Module theory over the exterior algebra with applications to combinatorics

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Introduction

In 1975 Stanley initialised a new trend in commutative algebra by using in a systematic way methods from commutative algebra to study simplicial complexes. He associated a certain quotient ring of a polynomial ring to a simplicial complex, the so-called Stanley-Reisner ring, see, e.g., the books by Miller and Sturmfels [**43**] or Stanley [**57**]. In the meantime a new perception arose: in the same manner one can associate a certain quotient ring of an exterior algebra to a simplicial complex, called the exterior face ring, see, e.g., [**28**]. This turned out to be more natural. For example, in a "dictionary" of algebraic properties of the Stanley-Reisner ring and combinatorial properties of the simplicial complex one always should take into account a duality (the so-called Alexander duality), while one has a direct translation for exterior face rings. However, there are far more methods known to study quotient rings of a polynomial ring than of an exterior algebra. This thesis collects the known methods to work over an exterior algebra and, based on these, develops parts of a module theory over an exterior algebra. Moreover, using the example of Orlik-Solomon algebras, it shows that the exterior algebra is helpful in the study of other combinatorial objects that have no access via the polynomial ring. For more uses of exterior algebra methods see, e.g., [**28**], [**44**], [**45**], [**47**].

The exterior algebra $E = K\langle e_1, \ldots, e_n \rangle$ in n variables over some field K is a skew-commutative $\mathbb{Z}$-graded K-algebra, with $\deg e_i = 1$. We work on the category $\mathscr{M}$ of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$. Although E is not commutative, it behaves like a commutative local ring or *local ring (cf. [**11**, Section 1.5]) in many cases. Therefore it is possible to translate many notions and relations between these to the exterior algebra; sometimes one has to adjust them a bit.

The exterior algebra and the polynomial ring on a dual basis to $e_1, \ldots, e_n$ are Koszul duals of each other, that is, the Ext-algebra $\operatorname{Ext}_E^*(K,K)$ can be identified with that polynomial ring. These two algebras are further related via the so-called Bernstein-Gel'fand-Gel'fand correspondence (see Section 2.4 for details or the book by Eisenbud [**18**]). For example, Eisenbud and Schreyer [**21, 22**] used it in their construction for the proof of the Boij-Söderberg conjectures [**9**], which, in particular, yield a proof of the multiplicity conjecture of Huneke and Srinivasan (see [**32**]).

The exterior algebra as a tool in commutative algebra (as, e.g., in the construction of the Koszul complex) is widely known. Chapter 1 introduces it as a stand-alone K-algebra and gives the essential concepts in this spirit that are used in the thesis. Furthermore, Section 1.2 collects all necessary notions related to simplicial complexes. A simplicial complex Δ on the vertex set $[n] = \{1, \ldots, n\}$ is a set of subsets of $[n]$ closed under taking subsets.

Chapter 2, on graded resolutions over the exterior algebra, has an introductory character, too. In the first two sections it presents projective and injective resolutions, the differences and relations between these. One important invariant related to projective resolutions is the *Castelnuovo-Mumford regularity* $\operatorname{reg}_E M = \max\{j \in \mathbb{Z} : \beta_{i,i+j}(M) \neq 0\}$ of a module $M \in \mathscr{M}$, where $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^E(K,M)_j$ are the *graded Betti numbers* of M. We say that M has a *d-linear projective resolution* if $\beta_{i,i+j}(M) = 0$ for all integers i and $j \neq d$. The *complexity* of M measures the growth rate of the Betti numbers of M and is defined as

$$\operatorname{cx}_E M = \inf\{c \in \mathbb{Z} : \beta_i(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\},$$

where $\beta_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(M)$ is the i-th *total Betti number*. Injective resolutions are the duals of projective resolutions via the duality functor $(-)^* = \operatorname{Hom}_E(-,E)$, which is exact on $\mathscr{M}$. The numbers $\mu_{i,j}(M) = \dim_K \operatorname{Ext}_E^i(K,M)_j$ are called the *graded Bass numbers* of M. We say that M has a *d-linear injective resolution* if $\mu_{i,j-i}(M) = 0$ for all $j \neq d$. The Bass numbers of M and the Betti numbers of its dual M^* are related via $\mu_{i,j}(M) = \beta_{i,n-j}(M^*)$ by [**3**, Proposition 5.2]. In particular, M has a d-linear injective resolution if and only if M^* has an $(n-d)$-linear projective resolution. A particular projective resolution of K is the Cartan complex, an injective the Cartan cocomplex, which are constructed in Section 2.3. They play a similar role as the Koszul complex over the polynomial ring.

An important tool to study ideals in a polynomial ring is the Gröbner basis theory. Aramova, Herzog and Hibi [**3**] establish a Gröbner basis theory in the exterior algebra. In doing so they lie the foundations to work with generic initial ideals in the exterior algebra. Generic initial ideals preserve much information of the original ideal and, additionally, are strongly stable. Therefore in many situations it is a successful strategy to pass on to the generic initial ideal and then exploit the special properties of strongly stable ideals. Accordingly, first these properties are studied in Section 3.1, followed by a presentation of initial and generic initial ideals in Section 3.2, including the corresponding operation on simplicial complexes, the so-called *exterior algebraic shifting*.

Let $M \in \mathscr{M}$. Following [**1**] we call an element $v \in E_1$ *regular* on M (or *M-regular*) if the annihilator $0 :_M v$ of v in M is the smallest possible, that is, the submodule vM. Now *M-regular sequences* can be defined in the obvious way. The maximal ones have all the same length, which is called the *depth* of M over E and denoted by $\operatorname{depth} M$. This invariant is studied extensively in Chapter 4. We show in Theorem 4.1.12 that

$$\operatorname{depth}_E M = \operatorname{depth}_E M^* \quad \text{and} \quad \operatorname{cx}_E M = \operatorname{cx}_E M^*,$$

where the second equality follows from the relation $\operatorname{depth}_E M + \operatorname{cx}_E M = n$ (provided that K is infinite, see [**1**, Theorem 3.2]). Let Δ be a simplicial complex and $K\{\Delta\} = E/(e_F : F \notin \Delta)$ be its face ring, $K[\Delta] = S/(x_F : F \notin \Delta)$ be its Stanley-Reisner ring, where $e_F = \prod_{i\in F} e_i$, $x_F = \prod_{i\in F} x_i$ and $S = K[x_1,\ldots,x_n]$. We prove in Theorem 4.3.6 that

$$\operatorname{depth}_E K\{\Delta\} \leq \operatorname{depth}_S K[\Delta],$$

even in the more general situation of squarefree modules. In Section 4.4 the generic exterior annihilator numbers $\alpha_{i,j}(E/J)$ are studied. These are in some sense a refinement of the depth. They can be used to compute the Cartan-Betti numbers, introduced in [**46**], and, in particular, the Betti numbers of E/J. We further show in Proposition 4.4.9 that, if

$|K| = \infty$, then

$$\beta^S_{i,i+j}(K[\Delta^e]) = \sum_{l=1}^{n} \binom{n-l-j+2}{i-1} \alpha_{l,j}(K\{\Delta\})$$

for the exterior shifting Δ^e of Δ.

Chapter 5 is devoted to module-theoretic properties. Modules with linear injective resolutions behave quite nice. For example, let J be a graded ideal in E such that E/J has a d-linear injective resolution, $|K| = \infty$ and $H(M,t) = \sum_{i\in\mathbb{Z}} \dim_K M_i t^i$ denote the *Hilbert series* of $M \in \mathscr{M}$. Then we show in Theorems 5.1.6 and 5.1.3 that

$$\operatorname{reg}_E(E/J) + \operatorname{depth}_E(E/J) = d$$

and there exists a polynomial $Q(t) \in \mathbb{Z}[t]$ with non-negative coefficients such that

$$H(E/J,t) = Q(t) \cdot (1+t)^{\operatorname{depth}_E(E/J)} \quad \text{and} \quad Q(-1) \neq 0.$$

In Theorem 5.1.8 we characterise in this situation the ideals J that have a linear projective resolution. A face ring has a linear injective resolution if and only if the associated simplicial complex is Cohen-Macaulay, a direct consequence of the Eagon-Reiner theorem [**17**].

A module $M \in \mathscr{M}$ is called *componentwise linear* if the submodule $M_{<j>}$ of M, which is generated by all homogeneous elements of degree j belonging to M, has a j-linear resolution. We call M *componentwise injective linear* if $M^{<j>}$ has a linear injective resolution for all $j \in \mathbb{Z}$, with $M^{<j>} = M/N$, where N is the biggest submodule of M such that $N_j = 0$. These are dual notions in the sense that M is componentwise linear if and only if M^* is componentwise injective linear. A face ring is componentwise injective linear if and only if the simplicial complex is sequentially Cohen-Macaualy, a result whose dual version was known before by [**31**].

Furthermore, M is said to have *linear quotients* with respect to a homogeneous system of generators $m_1,\dots,m_t$ if $(m_1,\dots,m_{i-1}) :_E m_i$ is an ideal in E generated by linear forms for $i = 1,\dots,t$. A graded ideal that has linear quotients w.r.t. a minimal system of generators is componentwise linear. Dually, E/J has *pure decomposable quotients* w.r.t. a decomposition $J = J_1 \cap \ldots \cap J_m \subset E$ for graded ideals J_i such that $\dim_K \operatorname{soc}(E/J_i) = 1$ and $\bigcap_{j=1}^{i-1} J_j / \bigcap_{j=1}^{i} J_j$ is isomorphic to an ideal that is generated by a product of linear forms, shifted in degrees by $n - \max\{j : (E/J_i)_j \neq 0\}$, for $i = 1,\dots,m$. A face ring satisfies this property if and only if the simplicial complex is shellable; again the dual version was known before by [**33**].

In Chapter 6 we apply our results on Orlik-Solomon algebras. The singular cohomology $H^\bullet(X;K)$ of the complement X of an essential central affine hyperplane arrangement $\mathscr{A}$ in $\mathbb{C}^m$ is isomorphic to E/J where J is the *Orlik-Solomon ideal* of X which is generated by all

$$\partial e_S = \sum_{i=1}^{t} (-1)^{i-1} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_i}} \wedge \cdots \wedge e_{j_t} \text{ for } S = \{j_1,\dots,j_t\} \subseteq [n]$$

where $\{H_{j_1},\dots,H_{j_t}\}$ is a dependent set of hyperplanes of $\mathscr{A}$, i.e. their defining linear forms are linearly dependent. The definition of E/J depends only on the matroid of $\mathscr{A}$, so we study more generally the Orlik-Solomon algebra of a matroid. For the convenience of the reader we start with all necessary matroid notions. In Theorem 6.2.3 and Corollary 6.3.1 we determine the depth and the regularity of an Orlik-Solomon algebra. More precisely, if k is the number of connected components of the matroid and l is its rank, then

$$\operatorname{depth}_E(E/J) = k \quad \text{and} \quad \operatorname{reg}_E(E/J) = l - k.$$

Note: Most of the results in Sections 3.1, 4.1, 5.1 and Chapter 6 have been published in [**40**], most of the results in Sections 4.3 and 4.4 will appear in [**39**]; see also [**38**] for an overview article.

CHAPTER 1

Basic notions

This chapter introduces the basic notions and tools that are used in the thesis. It is assumed that the reader is familiar in the field of commutative algebra, as, e.g., found in the book of Matsumura [**42**], in particular with the homological methods used there and the theory of graded rings and modules. An extensive introduction into the homological methods gives Weibel [**60**], a good overview on graded module theory is contained in the book of Bruns and Herzog [**11**]. The forthcoming book by Herzog and Hibi [**30**] has an introductory section on the exterior algebra.

1.1. Exterior algebra

Let $E = K\langle e_1, \dots, e_n\rangle$ be the exterior algebra over an n-dimensional vector space V with fixed basis $e_1, \dots, e_n$ over some field K; see, e.g., Bourbaki [**10**] for details on this construction. It is a K-algebra with defining relations $v \wedge v = 0$ for all $v \in V$. Then it follows that $v \wedge w = -w \wedge v$ for all $v, w \in V$. Setting $\deg e_i = 1$ induces a $\mathbb{Z}$-grading on E. In the following a grading is a $\mathbb{Z}$-grading unless otherwise specified. The elements $v \in E$ of degree one are called *linear forms*. For a subset $A = \{i_1, \dots, i_r\} \subseteq [n] = \{1, \dots, n\}$ with $i_1 < \dots < i_r$ we write $e_A = e_{i_1} \cdots e_{i_r}$ for the corresponding *monomial* $e_{i_1} \wedge \dots \wedge e_{i_r}$. (From now on we omit the $\wedge$ in products.) There is a one-to-one correspondence between subsets of $[n]$ and monomials in E. As a K-vector space E has a basis consisting of all monomials. In particular, its vector space dimension is finite.

We only consider graded ideals in E. Then every left or right ideal is a two-sided ideal. For non-graded ideals this is not the case, e.g., over a field of characteristic $\neq 2$ the left ideal $I = (gf : g \in E)$ and the right ideal $J = (fg : g \in E)$ in $E = K\langle e_1, \dots, e_4\rangle$, generated by $f = e_1 + e_2e_3$, are not equal, because $e_4 f = -e_1e_4 + e_2e_3e_4 \in I \setminus J$.

An ideal in E is called *monomial* if it is generated by monomials. The only graded maximal ideal in E is $\mathfrak{m} = (e_1, \dots, e_n)$. The category of modules we work on is the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$. We denote this category by $\mathscr{M}$. For example if $J \subseteq E$ is a graded ideal, then E/J belongs to $\mathscr{M}$. In particular, E is an object in $\mathscr{M}$. The set of all homogeneous elements of degree k belonging to M is denoted by M_k.

There are three different ways of regarding the field K. First, we have $K = E_0$ as a subset of E, but this is not a submodule. Second, there is the E-module $K \cong E_n = Ke_{[n]}$. But this is not a graded isomorphism, so we prefer to write $K(-n)$ instead. Third, we can regard $K \cong E/\mathfrak{m}$. This is the usual way when considering K as E-module, unless otherwise stated.

Let $(-)^*$ denote the duality functor on $\mathscr{M}$ given by $M^* = \operatorname{Hom}_E(M,E)$ for $M \in \mathscr{M}$. It is an exact contravariant functor on $\mathscr{M}$, which has been proved by different means in [**3**, Proposition 5.1] or [**18**, Proposition 7.19]. For a graded ideal $J \subseteq E$ one easily sees that

$$(E/J)^* = \operatorname{Hom}_E(E/J,E) \cong 0:_E J$$

as graded E-modules (cf. [**3**, Section 5]), where $0:_E J = \{f \in E : fJ = 0\}$. In particular, there is a graded isomorphism of E-modules

$$K^* = \operatorname{Hom}_E(E/\mathfrak{m},E) \cong K(-n),$$

since the annihilator of $\mathfrak{m}$ is exactly $Ke_1 \cdots e_n$. This is also the socle of E. The *socle* of an E-module M is the set of elements belonging to M that are annihilated by the maximal ideal $\mathfrak{m}$.

For a K-vector space W let $W^\vee = \operatorname{Hom}_K(W,K)$ be the K-dual of W. Note that the symbols $*$ and $\vee$ are used in various ways in the literature.

Lemma 1.1.1. [**3**, Proposition 5.1] *Let $M \in \mathscr{M}$. Then $(M^*)_i \cong (M_{n-i})^\vee$ as K-vector spaces for all $i \in \mathbb{Z}$.*

More precisely,

$$M^* = \operatorname{Hom}_E(M,E) \cong \operatorname{Hom}_K(M,K(-n)) = M^\vee(-n),$$

since $\operatorname{Hom}_E(K,E) \cong K(-n)$ and this is even a functorial isomorphism.

Every module $M \in \mathscr{M}$ has only finitely many non-zero homogeneous components. The highest degree in which a non-zero module M is not zero is denoted by $d(M)$, i.e.,

$$d(M) = \max\{i \in \mathbb{Z} : M_i \neq 0\} = n - \min\{i \in \mathbb{Z} : (M^*)_i \neq 0\},$$

where the duality version results from Lemma 1.1.1.

The function $H(-,M)$ with

$$H(i,M) = \dim_K M_i$$

which maps i to the vector space dimension of the i-th homogeneous component, is called the *Hilbert function* of M.

The *Hilbert series* of M

$$H(M,t) = \sum_{i \in \mathbb{Z}} H(i,M) t^i$$

has only finitely many summands. In particular, if $M \neq 0$ is $\mathbb{N}$-graded, i.e., $M_i = 0$ for $i < 0$ then $H(M,t)$ is a polynomial of degree $d(M)$.

We say that for $a \in \mathbb{Z}$ the graded module $M(-a)$, whose i-th graded component is M_{i-a}, is the module M *shifted in degrees by* $-a$. The highest degree of a non-zero element in $M(-a)$ is $d(M(-a)) = d(M) + a$ and its Hilbert series is $H(M(-a),t) = t^a H(M,t)$.

For a linear form $v \in E_1$ and $M \in \mathscr{M}$ the finite sequence

$$(M,v): \qquad \ldots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \ldots$$

is a complex since $v^2 = 0$. The j-th cohomology of this complex is $H^j(M,v) = \left(\frac{0:_M v}{vM}\right)_j$.

1.2. Simplicial complexes

Let Δ be a simplicial complex on the vertex set $[n]$, i.e., Δ is a collection of subsets of $[n]$ such that whenever $F \in \Delta$ and $G \subseteq F$ it holds that $G \in \Delta$. The elements of Δ are called the *faces* of Δ, the maximal faces *facets*. A subset of $[n]$ not in Δ is called a *non-face*. We say that a face of cardinality i has *dimension* $i-1$. The *dimension* of a non-empty complex Δ is the maximum of the dimension of its faces. The simplicial complex $2^{[n]} = \{F \subseteq [n]\}$, that consists of all subsets of $[n]$, is called the $(n-1)$-simplex. The number $f_i = f_i(\Delta)$ of faces of dimension i is encoded in the so-called *f-vector* $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$, where $d-1 = \dim \Delta$. Of interest is the number

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i(\Delta)$$

which is called the *reduced Euler characteristic* of Δ.

A new technique in studying simplicial complexes was introduced by Stanley [**56**]. He considers the following commutative ring associated to a simplicial complex Δ: let $I_\Delta = (x_F : F \not\in \Delta)$ be the ideal in the polynomial ring $S = K[x_1, \dots, x_n]$ generated by the squarefree monomials x_F corresponding to non-faces of Δ. It is called the *Stanley-Reisner ideal* of Δ and the quotient ring $K[\Delta] = S/I_\Delta$ is called the *Stanley-Reisner ring* of Δ.

