E}\, \mathrm{PV}_t(K_r)^2 \\ &\leq (1 - \exp(-t\, \mathrm{Vol}(K_r)))\, \mathbb{E}\, \widetilde{\mathrm{PV}}_t(W)^2. \end{aligned}$$

For $r \to 0$ the right-hand side has order r^d, whereas $V_{d-1}(K_r)$ is only of order r^{d-1}.

8.3 Proof of Theorem 8.1

In this section, we use Corollary 5.14 to prove Theorem 8.1. Since it follows from formula (8.9) that condition (5.31) is satisfied, it only remains to check condition (5.32), which requires

$$\lim_{t\to\infty} \frac{t^{|\sigma|}}{(\mathrm{Var}\, \mathrm{PV}_t(K))^2} \underbrace{\int_{(\mathbb{R}^d)^{|\sigma|}} |(f_{i,t} \otimes f_{i,t} \otimes f_{j,t} \otimes f_{j,t})_\sigma(y_1, \ldots, y_{|\sigma|})| \, \mathrm{d}y_1 \ldots \mathrm{d}y_{|\sigma|}}_{=:M_{\sigma,t}} = 0 \tag{8.13}$$

for all $\sigma \in \widetilde{\Pi}_{\geq 2}(i, i, j, j)$ and $i, j \in \mathbb{N}$. In order to prove this behaviour, we compute upper bounds for $M_{\sigma,t}$.

We define functions $g_n : \left(\mathbb{R}^d\right)^n \to \mathbb{R}, n \in \mathbb{N}$, as

$$g_n(x_1, \ldots, x_n) = \max\left\{\mathrm{diam}(x_1, \ldots, x_n), \max_{i=1,\ldots,n} \mathrm{dist}(x_i, \partial K)\right\},$$

where $\mathrm{diam}(x_1, \ldots, x_n)$ stands for the diameter of $x_1, \ldots, x_n$. Using this notation, we can state the following upper bound for $f_{n,t}$:

Lemma 8.9 *Let $x_1, \ldots, x_n \in \mathbb{R}^d$ and $\delta = g_n(x_1, \ldots, x_n)$. Then*

$$|f_{n,t}(x_1, \ldots, x_n)| \leq \frac{2}{n!\, t} \exp(-t\kappa_d \delta^d / 4^d) =: \tilde{f}_{n,t}(\delta).$$

Proof. Without loss of generality we can assume $\mathrm{dist}(x_1, \partial K) = \delta$ or $\mathrm{dist}(x_1, x_2) = \delta$. For the first case, let $\tilde{x} = \frac{1}{2}(x_1 + \mathrm{proj}_{\partial K}(x_1))$, where $\mathrm{proj}_{\partial K}(x_1)$ is the projection of x_1 on the boundary of K. If the projection is not unique (this can happen for $x_1 \in K$), it does not matter which of the points is taken. Then, it holds that

$$\frac{\delta}{4} \leq \mathrm{dist}(y, \tilde{x}) \leq \frac{3}{4}\delta \leq \mathrm{dist}(y, \partial K) \quad \text{for all} \quad y \in B^d(x_1, \delta/4).$$

Hence, it follows from Lemma 8.3 and a straightforward computation as in the proof of Lemma 8.4 that

$$\begin{aligned} |f_{n,t}(x_1,\dots,x_n)| &\leq \frac{1}{n!}\int_{\mathbb{R}^d\setminus B^d(x_1,\delta/4)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,||y-x_1||))\,\mathrm{d}y \\ &\quad +\frac{1}{n!}\int_{B^d(x_1,\delta/4)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,\mathrm{dist}(y,\partial K)))\,\mathrm{d}y \\ &\leq \frac{1}{n!}\int_{\mathbb{R}^d\setminus B^d(x_1,\delta/4)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,||y-x_1||))\,\mathrm{d}y \\ &\quad +\frac{1}{n!}\int_{\mathbb{R}^d\setminus B^d(\tilde{x},\delta/4)} \mathbb{P}(z(y,\eta_t)\notin B^d(y,||y-\tilde{x}||))\,\mathrm{d}y \\ &= \frac{2}{n!\,t}\exp(-t\kappa_d\delta^d/4^d). \end{aligned}$$

In the case $\mathrm{dist}(x_1,x_2)=\delta$, we replace $\tilde{x}$ by x_2 and obtain the same bound. □

We prepare the application of the coarea formula by showing the following properties of g_n :

Lemma 8.10 *a) g_n is a Lipschitz function with $||\nabla g_n|| \geq 1$ almost everywhere.*

b) There is a constant $c_d > 0$ only depending on the dimension d such that

$$\begin{aligned} \mathcal{H}^{nd-1}(g_n^{-1}(\delta)) &\leq n(n-1)\sum_{i=0}^{d-1}\kappa_{d-i}V_i(K)\,\delta^{d-i}d\kappa_d\,\delta^{d-1}(\kappa_d\delta^d)^{n-2} \\ &\quad +2n\sum_{i=0}^{d-1}(d-i)\kappa_{d-i}V_i(K)\,\delta^{d-1-i}(\kappa_d\delta^d)^{n-1} \\ &\leq c_d n^2\sum_{i=0}^{d-1}\kappa_{d-i}V_i(K)(\kappa_d\delta^d)^{n-\frac{i+1}{d}} \end{aligned} \tag{8.14}$$

for $\delta \geq 0$.

Proof. $g_n(x_1,\dots,x_n)$ is always given by the distance of two points or by the distance of a point to the boundary of K. If we move one of these points exactly in the opposite direction v of the second point or the boundary of K, the directional derivative is $\nabla_v g_n(x_1,\dots,x_n) = ||v||$. Now

$$|\nabla_v g_n(x_1,\dots,x_n)| \leq ||\nabla g_n(x_1,\dots,x_n)||\;||v||$$

implies $||\nabla g_n(x_1,\dots,x_n)|| \geq 1$ and thus a).

For the proof of b) we consider the same situations as in the proof of a). If there are two points $x_i, x_j \in \mathbb{R}^d$ such that $\mathrm{dist}(x_i,x_j)=\delta$, x_i must be in $(\partial K)_\delta$, x_j in a sphere around x_i with radius δ and the remaining $n-2$ points must be in a ball with radius δ and centre x_i. If $\mathrm{dist}(x_i,\partial K)=\delta$, x_i must be in the set $\{y\in\mathbb{R}^d : \mathrm{dist}(y,\partial K)=\delta\}$ and the remaining points are in a ball with radius δ and centre x_i. Combining these considerations with the Steiner formula (see Proposition 2.2) yields formula (8.14). $\square$

Recall that $\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)$. For $\ell = 1,2,3,4$ let $\sigma_\ell(y_1,\dots,y_{|\sigma|}) \subset \{y_1,\dots,y_{|\sigma|}\}$ be the new variables that occur in the ℓ-th function of $(f_{i,t} \otimes f_{i,t} \otimes f_{j,t} \otimes f_{j,t})_\sigma$ and let $|\sigma_\ell|$ stand for the number of these variables. We set $r = g_{|\sigma|}(y_1,\dots,y_{|\sigma|})$ and

$$\delta_1 = g_{|\sigma_1|}(\sigma_1(y_1,\dots,y_{|\sigma|})),\dots,\delta_4 = g_{|\sigma_4|}(\sigma_4(y_1,\dots,y_{|\sigma|})).$$

Since $\sigma_\ell(y_1,\dots,y_{|\sigma|}) \subset \{y_1,\dots,y_{|\sigma|}\}$, it is easy to see that $\delta_\ell = g_{|\sigma_\ell|}(\sigma_\ell(y_1,\dots,y_{|\sigma|})) \leq g_{|\sigma|}(y_1,\dots,y_{|\sigma|}) = r$ for $\ell = 1,2,3,4$. If there is a y_j with $\operatorname{dist}(y_j,\partial K) = r$, we have at least two $\ell_1,\ell_2 \in \{1,2,3,4\}$ such that $y_j \in \sigma_{\ell_1}(y_1,\dots,y_{|\sigma|})$ and $y_j \in \sigma_{\ell_2}(y_1,\dots,y_{|\sigma|})$, which implies $\delta_{\ell_1} = \delta_{\ell_2} = r$. The other case is that there are y_{j_1} and y_{j_2} such that $\operatorname{dist}(y_{j_1},y_{j_2}) = r$. If there is an $\ell \in \{1,2,3,4\}$ with $y_1,y_2 \in \sigma_\ell(y_1,\dots,y_{|\sigma|})$, it follows directly $\delta_\ell = r$. Otherwise, $\sigma \in \widetilde{\Pi}_{\geq 2}(i,i,j,j)$ implies that we have a y_{j_3} and $\ell_1,\ell_2 \in \{1,2,3,4\}$ with $y_{j_1},y_{j_3} \in \sigma_{\ell_1}(y_1,\dots,y_{|\sigma|})$ and $y_{j_2},y_{j_3} \in \sigma_{\ell_2}(y_1,\dots,y_{|\sigma|})$. By the inequality

$$r = \operatorname{dist}(y_{j_1},y_{j_2}) \leq \operatorname{dist}(y_{j_1},y_{j_3}) + \operatorname{dist}(y_{j_3},y_{j_2}),$$

it follows that

$$\max\{\delta_{\ell_1},\delta_{\ell_2}\} \geq \max\{\operatorname{dist}(y_{j_1},y_{j_3}),\operatorname{dist}(y_{j_3},y_{j_2})\} \geq r/2.$$