In the same way one associates to Δ the *exterior face ring* $K\{\Delta\} = E/J_\Delta$, where J_Δ is the *exterior face ideal* $J_\Delta = (e_F : F \not\in \Delta)$, see, e.g., [**28**] for a good overview. It is minimially generated by the monomials corresponding to the minimal non-faces of Δ.

The i-th graded component $K\{\Delta\}_i$ is generated as K-vector space by the monomials of degree i, that correspond to faces of dimension $i-1$ of Δ. Thus

$$H(i, K\{\Delta\}) = f_{i-1}(\Delta),$$

whence the Hilbert function of $K\{\Delta\}$ reflects the f-vector of Δ. In particular, if $d-1$ is the dimension of Δ, then $d(K\{\Delta\}) = d$.

The *Alexander dual* of Δ is the simplicial complex

$$\Delta^* = \{F \subseteq [n] : [n] \setminus F \not\in \Delta\}.$$

So the facets of Δ^* are the complements of the minimal non-faces of Δ.

This combinatorial duality can be expressed via the algebraic duality on $\mathscr{M}$. This is a well-known fact, for the convenience of the reader we give a short proof.

Lemma 1.2.1. *Let Δ be a simplicial complex. Then*

$$(K\{\Delta\})^* \cong J_{\Delta^*}.$$

PROOF. It is enough to show that $J_{\Delta^*} = 0 :_E J_\Delta$ since we already know that $(K\{\Delta\})^* \cong 0 :_E J_\Delta$. Let $F_1, \dots, F_t$ be the minimal non-faces of Δ, i.e., $J_\Delta = (e_{F_1}, \dots, e_{F_t})$. A subset of $[n]$ is a face of Δ if and only if it contains no minimal non-face. Therefore it holds that

$$\begin{aligned} J_{\Delta^*} &= (e_F : F \not\in \Delta^*) = (e_F : [n] \setminus F \in \Delta) = (e_F : F_i \not\subseteq [n] \setminus F \text{ for all } i = 1, \dots, t) \\ &= (e_F : F_i \cap F \neq \emptyset \text{ for all } i = 1, \dots, t) = 0 :_E (e_{F_1}, \dots, e_{F_t}) = 0 :_E J_\Delta. \end{aligned}$$

□

Let Δ be a simplicial complex of dimension $d-1$. Consider the complex $\tilde{\mathscr{C}}$

$$0 \longrightarrow \bigoplus_{\substack{F\in\Delta,\\ \dim F=d-1}} \mathbb{Z}F \longrightarrow \bigoplus_{\substack{F\in\Delta,\\ \dim F=d-2}} \mathbb{Z}F \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{F\in\Delta,\\ \dim F=0}} \mathbb{Z}F \longrightarrow \bigoplus_{\substack{F\in\Delta,\\ \dim F=-1}} \mathbb{Z}F \longrightarrow 0$$

with differential induced by

$$F = \{j_0,\dots,j_i\} \mapsto \sum_{k=0}^{i} (-1)^k \{j_0,\dots,\widehat{j_k},\dots,j_i\}$$

for $F \in \Delta$, $\dim F = i$, where $\widehat{j_k}$ means that j_k is omitted.

The *i-th reduced simplicial homology of* Δ *with coefficients in* K is defined to be

$$\tilde{H}_i(\Delta;K) = H_i(\tilde{\mathscr{C}} \otimes_{\mathbb{Z}} K), \qquad i = -1,\dots,d-1.$$

The *i-th reduced simplicial cohomology of* Δ *with coefficients in* K is defined to be

$$\tilde{H}^i(\Delta;K) = H^i(\mathrm{Hom}_{\mathbb{Z}}(\tilde{\mathscr{C}},K)), \qquad i = -1,\dots,d-1.$$

It is not hard to show that there exist natural isomorphisms of K-vector spaces

$$\tilde{H}^i(\Delta;K) \cong \mathrm{Hom}_K(\tilde{H}_i(\Delta;K),K), \qquad \tilde{H}_i(\Delta;K) \cong \mathrm{Hom}_K(\tilde{H}^i(\Delta;K),K).$$

A simplicial complex Δ is called *acyclic* if $\tilde{H}^i(\Delta;K) = 0$ for all i. It is easy to see that the complex $\mathrm{Hom}_{\mathbb{Z}}(\tilde{\mathscr{C}},K)$ is the same as the complex $(K\{\Delta\},\sum_{j=1}^n e_j)$, introduced at the end of Section 1.1, only shifted by one in homological position because an $(i-1)$-dimensional face corresponds to a monomial of degree i.

Corollary 1.2.2. [**2**, Lemma 3.3] *Let* Δ *be a non-empty simplicial complex of dimension* $d-1 > -1$. *Then for all* $i = 0,\dots,d$ *we have*

$$H^i(K\{\Delta\},\sum_{j=1}^n e_j) \cong \tilde{H}^{i-1}(\Delta;K)$$

as K*-vector spaces.*

A simplicial complex Δ is called *Cohen-Macaulay* (over K) if the Stanly-Reisner ring $K[\Delta]$ is a Cohen-Macaulay ring as a module over the polynomial ring $K[x_1,\dots,x_n]$. This property can be expressed using simplicial homology, as was proved by Reisner (see [**50**]).

Proposition 1.2.3. *Let* Δ *be a simplicial complex. The following conditions are equivalent:*

(i) Δ *is Cohen-Macaulay over* K*;*
(ii) $\tilde{H}_i(\mathrm{lk}_\Delta F;K) = 0$ *for all* $F \in \Delta$ *and all* $i < \dim \mathrm{lk}_\Delta F$.

Here $\mathrm{lk}_\Delta F = \{G : G \cup F \in \Delta, G \cap F = \emptyset\}$ is the *link* of F in Δ.

CHAPTER 2

Resolutions

We study projective and injective resolutions over the exterior algebra. Observe that injective resolutions behave very special since E is an injective E-module. In Section 2.1 we introduce all necessary notions related to projective resolutions. In Section 2.2 the same is done for injective resolutions, and the relations between these two types of resolutions are presented. Section 2.3 is devoted to the Cartan complex and cocomplex. These are a useful projective and a useful injective resolution of the residue field K and play a similar role as the Koszul complex for modules over a polynomial ring. The last section, Section 2.4, gives a short introduction into the BGG correspondence, which is an equivalence of categories between bounded complexes of coherent sheaves on projective space and doubly infinite free resolutions over the exterior algebra.

2.1. Projective resolutions

An analogous version of Nakayama's Lemma for graded modules over E almost trivially holds because the maximal ideal $\mathfrak{m}$ is nilpotent. Therefore many proofs for (graded) modules over a commutative local or *local ring (cf. [**11**, Chapter 1.5]) can be translated to modules over an exterior algebra. So for example it follows that every projective E-module is free over E (see, e.g., [**60**, Proposition 4.3.11] for a proof over a commutative local ring).

Also every E-module $M \in \mathscr{M}$ has a graded free resolution, which can be constructed as follows: Let $m_1, \dots, m_t$ be a system of homogeneous generators for M. The homomorphism $F_0 = \bigoplus_{i=1}^{t} E(-\deg m_i) \longrightarrow M$ mapping the i-th generator of the free module to m_i is a surjection on M. Its kernel is in $\mathscr{M}$ (note that E is Noetherian) whence one can repeat this step with a system of generators of the kernel. In this way one obtains an exact complex

$$\dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with finitely generated graded free modules F_i, called a *graded free resolution* of M. To distinguish it from the injective resolution of M, we sometimes call this resolution *projective*. If the systems of generators chosen are minimal in each step, then the resolution is called *minimal*. An equivalent condition to this is that all entries in the matrices representing the maps are elements in $\mathfrak{m}$. One can show that two minimal resolutions are isomorph as complexes. Then the minimal graded projective resolution is "uniquely" determined and the ranks and degree shifts in a minimal resolution depend only on M. So if

$$F_{\bullet}: \dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta_{1,j}(M)} \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

is a minimal graded projective resolution of M, then the exponents $\beta_{i,j}(M)$ are called the *graded Betti numbers* of M and the ranks $\beta_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(M)$are called the *(total) Betti numbers* of M. In particular, $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^E(K,M)_j = \dim_K H_i(K \otimes_E F_\bullet)$.

The resolution is called *t-linear* if $\beta_{i,i+j}(M) = 0$ for all $j \neq t$. An equivalent condition is that all entries in the matrices representing the maps are of degree one and M is generated in degree t.

The Betti numbers of a graded module determine its Hilbert series, and if M has a linear projective resolution, then also the converse is true.

Example 2.1.1. A simple example of a linear projective resolution is given by

$$\ldots \longrightarrow E(-2) \xrightarrow{\cdot e_1} E(-1) \xrightarrow{\cdot e_1} E \longrightarrow E/(e_1) \longrightarrow 0.$$

A (minimal) graded projective resolution of an E-module M has always infinite length unless the module is free. Therefore the projective dimension is not significant. Instead one measures the growth rate of the Betti numbers by the *complexity* which is defined as

$$\operatorname{cx}_E M = \inf\{c \in \mathbb{Z} : \beta_i(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\}.$$

Note that $\operatorname{cx}_E 0 = \operatorname{cx}_E E = -\infty$. The definition directly implies that $\operatorname{cx}_E(E/J) = \operatorname{cx}_E J$ for a graded ideal $J \subseteq E$, because if $F_\bullet \to J$ is the minimal graded projective resolution of J, then $F_\bullet \to E \to E/J$ is the minimal graded projective resolution of E/J.

Another important number related to free resolutions is the *(Castelnuovo-Mumford) regularity*

$$\operatorname{reg}_E M = \max\{j \in \mathbb{Z} : \beta_{i,i+j}(M) \neq 0 \text{ for some } i \geq 0\}$$

of a non-zero module M. We set $\operatorname{reg}_E 0 = -\infty$. Obviously the regularity is greater or equal to the lowest degree of a generator in a minimal set of generators. From above it is bounded by $d(M) = \max\{i : M_i \neq 0\}$ which can be seen when computing $\operatorname{Tor}_i^E(M,K)$ via the Cartan complex (see Section 2.3 for details on this special resolution of K), so this is always a well-defined and finite number.

A module that is generated in one degree has a linear resolution if and only if its regularity equals this degree. A short exact sequence $0 \to U \to M \to N \to 0$ induces a long exact sequence of Tor-modules

$$\ldots \longrightarrow \operatorname{Tor}_{i+1}^E(N,K)_{i+1+(j-1)} \longrightarrow \operatorname{Tor}_i^E(U,K)_{i+j} \longrightarrow \operatorname{Tor}_i^E(M,K)_{i+j} \longrightarrow$$
$$\operatorname{Tor}_i^E(N,K)_{i+j} \longrightarrow \operatorname{Tor}_{i-1}^E(U,K)_{i-1+(j+1)} \longrightarrow \ldots$$

Thus the regularities of modules in a short exact sequence are related to each other in the following way.

Lemma 2.1.2. *Let $0 \to U \to M \to N \to 0$ be a short exact sequence of non-zero modules in $\mathcal{M}$. Then:*

(i) $\operatorname{reg}_E U \leq \max\{\operatorname{reg}_E M, \operatorname{reg}_E N + 1\}$.
(ii) $\operatorname{reg}_E M \leq \max\{\operatorname{reg}_E U, \operatorname{reg}_E N\}$.
(iii) $\operatorname{reg}_E N \leq \max\{\operatorname{reg}_E U - 1, \operatorname{reg}_E M\}$.

Since the minimal graded projective resolutions of an ideal and its quotient ring are the same except for the beginning, the regularities of both can be compared.

Corollary 2.1.3. *For a graded ideal $0 \neq J \subset E$ it holds that* $\operatorname{reg}_E J = \operatorname{reg}_E(E/J) + 1$.

Open problem 2.1.4. *Over the polynomial ring it is true that the Castelnuovo-Mumford regularity of the intersection of d linear ideals (i.e., ideals generated by linear forms) is less or equal to d (cf.* [**15**]*). Is the same true over the exterior algebra?*

2.2. Injective resolutions

The exactness of the duality functor $M \mapsto M^*$ and $M = M^{**}$ imply that a module is injective in $\mathscr{M}$ if and only if its dual is projective in $\mathscr{M}$. The dual of a free module is a free module and so the injective modules are exactly the free modules, just as the projective ones.

Consequently every module in $\mathscr{M}$ has a graded injective resolution. The resolution is called *minimal* if the $(i+1)$-th module is the injective hull of the cokernel of the $(i-1)$-th map. A graded injective resolution of M is minimal if and only if the dualized resolution is a minimal graded free resolution of M^*. This is satisfied if and only if the entries of the matrices of the maps in a minimal graded injective resolution of M are contained in the maximal ideal $\mathfrak{m}$ of E. In particular, M has a unique minimal graded injective resolution up to isomorphism.

We define the *graded Bass numbers* to be the numbers

$$\mu_{i,j}(M) = \dim_K \operatorname{Ext}^i_E(K,M)_j$$

and the *total Bass numbers* to be the numbers $\mu_i(M) = \sum_{j\in\mathbb{Z}} \mu_{i,j}(M)$. If I^0 is the 0-th module in the minimal graded injective resolution of M, there is a graded isomorphism $\operatorname{soc} M \cong \operatorname{soc} I^0$, since

$$\operatorname{soc} M \cong \operatorname{Hom}_E(K,M) \cong \operatorname{Ext}^0_E(K,M) \cong \operatorname{Hom}_E(K,I^0) \cong \operatorname{soc} I^0.$$

In particular, the dimension of the socle of M is the 0-th Bass number $\mu_0(M)$. That is because the K-vector space dimension of the socle of a free E-module is equal to its rank, as for every copy of E the socle has a copy of $Ke_{[n]}$.

Let $I^i = \bigoplus_{j\in\mathbb{Z}} E(-j)^{c_{ij}}$ be the i-th module in the minimal graded injective resolution of M. The maps in the complex obtained by applying $\operatorname{Hom}_E(K,-)$ to the minimal graded injective resolution of M are all zero and thus

$$\operatorname{Ext}^i_E(K,M) \cong \operatorname{Hom}_E(K,I^i) \cong \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_E(K,E)(-j)^{c_{ij}} \cong \bigoplus_{j\in\mathbb{Z}} K(-n-j)^{c_{ij}},$$

because $\operatorname{Hom}_E(K,E) \cong K(-n)$. Hence, $\mu_{i,k}(M) = \dim_K \bigoplus_{j\in\mathbb{Z}} K(-n-j)^{c_{ij}}_k = c_{i,k-n}$ and the minimal graded injective resolution of M is of the form

$$0 \longrightarrow M \longrightarrow \bigoplus_{j\in\mathbb{Z}} E(n-j)^{\mu_{0,j}} \longrightarrow \bigoplus_{j\in\mathbb{Z}} E(n-j)^{\mu_{1,j}} \longrightarrow \bigoplus_{j\in\mathbb{Z}} E(n-j)^{\mu_{2,j}} \longrightarrow \dots$$

So in contrast to the minimal graded projective resolution one has to take into account this shift by n, as for example in the following definition.

Definition 2.2.1. A module $M \in \mathscr{M}$ has a *d-linear injective resolution*, if it has a graded injective resolution of the form

$$0 \longrightarrow M \longrightarrow E(n-d)^{\mu_{0,d}} \longrightarrow E(n-d+1)^{\mu_{1,d-1}} \longrightarrow E(n-d+2)^{\mu_{0,d-2}} \longrightarrow \dots,$$

i.e., if and only if $\mu_{i,j-i}(M) = 0$ for all $j \neq d$.

Obviously a graded injective resolution is linear if and only if the entries in the matrices representing the maps are all of degree one.

A d-linear injective resolution implies that $d(M) = \max\{i : M_i \neq 0\} = d$ because the socle of M is isomorphic to the socle of $E(n-d)^{\mu_0(M)}$ which lies in degree d.

As for projective resolutions the Bass numbers of a graded module determine its Hilbert series, and if M has a linear injective resolution, then also the converse is true.

Example 2.2.2. A simple example of a linear injective resolution is given by

$$0 \longrightarrow (e_1) \longrightarrow E \xrightarrow{\cdot e_1} E(+1) \xrightarrow{\cdot e_1} E(+2) \longrightarrow \dots$$

The duality between projective and injective resolutions implies the following relation between the Bass numbers of a module and the Betti numbers of its dual.

Lemma 2.2.3. [**3**, Proposition 5.2] *Let $M \in \mathscr{M}$. Then*

$$\mu_{i,j}(M) = \beta_{i,n-j}(M^*).$$

In particular, M has a d-linear injective resolution if and only if M^* has an $(n-d)$-linear projective resolution.

Proof. Let $F_i = \bigoplus_{j\in\mathbb{Z}} E(-j)^{\beta_{i,j}}$ be the i-th module in the minimal projective resolution of M^*, $\beta_{i,j} = \beta_{i,j}(M^*)$.

Then its dual is

$$F_i^* = \operatorname{Hom}_E(F_i, E) \cong \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_E(E,E)(+j)^{\beta_{i,j}} \cong \bigoplus_{j\in\mathbb{Z}} E(+j)^{\beta_{i,j}}.$$

Thus computing $\operatorname{Ext}_E^i(K,M)$ via the dualized resolution yields

$$\operatorname{Ext}_E^i(K,M) \cong \operatorname{Hom}_E(K,F_i^*) \cong \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_E(K,E)(+j)^{\beta_{i,j}} \cong \bigoplus_{j\in\mathbb{Z}} K(-n+j)^{\beta_{i,j}},$$

where $\operatorname{Hom}_E(K,E) \cong K(-n)$ is used. So finally

$$\begin{aligned}\mu_{i,k}(M) = \dim_K \operatorname{Ext}_E^i(K,M)_k &= \dim_K \Big(\bigoplus_{j\in\mathbb{Z}} K(-n+j)^{\beta_{i,j}}\Big)_k \\ &= \dim_K \Big(\bigoplus_{j\in\mathbb{Z}} K_{-n+j+k}^{\beta_{i,j}}\Big) = \sum_{k=n-j} \beta_{i,j} = \beta_{i,n-k}.\end{aligned}$$

□

2.3. Cartan complex and cocomplex

A very useful instrument is the Cartan complex which plays a similar role as the Koszul complex for the polynomial ring. As this one it is a linear projective resolution of the residue class field and thus its dual, the Cartan cocomplex, is a linear injective resolution of it. We follow the exposition of Aramova and Herzog in [**2**, Section 4].