Hence, it holds that $r/2 \leq \max_{\ell=1,\dots,4}\delta_\ell \leq r$. Together with the coarea formula (see Proposition 2.4), Lemma 8.9, and Lemma 8.10, we obtain

$$\begin{aligned}
M_{\sigma,t} &\leq t^{|\sigma|}\int_{(\mathbb{R}^d)^{|\sigma|}} \tilde{f}_{i,t}(\delta_1)\tilde{f}_{i,t}(\delta_2)\tilde{f}_{j,t}(\delta_3)\tilde{f}_{j,t}(\delta_4)\,\mathrm{d}y_1\dots\mathrm{d}y_{|\sigma|}\\
&\leq t^{|\sigma|}\int_{(\mathbb{R}^d)^{|\sigma|}} \tilde{f}_{i,t}(\delta_1)\tilde{f}_{i,t}(\delta_2)\tilde{f}_{j,t}(\delta_3)\tilde{f}_{j,t}(\delta_4)\;||\nabla g_{|\sigma|}||\,\mathrm{d}y_1\dots\mathrm{d}y_{|\sigma|}\\
&= t^{|\sigma|}\int_0^\infty\int_{g_{|\sigma|}^{-1}(r)} \tilde{f}_{i,t}(\delta_1)\tilde{f}_{i,t}(\delta_2)\tilde{f}_{j,t}(\delta_3)\tilde{f}_{j,t}(\delta_4)\;\mathcal{H}^{|\sigma|d-1}(d(y_1,\dots,y_{|\sigma|}))\,\mathrm{d}r\\
&\leq t^{|\sigma|-4}\frac{16}{(i!\,j!)^2}\int_0^\infty \exp(-t\kappa_d r^d/8^d)c_d|\sigma|^2\sum_{i=0}^{d-1}\kappa_{d-i}V_i(K)(\kappa_d r^d)^{|\sigma|-\frac{i+1}{d}}\,\mathrm{d}r.
\end{aligned}$$

By substitution and the definition of the Gamma function, we have

$$\begin{aligned}
M_{\sigma,t} &\leq \frac{16c_d|\sigma|^2}{(i!\,j!)^2}\sum_{i=0}^{d-1}\kappa_{d-i}V_i(K)8^{|\sigma|d-i-1}t^{\frac{i+1}{d}-4}\int_0^\infty \exp(-t\kappa_d r^d/8^d)(t\kappa_d r^d/8^d)^{|\sigma|-\frac{i+1}{d}}\,\mathrm{d}r\\
&= \frac{16c_d|\sigma|^2}{d(i!\,j!)^2}\sum_{i=0}^{d-1}\kappa_{d-i}V_i(K)t^{\frac{i}{d}-4}8^{|\sigma|d-i}\kappa_d^{-\frac{1}{d}}\int_0^\infty \exp(-y)\,y^{|\sigma|-1-\frac{i}{d}}\,\mathrm{d}y\\
&= \frac{16c_d|\sigma|^2}{d(i!\,j!)^2}\sum_{i=0}^{d-1}\kappa_{d-i}V_i(K)t^{\frac{i}{d}-4}8^{|\sigma|d-i}\kappa_d^{-\frac{1}{d}}\Gamma(|\sigma|-i/d).
\end{aligned}$$

Thus, each $M_{\sigma,t}$ has order $t^{-3-\frac{1}{d}}$ or less. Since $\operatorname{Var}\mathrm{PV}_t(K)$ is of order $t^{-1-\frac{1}{d}}$, this proves formula (8.13). Now all assumptions of Corollary 5.14 are satisfied for the volume of the Poisson-Voronoi approximation, and Theorem 8.1 is a direct consequence.

In Theorem 8.1 and Theorem 8.2, we assume that the approximated set K is convex. But the convexity is not necessary for the construction of the Poisson-Voronoi approximation so that it is a natural question if one can extend our results to more general set classes. The convexity assumption is only needed to bound the volume and the surface area of the parallel sets $(\partial K)_r$ by the Steiner formula in the proofs of Theorem 8.2 and Theorem 8.1 and to apply Lemma 4 from [84] in the proof of Lemma 8.8. Hence, one can extend the results to compact sets $M \subset \mathbb{R}^d$ that satisfy the following additional assumptions:

(S1) There are constants $c_M^{(i)}, i = 1, \dots, d$, depending on M such that

$$\operatorname{Vol}((\partial M)_r) \leq \sum_{i=1}^{d} c_M^{(i)} r^i \quad \text{and} \quad \mathcal{H}^{d-1}(\partial((\partial M)_r)) \leq \sum_{i=1}^{d} c_M^{(i)} r^{i-1}$$

with $(\partial M)_r = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial M) \leq r\}$ for $r > 0$.

(S2) There is a constant $\gamma > 0$ such that

$$\liminf_{r \to 0} \operatorname{Vol}(\tilde{M}_{\gamma,r})/r > 0$$

with $\tilde{M}_{\gamma,r} = \{x \in M^C : \operatorname{dist}(x, M) \leq r, \operatorname{Vol}((B^d(x, 2r) \setminus B^d(x, r)) \cap M) \geq \gamma r^d\}$.

Assumption **(S1)** allows us to bound the volume and the surface area of the parallel sets $(\partial M)_r$ by a kind of Steiner formula. In the upper bound in Theorem 8.2, the intrinsic volumes must be replaced by the constants $c_M^{(i)}, i = 1, \dots, d$. Our proof of the lower bound in Theorem 8.2 requires assumption **(S2)**, which replaces a rolling ball result for convex sets from [84]. Then the constant $\underline{C}$ and the lower bound for t in formula (8.1) depend on the constant γ and the limit inferior in **(S2)**.

Since **(S1)** and **(S2)** are obviously true for convex sets, they still hold for polyconvex sets and, of course, for all polytopes.

Notes: The content of this chapter is published in *Schulte 2012a*.

Chapter 9

Central limit theorems for Boolean models

In this chapter, we investigate a class of random closed sets in $\mathbb{R}^d$, so-called Boolean models. We prove univariate and multivariate central limit theorems for their intrinsic volumes within increasing observation windows. The problem and the results are presented in the first section. The second section contains the proofs that make use of the Wiener-Itô chaos expansion and the Malliavin-Stein method.

9.1 Introduction and results

Let η be a stationary Poisson point process on $\mathcal{K}^d$. The intensity measure Λ of η is of the form

$$\Lambda(\cdot) = \gamma \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \mathbb{1}(x + K \in \cdot) \, \mathrm{d}\mathbb{Q}(K) \, \mathrm{d}x$$

where $\gamma > 0$ and $\mathbb{Q}$ is a probability measure on $\mathcal{K}^d$ that is concentrated on the non-empty compact convex sets whose circumcentre is the origin (see [79, Theorem 4.1.1]). Now the Boolean model

$$Z = \bigcup_{K \in \eta} K$$

is the union of all compact convex sets that belong to η. A random compact convex set Z_0 with probability distribution $\mathbb{Q}$ is called typical grain.

One can also think of η as a stationary marked Poisson point process in $\mathbb{R}^d$ such that every point has a random compact convex set with distribution $\mathbb{Q}$ as a mark. Then the Boolean model is the union of the Minkowski sums of the points of η and their marks.

In the following, we are interested in the intrinsic volumes of the Boolean model Z within an observation window $W \in \mathcal{K}_0^d$, i.e. we consider the Poisson functionals $V_k(Z \cap W)$ for $k = 0, \dots, d$. In order to investigate their asymptotic behaviour, we take as observation windows a sequence $(W_m)_{m \in \mathbb{N}}$ in $\mathcal{K}_0^d$ such that the inradii satisfy $\lim_{m \to \infty} r(W_m) = \infty$.