For a sequence $\mathbf{v} = v_1,\dots,v_m$ of linear forms in E_1 let $C_\bullet(\mathbf{v};E) = C_\bullet(v_1,\dots,v_m;E)$ be the free divided power algebra $E\langle x_1,\dots,x_m\rangle$. It is generated by the divided powers $x_i^{(j)}$ for $i = 1,\dots,m$ and $j \geq 0$ which satisfy the relations $x_i^j x_i^k = ((j+k)!/(j!k!))x_i^{j+k}$. Thus

$C_i(\mathbf{v};E)$ is a free E-module with basis $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$, $a \in \mathbb{N}^m$, $|a| = i$. The E-linear differential on $C_\bullet(v_1,\dots,v_m;E)$ is

$$\partial_i : C_i(v_1,\dots,v_m;E) \longrightarrow C_{i-1}(v_1,\dots,v_m;E), \qquad x^{(a)} \mapsto \sum_{a_j>0} v_j x_1^{(a_1)} \cdots x_j^{(a_j-1)} \cdots x_m^{(a_m)}.$$

One easily sees that $\partial \circ \partial = 0$ (regarding $v_j v_k = -v_k v_j$), so this is indeed a complex.

Definition 2.3.1. The complexes

$$C_\bullet(\mathbf{v};M) = C_\bullet(\mathbf{v};E) \otimes_E M \quad \text{and} \quad C^\bullet(\mathbf{v};M) = \operatorname{Hom}_E(C_\bullet(\mathbf{v};E),M)$$

are called the *Cartan complex* and *Cartan cocomplex* of $\mathbf{v}$ with coefficients in M. The corresponding homology modules

$$H_i(\mathbf{v};M) = H_i(C_\bullet(\mathbf{v};M)) \quad \text{and} \quad H^i(\mathbf{v};M) = H^i(C^\bullet(\mathbf{v};M))$$

are called the *Cartan homology* and *Cartan cohomology* of $\mathbf{v}$ with coefficients in M.

The elements of $C^i(\mathbf{v};M)$ can be identified with homogeneous polynomials $\sum m_a y^a$ with $m_a \in M$, $a \in \mathbb{N}^m$, $|a| = i$ and

$$m_a y^a(x^{(b)}) = \begin{cases} m_a & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

Moreover, $C^\bullet(\mathbf{v};E)$ can be identified with the polynomial ring $S = K[y_1,\dots,y_m]$. After this identifications the differential on $C^\bullet(\mathbf{v};M)$ is simply the multiplication with $\sum_{i=1}^m v_i y_i$. Setting $\deg x_i = 1$ and $\deg y_i = -1$ induces a grading on the complexes and their homologies.

Aramova, Avramov and Herzog [**1**] use product structures in cohomology to study graded modules over the exterior algebra. We do not describe this in detail, see [**1**, Construction 3.5] for the construction, Mac Lane [**41**] or Bourbaki [**10**] for a more general situation. The essential content we need is that $\operatorname{Ext}_E^*(K,K) = \bigoplus_{i\geq 0} \operatorname{Ext}_E^i(K,K)$ identifies naturally with the polynomial ring $S = \operatorname{Sym}_K^*((E_1)^\vee)$, and this is the same identification as in the preceding paragraph. They further show that $\operatorname{Ext}_E^*(K,M)$ has a structure as an S-module. If M is finitely generated as an E-module, then so is $\operatorname{Ext}_E^*(K,M)$ over S. Applying the Hilbert-Serre theorem, see, e.g., [**11**, Corollary 4.1.8], to the S-module $\operatorname{Ext}_E^*(K,M)$ yields $\dim_S \operatorname{Ext}_E^*(K,M) = \operatorname{cx}_E M^*$ (which is equal to $\operatorname{cx}_E M$ as we will see in Section 4.1).

Cartan homology and Cartan cohomology are related as follows:

Proposition 2.3.2. [**2**, Proposition 4.2] *Let $M \in \mathscr{M}$ and $\mathbf{v} = v_1,\dots,v_m$ be a sequence in E_1. Then $H_i(\mathbf{v};M)^* \cong H^i(\mathbf{v};M^*)$ as graded E-modules for all i.*

In [**3**, Theorem 2.2] is proved that the Cartan complex $C_\bullet(v_1,\dots,v_m;E)$ with coefficients in E is exact except in homological position 0 and hence it is a free resolution of $E/(v_1,\dots,v_m)$ over E. Thus it can be used to compute $\operatorname{Tor}_i^E(E/(v_1,\dots,v_m),-)$ and $\operatorname{Ext}_E^i(E/(v_1,\dots,v_m),-)$.

Proposition 2.3.3. [**3**, Theorem 2.2] *Let $M \in \mathscr{M}$ and $\mathbf{v} = v_1,\dots,v_m$ be a sequence in E_1. There are isomorphism of graded E-modules*

$$\operatorname{Tor}_i^E(E/(v_1,\dots,v_m),M) \cong H_i(\mathbf{v};M) \quad \text{and} \quad \operatorname{Ext}_E^i(E/(v_1,\dots,v_m),M) \cong H^i(\mathbf{v};M).$$

As over a polynomial ring one can deduce useful facts on $\mathrm{Tor}_i^E(K,M)$ and $\mathrm{Ext}_E^i(K,M)$, and hence on the Betti and Bass numbers of a module M, when computing these via the Cartan (co)complex. For example one obtains that $M_{\geq q} = \bigoplus_{j\geq q} M_j$ has a q-linear resolution if and only if $\beta_{i,i+j}(M) = 0$ for $j > q$. In particular, the regularity of M is the maximal number q such that $M_{\geq q}$ has a q-linear resolution, and is bounded from above by $d(M)$. Another useful conclusion is the following corollary. Here the truncation $M_{\leq k}$ of a module M is the submodule generated by all homogeneous elements of degree $\leq k$ belonging to M.

Corollary 2.3.4. *Let $M \in \mathscr{M}$. Then, for all $k \in \mathbb{Z}$ and all $j \leq k$, we have $\beta_{i,i+j}(M) = \beta_{i,i+j}(M_{\leq k})$.*

PROOF. Any homogeneous cycle of degree $i+j$ representing a homology class in $H_i(\mathbf{v};M)$ for some basis $\mathbf{v}$ of E_1 is of the form $\sum m_a x^{(a)}$ with $|a| = i$ and $\deg m_a = j$. Hence, for $j \leq k$, it also represents a homology class in $H_i(\mathbf{v};M_{\leq k})$ and vice versa. In the same way we see that the boundaries coincide for $j \leq k$. □

Cartan (co)homology can be used inductively as there are long exact sequences connecting the (co)homologies of $v_1,\dots,v_j$ and $v_1,\dots,v_{j+1}$.

To begin with there exists an exact sequence of complexes

$$0 \longrightarrow C_\bullet(v_1,\dots,v_j;M) \xrightarrow{\iota} C_\bullet(v_1,\dots,v_{j+1};M) \xrightarrow{\tau} C_{\bullet-1}(v_1,\dots,v_{j+1};M)(-1) \longrightarrow 0,$$

where ι is the natural inclusion map and τ is given by

$$\tau(g_0 + g_1 x_{j+1} + \dots + g_k x_{j+1}^{(k)}) = g_1 + g_2 x_{j+1} + \dots + g_k x_{j+1}^{(k-1)},$$

where the g_i belong to $C_{k-i}(v_1,\dots,v_j;M)$.

This exact sequence induces a long exact sequence of homology modules.

Proposition 2.3.5. ([**2**, Proposition 4.1]) *Let $M \in \mathscr{M}$ and $\mathbf{v} = v_1,\dots,v_m$ be a sequence in E_1. For all $1 \leq j \leq m$ there exists a long exact sequence of graded E-modules*

$$\dots \longrightarrow H_i(v_1,\dots,v_j;M) \xrightarrow{\alpha_i} H_i(v_1,\dots,v_{j+1};M) \xrightarrow{\beta_i} H_{i-1}(v_1,\dots,v_{j+1};M)(-1)$$

$$\xrightarrow{\delta_{i-1}} H_{i-1}(v_1,\dots,v_j;M) \longrightarrow H_{i-1}(v_1,\dots,v_{j+1};M) \longrightarrow \dots.$$

Here α_i is induced by the inclusion map ι, β_i by τ and δ_{i-1} is the connecting homomorphism, which acts as follows: if $z = g_0 + g_1 x_{j+1} + \dots + g_{i-1} x_{j+1}^{(i-1)}$ is a cycle in $C_{i-1}(v_1,\dots,v_{j+1};M)$, then $\delta_{i-1}(\overline{z}) = \overline{g_0 v_{j+1}}$.

Dualizing this long sequence yields a long exact sequence of cohomology.

Proposition 2.3.6. [**2**, Proposition 4.3] *Let $M \in \mathscr{M}$ and $\mathbf{v} = v_1,\dots,v_m$ be a sequence in E_1. For all $1 \leq j \leq m$ there exists a long exact sequence of graded E-modules*

$$\dots \longrightarrow H^{i-1}(v_1,\dots,v_{j+1};M) \longrightarrow H^{i-1}(v_1,\dots,v_j;M) \longrightarrow H^{i-1}(v_1,\dots,v_{j+1};M)(+1)$$

$$\xrightarrow{\cdot y_{j+1}} H^i(v_1,\dots,v_{j+1};M) \longrightarrow H^i(v_1,\dots,v_j;M) \longrightarrow \dots$$

2.4. The BGG correspondence

The Bernstein-Gel'fand-Gel'fand (BGG) correspondence is an equivalence between the category of bounded complexes of coherent sheaves on projective space and the category of doubly infinite free resolutions over the exterior algebra. It was introduced in [**5**]; Eisenbud, Fløystad and Schreyer derive in [**19**] an explicit version of this correspondence. Its essential content is a functor $\mathbf{L}$ from complexes of graded E-modules to complexes of graded S-modules, and its adjoint $\mathbf{R}$.

We need a special case of a broad result of [**20**] whose proof is based on the BGG correspondence. We extract the necessary arguments from [**20**] and [**19**] and adapt them to our notations, e.g., we consider a positive grading on E while the authors in [**19**] consider $\deg e_j = -1$.

Let S be the polynomial ring $S = K[y_1,\dots,y_n] \cong \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_E(K,K)$ with $\deg y_j = -1$, where $y_1,\dots,y_n$ is the dual basis of $e_1,\dots,e_n$ of E_1.

We give the definition of the functor $\mathbf{L}$ on graded modules. It can be expanded on complexes in a natural way by taking the total complex of a certain double complex.

Definition 2.4.1. Let $M \in \mathscr{M}$. We define $\mathbf{L}(M)$ to be the complex of S-modules

$$\dots \longrightarrow S \otimes_K M_i \longrightarrow S \otimes_K M_{i+1} \longrightarrow \dots$$

with differential induced by

$$s \otimes m \mapsto \sum_{j=1}^{n} y_j s \otimes e_j m.$$

Here $S \otimes_K M_i$ has cohomological degree i and $1 \otimes m$ has degree i for $m \in M_i$.

The homology of this complex can be expressed in more familiar terms.

Proposition 2.4.2. [**19**, Proposition 2.3] *Let M be in $\mathscr{M}$. Then*

$$H^i(\mathbf{L}(M))_j \cong \operatorname{Ext}^{i-j}_E(K,M)_j.$$

PROOF. The j-th homogeneous component of $H^i(\mathbf{L}(M))$ can be computed as the i-th homology of the complex

$$\dots \longrightarrow S_{j-i} \otimes_K M_i \longrightarrow S_{j-(i+1)} \otimes_K M_{i+1} \longrightarrow \dots$$

From linear algebra it is known that $S_{j-i} \otimes_K M_i \cong \operatorname{Hom}_K((S_{j-i})^\vee, M_i)$. A basis of the K-vector space $(S_{j-i})^\vee$ is given by $\{x^a : a \in \mathbb{N}^n, |a| = i-j\}$ where $x^a(y^b) = 1$ if $a = b$ and 0 otherwise, a basis of the E-module $C_{i-j}(e_1,\dots,e_n;E)$ by $\{x^{(a)} : a \in \mathbb{N}^n, |a| = i-j\}$. Hence there is an isomorphism of K-vector spaces

$$\operatorname{Hom}_K((S_{j-i})^\vee, M_i) \cong \operatorname{Hom}_E(C_{i-j}(e_1,\dots,e_n;E),M)_j.$$

The module on the right-hand side is the $(i-j)$-th vector space of the j-th part of the Cartan cocomplex whose homology in this position is just $\operatorname{Ext}^{i-j}_E(K,M)_j$. The isomorphism induces an isomorphism of complexes whence the the assertion follows. □

Remark 2.4.3. The preceding Proposition 2.4.2 induces an S-module structure on $\operatorname{Ext}^*_E(K,M)$. The proof of the proposition also shows that this is the same structure as the product structure in cohomology as introduced in Section 2.3.

Corollary 2.4.4. *Let $M \in \mathscr{M}$. The i-th homology $H^i(L(M))$ of the complex $\mathbf{L}(M)$ vanishes for all $i > 0$ if and only if M has a 0-linear injective resolution.*

PROOF. $H^i(L(M))$ vanishes if and only if $H^i(L(M))_j = 0$ for all $i \neq 0$ and all $j \in \mathbb{Z}$. Proposition 2.4.2 implies that this is the case if and only if

$$\mathrm{Ext}_E^{i-j}(K,M)_j = \mathrm{Ext}_E^{i-j}(K,M)_{i-(i-j)} = 0$$

which is true if and only if M has 0-linear injective resolution. □

The complex $\mathbf{L}(M)$ helps to interpret the cohomology $H^i(M,v)$ of the complex

$$(M,v): \qquad \ldots \longrightarrow M_{i-1} \xrightarrow{\cdot v} M_i \xrightarrow{\cdot v} M_{i+1} \longrightarrow \ldots$$

Theorem 2.4.5. [**20**, Theorem 4.1(b)] *Let $M \in \mathscr{M}$ have a d-linear injective resolution. Then $H^i(M,v) = 0$ for all $i \in \mathbb{Z}$ if and only if $H^d(M,v) = 0$.*

PROOF. We may assume $v = \sum_{j=1}^n e_j$ since the homology of the above complex does not depend on the chosen basis of E_1, and, after a possible degree shift, $d = 0$. This implies $d(M) = \max\{i : M_i \neq 0\} = 0$. Thus $\mathbf{L}(M)$ has the form

$$\mathbf{L}(M) : \ldots \longrightarrow S \otimes_K M_{-i} \longrightarrow S \otimes_K M_{-i+1} \longrightarrow \ldots \longrightarrow S \otimes_K M_0 \longrightarrow 0.$$

By Corollary 2.4.4 it is a graded free resolution of some S-module P, that is generated in degree zero. We consider the Laurent polynomial ring $K[T,T^{-1}]$ in one variable T with $\deg T = -1$. Via the ring homomorphism $S \to K[T] \hookrightarrow K[T,T^{-1}]$ which sends y_j to T for all $j = 1,\ldots,n$, the ring $K[T,T^{-1}]$ has a graded S-module structure. We compare $H^i(M,v)$ with $\mathrm{Tor}_i^S(K[T,T^{-1}],P)$. The complex

$$\ldots \longrightarrow K[T,T^{-1}]_i \otimes_K M_{-i} \longrightarrow K[T,T^{-1}]_{i-1} \otimes_K M_{-i+1} \longrightarrow \ldots$$

with differential $s \otimes m \mapsto Ts \otimes vm$ has the same homology as (M,v). This is also the homology in degree zero of the complex

$$\ldots \longrightarrow K[T,T^{-1}] \otimes_K M_{-i} \longrightarrow K[T,T^{-1}] \otimes_K M_{-i+1} \longrightarrow \ldots$$

with the same differential. The modules of this complex can be written as

$$K[T,T^{-1}] \otimes_K M_{-i} \cong K[T,T^{-1}] \otimes_S S \otimes_K M_{-i} \cong K[T,T^{-1}] \otimes_S \mathbf{L}(M)_{-i}.$$

The differential, which was the multiplication with $\sum_{j=1}^n T \otimes e_j$, becomes under the first isomorphism the multiplication with $\sum_{j=1}^n T \otimes 1_S \otimes e_j$ which is the same map as the multiplication with $\sum_{j=1}^n 1_{K[T,T^{-1}]} \otimes y_j \otimes e_j$. This justifies writing $L(M)_{-i}$. So finally

$$H^i(M,v) \cong \mathrm{Tor}_i^S(K[T,T^{-1}],P)_0.$$

This isomorphism implies the claim as follows: If $H^i(M,v) = 0$ for all $i \in \mathbb{Z}$, then obviously $H^0(M,v) = 0$. If $H^0(M,v) = 0$, the above isomorphism gives

$$0 = H^0(M,v) = \mathrm{Tor}_0^S(K[T,T^{-1}],P)_0 \cong \big(K[T,T^{-1}] \otimes_S P\big)_0.$$

In particular, $K[T,T^{-1}]_0 \otimes_K P_0 \cong K \otimes_K P_0 \cong P_0 = 0$. Since P is generated in degree 0 this already implies $P = 0$. But then $\mathrm{Tor}_i^S(K[T,T^{-1}],P)_0 = 0$ for all $i \in \mathbb{Z}$ which concludes the proof. □

Open problem 2.4.6. *The statement of Theorem 2.4.5 is a statement purely on modules over the exterior algebra. Is there a proof for it without use of the BGG correspondence?*

CHAPTER 3

Generic initial ideals

Aramova, Herzog and Hibi develop in [**3**] a Gröbner basis theory for ideals in an exterior algebra, most of the time analogous to the known Gröbner basis theory for polynomial rings. They prove that for any ideal in an exterior algebra there exists a (the) generic initial ideal provided the field is infinite. Green also considers generic initial ideals over the exterior algebra independently in Chapter 5 of [**27**]. Generic initial ideals are strongly stable, independent of the characteristic of the field (in contrast to the polynomial ring). Therefore we first have a look at stable and strongly stable ideals before we turn to the generic initial ideals. The results on stable ideals already appeared in [**40**].

3.1. Stable ideals

Stable ideals are a nice class of ideals because their invariants can be read off directly from the minimal generating system.

Let $\operatorname{supp}(u) = \{i \in [n] : e_i | u\}$, $\max(u) = \max \operatorname{supp}(u)$ and $\min(u) = \min \operatorname{supp}(u)$ for a monomial u, $u \neq 1$, in E. Let $G(J)$ be the unique minimal system of monomials generators of a monomial ideal J and $G(J)_j$ denote the subset of $G(J)$ consisting of the elements of degree j.

Definition 3.1.1. Let $J \subset E$ be a monomial ideal.

(i) J is called *stable* if $e_j \frac{u}{e_{\max(u)}} \in J$ for every monomial $u \in J$ and $j < \max(u)$.

(ii) J is called *strongly stable* if $e_j \frac{u}{e_i} \in J$ for every monomial $u \in J$, $i \in \operatorname{supp}(u)$ and $j < i$.

Observe that saying that some ideal is (strongly) stable always implies that this is a monomial ideal $\neq E$.

The following results on (strongly) stable ideals are inspired by the chapter on squarefree strongly stable ideals in the polynomial ring in [**32**].

Aramova, Herzog and Hibi construct in [**3**] the minimal graded free resolution of a stable ideal. It is an analogue to the well-known Eliahou-Kervaire resolution [**23**] of stable ideals in a polynomial ring. The construction yields the following formula for the graded Betti numbers of stable ideals.

Lemma 3.1.2. [**3**, Corollary 3.3] *Let $0 \neq J \subset E$ be a stable ideal. Then*

$$\beta_{i,i+j}(J) = \sum_{u \in G(J)_j} \binom{\max(u)+i-1}{i} \quad \textit{for all } i \geq 0, j \in \mathbb{Z}.$$

A direct consequence of the construction is a formula for the regularity of a stable ideal.

Lemma 3.1.3. [**1**, Corollary 3.2] *Let* $0 \neq J \subset E$ *be a stable ideal. Then*

$$\operatorname{reg}_E(J) = \max\{\deg u : u \in G(J)\}.$$

In particular, if J is stable and generated in one degree, it has a linear projective resolution. An example for such an ideal is the maximal ideal $\mathfrak{m}$ of E and all its powers.

The above formula for the Betti numbers also gives a possibility to interpret the complexity of a stable ideal J in terms of $G(J)$.

Lemma 3.1.4. *Let* $0 \neq J \subset E$ *be a stable ideal. Then*

$$\operatorname{cx}_E(E/J) = \max\{\max(u) : u \in G(J)\}.$$

PROOF. This is evident from the formula for the Betti numbers of stable ideals in Lemma 3.1.2 since

$$\beta_i(J) = \sum_{k=1}^{n} m_k(J) \binom{k+i-1}{i}$$

where $m_k(J) = |\{u \in G(J) : \max(u) = k\}|$. The binomial coefficient in this sum is a polynomial in i of degree $k-1$ and the number $\max\{\max(u) : u \in G(J)\}$ is exactly the maximal k which appears in the sum, i.e., with $m_k(J) \neq 0$. □

For strongly stable ideals J also the number $d(E/J)$ has a meaning in terms of $G(J)$.

Lemma 3.1.5. *Let* $0 \neq J \subset E$ *be a strongly stable ideal. Then*

$$d(E/J) = n - \max\{\min(u) : u \in G(J)\}.$$

Observe that this maximum does not change when replacing $G(J)$ by J because J is strongly stable.

PROOF. Set $s = \max\{\min(u) : u \in G(J)\}$. We want to show that

$$s = n - d(E/J).$$

The right hand side is

$$n - d(E/J) = \min\{i : (E/J)^*_i \neq 0\},$$

which is a direct consequence of Lemma 1.1.1.

As $J \subseteq (e_1, \ldots, e_s)$ we obtain "$\geq$" immediately from the equivalence

$$J \subseteq (e_{i_1}, \ldots, e_{i_r}) \Leftrightarrow e_{i_1} \cdots e_{i_r} \in 0 :_E J \cong (E/J)^*$$

for $i_1, \ldots, i_r \in [n]$.

The other inequality follows if we show that $J \subseteq (e_{i_1}, \ldots, e_{i_r})$ implies $r \geq s$. We can assume that $s < n$. Otherwise, $s = n$ implies $e_n \in J$ and thus $J = \mathfrak{m}$, for which the assertion is trivial.

So first consider the case that $J \subseteq (e_{s+1})$. As J is strongly stable, $e_i \frac{u}{e_{s+1}} \in J$ for all monomials $u \in J$ and all $i \leq s$. But $e_i \frac{u}{e_{s+1}} \notin (e_{s+1})$ for $i \notin \operatorname{supp}(u)$, which is not possible. Therefore i must be contained in $\operatorname{supp}(u)$ for all $i \leq s$. That is, $J = (e_1 \cdots e_{s+1})$. We conclude $s = \max\{\min(u) : u \in G(J)\} = 1$ and hence $r \geq s$.

Now assume $J \not\subseteq (e_{s+1})$ and consider the ideal $\overline{J} = J + (e_{s+1})/(e_{s+1})$. This ideal is again strongly stable in the exterior algebra $\overline{E}$ in $n-1$ variables $e_1, \dots, e_s, e_{s+2}, \dots, e_n$. The position of e_i diminishes by one for $i > s+1$. We see that

$$\min(\overline{u}) = \begin{cases} \min(u) & \text{if } \min(u) \leq s \\ \min(u) - 1 & \text{if } \min(u) > s+1 \end{cases}$$

for the residue class of a monomial u with $e_{s+1} \not| u$. By the choice of s the second case does not occur for monomials in J which entails $\max\{\min(\overline{u}) : \overline{u} \in \overline{J}\} = s$. On the other hand $\overline{J} \subseteq (e_{i_1}, \dots, e_{i_r}) + (e_{s+1})/(e_{s+1})$ so by induction hypothesis $s \leq r$ if $s \notin \{i_1, \dots, i_r\}$ and $s \leq r-1$ otherwise. This concludes the proof. □

3.2. Initial and generic initial ideals

The monomial order on E considered in this thesis is always the reverse lexicographic order with $e_1 > e_2 > \dots > e_n$. The *initial ideal* of a graded ideal $J \subset E$ is the ideal generated by the initial terms $\text{in}(f)$, $f \in J$ with respect to this order, and is denoted by $\text{in}(J)$. The existence of the generic initial ideal $\text{gin}(J)$ of a graded ideal J in the exterior algebra over an infinite field is proved by Aramova, Herzog and Hibi in [**3**, Theorem 1.6]. That is, they prove the existence of a non-empty Zariski-open subset $U \subset \text{GL}(n;K)$ such that there is a monomial ideal $I \subset E$ with $I = \text{in}(g(J))$ for all $g \in U$. This ideal I is called the *generic initial ideal* of J, denoted by $\text{gin}(J)$. Generic initial ideals preserve much information of the original ideal.

The generic initial ideal of a graded ideal is strongly stable if it exists [**3**, Proposition 1.7]. This is independent of the characteristic of the field in contrast to ideals in the polynomial ring. In the following we investigate some relationships between a graded ideal in the exterior algebra and its (generic) initial ideal.

By [**3**, Corollary 1.2] the Hilbert functions of E/J and $E/\text{in}(J)$ coincide for any graded ideal $J \subseteq E$.

We present a proof of the following proposition since parts of it are used in the proof of the corollary afterwards.

Proposition 3.2.1. [**34**, Lemma 1.1] *Let* $|K| = \infty$ *and* $0 \neq J \subset E$ *be a graded ideal in* E. *Then*

$$\text{gin}((E/J)^*) \cong (E/\text{gin}(J))^*$$

as graded E*-modules, where* $(E/J)^*$ *is identified with the ideal* $0 :_E J$.

PROOF. It is enough to show

$$\text{in}(0 :_E J) = (0 :_E \text{in}(J))$$

since there exists a generic automorphism g with

$$\text{gin}((E/J)^*) = \text{gin}(0 :_E J) = \text{in}(g(0 :_E J)) = \text{in}(0 :_E g(J))$$

and

$$(E/\text{gin}(J))^* = (0 :_E \text{gin}(J)) = (0 :_E \text{in}(g(J))).$$

Let $f \in (0 :_E J)$, i.e., $fJ = 0$. In particular, $\text{in}(fh) = 0$ for all $h \in J$. If $\text{in}(f)\,\text{in}(h) \neq 0$ then $0 = \text{in}(fh) = \text{in}(f)\,\text{in}(h)$ so this case cannot occur. But then $\text{in}(f)\,\text{in}(h) = 0$ for all $h \in J$ and thus $\text{in}(f) \in (0 :_E \text{in}(J))$. This shows that $\text{in}(0 :_E J) \subseteq (0 :_E \text{in}(J))$.

Using the isomorphism $(M^*)_i \cong (M_{n-i})^\vee$ of K-vector spaces and the fact that E/J and $E/\operatorname{in}(J)$ have the same Hilbert function, one sees that $(0:_E \operatorname{in}(J)) \cong (E/\operatorname{in}(J))^*$ and $(0:_E J) \cong (E/J)^*$ have the same Hilbert function as well:

$$\begin{aligned}\dim_K(E/\operatorname{in}(J))^*_i &= \dim_K((E/\operatorname{in}(J))_{n-i})^\vee = \dim_K(E/\operatorname{in}(J))_{n-i}\\ &= \dim_K(E/J)_{n-i} = \dim_K((E/J)_{n-i})^\vee = \dim_K(E/J)^*_i.\end{aligned}$$

The same is true for $(0:_E J)$ and $\operatorname{in}(0:_E J)$, so $(0:_E \operatorname{in}(J))$ and $\operatorname{in}(0:_E J)$ have the same Hilbert function. This concludes the proof. □

The (Castelnuovo-Mumford) regularity of E/J and $E/\operatorname{gin}(J)$ coincide.

Theorem 3.2.2. [**2**, Theorem 5.3] *Let $|K| = \infty$ and $0 \neq J \subset E$ be a graded ideal in E. Then*

$$\operatorname{reg}_E(E/J) = \operatorname{reg}_E(E/\operatorname{gin}(J)).$$

We can compare the Bass numbers of a graded ideal with the Bass numbers of its initial ideal because we already know that $\beta_{i,j}(E/J) \leq \beta_{i,j}(E/\operatorname{in}(J))$ by [**3**, Proposition 1.8].

Corollary 3.2.3. *Let $|K| = \infty$ and $J \subset E$ be a graded ideal. Then*

$$\mu_{i,j}(E/J) \leq \mu_{i,j}(E/\operatorname{in}(J)) \quad \textit{and} \quad \mu_{i,j}(E/J) \leq \mu_{i,j}(E/\operatorname{gin}(J)).$$

In particular, if $E/\operatorname{in}(J)$ or $E/\operatorname{gin}(J)$ has a d-linear injective resolution, then E/J has a d-linear injective resolution as well.

PROOF. Exploiting the known inequality [**3**, Propositon 1.8]

$$\beta_{i,j}(E/J) \leq \beta_{i,j}(E/\operatorname{in}(J))$$

and (the proof of) Proposition 3.2.1 gives the following relationships

$$\mu_{i,j}(E/J) = \beta_{i,n-j}((E/J)^*) \leq \beta_{i,n-j}(\operatorname{in}((E/J)^*)) = \beta_{i,n-j}((E/\operatorname{in}(J))^*) = \mu_{i,j}(E/\operatorname{in}(J)).$$

From [**3**, Propositon 1.8] follows $\beta_{i,j}(E/J) \leq \beta_{i,j}(E/\operatorname{gin}(J))$ which implies the assertion on the generic initial ideal in the same way. □

For the generic initial ideal also the converse is true.

Corollary 3.2.4. *Let $|K| = \infty$ and $J \subset E$ be a graded ideal. If E/J has a d-linear injective resolution, then $E/\operatorname{gin}(J)$ has a d-linear injective resolution as well.*

PROOF. If E/J has a d-linear injective resolution, then its dual $(E/J)^*$ has an $(n-d)$-linear projective resolution. In particular, its Castelnuovo-Mumford regularity equals $n-d$. By Theorem 3.2.2 the regularity of $(E/J)^*$ and $\operatorname{gin}((E/J)^*)$ coincide. Using Proposition 3.2.1 we obtain

$$n-d = \operatorname{reg}_E(E/J)^* = \operatorname{reg}_E \operatorname{gin}((E/J)^*) = \operatorname{reg}_E(E/\operatorname{gin}(J))^*.$$

As $(E/J)^*$ is generated in degree $n-d$, its generic initial ideal $(E/\operatorname{gin}(J))^*$ is generated in degree $\geq n-d$. Thus $(E/\operatorname{gin}(J))^*$ has an $(n-d)$-linear projective resolution and so $E/\operatorname{gin}(J)$ has a d-linear injective resolution. □

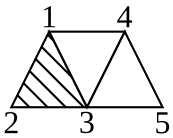
The passage from a monomial ideal in E to its generic initial ideal gives rise to a passage from a simplicial complex to another one. Let Δ be a simplicial complex on the vertex set $[n]$. The generic initial ideal of J_Δ is again a monomial ideal and thus there exists a simplicial complex Δ^e on $[n]$ such that $J_{\Delta^e} = \text{gin}(J)$. This new complex Δ^e is called the *exterior algebraic shifting* of Δ. Algebraic shifting has been introduced by Kalai in [**35**], [**36**] by other means (see also Björner and Kalai [**7**], [**8**]), see also the overview article [**37**]. Another important variant of algebraic shifting is related to taking the generic initial ideal in a polynomial ring. The name shifting comes from the fact that the so obtained simplicial complexes satisfy the following property.

Definition 3.2.5. A simplicial complex Δ is called *shifted* if for every face $F \in \Delta$, $i \in F$ and $j > i$ it holds that $(F \setminus \{i\}) \cup \{j\} \in \Delta$.

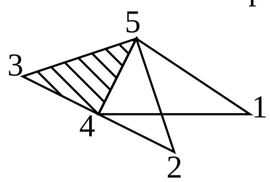
This is the combinatorial property corresponding to strongly stable ideals: the face ideal J_Δ is strongly stable if and only if Δ is a shifted complex.

As the generic initial ideal carries a lot of information about the original ideal the same is true for a simplicial complex and its shifting. For example both complexes have the same f-vector because the Hilbert functions of J and $\text{gin}(J)$ are equal. Further results in this direction are proved in Section 4.3, see also [**2**].

Example 3.2.6. Let Δ be the following simplicial complex:

$\Delta =$ 

The face ideal of Δ is $J_\Delta = (e_{15}, e_{24}, e_{25}, e_{134}, e_{345})$. Its generic initial ideal is $\text{gin}(J_\Delta) = (e_{12}, e_{13}, e_{23}, e_{245}, e_{145})$ and thus the exterior shifted complex Δ^e of Δ is

$\Delta^e =$  .

CHAPTER 4

Depth of graded E-modules

This chapter is devoted to an important invariant of an E-module, the depth, introduced by Aramova, Avramov and Herzog in [**1**]. It is defined similar to the depth of a module over a polynomial ring and closely related to the complexity, at least if the ground field is infinite.

We present some interesting results on the depth and useful lemmas to compute it. In Section 4.3 the depth of a squarefree module over an exterior algebra is compared with the depth of a certain corresponding squarefree module over a polynomial ring.

In the last section we study the (generic) annihilator numbers which are in some sense a refinement of the notion of depth. We derive some facts which relate these numbers to the Betti numbers of the corresponding module over the polynomial ring.

Most of the results in Section 4.1 appeared in [**40**], those in 4.3 and 4.4 will appear in [**39**].

4.1. Regular elements and depth

Let $M \in \mathscr{M}$. As every linear form of E is annihilated by itself, it is not useful to define the notion of a regular element exactly as in commutative algebra. Following [**1**] we call instead an element $v \in E_1$ *regular* on M (or *M-regular*) if the annihilator of v in M is the smallest possible, that is, the submodule vM.

An *M-regular sequence* is a sequence $v_1, \dots, v_s$ such that v_i is $M/(v_1, \dots, v_{i-1})M$-regular for $i = 1, \dots, s$ and $M/(v_1, \dots, v_s)M \neq 0$.

The following statements about regular elements are proved in [**1**]. Every M-regular sequence can be extended to a maximal one and all maximal regular sequences have the same length.

Definition 4.1.1. Let $M \in \mathscr{M}$. The length of a maximal M-regular sequence is called the *depth* of M over E and denoted by $\operatorname{depth}_E M$.

Similar to the well-known Auslander-Buchsbaum formula

$$\operatorname{proj}\dim_S N + \operatorname{depth}_S N = n$$

for a finitely generated module $N \neq 0$ over a polynomial ring S with n variables (see, e.g., [**11**, Theorem 1.3.3]) there is a formula relating the complexity and the depth of a finitely generated E-module.

Theorem 4.1.2. [**1**, Theorem 3.2] *Let $M \in \mathscr{M}$, $M \neq 0$, and $|K| = \infty$. Then*

$$\operatorname{cx}_E M + \operatorname{depth}_E M = n.$$

For a graded ideal $0 \neq J \subset E$ it holds that $\mathrm{cx}_E(E/J) = \mathrm{cx}_E(J)$. Thus if $|K| = \infty$ the above theorem implies also

$$\mathrm{depth}_E(E/J) = \mathrm{depth}_E(J).$$

Example 4.1.3.

(i) Every linear form $v \in E_1$ is regular on E. Therefore a sequence $v_1, \dots, v_s$ of linear forms is a regular sequence on E if and only if $v_1, \dots, v_s$ are linearly independent. In particular, $e_1, \dots, e_n$ is a maximal regular sequence on E. Thus $\mathrm{depth}_E E = n$.

(ii) A linear form v is regular on E/J if and only if $J : v = J + (v)$ because

$$\mathrm{ann}_{E/J}\, v = (J : v)/J \quad \text{and} \quad v \cdot (E/J) = (J + (v))/J.$$

For example $(e_{12}) : (e_3) = (e_3, e_{12})$ and therefore e_3 is regular on $E/(e_{12})$.

Let $J = (e_{F_1}, \dots, e_{F_t}) \neq 0$ be a monomial ideal. A variable e_i is regular on J if and only if $i \notin F_1 \cup \ldots \cup F_t$. All these variables form a regular sequence on J, but they need not to be a maximal regular sequence as the Example 4.1.5 shows. Nevertheless this gives a lower bound on the depth and, if the field is infinite, an upper bound on the complexity of monomial ideals, i.e.,

$$\mathrm{depth}_E(J) \geq n - |F_1 \cup \ldots \cup F_t| \qquad \text{and} \qquad \mathrm{cx}_E(J) \leq |F_1 \cup \ldots \cup F_t|.$$

For stable ideals these are equalities.