For first order properties of the Boolean model Z, we refer to the monograph [79] by Schneider and Weil and the references therein, where it was shown (see [79, Theorem

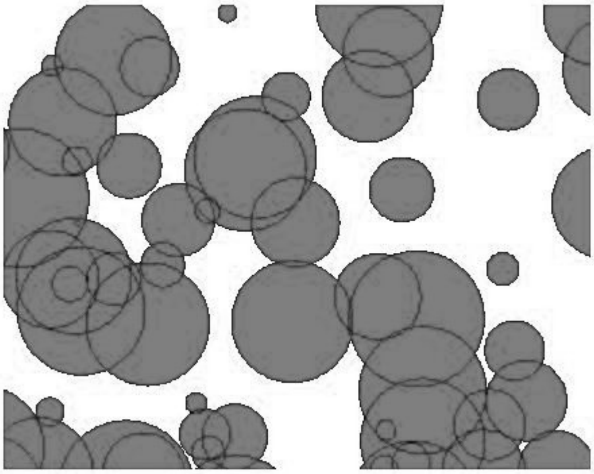

Figure 9.1: Boolean model with disks as typical grains

9.1.5]) that

$$\lim_{m\to\infty} \frac{\mathbb{E}V_k(Z\cap W_m)}{V_d(W_m)} = v_{k,Z}$$

for $k = 0,\dots,d$ with constants $v_{k,Z}$ depending on the dimension d, the intensity γ, and the distribution of the typical grain. For the volume in increasing observation windows central limit theorems were proven by Baddeley and Mase in [2, 48]. Heinrich obtained a rate of convergence and large deviation inequalities in [22]. The surface area was considered by Molchanov in [54]. This result was extended by Heinrich and Molchanov to more general functionals than the surface area, including so-called positive extensions of the intrinsic volumes (see [24]). Some formulas for the second order moments of intrinsic volumes were derived and evaluated for special cases by Mecke in [52]. Baryshnikov and Yukich, and Penrose proved central limit theorems for functionals related to the volume by stabilization techniques in [5, 72]. Using a similar method, Penrose derived central limit theorems for the numbers of occupied and vacant clusters in [68]. For limit theorems in a statistical context we refer to the works of Molchanov and Stoyan and Pantle, Schmidt, and Spodarev (see [55, 62]), where the asymptotic normality of estimators related to Boolean models was shown.

In contrast to the previous works focusing on volume or surface area, we prove univariate central limit theorems for all intrinsic volumes and a multivariate central limit theorem for the vector of all intrinsic volumes in this chapter. For both results we can provide rates of convergences depending on the inradius of the observation window.

Recall that $r(K)$ and $R(K)$ stand for the inradius and the circumradius of a compact convex set $K\in\mathcal{K}^d$. By $B(K)$ we denote the smallest closed ball containing the compact convex set $K\in\mathcal{K}^d$.

We begin with the following result for the asymptotic covariances of the intrinsic volumes:

Theorem 9.1 *Assume that the typical grain Z_0 satisfies $\mathbb{E}V_d(B(Z_0))^3<\infty$. For $k,\ell\in$*

$\{0,\dots,d\}$ the limit

$$\sigma_{k,\ell} = \lim_{m\to\infty} \frac{\operatorname{Cov}(V_k(Z\cap W_m), V_\ell(Z\cap W_m))}{V_d(W_m)}$$

exists for a sequence of convex bodies $(W_m)_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} r(W_m)=\infty$. Moreover, there are constants $c_{k,\ell}$ depending on the dimension d, the intensity parameter γ, and the typical grain Z_0 such that

$$\left| \frac{\operatorname{Cov}(V_k(Z\cap W), V_\ell(Z\cap W))}{V_d(W)} - \sigma_{k,\ell} \right| \leq \frac{c_{k,\ell}}{r(W)} \tag{9.1}$$

for $W \in \mathcal{K}_0^d$ with $r(W)\geq 1$.

If the typical grain Z_0 is a convex body with positive probability, the covariance matrix $\Sigma = (\sigma_{k,\ell})_{k,\ell=0,\dots,d}$ is positive definite.

The knowledge of the asymptotic covariance matrix allows us to state the following central limit theorem:

Theorem 9.2 *Assume that the typical grain Z_0 satisfies $\mathbb{E}V_d(B(Z_0))^4<\infty$.*

a) Let N be a standard Gaussian random variable. Then there are constants c_k, $k\in\{0,\dots,d\}$, depending on the dimension d, the intensity parameter γ, and the typical grain Z_0 such that

$$d_W\left(\frac{V_k(Z\cap W) - \mathbb{E}V_k(Z\cap W)}{\sqrt{\operatorname{Var} V_k(Z\cap W)}}, N \right) \leq \frac{c_k}{r(W)^{d/2}}$$

for $W\in\mathcal{K}_0^d$ with $r(W)\geq 1$.

b) Let $\mathcal{V}(W) = (V_0(Z\cap W),\dots,V_d(Z\cap W))$ for $W\in\mathcal{K}_0^d$ and let $\boldsymbol{N}(\Sigma)$ be a $(d+1)$-dimensional centred Gaussian random vector with the covariance matrix Σ defined in Theorem 9.1. Then there is a constant $c>0$ depending on the dimension d, the intensity parameter γ, and the typical grain Z_0 such that

$$d_3\left(\frac{1}{\sqrt{V_d(W)}} (\mathcal{V}(W) - \mathbb{E}\mathcal{V}(W)), \boldsymbol{N}(\Sigma) \right) \leq \frac{c}{r(W)}$$

for $W\in\mathcal{K}_0^d$ with $r(W)\geq 1$.

Comparing part a) and part b) of Theorem 9.2, one sees that for $d\geq 3$ the rate of convergence in the multivariate case is weaker than in the univariate case. This is caused by the slow rate of convergence in Theorem 9.1 since we need to bound

$$\sum_{k,\ell=0}^{d} \left| \sigma_{k,\ell} - \frac{\operatorname{Cov}(V_k(Z\cap W), V_\ell(Z\cap W))}{V_d(W)} \right|,$$

in order to apply the multivariate part of Theorem 5.15. In the univariate case, we normalize with the exact variance and do not have such a term. If we replace the

Gaussian random vector $\mathbf{N}(\Sigma)$ with the covariance matrix Σ by a centred Gaussian random vector $\mathbf{N}(\Sigma_W)$ having the covariance matrix Σ_W of $V_d(W)^{-1/2}\mathcal{V}(W)$, the sum above vanishes and we obtain

$$d_3\left(\frac{1}{\sqrt{V_d(W)}}(\mathcal{V}(W)-\mathbb{E}\mathcal{V}(W)),\mathbf{N}(\Sigma_W)\right)\leq\frac{c}{r(W)^{d/2}},$$

which is the same rate as in the univariate case. Using the triangle inequality for the distance d_3, we see that

$$\begin{aligned}&d_3(\mathbf{N}(\Sigma),\mathbf{N}(\Sigma_W))-d_3\left(\frac{1}{\sqrt{V_d(W)}}(\mathcal{V}(W)-\mathbb{E}\mathcal{V}(W)),\mathbf{N}(\Sigma_W)\right)\\&\leq d_3\left(\frac{1}{\sqrt{V_d(W)}}(\mathcal{V}(W)-\mathbb{E}\mathcal{V}(W)),\mathbf{N}(\Sigma)\right).\end{aligned}$$

Since $d_3(\mathbf{N}(\Sigma),\mathbf{N}(\Sigma_W))$ has at most the rate of

$$\sum_{k,\ell=0}^{d}\left|\sigma_{k,\ell}-\frac{\operatorname{Cov}(V_k(Z\cap W),V_\ell(Z\cap W))}{V_d(W)}\right|,$$

the rate in Theorem 9.2 cannot be better than the rate in Theorem 9.1.

If we rescale by $\sqrt{V_d(W)}$ in the univariate case and consider the Wasserstein distance to $\sqrt{\sigma_{k,k}}\,N$, we also obtain only the rate $r(W)^{-1}$.

9.2 Proofs

We begin with the computation of the Wiener-Itô chaos expansions of the Poisson functionals $V_k(Z\cap W)$, $k\in\{0,\dots,d\}$, whose kernels are denoted by $f_n^{(k)}$, $n\in\mathbb{N}$.

Lemma 9.3 *For $K_1,\dots,K_n\in\mathcal{K}^d$, $n\in\mathbb{N}$, and $k\in\{0,\dots,d\}$ we have*

$$f_n^{(k)}(K_1,\dots,K_n)=\frac{(-1)^n}{n!}(\mathbb{E}V_k(Z\cap K_1\cap\ldots\cap K_n\cap W)-V_k(K_1\cap\ldots\cap K_n\cap W)).$$

Proof. We show by induction the identity

$$\begin{aligned}&D_{K_1,\dots,K_n}V_k(Z\cap W)\\&=(-1)^n\left(V_k(Z\cap K_1\cap\ldots\cap K_n\cap W)-V_k(K_1\cap\ldots\cap K_n\cap W)\right).\end{aligned}\tag{9.2}$$

for the n-th iterated difference operator applied to $V_k(Z\cap W)$. For $n=1$ this follows from the pathwise definition of the difference operator in Equation (4.1) and the additivity of the intrinsic volumes. By the definition of the iterated difference operator in Equation (4.2), again the additivity of the intrinsic volumes, and the assumption of the induction, we obtain

$$\begin{aligned}
&D_{K_1,\dots,K_{n+1}}V_k(Z\cap W)\\
&=(-1)^n D_{K_{n+1}}\left(V_k(Z\cap K_1\cap\ldots\cap K_n\cap W)-V_k(K_1\cap\ldots\cap K_n\cap W)\right)\\
&=(-1)^n\left(V_k((Z\cup K_{n+1})\cap K_1\cap\ldots\cap K_n\cap W)-V_k(K_1\cap\ldots\cap K_n\cap W)\right)\\
&\quad-(-1)^n\left(V_k(Z\cap K_1\cap\ldots\cap K_n\cap W)-V_k(K_1\cap\ldots\cap K_n\cap W)\right)\\
&=(-1)^n\left(V_k((Z\cup K_{n+1})\cap K_1\cap\ldots\cap K_n\cap W)-V_k(Z\cap K_1\cap\ldots\cap K_n\cap W)\right)\\
&=(-1)^n\left(V_k(K_1\cap\ldots\cap K_n\cap K_{n+1}\cap W)-V_k(Z\cap K_1\cap\ldots\cap K_n\cap K_{n+1}\cap W)\right).
\end{aligned}$$

Combining identity (9.2) with the definition of the kernels in Equation (4.4) concludes the proof. □

We prepare for the proofs of our main results by some inequalities. Let $Q_1=[0,1]^d$ and let $N(Q_1)$ be the number of grains hitting Q_1. By φ_{Q_1} we denote the generating function of $N(Q_1)$.

Lemma 9.4 *For all $A\in\mathcal{K}^d$ it holds that*

$$|\mathbb{E}V_k(Z\cap A)|\le c_1(d)\,\varphi_{Q_1}(2)\,V_k(Q_1)\,(V_d(B(A))+V_0(A))$$

and

$$\mathbb{E}V_k(Z\cap A)^4\le c_2(d)\,\varphi_{Q_1}(16)\,V_k(Q_1)^4\sum_{i=0}^{d}V_i(A)^4$$

with constants $c_1(d),c_2(d)>0$ only depending on the dimension d.

Proof. Since the inequalities are obviously true for $A=\emptyset$, we assume $A\neq\emptyset$ in the following. We divide $\mathbb{R}^d$ in a grid of cubes of edge length one and denote the set of all these cubes intersecting A by $\mathcal{Q}(A)$. It follows from the inclusion-exclusion formula for intrinsic volumes that

$$|\mathbb{E}V_k(Z\cap A)|\le\mathbb{E}|V_k(Z\cap\bigcup_{Q\in\mathcal{Q}(A)}Q\cap A)|\le\sum_{I\subset\mathcal{Q}(A)}\mathbb{E}|V_k(Z\cap\bigcap_{Q\in I}Q\cap A)|.$$

For a set $D\in\mathcal{K}^d$, we denote the grains of the Boolean model hitting D by $Z_1,\dots,Z_{N(D)}$ and obtain by the properties of V_k

$$|V_k(Z\cap D)|=|V_k(D\cap\bigcup_{j=1,\dots,N(D)}Z_j)|\le\sum_{J\subset\{1,\dots,N(D)\}}|V_k(D\cap\bigcap_{j\in J}Z_j)|\le 2^{N(D)}V_k(D).$$

Consequently, we have

$$\mathbb{E}|V_k(Z\cap\bigcap_{Q\in I}Q\cap A)|\le\mathbb{E}2^{N(\bigcap_{Q\in I}Q\cap A)}\,V_k(\bigcap_{Q\in I}Q\cap A)\le\mathbb{E}2^{N(Q_1)}\,V_k(Q_1)=\varphi_{Q_1}(2)\,V_k(Q_1).$$

Due to the fact that the $Q\in\mathcal{Q}(A)$ form a grid, we know that

$$|\{I\subset\mathcal{Q}(A):\bigcap_{Q\in I}Q\neq\emptyset\}|\le\tilde{c}(d)\,|\mathcal{Q}(A)|$$

with a constant $\tilde{c}(d) > 0$ only depending on the dimension d. Altogether, we see that

$$|\mathbb{E}V_k(Z \cap A)| \leq \tilde{c}(d)\, \varphi_{Q_1}(2)\, V_k(Q_1)\, |\mathcal{Q}(A)|.$$

Combining this with the inequality

$$\begin{aligned} |\mathcal{Q}(A)| &\leq V_d(A + \sqrt{d}B^d) \leq V_d(B(A) + \sqrt{d}B^d) \\ &\leq \kappa_d\, (R(A) + \sqrt{d})^d \leq \kappa_d\, 2^{d-1}(R(A)^d + d^{\frac{d}{2}}) \\ &\leq 2^{d-1}(V_d(B(A)) + d^{\frac{d}{2}}\kappa_d) \leq 2^{d-1}(1 + d^{\frac{d}{2}}\kappa_d)\, (V_d(B(A)) + V_0(A)) \\ &\leq 2^d\, d^{\frac{d}{2}}(\kappa_d + 1)\, (V_d(B(A)) + V_0(A)), \end{aligned}$$

yields

$$|\mathbb{E}V_k(Z \cap A)| \leq c_1(d)\, \varphi_{Q_1}(2)\, V_k(Q_1)\, (V_d(B(A)) + V_0(A))$$

with $c_1(d) = 2^d d^{\frac{d}{2}}(\kappa_d + 1)\, \tilde{c}(d)$. The Steiner formula (see Proposition 2.1) gives us the upper bound

$$|\mathcal{Q}(A)| \leq V_d(A + \sqrt{d}B^d) = \sum_{i=0}^{d} \kappa_{d-i}\, d^{(d-i)/2}\, V_i(A) \leq \overline{c}(d) \sum_{i=0}^{d} V_i(A)$$

with a constant $\overline{c}(d) > 0$ only depending on the dimension d. Together with similar arguments as above and Jensen's inequality, we obtain

$$\begin{aligned} \mathbb{E}V_k(Z \cap A)^4 &\leq \mathbb{E}\left(\sum_{I \subset \mathcal{Q}(A), \bigcap_{Q \in I} Q \neq \emptyset} |V_k(\bigcap_{Q \in I} Q \cap Z \cap A)| \right)^4 \\ &\leq \tilde{c}(d)^3\, |\mathcal{Q}(A)|^3\, \mathbb{E} \sum_{I \subset \mathcal{Q}(A), \bigcap_{Q \in I} Q \neq \emptyset} V_k(\bigcap_{Q \in I} Q \cap Z \cap A)^4 \\ &\leq \tilde{c}(d)^4\, |\mathcal{Q}(A)|^4\, \mathbb{E}2^{4N(Q_1)}\, V_k(Q_1)^4 \\ &\leq \tilde{c}(d)^4\, \varphi_{Q_1}(16)\, V_k(Q_1)^4 \left(\overline{c}(d) \sum_{i=0}^{d} V_i(A) \right)^4 \\ &\leq (d+1)^3\, \tilde{c}(d)^4\, \overline{c}(d)^4\, \varphi_{Q_1}(16)\, V_k(Q_1)^4 \sum_{i=0}^{d} V_i(A)^4, \end{aligned}$$

which concludes the proof. □

The number of grains hitting the cube Q_1 follows a Poisson distribution with parameter $\Lambda\left(\{K \in \mathcal{K}^d : K \cap Q_1 \neq \emptyset\}\right)$. The assumption $\mathbb{E}V_d(B(Z_0))^3 < \infty$ ensures that this parameter is finite. This can be also deduced from weaker assumptions on the typical grain Z_0. Hence, we can regard $\varphi_{Q_1}(2)$ and $\varphi_{Q_1}(16)$ as constants.

The previous result allows us to derive bounds for the kernels of the Wiener-Itô chaos expansions of $V_k(Z \cap W)$ and the fourth moment of the difference operator applied to $V_k(Z \cap W)$. The advantage of these bounds is that they do not involve expectations.