Lemma 4.1.4. *Let $|K| = \infty$ and $0 \neq J \subset E$ be a stable ideal with $G(J) = \{e_{F_1}, \dots, e_{F_t}\}$. Then*

$$\mathrm{depth}_E(J) = n - |F_1 \cup \ldots \cup F_t| \quad \textit{and} \quad \mathrm{cx}_E(J) = |F_1 \cup \ldots \cup F_t|.$$

PROOF. We already computed the complexity of a stable ideal in Lemma 3.1.4:

$$\mathrm{cx}_E(J) = \max\{\max(u) : u \in G(J)\} = \max\{\max(F_i) : i = 1, \dots, t\} =: s.$$

Since J is stable, it holds that

$$F_1 \cup \ldots \cup F_t = \{i \in [n] : i \leq s\},$$

whence the assertion follows. □

In general these bounds are not attained:

Example 4.1.5. Let $J = (e_{12}, e_{23}, e_{34}) \subseteq K\langle e_1, \dots, e_4 \rangle$. The above bound for the depth is 0, but $v = e_1 + \ldots + e_4$ is regular on E/J and thus $\mathrm{depth}_E(J) = \mathrm{depth}_E(E/J) \geq 1$. To prove this we compute $J : v$ and $J + v$ degreewise. Let $f \in E$ be a homogeneous element such that $vf \in J$. Suppose $\deg f = 1$, say $f = \sum_{i=1}^{4} \alpha_i e_i$. Then

$$\begin{aligned} vf = {} & (\alpha_2 - \alpha_1)e_{12} + (\alpha_3 - \alpha_1)e_{13} + (\alpha_4 - \alpha_1)e_{14} + (\alpha_3 - \alpha_2)e_{23} \\ & + (\alpha_4 - \alpha_2)e_{24} + (\alpha_4 - \alpha_3)e_{34} \in J. \end{aligned}$$

Since J is monomial each term of vf must be in J. This gives equations

$$\alpha_3 - \alpha_1 = \alpha_4 - \alpha_1 = \alpha_4 - \alpha_2 = 0,$$

whence $f \in (v)$ follows. Thus $(J : v)_1 = (v)_1 = (J + (v))_1$ because $J_1 = 0$. Note that $J_3 = E_3$ and therefore $(J : v)_2 = E_2$. The computations

$$e_{13} = ve_3 - e_{23} + e_{34}, \ \ e_{24} = v(e_1 + e_4) + e_{12} + e_{13} - e_{34}, \ \ e_{14} = ve_4 - e_{24} - e_{34}$$

show that $(J + (v))_2 = E_2$. So v is indeed regular on E/J. Actually, it is even a maximal regular sequence and thus $\operatorname{depth}_E(E/J) = 1$.

Wether a sequence of linear forms is regular on a module can be detected by its Cartan complex:

Proposition 4.1.6. [**1**, Remark 3.4] *Let $M \in \mathscr{M}$, $M \neq 0$, and $\mathbf{v} = v_1, \dots, v_m$ be a sequence of linear forms. The following statements are equivalent:*

(i) $\mathbf{v}$ *is M-regular.*
(ii) $H_1(\mathbf{v}; M) = 0$.
(iii) $H_i(\mathbf{v}; M) = 0$ *for* $i \geq 1$.

In particular, permutations of regular sequences are regular sequences because

$$H_1(\mathbf{v}; M) \cong \operatorname{Tor}_1^E(E/(v_1, \dots, v_m), M)$$

by Proposition 2.3.3 and the vanishing of the module on the right-hand side does not depend on the order of the elements.

It is a well-known fact that being a regular sequence is a Zariski-open condition. However, for the convenience of the reader we include a proof of it.

Proposition 4.1.7. *Let $M \in \mathscr{M}$ and $\operatorname{depth}_E M = t > 0$. Then there exists a non-empty Zariski-open set $U \subseteq GL_n(K)$ such that $v_1, \dots, v_t$ is an M-regular sequence for all $\gamma \in U$, $v_i = \gamma(e_{n-i+1})$, $i = 1, \dots, n$.*

PROOF. We use Proposition 4.1.6. Let U be the non-empty Zariski-open set such that the rank of the matrices of the K-linear maps

$$C_2(v_1, \dots, v_t; M)_k \xrightarrow{\delta_2} C_1(v_1, \dots, v_t; M)_k \xrightarrow{\delta_1} C_0(v_1, \dots, v_t; M)_k$$

are maximal and constant for all $k \in \mathbb{Z}$ (note that $M_k = 0$ for almost all k). Thus the dimensions of the images of the maps w. r. t. a generic sequence are maximal and correspondingly the dimensions of the kernels are minimal. Let $c_{i,k}$ denote the dimension of the kernel of the map δ_i in degree k w.r.t. a generic sequence and let $b_{i,k}$ denote the dimension of the respective image. If δ_1 and δ_2 are the maps w.r.t. an arbitrary sequence, then

$$\dim_K(\operatorname{Ker} \delta_1)_k \geq c_{1,k} \geq b_{2,k} \geq (\operatorname{Im} \delta_2)_k.$$

Since $\operatorname{depth}_E M = t$ there exists an M-regular sequence of linear forms $v_1, \dots, v_t$. For this sequence it holds that $\dim_K(\operatorname{Ker} \delta_1)_k = (\operatorname{Im} \delta_2)_k$ for all k. Therefore it follows that $c_{1,k} = b_{2,k}$ for all k and thus every sequence induced by U is an M-regular sequence. □

For small examples one can compute this open set directly.

Example 4.1.8. Let $J = (e_{123}) \subseteq K\langle e_1, e_2, e_3, e_4\rangle$. A linear form $v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4$ is regular on J if and only if $\alpha_4 \neq 0$.

Reducing modulo a regular element does not change the minimal projective graded resolution of a module numerically.

Proposition 4.1.9. *Let $M \in \mathscr{M}$ and $v \in E_1$ be a regular element on M. If $F_\bullet$ is a projective graded resolution of M over E, then $F_\bullet/vF_\bullet$ is a projective graded resolution of M/vM over $E/(v)$ whose modules have the same ranks and degree shifts as the modules in $F_\bullet$. If in addition $F_\bullet$ is minimal, then $F_\bullet/vF_\bullet$ is minimal too and*

$$\beta_{i,j}^{E}(M) = \beta_{i,j}^{E/(v)}(M/vM).$$

PROOF. Let $F_\bullet$ be a projective graded resolution of M over E. The i-th module in $F_\bullet/vF_\bullet$ is a free $E/(v)$-module of the same rank as F_i, the i-th module of $F_\bullet$:

$$F_i/vF_i \cong E/(v) \otimes_E F_i \cong E/(v) \otimes_E \bigoplus_{j\in\mathbb{Z}} E(-j)^{b_{ij}} \cong \bigoplus_{j\in\mathbb{Z}} E/(v)(-j)^{b_{ij}}$$

The homology of the complex is

$$H_i(F_\bullet/vF_\bullet) \cong H_i(E/(v) \otimes_E F_\bullet) \cong \operatorname{Tor}_i^E(E/(v), M) \cong H_i(v;M),$$

where $H_i(v;M)$ is the i-th Cartan homology of v with coefficients in M (see Proposition 2.3.3). For $i \geq 1$ we have $H_i(v;M) = 0$ by Proposition 4.1.6 whence $F_\bullet/vF_\bullet$ is exact except at homological position 0. The 0-th homology is $E/(v) \otimes_E M \cong M/vM$ so this complex is indeed a projective graded resolution of M/vM over $E/(v)$.

If in addition $F_\bullet$ is minimal, the entries in the matrices representing the maps of $F_\bullet$ are of degree ≥ 1. This property remains unchanged when reducing modulo v. The exponents b_{ij} are then the graded Betti numbers of M. □

Before we can understand what happens when reducing an injective resolution by a regular element we first need to understand the interplay between regular elements and duality.

Obviously a linear form v is regular on $M \neq 0$ if and only if the complex

$$(M,v): \qquad \ldots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \ldots$$

is exact.

Remark 4.1.10. For $M = K\{\Delta\}$ and $v = e_1 + \ldots + e_n$ Aramova and Herzog identified the i-th cohomology of the above complex (M,v) as the $(i-1)$-th reduced simplicial cohomology of Δ (cf. Corollary 1.2.2). They further show that if there exists a regular element on $K\{\Delta\}$, then $e_1 + \ldots + e_n$ is regular. Thus we conclude that $\operatorname{depth}_E K\{\Delta\} = 0$ if and only if $\tilde{H}^i(\Delta;K) \neq 0$ for some i, i.e., if Δ is not acyclic.

We will need the following useful lemma.

Lemma 4.1.11. *Let $0 \to U \to M \to N \to 0$ be an exact sequence of non-zero modules in $\mathscr{M}$. If $v \in E_1$ is regular on two of the three modules, then it is regular on the third.*

PROOF. Recall that $H^i(M,v)$ is the homology of the complex (M,v). The short exact sequence induces a short exact sequence of complexes

$$0 \longrightarrow (U,v) \longrightarrow (M,v) \longrightarrow (N,v) \longrightarrow 0$$

which induces a long exact sequence of the homology modules

$$\dots \longrightarrow H^{i-1}(N,v) \longrightarrow H^i(U,v) \longrightarrow H^i(M,v) \longrightarrow H^i(N,v) \longrightarrow H^{i+1}(U,v) \longrightarrow \dots$$

Then the observation that v is regular on one of these modules, say M, if and only if the corresponding homology $H^i(M,v)$ is zero for all i concludes the proof. □

Let $v_1,\dots,v_m \in E_1$ and $M \in \mathscr{M}$. To simplify notation we define

$$H_i(k) = H_i(v_1,\dots,v_k;M) \text{ for } i>0,\ k=1,\dots,m$$

and

$$\tilde{H}_0(k) = \frac{0 :_{M/(v_1,\dots,v_{k-1})M} v_k}{v_k(M/(v_1,\dots,v_{k-1})M)}.$$

Analogously

$$H^i(k) = H^i(v_1,\dots,v_k;M) \text{ for } i>0,\ k=1,\dots,m$$

and

$$\tilde{H}^0(k) = \frac{0 :_{0:_M(v_1,\dots,v_{k-1})} v_k}{v_k(0 :_M (v_1,\dots,v_{k-1}))}.$$

Finally we set $H_i(0) = H^i(0) = 0$ for $i>0$. The modules $\tilde{H}_0(k)$ and $\tilde{H}^0(k)$ are not the 0-th Cartan homology and cohomology but defined such that the long exact sequences of Cartan homology and cohomology modules of Propositions 2.3.5, 2.3.6 induce exact sequences

$$\dots \longrightarrow H_2(k) \longrightarrow H_1(k)(-1) \longrightarrow H_1(k-1) \longrightarrow H_1(k) \longrightarrow \tilde{H}_0(k)(-1) \longrightarrow 0$$

and

$$0 \longrightarrow \tilde{H}^0(k)(+1) \longrightarrow H^1(k) \longrightarrow H^1(k-1) \longrightarrow H^1(k)(+1) \longrightarrow H^2(k) \longrightarrow \dots$$

Theorem 4.1.12. *Let $M \in \mathscr{M}$. A sequence $\mathbf{v} = v_1,\dots,v_m$ in E_1 is an M-regular sequence if and only if it is an M^*-regular sequence. In particular,* $\operatorname{depth}_E M = \operatorname{depth}_E M^*$ *and, if K is infinite,* $\operatorname{cx}_E M = \operatorname{cx}_E M^*$.

PROOF. To prove the assertion it is enough to show that if $\mathbf{v}$ is an M^*-regular sequence, then it is an M-regular sequence as well.

First of all we state two observations which will be used several times in the proof. Let $N,N' \in \mathscr{M}$ and $v \in E_1$.

$(*)$ If v is N'-regular and $vN' \subseteq vN$, then $vN \cap N' = vN'$.

This is obvious since $x \in vN \cap N'$ implies $x \in 0 :_{N'} v = vN'$.

$(**)$ If v is regular on N,N' and $N \cap N'$, then v is regular on $N+N'$.

This follows from the short exact sequence

$$0 \longrightarrow N \cap N' \longrightarrow N \oplus N' \longrightarrow N+N' \longrightarrow 0$$

and Lemma 4.1.11.

The main task is to show by an induction on $t \geq 0$ that v_k is regular on each module of the form $v_{i_1}\cdots v_{i_r}(v_{j_1},\dots,v_{j_t})M$ for $\{i_1,\dots,i_r,j_1,\dots,j_t\} \subseteq \{1,\dots,m\} \setminus \{k\}$ and all $k = 1,\dots,m$.

Then with $r = 0$, $t = k-1$ this means that v_k is $(v_1, \dots, v_{k-1})M$-regular, with $r = t = 0$ that v_k is M-regular. Hence the exact sequence

$$0 \longrightarrow (v_1, \dots, v_{k-1})M \longrightarrow M \longrightarrow M/(v_1, \dots, v_{k-1})M \longrightarrow 0$$

implies by Lemma 4.1.11 that v_k is $M/(v_1, \dots, v_{k-1})M$-regular for all $k = 1, \dots, m$.

For the induction let $t = 0$. For simplicity we show that v_k is $v_1 \cdots v_{k-1}M$-regular. But as permutations of regular sequences are regular sequences, the proof works for arbitrary elements of the sequence as well.

The Cartan homology of $\mathbf{v}$ with values in M^* vanishes which implies by Proposition 2.3.2 that the Cartan cohomology of $\mathbf{v}$ with values in M vanishes. In particular

$$\tilde{H}^0(k; M) = \frac{0 :_{0 :_M (v_1, \dots, v_{k-1})} v_k}{v_k(0 :_M (v_1, \dots, v_{k-1}))} = 0$$

for all $k = 1, \dots, m$. We show by induction on k that v_k is $v_{k-1} \cdots v_1 M$-regular. If $k = 1$ we have

$$\tilde{H}^0(1; M) = \frac{0 :_M v_1}{v_1 M} = 0.$$

Hence v_1 is M-regular. Now suppose that the assertion is known for $k-1$.

The module $0 :_M (v_1, \dots, v_{k-1})$ contains all elements of M that are annihilated by all v_i, $i = 1, \dots, k-1$. Since every v_i is M-regular (same argument as for v_1, since permutations of regular sequences are regular sequences), $0 :_M (v_1, \dots, v_{k-1}) = v_{k-1}M \cap \dots \cap v_1 M$.

We see $v_{l-1}M \cap \dots \cap v_1 M = v_{l-1} \cdots v_1 M$ by another induction on $l \leq k$. If $l \leq 2$ this is obvious. Now if $l > 2$ we have

$$v_{l-1}M \cap \dots \cap v_1 M = v_{l-1}M \cap (v_{l-2}M \cap \dots \cap v_1 M) = v_{l-1}M \cap v_{l-2} \cdots v_1 M \overset{(*)}{=} v_{l-1} \cdots v_1 M$$

where the induction hypothesis of the first induction on k is used. i.e., that v_{l-1} is regular on $v_{l-2} \cdots v_1 M$ since $l - 1 \leq k - 1$.

Then

$$0 = \tilde{H}^0(k; M) = \frac{0 :_{0 :_M (v_1, \dots, v_{k-1})} v_k}{v_k(0 :_M (v_1, \dots, v_{k-1}))} = \frac{0 :_{v_{k-1} \cdots v_1 M} v_k}{v_k(v_{k-1} \cdots v_1 M)}$$

implies that v_k is $v_{k-1} \cdots v_1 M$-regular.

We return to the induction on t and suppose $t > 0$. We decompose the module $v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_t})M$ in two parts. By induction hypothesis the linear form v_k is regular on $v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_{t-1}})M$ and on $v_{i_1} \cdots v_{i_r} v_{j_t} M$. Furthermore, the induction hypothesis gives that v_{j_t} is $v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_{t-1}})M$-regular. Thus with $(*)$ follows that the intersection of the two parts is

$$v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_{t-1}})M \cap v_{i_1} \cdots v_{i_r} v_{j_t} M = v_{j_t} v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_{t-1}})M.$$

Again by induction hypothesis v_k is regular on this intersection. So $(**)$ implies that v_k is regular on $v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_{t-1}})M + v_{i_1} \cdots v_{i_r} v_{j_t} M = v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_t})M$.

Finally it remains to show that $M/(v_1, \dots, v_m)M \neq 0$ for $v_1, \dots, v_m$ being an M-regular sequence. But this is evident since $M/(v_1, \dots, v_m)M = 0$ would imply

$$M = (v_1, \dots, v_m)M = (v_1, \dots, v_m)^2 M = \dots = (v_1, \dots, v_m)^{m+1} M = 0$$

and hence $M^* = 0$, which is not possible (unless $m = 0$). $\square$

One consequence of the preceding theorem (and its proof) is the following identity.

Proposition 4.1.13. *Let $M \in \mathscr{M}$ and $v_1, \dots, v_m$ be an M-regular sequence. Then*

$$(M/(v_1, \dots, v_m)M)^* \cong v_1 \cdots v_m M^*$$

as graded E-modules.

PROOF. We show by induction on m that $(M/(v_1, \dots, v_m)M)^* \cong v_m \cdots v_1 M^*$, using that $v_1, \dots, v_m$ is also an M^*-regular sequence by Theorem 4.1.12. If $m = 1$ this follows from the exact sequence

$$0 \longrightarrow 0 :_M v_1 \longrightarrow M \xrightarrow{\cdot v_1} M \longrightarrow M/v_1 M \longrightarrow 0$$

and the corresponding exact dual sequence

$$0 \longrightarrow (M/v_1 M)^* \longrightarrow M^* \xrightarrow{\cdot v_1} M^* \longrightarrow (0 :_M v_1)^* \longrightarrow 0$$

because here $(M/v_1 M)^*$ is the kernel of the multiplication with v_1 which is $v_1 M^*$ as v_1 is M^*-regular.

Now suppose $m > 1$ and the assertion is proved for regular sequences of length $< m$.