Corollary 9.5 *Let $W \in \mathcal{K}_0^d$. There are constants α_k, β_k, $k \in \{0, \ldots, d\}$, depending on the dimension d, the intensity parameter γ, and the typical grain Z_0 such that*

$$|f_n^{(k)}(K_1, \ldots, K_n)| \leq \frac{\alpha_k}{n!}(V_d(B(K_1 \cap \ldots \cap K_n \cap W)) + V_0(K_1 \cap \ldots \cap K_n \cap W))$$

for $K_1, \ldots, K_n \in \mathcal{K}^d$ and

$$\mathbb{E}(D_K V_k(Z \cap W))^4 \leq \beta_k \sum_{i=0}^{d} V_i(K \cap W)^4$$

for $K \in \mathcal{K}^d$.

Proof. First note that for $A \in \mathcal{K}^d$, we have

$$V_k(A) \leq V_k(B(A)) \leq V_k(B^d)\, R(A)^k \leq V_k(B^d)\,(1/\kappa_d + 1)\,(V_d(B(A)) + V_0(A)).$$

Now the first estimate follows from Lemma 9.3 and Lemma 9.4. For the second inequality we observe that

$$\begin{aligned} \mathbb{E}\left[(D_K V_k(Z \cap W))^4\right] &= \mathbb{E}\left(V_k(K \cap W) - V_k(Z \cap K \cap W)\right)^4 \\ &\leq 8\left(V_k(K \cap W)^4 + \mathbb{E}V_k(Z \cap K \cap W)^4\right) \\ &\leq \beta_k \sum_{i=0}^{d} V_i(K \cap W)^4, \end{aligned}$$

where Equation (9.2), Jensen's inequality, and Lemma 9.4 were used. □

The following lemma is helpful to bound integrals with respect to the measure Λ and is applied several times in the proofs of Theorem 9.1 and Theorem 9.2.

Lemma 9.6 *Let $A \in \mathcal{K}^d$. Then we have*

$$\int_{\mathbb{R}^d} V_d(B(A \cap (x+L))) + V_0(A \cap (x+L))\,\mathrm{d}x \leq 2^d\,(V_d(B(L)) + 1)\,(V_d(B(A)) + V_0(A)),$$

for every $L \in \mathcal{K}^d$ whose circumcentre is the origin, and

$$\int_{\mathcal{K}^d} V_d(B(A \cap K)) + V_0(A \cap K)\,\mathrm{d}\Lambda(K) \leq \tau\,(V_d(B(A)) + V_0(A))$$

with $\tau = \gamma\, 2^d\,(\mathbb{E}V_d(B(Z_0)) + 1)$.

Proof. We assume that $A \neq \emptyset$ since the inequalities obviously hold for $A = \emptyset$. First, we have

$$\begin{aligned} \int_{\mathbb{R}^d} V_d(B(A \cap (x+L)))\,\mathrm{d}x &\leq \min\{V_d(B(A)), V_d(B(L))\} \int_{\mathbb{R}^d} \mathbb{1}(A \cap (x+L) \neq \emptyset)\,\mathrm{d}x \\ &\leq \min\{V_d(B(A)), V_d(B(L))\}\, V_d(A + B(L)). \end{aligned}$$

Since

$$\begin{aligned} V_d(A+B(L)) &\leq V_d(B(A)+B(L)) = \kappa_d\,(R(A)+R(L))^d \\ &\leq 2^{d-1}\,(V_d(B(A))+V_d(B(L))) \leq 2^d \max\{V_d(B(A)), V_d(B(L))\}, \end{aligned}$$

we obtain

$$\int_{\mathbb{R}^d} V_d(B(A\cap(x+L)))\,\mathrm{d}x \leq 2^d\, V_d(B(L))\, V_d(B(A)).$$

Moreover, we have

$$\int_{\mathbb{R}^d} V_0(A\cap(x+L))\,\mathrm{d}x \leq V_d(A+B(L)) \leq 2^{d-1}\,(V_d(B(A))+V_d(B(L))).$$

Together this yields

$$\begin{aligned} &\int_{\mathbb{R}^d} V_d(B(A\cap(x+L))) + V_0(A\cap(x+L))\,\mathrm{d}x \\ &\leq 2^d\, V_d(B(L))V_d(B(A)) + 2^{d-1}(V_d(B(A))+V_d(B(L))) \\ &\leq 2^d\,(V_d(B(L))+1)\,(V_d(B(A))+V_0(A)), \end{aligned}$$

which proves the first claim. The second part follows from putting $L = Z_0$ and taking the expectation. □

The next lemma allows us to bound the ratio between intrinsic volumes and the volume of a convex body in dependence on its inradius.

Lemma 9.7 *Let $W \in \mathcal{K}_0^d$ have the inradius $r(W) > 0$ and let $k \in \{0, \ldots, d-1\}$. Then*

$$\frac{V_k(W)}{V_d(W)} \leq \frac{2^d-1}{\kappa_{d-k}}\, r(W)^{-(d-k)}.$$

Proof. A straightforward computation shows that

$$V_d(W+r(W)B^d) - V_d(W) \leq V_d(W+W) - V_d(W) = V_d(2W) - V_d(W) = (2^d-1)V_d(W).$$

Together with the Steiner formula and the fact that $V_i(W) \geq 0$ for $i = 0, \ldots, d-1$, we obtain

$$\begin{aligned} (2^d-1)V_d(W) &\geq V_d(W+r(W)B^d) - V_d(W) = \sum_{i=0}^{d-1} \kappa_{d-i}\, r(W)^{d-i}\, V_i(W) \\ &\geq \kappa_{d-k}\, r(W)^{d-k}\, V_k(W), \end{aligned}$$

which concludes the proof. □

The following observation about the intersection of translations of convex bodies is the reason for the positive definiteness of the covariance matrix if the typical grain is a convex body with positive probability.

Lemma 9.8 *For all $K_1, \dots, K_{d+1} \in \mathcal{K}_0^d$ there are constants $c_0, r_0 > 0$ such that*

$$\lambda_d^d\big(\{(x_2, \dots, x_{d+1}) \in (\mathbb{R}^d)^d : R(L) \le c_0\, r(L) \le r \text{ for } L = K_1 \cap \bigcap_{i=2}^{d+1}(x_i + K_i)\}\big) > 0$$

for all $0 < r \le r_0$.

Proof. Let $n_1, \dots, n_{d+1} \in \mathbb{R}^d$ be such that they have norm one, and their convex cone is $\mathbb{R}^d$. Then we can choose points $y_i \in \partial K_i$ such that n_i is an outer normal vector of K_i in y_i for $i = 1, \dots, d+1$. Let S_i be the support cone of K_i in y_i (this is the polar cone to the cone of all normal vectors in y_i). Since $K_1, \dots, K_{d+1}$ have interior points, there are translation vectors $v_1, \dots, v_{d+1} \in \mathbb{R}^d$ such that $0 \in \operatorname{int}(\bigcap_{i=1}^{d+1}(v_i + K_i))$. We define $S = \bigcap_{i=1}^{d+1}(v_i + y_i + S_i)$. Obviously, we have $0 \in \operatorname{int}(S)$. Moreover, S is bounded by the choice of $n_1, \dots, n_{d+1}$ and, thus, a convex body. For $t > 0$ and $v_{i,t} = tv_i + (t-1)y_i$, $i = 1, \dots, d+1$, we have $\bigcap_{i=1}^{d+1}(v_{i,t} + y_i + S_i) = tS$. The support cone S_i approximates K_i locally in y_i for $i = 1, \dots, d+1$ so that $\frac{1}{t}\bigcap_{i=1}^{d+1}(v_{i,t} + K_i)$ converges in the Hausdorff distance to S as $t \to \infty$.

Since inradius and circumradius are continuous with respect to the Hausdorff distance, the intersection $\bigcap_{i=1}^{d+1}(v_{i,t} + K_i)$ has a similar ratio between inradius and circumradius as S. By the continuity of translations, we obtain a similar ratio between inradius and circumradius if we move $v_{1,t}, \dots, v_{d+1,t} \in \mathbb{R}^d$ slightly. Because of translation invariance, we can put $v_{1,t} = 0$. □

Now we are prepared for the proof of Theorem 9.1:

Proof of Theorem 9.1: In order to simplify our notation, we use the abbreviation

$$g_k(A) = \mathbb{E}[V_k(Z \cap A)] - V_k(A)$$

for $k \in \{0, \dots, d\}$ and $A \in \mathcal{K}^d$. By substitution and translation invariance, we have

$$\begin{aligned}
&n! \int_{(\mathcal{K}^d)^n} f_n^{(k)}(K_1, \dots, K_n)\, f_n^{(\ell)}(K_1, \dots, K_n)\, \mathrm{d}\Lambda(K_1, \dots, K_n) \\
&= \frac{\gamma}{n!} \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} g_k((x_1 + K_1) \cap K_2 \cap \ldots \cap K_n \cap W) \\
&\qquad\qquad g_\ell((x_1 + K_1) \cap K_2 \cap \ldots \cap K_n \cap W)\, \mathrm{d}\Lambda(K_2, \dots, K_n)\, \mathrm{d}\mathbb{Q}(K_1)\, \mathrm{d}x_1 \\
&= \frac{\gamma}{n!} \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} g_k(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W - y)) \\
&\qquad\qquad g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W - y))\, \mathrm{d}\Lambda(K_2, \dots, K_n)\, \mathrm{d}\mathbb{Q}(K_1)\, \mathrm{d}y.
\end{aligned}$$

Thus, we have for

$$J_{n,k,\ell} = \frac{\gamma}{n!} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} g_k(K_1 \cap \ldots \cap K_n)\, g_\ell(K_1 \cap \ldots \cap K_n)\, \mathrm{d}\Lambda(K_2, \dots, K_n)\, \mathrm{d}\mathbb{Q}(K_1)$$

that

$$\big|n!\, \langle f_n^{(k)}, f_n^{(\ell)} \rangle_{L_s^2(\Lambda^n)} - V_d(W) J_{n,k,\ell}\big| \le \frac{\gamma}{n!}\,(R_1 + R_2) \tag{9.3}$$

with

$$R_1 = \Big| \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \mathbb{1}\,(y \notin W)\; g_k(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y)) \\ g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))\, \mathrm{d}\Lambda(K_2,\ldots,K_n)\, \mathrm{d}\mathbb{Q}(K_1)\, \mathrm{d}y \Big|$$

and

$$R_2 = \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \mathbb{1}\,(y \in W) \\ |g_k(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))\, g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y)) \\ - g_k(K_1 \cap K_2 \cap \ldots \cap K_n)\, g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n)| \\ \mathrm{d}\Lambda(K_2,\ldots,K_n)\, \mathrm{d}\mathbb{Q}(K_1)\, \mathrm{d}y.$$

In the following, we bound R_1 and R_2. Let $K_1,\ldots,K_n \in \mathcal{K}^d$ and $y \in \mathbb{R}^d$ be fixed, for the moment. It follows from Lemma 9.3 and Corollary 9.5 that

$$|g_k(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))\, g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))| \\ \leq \alpha_k\, \alpha_\ell\, (V_d(B(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))) + V_0(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y)))^2$$

with the constants α_k and α_ℓ from Corollary 9.5. By the monotonicity of the intrinsic volumes and by applying Lemma 9.6 for the integration with respect to $K_2,\ldots,K_n$, we obtain

$$\int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} (V_d(B(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))) \\ + V_0(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y)))^2\, \mathrm{d}\Lambda(K_2,\ldots,K_n)\, \mathrm{d}\mathbb{Q}(K_1) \\ \leq \int_{\mathcal{K}^d} \big(V_d(B(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))) + V_0(K_1 \cap K_2 \cap \ldots \cap K_n \cap (W-y))\big) \\ (V_d(B(K_1)) + 1)\, \mathrm{d}\Lambda(K_2,\ldots,K_n)\, \mathrm{d}\mathbb{Q}(K_1) \\ \leq \tau^{n-1} \int_{\mathcal{K}^d} \big(V_d(B(K_1 \cap (W-y))) + V_0(K_1 \cap (W-y))\big)(V_d(B(K_1)) + 1)\, \mathrm{d}\mathbb{Q}(K_1)$$

with the constant τ defined in Lemma 9.6. Hence, we deduce that

$$R_1 \leq \alpha_k\, \alpha_\ell\, \tau^{n-1} \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \mathbb{1}\,(y \notin W)\; (V_d(B(K_1 \cap (W-y))) + V_0(K_1 \cap (W-y))) \\ (V_d(B(K_1)) + 1)\, \mathrm{d}\mathbb{Q}(K_1)\, \mathrm{d}y \\ \leq \alpha_k\, \alpha_\ell\, \tau^{n-1} \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \mathbb{1}\,(y \notin W, (y+K_1) \cap W \neq \emptyset)\; (V_d(B(K_1)) + 1)^2\, \mathrm{d}\mathbb{Q}(K_1)\, \mathrm{d}y \\ \leq \alpha_k\, \alpha_\ell\, \tau^{n-1} \int_{\mathcal{K}^d} V_d((W + B(K_1)) \setminus W)\, (V_d(B(K_1)) + 1)^2\, \mathrm{d}\mathbb{Q}(K_1).$$

Since by the Steiner formula

$$V_d((W + B(K_1) \setminus W) = \sum_{i=0}^{d-1} \kappa_{d-i} V_i(W) R(K_1)^{d-i}$$

and by our moment assumption $\mathbb{E}[V_d(B(K_1))^3] < \infty$, we finally get

$$R_1 \leq c\,\alpha_k\,\alpha_\ell\,\tau^{n-1} \sum_{i=0}^{d-1} \kappa_{d-i} V_i(W)$$

with $c = \max_{i=0,\dots,d-1} \mathbb{E}\left[R(Z_0)^{d-i}\,(V_d(B(Z_0))+1)^2\right]$. By a similar argument, we have

$$\begin{aligned}
R_2 \;\leq\;& \alpha_k\,\alpha_\ell \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \mathbb{1}\,(y \in W, K_1 \not\subset W-y)\;(V_d(B(K_1))+1) \\
& \quad \big(V_d(B(K_1\cap K_2\cap\ldots\cap K_n\cap(W-y))) + V_d(B(K_1\cap K_2\cap\ldots\cap K_n)) \\
& \quad +V_0(K_1\cap K_2\cap\ldots\cap K_n\cap(W-y)) + V_0(K_1\cap K_2\cap\ldots\cap K_n)\big) \\
& \quad \mathrm{d}\Lambda(K_2,\ldots,K_n)\,\mathrm{d}\mathbb{Q}(K_1)\,\mathrm{d}y \\
\leq\;& \alpha_k\,\alpha_\ell\,\tau^{n-1} \int_{\mathbb{R}^d}\int_{\mathcal{K}^d} \mathbb{1}\,(y\in W, y+K_1\not\subset W)\;(V_d(B(K_1))+1) \\
& \qquad (V_d(B(K_1\cap(W-y))) + V_d(B(K_1)) + 2)\,\mathrm{d}\mathbb{Q}(K_1)\,\mathrm{d}y \\
\leq\;& 2\,\alpha_k\,\alpha_\ell\,\tau^{n-1} \int_{\mathbb{R}^d}\int_{\mathcal{K}^d} \mathbb{1}\left(y\in W\cap\partial W_{R(K_1)}\right)\;(V_d(B(K_1))+1)^2\;\mathrm{d}\mathbb{Q}(K_1)\,\mathrm{d}y \\
\leq\;& 2\,\alpha_k\,\alpha_\ell\,\tau^{n-1} \int_{\mathcal{K}^d} \sum_{i=0}^{d-1} \kappa_{d-i} V_i(W) R(K_1)^{d-i}\,(V_d(B(K_1))+1)^2\,\mathrm{d}\mathbb{Q}(K_1) \\
\leq\;& 2\,c\,\alpha_k\,\alpha_\ell\,\tau^{n-1} \sum_{i=0}^{d-1} \kappa_{d-i} V_i(W)
\end{aligned}$$

Here, we used that if $y \in W$ and $y + K_1 \not\subset W$, then $y \in W \cap \partial W_{R(K_1)}$ with $\partial W_{R(K_1)} = \{z \in \mathbb{R}^d : \mathrm{dist}(z, \partial W) \leq R(K_1)\}$. Now the Steiner formula (see Proposition 2.1) implies that

$$\begin{aligned}
& V_d\left(W\cap\partial W_{R(K_1)}\right) \\
& \leq V_d\left(W^C\cap\partial W_{R(K_1)}\right) = V_d(W+B(K_1)) - V_d(W) = \sum_{i=0}^{d-1}\kappa_{d-i}V_i(W)R(K_1)^{d-i}.
\end{aligned}$$