The proof of Theorem 4.1.12 shows that v_k is regular on $v_{i_1} \cdots v_{i_r}(v_{j_1}, \dots, v_{j_t})M^*$ for $\{i_1, \dots, i_r, j_1, \dots, j_t\} \subseteq [m] \setminus \{k\}$ for $k = 1, \dots, m$. In particular, v_m is regular on $v_{m-1} \cdots v_1 M^*$, thus we can use the induction hypothesis on it which gives

$$v_m \cdots v_1 M^* = v_m(v_{m-1} \cdots v_1 M^*) \cong v_m(M/(v_{m-1}, \dots, v_1)M)^*.$$

By assumption v_m is regular on $M/(v_{m-1}, \dots, v_1)M$ and hence by Theorem 4.1.12 also on its dual $(M/(v_{m-1}, \dots, v_1)M)^*$. Thus another application of the induction hypothesis gives

$$\begin{aligned} v_m(M/(v_{m-1}, \dots, v_1)M)^* &\cong \big(M/(v_{m-1}, \dots, v_1)M/v_m(M/(v_{m-1}, \dots, v_1)M)\big)^* \\ &\cong (M/(v_m, v_{m-1}, \dots, v_1)M)^*. \end{aligned}$$

□

The relation $H_i(\mathbf{v}; M^*) = H^i(\mathbf{v}; M)^*$ between Cartan homology and Cartan cohomology in Proposition 2.3.2 provides the following dual version of Proposition 4.1.6.

Corollary 4.1.14. *Let $M \in \mathscr{M}$, $M \neq 0$ and $\mathbf{v} = v_1, \dots, v_m$ be a sequence in E_1. Then the following statements are equivalent:*

(i) *$v_1, \dots, v_m$ is M-regular.*
(ii) $H^1(\mathbf{v}; M) = 0$.
(iii) $H^i(\mathbf{v}; M) = 0$ *for all* $i > 0$.

PROOF. An E-module is zero if and only if its dual is zero. Thus the equality of the conditions follows from Theorem 4.1.12, Propositon 2.3.2 and Proposition 4.1.6. □

Now we can use the same argumentation for reducing injective resolutions by a regular element as for projective resolutions.

Proposition 4.1.15. *Let $M \in \mathscr{M}$, $v \in E_1$ be an M-regular element and $I^\bullet$ a graded injective resolution of M over E. Then $\operatorname{Hom}_E(E/(v), I^\bullet)$ is a graded injective resolution of vM over $E/(v)$ with the same ranks and degree shifts as $I^\bullet$. If in addition $I^\bullet$ is minimal, then $\operatorname{Hom}_E(E/(v), I^\bullet)$ is minimal too and*

$$\mu_{i,j}^E(M) = \mu_{i,j}^{E/(v)}(vM).$$

PROOF. Let $I^\bullet$ be a graded injective resolution of M. The homology of the complex $\operatorname{Hom}_E(E/(v), I^\bullet)$ is isomorphic to the Cartan cohomology $H^i(v;M)$ of M with respect to v by Proposition 2.3.3. As v is M-regular, Corollary 4.1.14 implies that $H^i(v;M) = 0$ for $i > 0$.

The i-th module in this resolution is of the form

$$\begin{aligned}\operatorname{Hom}_E(E/(v), \bigoplus_{j\in\mathbb{Z}} E(n-j)^{c_{i,j}(M)}) &\cong \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_E(E/(v), E)(n-j)^{c_{i,j}(M)} \\ &\cong \bigoplus_{j\in\mathbb{Z}} (0 :_E v)(n-j)^{c_{i,j}(M)} \\ &\cong \bigoplus_{j\in\mathbb{Z}} (v)(n-j)^{c_{i,j}(M)} \\ &\cong \bigoplus_{j\in\mathbb{Z}} E/(v)(n-1-j)^{c_{i,j}(M)}.\end{aligned}$$

Thus $\operatorname{Hom}_E(E/(v), I^\bullet)$ is indeed a graded injective resolution of vM with the same ranks and degree shifts (bear in mind that $E/(v)$ is an exterior algebra with $n-1$ variables). The minimality is preserved because an injective resolution over E is minimal if and only if all entries in the matrices of the maps are in the maximal ideal. This property is not touched by applying $\operatorname{Hom}_E(E/(v), \cdot)$. Then the exponents $c_{i,j}$ are the graded Bass numbers $\mu_{i,j}$. □

Many statements over the exterior algebra are analogous to statements over a polynomial ring. However, one has to be careful. For example, Lemma 4.1.11 applied on the sequence $0 \to J \to E \to E/J \to 0$ implies that a linear form v is regular on J if and only if it is regular on E/J. Inductively it follows that even a sequence $v_1, \dots, v_m$ of linear forms is a J-regular sequence if and only if it is a regular sequence on E/J. There is no such statement over the polynomial ring.

On the other hand, the situation over the exterior algebra can also be not as nice as over the polynomial ring. For example the annihilator of an element in E can be arbitrary complicated. Therefore elements whose annihilators have a simple shape in some sense are distinguished. Following [**14**] we call a homogeneous element in E of positive degree *pure decomposable* if it is a product of linear forms. The annihilator of such an element is generated by linear forms:

Lemma 4.1.16. *Let $f \neq 0$ be a homogeneous element in E of degree $m > 0$. Then $\operatorname{ann}_E f = (v_1, \dots, v_m)$, where $v_1, \dots, v_m$ are linearly independent linear forms, if and only if $f = \alpha \cdot v_1 \cdots v_m$ for some $\alpha \in K$.*

PROOF. Suppose $\operatorname{ann}_E f = (v_1, \dots, v_m)$ where $v_1, \dots, v_m$ are linearly independent linear forms. After a change of coordinates we may assume that $v_i = e_i$ for $i = 1, \dots, m$. Then

$$(f) = 0 : (0 : f) = 0 : (e_1, \dots, e_m) = (e_1 \cdots e_m)$$

and hence $f = \alpha e_1 \cdots e_m$ for some $\alpha \in K$.

Now let $f = \alpha \prod_{i=1}^{m} v_i$ for some $\alpha \in K$. The linear forms $v_1, \ldots, v_m$ must be linearly independent because otherwise f would be zero. Then $v_m, \ldots, v_1$ is a regular sequence on E. We claim that $\operatorname{ann}_E f = (v_1, \ldots, v_m)$. To see the non-trivial inclusion let $g \in E$ be a homogeneous element with $gf = 0$. That is, $gv_1 \cdots v_m = 0$. Since v_m is regular on E it follows that $gv_1 \cdots v_{m-1} \in (v_m)$. Now v_{m-1} is regular on $E/(v_m)$ whence $gv_1 \cdots v_{m-2} \in (v_{m-1}, v_m)$ and, inductively, $g \in (v_1, \ldots, v_m)$. □

In view of the preceding lemma we extend the definition of a pure decomposable element as follows: a homogeneous element m in a graded E-module is called *pure decomposable* if its annihilator $0 :_E m$ is generated by linear forms. A *pure decomposable module* is a module that is generated by pure decomposable elements.

Proposition 4.1.17. *Let $m \in M \in \mathscr{M}$ be a homogeneous non-zero element of degree d. Then m is pure decomposable if and only if the regularity of the cyclic submodule generated by m is d, i.e., if* $\operatorname{reg}_E \langle m \rangle = d$.

PROOF. Let m be pure decomposable. The minimal free resolution of $\langle m \rangle$ over E starts with the map $E(-d) \to \langle m \rangle$, $1 \mapsto m$. The kernel of this map is the annihilator of m which is generated by linear forms by definition. Thus the resolution can be continued with the Cartan complex with respect to this linear forms, shifted in degrees by $-d$, which is a d-linear resolution.

If m is not pure decomposable, then there exists a minimal generator of the annihilator of m of degree $j > 1$. Thus $\beta_{1,d+j}(\langle m \rangle) \neq 0$ and therefore $\operatorname{reg}\langle m \rangle \geq d + j > d$. □

4.2. Depth and the generic initial ideal

Recall that the monomial order considered is always the reverse lexicographic order with $e_1 > e_2 > \ldots > e_n$.

As the regularity, also the depth is preserved by the passage to the generic initial ideal. This result is already known. For the convenience of the reader we present a proof.

Theorem 4.2.1. [**34**, Proposition 2.3] *Let $|K| = \infty$ and $J \subset E$ be a graded ideal in E. Then*

$$\operatorname{depth}_E(E/J) = \operatorname{depth}_E(E/\operatorname{gin}(J)) \quad \textit{and} \quad \operatorname{cx}_E(E/J) = \operatorname{cx}_E(E/\operatorname{gin}(J)).$$

PROOF. The inequality

$$\beta_{i,j}(E/J) \leq \beta_{i,j}(E/\operatorname{gin}(J))$$

follows from [**3**, Proposition 1.8]. Thus $\operatorname{cx}_E(E/J) \leq \operatorname{cx}_E(E/\operatorname{gin}(J))$. It follows from the equation

$$\operatorname{cx}_E M + \operatorname{depth}_E M = n$$

in Theorem 4.1.2 that

$$\operatorname{depth}_E(E/J) \geq \operatorname{depth}_E(E/\operatorname{gin}(J)).$$

Being a regular sequence is an open condition by Proposition 4.1.7. Hence there exists an E/J-regular sequence $v_1, \ldots, v_t$, $t = \operatorname{depth}_E(E/J)$, of linear forms such that there is a generic automorphism g mapping v_i to e_{n-i+1} with $\operatorname{gin}(J) = \operatorname{in}(g(J))$. Since the considered monomial order is the revlex order, we have $\mathrm{in}(g(J)+e_n)=\mathrm{in}(g(J))+e_n$ and $\mathrm{in}(g(J):e_n)=\mathrm{in}(g(J)):e_n$ by [**2**, Proposition 5.1]. Then the Hilbert functions

$$\begin{aligned}H(-,E/(\mathrm{gin}(J+v_1))) &= H(-,E/(J+v_1)) = H(-,E/g(J+v_1))\\ &= H(-,E/(g(J)+e_n)) = H(-,E/\mathrm{in}(g(J)+e_n)) = H(-,E/(\mathrm{gin}(J)+e_n))\end{aligned}$$

are equal. Analogously one sees $H(-,E/\mathrm{gin}(J:v_1)) = H(-,E/(\mathrm{gin}(J):e_n))$. On the other hand $J+v_1 = J:v_1$ as v_1 is E/J-regular. Hence

$$H(-,E/\mathrm{gin}(J+v_1)) = H(-,E/\mathrm{gin}(J:v_1)).$$

Thus the Hilbert functions of $\mathrm{gin}(J)+e_n$ and $\mathrm{gin}(J):e_n$ coincide as well which already implies $\mathrm{gin}(J)+e_n = \mathrm{gin}(J):e_n$ because $\mathrm{gin}(J)+e_n \subseteq \mathrm{gin}(J):e_n$ is always true. This shows that e_n is $E/\mathrm{gin}(J)$-regular.

The above computations can be generalised to a longer regular sequence $v_1,\dots,v_i$, $i\le t$ so inductively we obtain that e_{n-i+1} is $E/(\mathrm{gin}(J)+(e_n,\dots,e_{n-i+2}))$-regular for $i=1,\dots,t$. The quotients do not vanish because the corresponding quotient rings $E/(J+(v_1,\dots,v_{i-1}))$ do not vanish. So this means that $e_n,\dots,e_{n-t+1}$ is an $E/\mathrm{gin}(J)$-regular sequence.

This proves the reverse inequality

$$\mathrm{depth}_E(E/J) \le \mathrm{depth}_E(E/\mathrm{gin}(J)).$$

Therefore the depths, and, with Theorem 4.1.2, also the complexities are equal. □

We extract one fact proved along the way in the proof of Theorem 4.2.1, taking into account that $J=\mathrm{gin}(J)$ for stable ideals J.

Corollary 4.2.2. *Let $J\subset E$ be a stable ideal, $|K|=\infty$ and $\mathrm{depth}_E(E/J)=t$. Then $e_n,\dots,e_{n-t+1}$ is a regular sequence on E/J.*

For a simplicial complex Δ Theorem 4.2.1 says that the depths of its face ring $K\{\Delta\}$ and the face ring of its exterior shifting $K\{\Delta^e\}$ are equal (cf. Section 3.2 for the definition of exterior shifting). It is possible to read off the depth of the face ring of a shifted complex from the structure of the complex.

The *join* of two simplicial complexes Δ and Γ is the simplicial complex

$$\Delta*\Gamma = \{F\cup G : F\in\Delta, G\in\Gamma\},$$

whose faces are the unions of a face of Δ and a face of Γ.

Theorem 4.2.3. *Let Δ be a simplicial complex on the vertex set $[n]$ and $|K|=\infty$. Then the following statements are equivalent:*

(i) $\mathrm{depth}_E(K\{\Delta\})=t$;
(ii) $\Delta^e = \Gamma * 2^{[t]}$, *where Γ is a non-acyclic, $(\dim\Delta - t)$-dimensional simplicial complex and $2^{[t]}$ is the $(t-1)$-simplex on the vertex set $\{n-t+1,\dots,n\}$.*

PROOF. Recall that $\mathrm{depth}_E(K\{\Delta\}) = \mathrm{depth}_E(K\{\Delta^e\})$ by Theorem 4.2.1.

(i)⇒(ii) We first assume that $\mathrm{depth}_E(K\{\Delta\})=t$. In order to prove (ii) we show that for $F\in\Delta^e$ it holds that $F\cup\{n-r+1,\dots,n\}$ is a face of Δ^e for any $r\le t$. Let $F\in\Delta^e$. Without loss of generality we may assume that $F\cap\{n-r+1,\dots,n\}=\emptyset$. The sequence $e_n,\dots,e_{n-r+1}$ is regular on $K\{\Delta^e\} = E/\mathrm{gin}(J_\Delta)$ by Corollary 4.2.2. Suppose that $F\cup$

$\{n-r+1,\dots,n\} \notin \Delta^e$. Thus, $e_n \cdots e_{n-r+1} e_F = 0$ in $K\{\Delta^e\}$. By the definition of a regular sequence it follows that

$$e_{n-r+2} \cdots e_n e_F \in J_{\Delta^e} + (e_{n-r+1}).$$

Therefore, $e_{n-r+2} \cdots e_n e_F = 0 \in E/(\text{gin}(J_\Delta) + (e_{n-r+1}))$. Inductively, we get $e_F = 0 \in E/(\text{gin}(J_\Delta) + (e_n, \dots, e_{n-r+1}))$. Since $F \cap \{n-r+1,\dots,n\} = \emptyset$ by assumption, we have $e_F \in \text{gin}(J_\Delta)$, i.e., $F \notin \Delta^e$. This is a contradiction. This shows $\Delta^e = \Gamma * 2^{[t]}$ for some $(\dim \Delta - t)$-dimensional simplicial complex Γ with $J_\Gamma = \text{gin}(J_\Delta) + (e_n, \dots, e_{n-t+1})$. From the definition of depth and Corollary 4.2.2 it follows that

$$\text{depth}_E(K\{\Gamma\}) = \text{depth}_E(K\{\Delta^e\}) - t = 0.$$

We therefore conclude from Remark 4.1.10 that $\widetilde{H}_i(\Gamma; K) \neq 0$ for some $0 \leq i \leq \dim(\Gamma)$, i.e., Γ is non-acyclic.

(ii)⇒(i) We have to show that $\text{depth}_E(K\{\Delta^e\}) = t$. By assumption, we have $\Delta^e = \Gamma * 2^{[t]}$, where $2^{[t]}$ is the $(t-1)$-simplex on the vertex set $\{n-t+1,\dots,n\}$ and Γ is a non-acyclic simplicial complex on the vertex set $[n-t]$. On the level of face rings this means

$$K\{\Delta^e\} = K\{\Gamma\} \otimes_K K\{2^{[t]}\}.$$

The face ring of the simplex is $K\{2^{[t]}\} = K\langle e_{n-t+1}, \dots, e_n \rangle$. The sequence $e_n, \dots, e_{n-t+1}$ is regular on $K\{2^{[t]}\}$ and thus also on $K\{\Delta^e\}$. This implies $\text{depth}_E(K\{\Delta^e\}) \geq t$. It further holds that

$$K\{\Delta^e\}/(e_n, \dots, e_{n-t+1}) \cong K\{\Gamma\}.$$

Since Γ is non-acyclic it follows from Remark 4.1.10 that $\text{depth}_E(K\{\Gamma\}) = 0$. Therefore $e_n, \dots, e_{n-t+1}$ is a maximal regular sequence on $K\{\Delta^e\}$ and thus $\text{depth}_E(K\{\Delta^e\}) = t$. □

Example 4.2.4. In Example 3.2.6 the exterior shifting of the simplicial complex Δ with $J_\Delta = (e_{15}, e_{24}, e_{25}, e_{134}, e_{345})$ is computed. The face ideal of the shifting is the ideal

$$J_{\Delta^e} = (e_{12}, e_{13}, e_{23}, e_{245}, e_{145}).$$

It is not a cone over any vertex, hence the simplex as in Theorem 4.2.3 is the empty one and we conclude $\text{depth}_E K\{\Delta\} = \text{depth}_E K\{\Delta^e\} = 0$. If also the two non-filled triangles 245 and 145 of Δ^e were filled, one could split the complex into the edge 45, a 1-simplex, and three points. Hence, in that case, the depth would be 2.

4.3. Squarefree modules over the exterior algebra and the polynomial ring

Yanagawa introduces squarefree modules over the polynomial ring $S = K[x_1, \dots, x_n]$ as a generalisation of squarefree monomial ideals in [**62**]. In this section S is always considered to be positively graded, i.e., $\deg x_i = 1$ for all $i = 1, \dots, n$.

For $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ we say that a is *squarefree* if $0 \leq a_i \leq 1$ for $i = 1, \dots, n$. We set $|a| = a_1 + \ldots + a_n$ and $\text{supp}(a) = \{i \ : \ a_i \neq 0\}$.

A finitely generated $\mathbb{N}^n$-graded S-module $N = \oplus_{a \in \mathbb{N}^n} N_a$ is called *squarefree* if the multiplication map $N_a \to N_{a+\varepsilon_i}$, $y \mapsto x_i y$ is bijective for any $a \in \mathbb{N}^n$ and for all $i \in \text{supp}(a)$, where $\varepsilon_i \in \mathbb{N}^n$ is the vector with 1 at the i-th position and zero otherwise.

Römer defines in [**53**, Definition 1.4] a finitely generated $\mathbb{N}^n$-graded E-module $M = \oplus_{a \in \mathbb{N}^n} M_a$ to be *squarefree* if it has only squarefree non-zero components.

Aramova, Avramov and Herzog [**1**] and Römer [**53**] construct a squarefree E-module N_E and its minimal free resolution starting from a squarefree S-module N and its minimal free resolution.

Theorem 4.3.1. [**53**, Theorem 1.2] *The assignment $N \mapsto N_E$ induces an equivalence between the categories of squarefree S-modules and squarefree E-modules (where the morphisms are the $\mathbb{N}^n$-graded homomorphisms).*

Example 4.3.2. The Stanley-Reisner ring $K[\Delta]$ is a typical example of a squarefree S-module, the exterior face ring $K\{\Delta\}$ of a squarefree E-module.