Combining the previous inequalities for R_1 and R_2 with formula (9.3), we see that

$$\begin{aligned}
& \Big| \frac{n!}{V_d(W)} \int_{(\mathcal{K}^d)^n} f_n^{(k)}(K_1,\ldots,K_n)\, f_n^{(\ell)}(K_1,\ldots,K_n)\,\mathrm{d}\Lambda(K_1,\ldots,K_n) \\
& - \frac{\gamma}{n!} \int_{\mathcal{K}^d}\int_{(\mathcal{K}^d)^{n-1}} g_k(K_1\cap\ldots\cap K_n)\, g_\ell(K_1\cap\ldots\cap K_n)\,\mathrm{d}\Lambda(K_2,\ldots,K_n)\,\mathrm{d}\mathbb{Q}(K_1)\Big| \\
& \leq \frac{3\,\gamma\,c\,\alpha_k\,\alpha_\ell\,\tau^{n-1}}{n!} \sum_{i=0}^{d-1} \kappa_{d-i}\frac{V_i(W)}{V_d(W)}.
\end{aligned}$$

Now Lemma 9.7 and the formula

$$\frac{\mathrm{Cov}(V_k(Z\cap W), V_\ell(Z\cap W))}{V_d(W)} = \sum_{n=1}^{\infty} \frac{n!}{V_d(W)}\,\langle f_n^{(k)}, f_n^{(\ell)}\rangle_{L^2_s(\Lambda^n)}$$

prove that

$$\sigma_{k,\ell} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} g_k(K_1 \cap K_2 \cap \ldots \cap K_n)\, g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n) \, \mathrm{d}\Lambda(K_2, \ldots, K_n)\, \mathrm{d}\mathbb{Q}(K_1)$$

for $k, \ell = 0, \ldots, d$ and that the inequality (9.1) holds.

Next we show that the asymptotic covariance matrix $\Sigma = (\sigma_{k,\ell})_{k,\ell=0,\ldots,d}$ is positive definite if $\mathbb{P}(V_d(Z_0) > 0) > 0$. For a vector $a = (a_0, \ldots, a_d) \in \mathbb{R}^{d+1}$ we have

$$\begin{aligned}
&a^T \Sigma\, a \\
&= \gamma \sum_{k,\ell=0}^{d} a_k\, a_\ell \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} g_k(K_1 \cap K_2 \cap \ldots \cap K_n)\, g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n) \\
&\qquad \mathrm{d}\Lambda(K_2, \ldots, K_n)\, \mathrm{d}\mathbb{Q}(K_1) \\
&= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k,\ell=0}^{d} a_k\, a_\ell \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} g_k(K_1 \cap K_2 \cap \ldots \cap K_n)\, g_\ell(K_1 \cap K_2 \cap \ldots \cap K_n) \\
&\qquad \mathrm{d}\Lambda(K_2, \ldots, K_n)\, \mathrm{d}\mathbb{Q}(K_1) \\
&= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \left(\sum_{k=0}^{d} a_k\, g_k(K_1 \cap K_2 \cap \ldots \cap K_n) \right)^2 \mathrm{d}\Lambda(K_2, \ldots, K_n)\, \mathrm{d}\mathbb{Q}(K_1).
\end{aligned}$$

Since each summand is non-negative, the matrix Σ is positive definite if we can prove that one summand is greater than zero for all $a \in \mathbb{R}^{d+1}$ with $a \neq 0$.

For $L \in \mathcal{K}^d$ we denote by $N_1(L)$ the number of grains of η that intersect L but do not cover it and by $N_2(L)$ the number of grains of η that cover L. Both random variables are independent and follow Poisson distributions with the parameters

$$s_1(L) = \Lambda(\{K \in \mathcal{K}^d : K \cap L \neq \emptyset \text{ and } L \not\subset K\})$$

and

$$s_2(L) = \Lambda(\{K \in \mathcal{K}^d : L \subset K\}).$$

If $N_2(L) \neq 0$, we have

$$V_k(Z \cap L) - V_k(L) = V_k(L) - V_k(L) = 0.$$

For $N_1(L) = N_2(L) = 0$ we obtain

$$V_k(Z \cap L) - V_k(L) = 0 - V_k(L) = -V_k(L).$$

In the case $N_1(L) \neq 0$ and $N_2(L) = 0$, we have the inequality

$$|V_k(Z \cap L) - V_k(L)| = \Big| \sum_{I \subset \{1, \ldots, N_1(L)\}} (-1)^{|I|+1}\, V_k(\bigcap_{i \in I} Z_i \cap L) - V_k(L) \Big| \leq 2^{N_1(L)} V_k(L),$$

where $Z_1, \ldots, Z_{N_1(L)}$ are the grains that intersect L. Altogether, we find

$$\mathbb{E} V_k(Z \cap L) - V_k(L) = -\exp(-s_1(L) - s_2(L))\, V_k(L) + \widetilde{R}(L)$$

with

$$\begin{aligned} |\widetilde{R}(L)| &\leq \exp(-s_2(L)) \sum_{n=1}^{\infty} \frac{s_1(L)^n}{n!} \exp(-s_1(L))\, 2^n\, V_k(L) \\ &= \exp(-s_2(L))\, (\exp(s_1(L)) - \exp(-s_1(L)))\; V_k(L) \\ &\leq \exp(-s_2(L) + s_1(L))\, 2s_1(L)\, V_k(L) \end{aligned}$$

so that

$$g_k(L) = \exp(-s_2(L))\, (-\exp(-s_1(L)) + \exp(s_1(L))\, s_1(L)\, R(L))\; V_k(L)$$

with $|R(L)| \leq 2$. By the mean value theorem, we can rewrite this as

$$g_k(L) = -\exp(-s_2(L))\, (1 + c_k(L)\, s_1(L))\, V_k(L)$$

with $|c_k(L)| \leq 9$ for $s_1(L) \leq 1$.

From now on, we consider the case $n = d+1$. Fix arbitrary $K_1, \dots, K_{d+1} \in \mathcal{K}^d$ with non-empty interior and let $L = K_1 \cap (x_2 + K_2) \cap \dots \cap (x_{d+1} + K_{d+1})$ for $x_2, \dots, x_{d+1} \in \mathbb{R}^d$. Now it follows that

$$\sum_{k=0}^{d} a_k\, g_k(L) = \exp(-s_2(L)) \sum_{k=0}^{d} a_k\, (1 + c_k(L)\, s_1(L))\, V_k(L)$$

if L is sufficiently small (and hence $s_1(L) \leq 1$). Let $k_0 = \min\{k = 0, \dots, d : a_k \neq 0\}$ (such a $k_0 \in \{0, \dots, d\}$ exists for $a \neq 0$). Now we can choose $x_2, \dots, x_{d+1} \in \mathbb{R}^d$ according to Lemma 9.8. Then, we have

$$V_k(B^d)\, r(L)^k \leq V_k(L) \leq c_0^k\, V_k(B^d)\, r(L)^k$$

for $k = 0, \dots, d$. If we choose the parameter $r > 0$ in Lemma 9.8 sufficiently small, the k_0-th summand, which cannot be zero, dominates the sum. This implies that

$$\int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^d} \left(\sum_{k=0}^{d} a_k\, g_k(K_1 \cap K_2 \cap \dots \cap K_{d+1}) \right)^2 \mathrm{d}\Lambda(K_2, \dots, K_{d+1})\, \mathrm{d}\mathbb{Q}(K_1) > 0$$

for $a \neq 0$ so that the covariance matrix Σ is indeed positive definite. □

Using Theorem 5.15 and the same inequalities as in the proof of Theorem 9.1, we can prove Theorem 9.2.

Proof of Theorem 9.2: Let $i, j \geq 1$, $0 \leq k, \ell \leq d$ and $\sigma \in \widetilde{\Pi}_{\geq 2}^{(1)}(i,i,j,j)$. From Corollary 9.5, it follows that

$$\begin{aligned} &\int_{(\mathcal{K}^d)^{|\sigma|}} |(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(\ell)} \otimes f_j^{(\ell)})_\sigma|\, \mathrm{d}\Lambda^{|\sigma|} \\ &\leq \frac{(\alpha_k\, \alpha_\ell)^2}{(i!\, j!)^2} \int_{(\mathcal{K}^d)^{|\sigma|}} \prod_{m=1}^{4} \left(V_d(B(\bigcap_{n \in N_m(\sigma)} K_n \cap W)) + V_0(\bigcap_{n \in N_m(\sigma)} K_n \cap W) \right) \mathrm{d}\Lambda(K_1, \dots, K_{|\sigma|}) \end{aligned}$$

with constants α_k and α_ℓ as in Corollary 9.5 and sets $N_m(\sigma) \subset \{1,\dots,|\sigma|\}$, $m = 1,\dots,4$, depending on σ. Every $1 \leq n \leq |\sigma|$ is contained in at least two of these sets. By removing the index n from the sets until it occurs only in one set, we increase the integral and can use Lemma 9.6 to integrate over K_n. Due to the special structure of σ, we obtain by iterating this step and using the abbreviation $h_W(A) = V_d(B(A \cap W)) + V_0(A \cap W)$

$$\begin{aligned}
&\int_{(\mathcal{K}^d)^{|\sigma|}} |(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(\ell)} \otimes f_j^{(\ell)})_\sigma| \, \mathrm{d}\Lambda^{|\sigma|} \\
&\leq \frac{(\alpha_k\, \alpha_\ell)^2}{(i!\, j!)^2}\, \tau^{|\sigma|-3} \int_{(\mathcal{K}^d)^3} h_W(K_1)\, h_W(K_2 \cap K_3)\, h_W(K_1 \cap K_2)\, h_W(K_3)\, \mathrm{d}\Lambda(K_1, K_2, K_3) \\
&= \frac{(\alpha_k\, \alpha_\ell)^2}{(i!\, j!)^2}\, \tau^{|\sigma|-3} \int_{\mathcal{K}^d} \left(\int_{\mathcal{K}^d} h_W(K_1)\, h_W(K_1 \cap K_2)\, \mathrm{d}\Lambda(K_1) \right)^2 \mathrm{d}\Lambda(K_2)
\end{aligned}$$

with the constant τ from Lemma 9.6. For a fixed $K_2 \in \mathcal{K}^d$ Lemma 9.6 implies that

$$\begin{aligned}
&\int_{\mathcal{K}^d} h_W(K_1)\, h_W(K_1 \cap K_2)\, \mathrm{d}\Lambda(K_1) \\
&\leq \gamma\, \mathbb{E}\left[(V_d(B(Z_0) + 1) \int_{\mathbb{R}^d} V_d(B((x+Z_0) \cap K_2 \cap W)) + V_0((x+Z_0) \cap K_2 \cap W)\, \mathrm{d}x \right] \\
&\leq \gamma\, 2^d\, \mathbb{E}\left[(V_d(B(Z_0)) + 1)^2 \right] (V_d(B(K_2 \cap W)) + V_0(K_2 \cap W)) \,.
\end{aligned}$$

Putting $c_1 = \gamma\, 2^d\, \mathbb{E}(V_d(B(Z_0)) + 1)^2$ and applying the Steiner formula (see Proposition 2.1) and the moment assumption $\mathbb{E}[V_d(B(Z_0))^4] < \infty$ yield

$$\begin{aligned}
&\int_{\mathcal{K}^d} \left(\int_{\mathcal{K}^d} h_W(K_1)\, h_W(K_1 \cap K_2)\, \mathrm{d}\Lambda(K_1) \right)^2 \mathrm{d}\Lambda(K_2) \\
&\leq c_1^2 \int_{\mathcal{K}^d} (V_d(B(K_2 \cap W)) + V_0(K_2 \cap W))^2 \, \mathrm{d}\Lambda(K_2) \\
&\leq \gamma\, c_1^2\, \mathbb{E}\left[(V_d(B(Z_0)) + 1)^2 \int_{\mathbb{R}^d} \mathbb{1}\{(x+Z_0) \cap W \neq \emptyset\}\, \mathrm{d}x \right] \\
&\leq \gamma\, c_1^2\, \mathbb{E}\left[(V_d(B(Z_0)) + 1)^2 \sum_{i=0}^d \kappa_{d-i} V_i(W) R(Z_0)^{d-i} \right] \leq c_2 \sum_{i=0}^d V_i(W)
\end{aligned}$$

with a constant c_2 depending on the dimension d, the intensity parameter γ, and the typical grain Z_0. Finally, we have

$$\int_{(\mathcal{K}^d)^{|\sigma|}} |(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(\ell)} \otimes f_j^{(\ell)})_\sigma| \, \mathrm{d}\Lambda^{|\sigma|} \leq \frac{c_2\, (\alpha_k\, \alpha_\ell)^2}{(i!\, j!)^2}\, \tau^{|\sigma|-3} \sum_{i=0}^d V_i(W) \leq \frac{a\, b^{i+j}}{(i!\, j!)^2} \tag{9.4}$$

with $a = \max\limits_{0 \leq k,\ell \leq d} \frac{c_2\, (\alpha_k\, \alpha_\ell)^2}{\tau^3} \sum_{i=0}^d V_i(W)$ and $b = \max(\tau, 1)$. Now formula (9.4) has exactly the form required in the conditions (5.35) and (5.36) in Theorem 5.15, and it follows from Lemma 9.7 that there are constants $c_3, c_4 > 0$ such that

$$\frac{a}{V_d(W)^2} = \max_{0 \leq k,\ell \leq d} \frac{c_2\, (\alpha_k\, \alpha_\ell)^2}{\tau^3} \sum_{i=0}^d \frac{V_i(W)}{V_d(W)^2} \leq \frac{c_3}{V_d(W)} \leq \frac{c_4}{r(W)^d} \tag{9.5}$$

for all $W \in \mathcal{K}_0^d$ satisfying $r(W) \geq 1$.

By Corollary 9.5, we obtain

$$\begin{aligned}
&\int_{\mathcal{K}^d} \mathbb{E}(D_K V_k(Z \cap W))^4 \,\mathrm{d}\Lambda(K) \\
&\leq \gamma\, \beta_k\, \mathbb{E} \int_{\mathbb{R}^d} \sum_{i=0}^{d} V_i((x+Z_0) \cap W)^4 \,\mathrm{d}x \\
&\leq \gamma\, \beta_k\, \mathbb{E} \sum_{i=0}^{d} \min\{V_i(Z_0), V_i(W)\}^4 \int_{\mathbb{R}^d} \mathbb{1}((x+Z_0) \cap W \neq \emptyset) \,\mathrm{d}x \\
&\leq \gamma\, \beta_k\, \mathbb{E} \sum_{i=0}^{d} \min\{V_i(B^d) R(Z_0)^i, V_i(W)\}^4 \sum_{j=0}^{d} \kappa_{d-j} V_j(W) R(Z_0)^{d-j} \\
&\leq c_5\, \mathbb{E} \sum_{i,j=0}^{d} R(Z_0)^{\min\{3d+j,4i\}} V_i(W)^{\max\{4-(3d+j)/i,0\}} V_j(W) R(Z_0)^{d-j}
\end{aligned}$$

with a constant $c_5 > 0$ depending on the dimension d, the intensity γ, the typical grain Z_0, and k. For $i = 0$ we put $\max\{4 - (3d+j)/i, 0\} = 0$. Now the moment assumption $\mathbb{E}V_d(B(Z_0))^4 < \infty$ implies that

$$\int_{\mathcal{K}^d} \mathbb{E}(D_K V_k(Z \cap W))^4 \,\mathrm{d}\Lambda(K) \leq c_6 \sum_{i,j=0}^{d} V_i(W)^{\max\{4-(3d+j)/i,0\}} V_j(W)$$

with $c_6 = c_5 \max_{i=1,\dots,4d} \mathbb{E}R(Z_0)^i$. Together with Lemma 9.7, this yields

$$\begin{aligned}
&\frac{1}{V_d(W)^2} \int_{\mathcal{K}^d} \mathbb{E}(D_K V_k(Z \cap W))^4 \,\mathrm{d}\Lambda(K) \\
&\leq c_6 \sum_{i,j=0}^{d} \left(\frac{V_i(W)}{V_d(W)}\right)^{\max\{4-(3d+j)/i,0\}} \left(\frac{1}{V_d(W)}\right)^{1-\max\{4-(3d+j)/i,0\}} \frac{V_j(W)}{V_d(W)} \\
&\leq c_7 \sum_{i,j=0}^{d} \left(\frac{1}{r(W)^{d-i}}\right)^{\max\{4-(3d+j)/i,0\}} \left(\frac{1}{r(W)^{d}}\right)^{1-\max\{4-(3d+j)/i,0\}} \frac{1}{r(W)^{d-j}} \\
&= c_7 \sum_{i,j=0}^{d} r(W)^{-2d+j+\max\{4i-3d-j,0\}} \leq \frac{c_8}{r(W)^d}
\end{aligned} \tag{9.6}$$

for $r(W) \geq 1$.

Combining the inequalities (9.1), (9.5), and (9.6) with Theorem 5.15 b) yields the multivariate part of Theorem 9.2. The bound for the Wasserstein distance is a direct consequence of Theorem 5.15 a) and the inequalities (9.5) and (9.6). □

Notes: The results and proofs of this chapter are part of *Hug, Last, and Schulte 2012*.

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