These two correspond to each other under the equivalence of categories, i.e.,

$$(K[\Delta])_E \cong K\{\Delta\}$$

(see [**1**, Theorem 1.3]).

As the equivalence of the categories of squarefree S- and squarefree E-modules bases on the construction of resolutions, one also gets the following relation between the Betti numbers of a squarefree S-module and its corresponding E-module. Let $\beta^S_{i,j}$ and $\beta^E_{i,j}$ denote the graded Betti numbers over S and E, respectively. Recall that an S-module N has a t-linear resolution if and only if $\beta^S_{i,i+j}(N) = 0$ for all $j \neq t$.

Corollary 4.3.3. [**53**, Corollary 1.3] *Let N be a squarefree S-module and let N_E be the associated squarefree E-module. Then*

$$\beta^E_{i,i+j}(N_E) = \sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \beta^S_{k,k+j}(N).$$

In particular, N has a t-linear projective resolution over S if and only if N_E has a t-linear projective resolution over E.

The formula for the Betti numbers implies that the regularity of a squarefree S-module and its corresponding E-module coincide.

Lemma 4.3.4. *Let N be a squarefree S-module and let N_E be the associated squarefree E-module. Then*

$$\operatorname{reg}_S(N) = \operatorname{reg}_E(N_E).$$

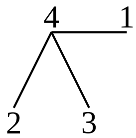
PROOF. We plug in the formula for the Betti numbers into the definition of regularity:

$$\begin{aligned}
\operatorname{reg}_E(N_E) &= \max\{j \in \mathbb{Z} : \beta^E_{i,i+j}(N_E) \neq 0, i \geq 0\} \\
&= \max\{j \in \mathbb{Z} : \sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \beta^S_{k,k+j}(N) \neq 0, i \geq 0\} \\
&= \max\{j \in \mathbb{Z} : \beta^S_{k,k+j}(N) \neq 0 \text{ for some } 0 \leq k \leq i, i \geq 0\} \\
&= \max\{j \in \mathbb{Z} : \beta^S_{k,k+j}(N) \neq 0, k \geq 0\} \\
&= \operatorname{reg}_S(N).
\end{aligned}$$

□

One could tend to think that also the two notions of depth (over the exterior algebra and the polynomial ring) coincide. But this is not the case (for simplicial complexes this is already clear from Theorem 4.2.3 because a non-acyclic complex does not need to have depth zero over S). In the following example the depth over S is indeed greater than the depth over E.

Example 4.3.5. Let Δ be the simplicial complex on the vertex set $[4]$ consisting of three edges intersecting in a common vertex 4.

$\Delta =$ 

The face ideal is the ideal $J_\Delta = (e_{12}, e_{13}, e_{23})$. It is a shifted complex whence the depth of $K\{\Delta\}$ is 1 by Theorem 4.2.3. For example, e_4 is a maximal regular sequence on $K\{\Delta\}$. On the other hand, with the help of Macaualy 2 [**26**], we compute that $x_4, x_1 + x_2 + x_3$ is a maximal regular sequence on $K[\Delta]$.

This relation between the depths is true in general.

Theorem 4.3.6. *Let $|K| = \infty$ and $N \neq 0$ be a finitely generated $\mathbb{N}^n$-graded squarefree S-module. Then*

$$0 \leq \operatorname{depth}_S(N) - \operatorname{depth}_E(N_E) = \operatorname{cx}_E(N_E) - \operatorname{proj\,dim}_S(N) \leq \operatorname{reg}_S(N).$$

PROOF. A comparison of Theorem 4.1.2,

$$\operatorname{depth}_E N_E + \operatorname{cx}_E N_E = n,$$

and the Auslander-Buchsbaum formula just above Theorem 4.1.2,

$$\operatorname{depth}_S N + \operatorname{proj\,dim}_S N = n,$$

shows that the differences between the depths and between the complexity and the projective dimension are equal. We show that the inequalities hold for the latter difference.

Corollary 4.3.3 implies that

$$\beta_i^E(N_E) = \sum_{j \geq 0} \beta_{i,i+j}^E(N_E) = \sum_{j \geq 0} \sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \beta_{k,k+j}^S(N).$$

Let $m_j^{(i)} = \max\{k + j \,:\, \beta_{k,k+j}^S(N) \neq 0,\ 0 \leq k \leq i\}$. Then $\beta_{i,i+j}^E(N_E)$ is a polynomial in i of degree $m_j^{(i)} - 1$. Therefore, it follows that $\beta_i^E(N_E)$ is a polynomial in i of degree $m^{(i)} - 1$, where $m^{(i)} = \max\{m_j^{(i)} \,:\, j \geq 0, \beta_{i,i+j}^E(N_E) \neq 0\}$ (here we can write max instead of sup because the modules in the minimal free E-resolution of a finitely generated module are finitely generated). This yields for the complexity

$$\begin{aligned} \operatorname{cx}_E(N_E) &= \sup\{m^{(i)} \,:\, i \geq 0\} \\ &= \sup\{k + j \,:\, \beta_{k,k+j}^S(N) \neq 0,\ k \geq 0,\ j \geq 0\} \qquad (1) \\ &= \max\{l \,:\, \beta_{i,l}^S(N) \neq 0,\ i \geq 0,\ l \geq 0\}, \end{aligned}$$

where the last equality holds because N has a finite S-resolution. Let $p = \operatorname{proj}\dim_S(N)$ and $i, j \in \mathbb{Z}$ with $\beta^S_{i,i+j}(N) \neq 0$ and $\operatorname{cx}_E(N_E) = i + j$. Then $i \leq p$ and so $\operatorname{cx}_E(N_E) - p \leq \operatorname{cx}_E(N_E) - i = j \leq \operatorname{reg}_S(N)$. On the other hand there exists some k, $0 \leq k \leq \operatorname{reg}_S(N)$ such that $\beta^S_{p,p+k}(N) \neq 0$. This implies $\operatorname{cx}_E(N_E) \geq p + k$ by (1) and thus $\operatorname{cx}_E(N_E) - \operatorname{proj}\dim_S(N) \geq k \geq 0$. □

Remark 4.3.7.

(i) The relations in Theorem 4.3.6 in particular show that an exterior version of Terai's theorem [**58**],

$$\operatorname{proj}\dim_S I_\Delta = \operatorname{reg}_S K[\Delta^*],$$

does not hold, i.e., in general it is not true that $\operatorname{cx}_E J_\Delta = \operatorname{reg}_E K\{\Delta^*\}$, since $\operatorname{reg}_E K\{\Delta^*\} = \operatorname{reg}_S K[\Delta^*]$ by Lemma 4.3.4. Thus the class of simplicial complexes for which this equation holds are exactly those with $\operatorname{depth}_S(K[\Delta]) = \operatorname{depth}_E(K\{\Delta\})$ (see also Open problem 4.3.11).

(ii) For simplicial complexes the relation between the complexity and the Betti numbers over S in (1) has been noticed already in [**1**, Theorem 4.2].

An interesting question to ask is if there are classes of squarefree modules for which equality holds in the second inequality in Theorem 4.3.6. In the special case of Stanley-Reisner rings of simplicial complexes we can identify at least two such classes.

Lemma 4.3.8. *Let Δ be a simplicial complex of dimension $d-1 \geq 0$ and $|K| = \infty$. If J_Δ has a linear projective resolution or if Δ is Cohen-Macaulay, then*

$$\operatorname{depth}_S(K[\Delta]) - \operatorname{depth}_E(K\{\Delta\}) = \operatorname{cx}_E(K\{\Delta\}) - \operatorname{proj}\dim_S(K[\Delta]) = \operatorname{reg}_S(K[\Delta]).$$

PROOF. Examining the proof of Theorem 4.3.6 we see that we have an equality in the second inequality in Theorem 4.3.6 if and only if $\beta^S_{p,p+r}(K[\Delta]) \neq 0$ where $p = \operatorname{proj}\dim_S(K[\Delta])$ and $r = \operatorname{reg}_S(K[\Delta])$. This is the most right lower corner in the corresponding Betti diagram. It is obviously the case if I_Δ (equivalently J_Δ) has a linear resolution.

If Δ is Cohen-Macaulay, $K[\Delta]$ has depth d over S if $\dim\Delta = d-1$ and the face ring $K\{\Delta\}$ has a d-linear injective resolution over E (see, e.g., [**2**, Corollary 7.6] or Example 5.1.1). Following Theorem 5.1.6 (which is independent from the results in this section) it holds that

$$d = \operatorname{depth}_E(K\{\Delta\}) + \operatorname{reg}_E(K\{\Delta\}).$$

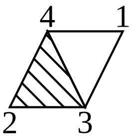
By Lemma 4.3.4 the regularity over E is the same as the regularity over S, hence

$$\operatorname{reg}_S(K[\Delta]) = \operatorname{reg}_E(K\{\Delta\}) = d - \operatorname{depth}_E(K\{\Delta\}) = \operatorname{depth}_S(K[\Delta]) - \operatorname{depth}_E(K\{\Delta\}).$$

As an alternative proof of this case it is possible to show that if Δ is Cohen-Macaulay then $K[\Delta]$ has only one so-called extremal Betti number. □

The following example shows that in general the converse of Lemma 4.3.8 is not true. A monomial ideal I in S is called *squarefree stable* if for all squarefree monomials $x_A \in I$ and all $i < \max(A)$ with $i \notin A$ one has $x_i x_{A \setminus \max(A)} \in I$.

Example 4.3.9. Let Δ be the simplicial complex on the vertex set $[4]$ consisting of two triangles – one of them filled, the other one missing – glued together along one edge.

$\Delta =$ 

The face ideal is $J_\Delta = (e_{12}, e_{134})$. It is not generated in one degree and thus does not have a linear projective resolution. On the other hand Δ is not pure so it cannot be Cohen-Macaulay. J_Δ is stable whence we can compute the regularity and the depth of $K\{\Delta\}$ from the known formulas Lemma 3.1.3 and Lemma 3.1.4. Also I_Δ is squarefree stable. Then the depth can be computed by the formula

$$\operatorname{depth}_S K[\Delta] = n - \max\{\max(u) - \deg(u) : u \in G(I_\Delta)\} - 1$$

which is proved in [**4**, Corollary 2.4]. Here we then compute

$$\begin{aligned} \operatorname{depth}_S(K[\Delta]) &= 4 - \max\{0,1\} - 1 = 2, \\ \operatorname{reg}_E(K\{\Delta\}) &= \max\{2,3\} - 1 = 2, \\ \operatorname{depth}_E(K\{\Delta\}) &= 4 - \max\{2,4\} = 0. \end{aligned}$$

Hence, $\operatorname{depth}_S(K[\Delta]) - \operatorname{depth}_E(K\{\Delta\}) = \operatorname{reg}_E(K\{\Delta\})$.

A natural question to ask is which triples of numbers (t,s,r) with $r \geq s-t \geq 0$ can occur such that there exists a squarefree S-module N with $\operatorname{depth}_E(N_E) = t$, $\operatorname{depth}_S(N) = s$ and $\operatorname{reg}_E(N_E) = r$. We can answer this issue by showing that all triples of numbers are possible.

Example 4.3.10. Let (t,s,r) be as above and let $n = r+t+3$. Let I be the ideal in $K[x_1,\dots,x_n]$ generated by the monomials

$$\begin{aligned} &x_1 x_2 \cdots x_{s-t} x_i && \text{for } i = s-t+1,\dots,n-t, \\ &x_1 x_2 \cdots \widehat{x_j} \cdots x_{r+2} && \text{for } j = 1,\dots,s-t \end{aligned}$$

(here $\widehat{x_j}$ means that x_j is omitted). Finally set $J = I_E$, i.e., J is the ideal in $K\langle e_1,\dots,e_n\rangle$ generated by the above monomials with x replaced by e. By construction I is squarefree stable and J is stable. Thus we can compute the invariants of S/I and E/J with the known formulas. First the depth over S of S/I is, as in the preceding Example 4.3.9,

$$\begin{aligned} \operatorname{depth}_S(S/I) &= n - \max\{\max(u) - \deg(u) : u \in G(I)\} - 1 \\ &= n - 1 - \max\{n - t - (s-t+1), r+2-(r+1)\} \\ &= n - 1 - \max\{n-s-1, 1\} = n - 1 - (n-s-1) = s, \end{aligned}$$

where we use $n = r+t+3 \geq s+3$. The depth of E/J over E is, computed with Lemma 3.1.4 and Theorem 4.1.2,

$$\begin{aligned} \operatorname{depth}_E(E/J) &= n - \max\{\max(u) : u \in G(J)\} = n - \max\{n-t, r+2\} \\ &= n - \max\{r+3, r+2\} = t. \end{aligned}$$

Finally the regularity of E/J is, by Lemma 3.1.3,

$$\operatorname{reg}_E(E/J) = \max\{\deg(u) : u \in G(J)\} - 1 = \max\{s-t+1, r+1\} - 1 = r+1-1 = r$$

where we use $r \geq s-t$.

Open problem 4.3.11.

(i) *Is there a characterisation of the simplicial complexes* Δ *with* $\mathrm{depth}_S(K[\Delta]) - \mathrm{depth}_E(K\{\Delta\}) = \mathrm{reg}_S(K[\Delta])$*?*

(ii) *Is there a characterisation of the simplicial complexes* Δ *with* $\mathrm{depth}_S(K[\Delta]) - \mathrm{depth}_E(K\{\Delta\}) = 0$*?*

4.4. Annihilator numbers over the exterior algebra

In this section the so-called exterior generic annihilator numbers of an E-module are introduced. They are the exterior analogue of the generic annihilator numbers of a module over a polynomial ring introduced first by Trung in [**59**] and studied later by Conca, Herzog and Hibi (see [**12**]).

Recall that $H^j(M,v) = \left(\frac{0:_M v}{vM}\right)_j$ is the homology of the complex

$$(M,v): \qquad \ldots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \ldots$$

for a module $M \in \mathscr{M}$ and a linear form $v \in E_1$.

Definition 4.4.1. Let $v_1,\ldots,v_n$ be a basis of E_1 and let $M \in \mathscr{M}$. The numbers

$$\alpha_{i,j}(v_1,\ldots,v_n;M) = \dim_K H^j(M/(v_1,\ldots,v_{i-1})M, v_i)$$

for $j \in \mathbb{Z}$ and $1 \leq i \leq n$ are called the *exterior annihilator numbers* of M with respect to $v_1,\ldots,v_n$.

This definition depends on the chosen basis. The following theorem justifies the definition of generic annihilator numbers which are independent of the basis.

Theorem 4.4.2. *Let* $J \subset E$ *be a graded ideal and* $|K| = \infty$*. Then there exists a non-empty Zariski-open set* $U \subseteq \mathrm{GL}_n(K)$ *such that*

$$\alpha_{i,j}(v_1,\ldots,v_n;E/J) = \alpha_{i,j}(e_n,\ldots,e_1;E/\mathrm{gin}(J))$$

for all $\gamma \in U$*,* $v_i = \gamma(e_{n-i+1})$*,* $i = 1,\ldots,n$*.*

PROOF. Let $U' = \{\varphi \in \mathrm{GL}_n(K) \ : \ \mathrm{in}(\varphi(J)) = \mathrm{gin}(J)\}$ be a non-empty Zariski-open set of linear transformations that can be used to compute the generic initial ideal of J. Set $U = \{\varphi^{-1} \ : \ \varphi \in U'\}$, which is Zariski-open and non-empty as well. Let $\gamma = \varphi^{-1} \in U$ and set $v_i = \gamma(e_{n-i+1})$ for $1 \leq i \leq n$, i.e., $\varphi(v_i) = e_{n-i+1}$. As φ is an automorphism, $E/(J+(v_1,\ldots,v_i))$ and $E/(\varphi(J)+(e_n,\ldots,e_{n-i+1}))$ have the same Hilbert function. [**2**, Proposition 5.1] implies that

$$\mathrm{in}(\varphi(J)+(e_n,\ldots,e_{n-i+1})) = \mathrm{gin}(J)+(e_n,\ldots,e_{n-i+1}),$$

because we use the revlex order. Therefore also the two quotient rings $E/(J+(v_1,\ldots,v_i))$ and $E/(\mathrm{gin}(J)+(e_n,\ldots,e_{n-i+1}))$ have the same Hilbert function. The sequences

$$0 \longrightarrow H^j(E/(J+(v_1,\ldots,v_{i-1})),v_i) \longrightarrow (E/(J+(v_1,\ldots,v_i)))_j$$
$$\xrightarrow{\cdot v_i} (E/(J+(v_1,\ldots,v_{i-1})))_{j+1} \longrightarrow (E/(J+(v_1,\ldots,v_i)))_{j+1} \longrightarrow 0$$

and

$$0 \longrightarrow H^j(E/(\text{gin}(J)+(e_n,\dots,e_{n-i+2})),e_{n-i+1}) \longrightarrow (E/(\text{gin}(J)+(e_n,\dots,e_{n-i+1})))_j$$
$$\stackrel{\cdot e_{n-i+1}}{\longrightarrow} (E/(\text{gin}(J)+(e_n,\dots,e_{n-i+2})))_{j+1} \longrightarrow (E/(\text{gin}(J)+(e_n,\dots,e_{n-i+1})))_{j+1} \longrightarrow 0$$

are exact sequences of K-vector spaces. The vector space dimensions of the three latter vector spaces in the two sequences coincide, hence it holds that

$$\begin{aligned}\alpha_{i,j}(v_1,\dots,v_n;E/J) &= \dim_K H^j(E/(J+(v_1,\dots,v_{i-1})),v_i)\\ &= \dim_K H^j(E/(\text{gin}(J)+(e_n,\dots,e_{n-i+2})),e_{n-i+1})\\ &= \alpha_{i,j}(e_n,\dots,e_1;E/\text{gin}(J)).\end{aligned}$$

□

As mentioned above, now the following definition makes sense.

Definition 4.4.3. Let $J \subset E$ be a graded ideal and $|K| = \infty$. The numbers

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(e_n,\dots,e_1;E/\text{gin}(J))$$

for $j \in \mathbb{Z}$ and $1 \le i \le n$ are called the *exterior generic annihilator numbers* of E/J.

Remark 4.4.4. Let $J \subset E$ be a graded ideal and $|K| = \infty$. By definition of the $\alpha_{i,j}$ and the fact that $\text{gin}(\text{gin}(J)) = \text{gin}(J)$, it holds that

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(E/\text{gin}(J)).$$

Set $\alpha_i(E/J) = \sum_{j\in\mathbb{Z}} \alpha_{i,j}(E/J)$ and $1 \le r \le n$. Then $\alpha_i(E/J) = 0$ for all $i \le r$ if and only if $r \le \text{depth}_E(E/J)$. This is an easy consequence of the fact that $e_n,\dots,e_{n-i+1}$ is a regular sequence on $E/\text{gin}(J)$ if and only if $i \le \text{depth}_E(E/\text{gin}(J)) = \text{depth}_E(E/J)$ (see Corollary 4.2.2).

We have already given a proof of the fact that being a regular sequence is a Zariski-open condition in Proposition 4.1.7. But it is also possible to prove this for quotient rings using the generic annihilator numbers which gives a shorter proof.

Proposition 4.4.5. *Let $|K| = \infty$, $J \subset E$ be a graded ideal and* $\text{depth}_E(E/J) = t > 0$. *Then there exists a non-empty Zariski-open set $U \subseteq GL_n(K)$ such that $v_1,\dots,v_t$ is an M-regular sequence for all $\gamma \in U$, $v_i = \gamma(e_{n-i+1})$, $i = 1,\dots,n$.*

PROOF. Let U be the non-empty Zariski-open set as in Theorem 4.4.2, i.e., such that the annihilator numbers with respect to sequences $v_1,\dots,v_n$ induced by U equal the generic annihilator numbers. Following Corollary 4.2.2 $e_n,\dots,e_{n-t+1}$ is a regular sequence on $E/\text{gin}(J)$ and therefore

$$\alpha_{i,j}(v_1,\dots,v_n;E/J) = \alpha_{i,j}(e_n,\dots,e_1;E/\text{gin}(J)) = 0$$

for $i \le t$. Thus $v_1,\dots,v_t$ is regular on E/J. □

The numbers $\alpha_{i,j}$ count residue classes of monomials in $E/\text{gin}(J)$ with certain properties.

Theorem 4.4.6. *Let $J \subset E$ be a graded ideal and $|K| = \infty$. Then*

$$\alpha_{i,j}(E/J) = |\{\overline{e_F} \in E/\text{gin}(J) \,:\, \deg \overline{e_F} = j,\ \max F \le n-i,\ \overline{e_F} \neq 0,\ \overline{e_{n-i+1}e_F} = 0\}|.$$

Here $\overline{e_F}$ denotes the residue class of $e_F \in E$ in $E/\text{gin}(J)$.

PROOF. Since $\alpha_{i,j}(E/J) = \alpha_{i,j}(E/\operatorname{gin}(J))$ we may assume that $J = \operatorname{gin}(J)$. Then $\alpha_{i,j}(E/J)$ can be computed using the sequence $e_n,\dots,e_1$, i.e.,

$$\begin{aligned}\alpha_{i,j}(E/J) &= \dim_K H^j\big(E/(J+(e_n,\dots,e_{n-i+2})), e_{n-i+1}\big)\\ &= \dim_K \left(\frac{(J+(e_n,\dots,e_{n-i+2})) : e_{n-i+1}}{J+(e_n,\dots,e_{n-i+1})}\right)_j.\end{aligned}$$

Thus $\alpha_{i,j}(E/J)$ is the number of monomials of degree j in $(J+(e_n,\dots,e_{n-i+2})) : e_{n-i+1}$, that are not contained in $J+(e_n,\dots,e_{n-i+1})$. We show that this number is exactly the cardinality of the set

$$\{\overline{e_F} \in E/\operatorname{gin}(J) \;:\; \deg \overline{e_F} = j,\ \max F \leq n-i,\ \overline{e_F} \neq 0,\ \overline{e_{n-i+1}e_F} = 0\}.$$

To this end let $e_F \in (J+(e_n,\dots,e_{n-i+2})) : e_{n-i+1}$ be of degree j. Then $e_{n-i+1}e_F \in J+(e_n,\dots,e_{n-i+2})$. Since J is a monomial ideal, either $e_{n-i+1}e_F \in (e_n,\dots,e_{n-i+2})$ or $e_{n-i+1}e_F \in J$. In the first case it follows that $e_F \in (e_n,\dots,e_{n-i+1})$, because e_{n-i+1} is regular on E, so that we do not need to count it. In the second case, $e_F \notin J+(e_n,\dots,e_{n-i+1})$ is equivalent to $e_F \notin J$ and $e_F \notin (e_n,\dots,e_{n-i+1})$, or equivalently $\overline{e_F} \neq 0$ and $\max F \leq n-i$. □

In the special case of the face ring of a simplicial complex Δ, Theorem 4.4.6 yields the following combinatorial description of the exterior generic annihilator numbers.

Corollary 4.4.7. *Let Δ be a simplicial complex, Δ^e be its exterior shifting and $|K| = \infty$. Then*

$$\alpha_{i,j}(K\{\Delta\}) = |\{F \in \Delta^e \;:\; |F| = j,\ F \subseteq [n-i],\ F \cup \{n-i+1\} \notin \Delta^e\}|.$$

Using this description we are able to express the Betti numbers of the Stanley-Reisner ring of the exterior shifting of a simplicial complex as a linear combination of certain generic annihilator numbers. One of the key ingredients for the proof is the Eliahou-Kervaire formula for the Betti numbers of squarefree stable ideals in a polynomial ring. We recall this formula before giving the result.

Proposition 4.4.8. [**4**, Corollary 2.3] *Let $0 \neq I \subset S$ be a squarefree stable ideal. Then*

$$\beta_{i,i+j}(S/I) = \sum_{u \in G(I)_{j+1}} \binom{\max(u)-j+1}{i-1}.$$

For a simplicial complex Δ, as Δ^e is shifted, the Stanley-Reisner ideal I_{Δ^e} of the exterior shifting is a squarefree stable ideal. We need this property in the following.

Our result expresses the annihilator numbers in terms of the minimal generators of J_{Δ^e} in the exterior algebra.

Proposition 4.4.9. *Let Δ be a simplicial complex, Δ^e be its exterior shifting and $|K| = \infty$. Then*

$$\alpha_{l,j}(K\{\Delta\}) = |\{u \in G(J_{\Delta^e})_{j+1} \;:\; \max(u) = n-l+1\}|.$$

In particular,

$$\beta^S_{i,i+j}(K[\Delta^e]) = \sum_{l=1}^{n} \binom{n-l-j+2}{i-1} \alpha_{l,j}(K\{\Delta\}).$$

PROOF. As shown in Corollary 4.4.7 the number $\alpha_{l,j}(E/J_\Delta)$ counts the cardinality of the set

$$\mathscr{A} = \{A \in \Delta^e \ : \ |A| = j,\ A \subseteq [n-l],\ A \cup \{n-l+1\} \notin \Delta^e\}.$$

The minimal generators of J_{Δ^e} are the monomials corresponding to minimal non-faces of Δ^e. Thus the elements of $\{u \in G(J_{\Delta^e})_{j+1} \ : \ \max(u) = n-l+1\}$ are the monomials e_B such that B lies in the set

$$\mathscr{B} = \{B \notin \Delta^e \ : \ |B| = j+1,\ \max(B) = n-l+1,\ \partial(B) \subseteq \Delta^e\},$$

where $\partial(B) = \{F \subset B \ : \ F \neq B\}$ denotes the boundary of B.

We show that there is a one-to-one correspondence between $\mathscr{A}$ and $\mathscr{B}$. Let $B \in \mathscr{B}$. Then $n-l+1 \in B$ and $A = B \setminus \{n-l+1\}$ is an element in $\mathscr{A}$. Conversely if $A \in \mathscr{A}$ then $B = A \cup \{n-l+1\} \in \mathscr{B}$. The only non-trivial point here is to see that the boundary of B is contained in Δ^e. So let $F \in \partial B$, i.e., $F \subseteq B$ and there exists $j \in B \setminus F$. If $j = n-l+1$ then F is a subset of $A \in \Delta^e$ and thus $F \in \Delta^e$. If $j \neq n-l+1$ then $j < n-l+1$ since $\max B = n-l+1$. If we exchange j and $n-l+1$ we are again in the first case and obtain $F \setminus \{n-l+1\} \cup \{j\} \in \Delta^e$. As Δ^e is shifted, we can substitute j back by $n-l+1$ and obtain again a face of Δ^e, namely F.

The statement about the Betti numbers then follows from the Eliahou-Kervaire formula for squarefree stable ideals (Proposition 4.4.8) and the simple fact that the minimal generators of I_Δ are in a one-to-one correspondence to the minimal generators of J_Δ. □

The exterior generic annihilator numbers can also be used to compute the Betti numbers over the exterior algebra. This is analogous to a result over the polynomial ring, see [**12**, Corollary 1.2]. To this end we use the *Cartan-Betti numbers* introduced by Nagel, Römer and Vinai in [**46**].

Definition 4.4.10. Let $J \subset E$ be a graded ideal, $|K| = \infty$ and let $v_1, \dots, v_n$ be a basis of E_1. We set

$$\beta_{i,j,r}(v_1, \dots, v_n; E/J) = \dim_K H_i(v_1, \dots, v_r; E/J)_j,$$

where $H_i(v_1, \dots, v_n; E/J)$ denotes the i-th Cartan homology.

Nagel, Römer and Vinai remark that there exists a non-empty Zariski-open set W such that the numbers $\beta_{i,j,r}$ are constant on it. Therefore they define:

Definition 4.4.11. Let $J \subset E$ be a graded ideal, $|K| = \infty$ and let $v_1, \dots, v_n$ be a sequence induced by W, for W as above. The numbers

$$\beta_{i,j,r}(E/J) = \beta_{i,j,r}(v_1, \dots, v_n; E/J)$$

are called the *Cartan-Betti numbers* of E/J.

For $r = n$, we obtain from Proposition 2.3.3 that the Cartan-Betti numbers of E/J are the usual exterior graded Betti numbers of E/J, i.e., $\beta_{i,j,n}(E/J) = \beta^E_{i,j}(E/J)$. We formulate and prove the following result using the generic annihilator numbers. Plugging in the description of the generic annihilator numbers in terms of the minimal generators of J_{Δ^e}, our result is the same as [**46**, Theorem 2.4(i)] which is a direct consequence of the construction of the Cartan homology for stable ideals in [**3**, Proposition 3.1].

Theorem 4.4.12. *Let $J \subset E$ be a graded ideal, $|K| = \infty$ and $v_1, \dots, v_n$ a basis of E_1. Then*

$$\beta_{i,i+j,r}(v_1,\dots,v_n;E/J) \leq \sum_{k=1}^{r} \binom{r+i-k-1}{i-1} \alpha_{k,j}(v_1,\dots,v_n;E/J) \quad \text{for } i \geq 1, j \geq 0.$$

PROOF. Set $\beta_{i,i+j,r}(v_1,\dots,v_n;E/J) = \beta_{i,i+j,r}$, $\alpha_{i,j} = \alpha_{i,j}(v_1,\dots,v_n;E/J)$ and

$$\tilde{H}_0(i) = \ker\Big(E/\left(J+(v_1,\dots,v_i)\right) \xrightarrow{\cdot v_i} E/\left(J+(v_1,\dots,v_{i-1})\right)\Big)$$

for $1 \leq i \leq n$. Then $\tilde{H}_0(i)$ is a graded E-module and the K-vector space dimension of the j-th graded piece equals $\alpha_{i,j}$. Recall from Section 4.1 that $\tilde{H}_0(i)$ is constructed such that there are long exact Cartan homology sequences

$$\dots \longrightarrow H_2(i) \longrightarrow H_1(i)(-1) \longrightarrow H_1(i-1) \longrightarrow H_1(i) \longrightarrow \tilde{H}_0(i)(-1) \longrightarrow 0.$$

Thus for $i = 1$ and $r = 1$ we obtain from the long exact Cartan homology sequence the following exact sequence

$$H_1(1)(-1)_{j+1} \to H_1(0)_{j+1} \to H_1(1)_{j+1} \to \tilde{H}_0(1)(-1)_{j+1} \to 0.$$

Since $H_i(0) = 0$ for $i \geq 1$ this yields

$$\beta_{1,j+1,1} = \alpha_{1,j}.$$

For $r \geq 1$ we have the exact sequence

$$H_1(r+1)(-1)_{j+1} \to H_1(r)_{j+1} \to H_1(r+1)_{j+1} \to \tilde{H}_0(r+1)(-1)_{j+1} \to 0.$$

From this sequence we conclude by induction hypothesis on r

$$\begin{aligned} \beta_{1,j+1,r+1} &\leq \alpha_{r+1,j} + \beta_{1,j+1,r} \\ &\leq \alpha_{r+1,j} + \sum_{k=1}^{r} \binom{r-k}{0} \alpha_{k,j} \\ &= \sum_{k=1}^{r+1} \alpha_{k,j}. \end{aligned}$$

Now let $i > 1$. For $r = 1$ there is the exact sequence

$$H_i(0)_{i+j} \to H_i(1)_{i+j} \to H_{i-1}(1)(-1)_{i+j} \to H_{i-1}(0)_{i+j}.$$

The outer spaces in the sequence are zero, hence

$$\beta_{i,i+j,1} = \beta_{i-1,i+j-1,1} \leq \alpha_{1,j}$$

by induction hypothesis on i.

Let now $r \geq 1$. There is the exact sequence

$$H_i(r)_{i+j} \to H_i(r+1)_{i+j} \to H_{i-1}(r+1)(-1)_{i+j} \to H_{i-1}(r)_{i+j}.$$

We conclude by induction hypothesis on r and i

$$
\begin{aligned}
\beta_{i,i+j,r+1} &\le \beta_{i,i+j,r} + \beta_{i-1,i-1+j,r+1} \\
&\le \sum_{k=1}^{r} \binom{r+i-k-1}{i-1} \alpha_{k,j} + \sum_{k=1}^{r+1} \binom{r+1+i-1-k-1}{i-2} \alpha_{k,j} \\
&= \sum_{k=1}^{r} \left(\binom{r+i-k-1}{i-1} + \binom{r+i-k-1}{i-2} \right) \alpha_{k,j} + \binom{i-2}{i-2} \alpha_{r+1,j} \\
&= \sum_{k=1}^{r+1} \binom{r+i-k}{i-1} \alpha_{k,j}.
\end{aligned}
$$

□

We formulate the above theorem for generic sequences of linear forms.

Corollary 4.4.13. *Let $J \subset E$ be a graded ideal and $|K| = \infty$. Then*

$$
\beta_{i,i+j,r}(E/J) \le \sum_{k=1}^{r} \binom{r+i-k-1}{i-1} \alpha_{k,j}(E/J) \quad \textit{for } i \ge 1, j \ge 0,
$$

and equality holds for all $i \ge 1$ and $1 \le r \le n$ if and only if J is componentwise linear (see Section 5.3 for a definition).

PROOF. Let $v_1, \dots, v_n$ be a sequence of linear forms that can be used to compute the exterior generic annihilator numbers and the Cartan-Betti numbers of E/J as well. Such a sequence exists as both conditions are Zariski-open and the intersection of two non-empty Zariski-open sets remains Zariski-open and non-empty. The inequality then follows from the above Theorem 4.4.12 with this sequence.

The inequalities in the proof of Theorem 4.4.12 are all equalities if and only if the long exact sequence is split exact. In this case the sequence $v_1, \dots, v_n$ is called a *proper sequence* for E/J. In [**46**, Theorem 2.10] it is shown that this is the case for a generic sequence if and only if J is a componentwise linear ideal. □

A natural question to ask is whether the exterior generic annihilator numbers play a special role among the exterior annihilator numbers of E/J with respect to an arbitrary sequence. Herzog posed the question if they are the minimal ones among all the annihilator numbers. In the attempt of proving this conjecture it turned out to be wrong. For the sake of completeness we first state the conjecture.

Question 4.4.14. *Let $J \subset E$ be a graded ideal and $|K| = \infty$. For any basis $v_1, \dots, v_n$ of E_1 it holds that*

$$
\alpha_{i,j}(E/J) \le \alpha_{i,j}(v_1, \dots, v_n; E/J)
$$

for $1 \le i \le n$ and $j \ge 0$.

Thus, our aim would be to prove that the annihilator numbers are minimal on a non-empty Zariski-open set. For $i = 1$ this is known to be true. Just take the non-empty Zariski-open set such that the ranks of the matrices of the maps of the complex

$$
(M, v): \qquad \dots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \dots
$$

are maximal (note that $M_j = 0$ for almost all j). The difficulty for a proof for $i > 1$ is that in the calculation of the annihilator numbers appear alternating sums. The following example gives a negative answer to Question 4.4.14.

Example 4.4.15. Let $1 \leq i, j \leq n$, $i + j \leq n$ and let $J = (e_{l_1} \cdots e_{l_{j+1}} \ :\ 1 \leq l_1 < l_2 < \ldots < l_{j+1} \leq n - i + 1) \subseteq E$ be a graded ideal. By construction, J is strongly stable and therefore $\mathrm{gin}(J) = J$. Then

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(e_n, \ldots, e_1; E/\mathrm{gin}(J)) = \alpha_{i,j}(e_n, \ldots, e_1; E/J),$$

i.e., we can use the sequence $e_n, \ldots, e_1$ to compute the generic annihilator numbers of E/J.

From the exact sequence

$$0 \longrightarrow H^j(E/(J + (e_n, \ldots, e_{n-i+2})), e_{n-i+1}) \longrightarrow (E/\,(J + (e_n, \ldots, e_{n-i+1})))_j$$
$$\stackrel{\cdot e_{n-i+1}}{\longrightarrow} (E/(J + (e_n, \ldots, e_{n-i+2})))_{j+1} \longrightarrow (E/(J + (e_n, \ldots, e_{n-i+1})))_{j+1} \longrightarrow 0$$

we deduce

$$\begin{aligned}\alpha_{i,j} = &\dim_K (E/(J + (e_n, \ldots, e_{n-i+1})))_j \\ &- \dim_K (E/(J + (e_n, \ldots, e_{n-i+2})))_{j+1} + \dim_K (E/(J + (e_n, \ldots, e_{n-i+1})))_{j+1}).\end{aligned}$$

In the following we consider the sequence $\mathbf{e} = e_n, \ldots, e_{n-i+3}, e_{n-i+1}, e_{n-i+2}, e_{n-i}, \ldots, e_1$ where e_{n-i+2} and e_{n-i+1} changed positions. We compute the exterior annihilator numbers of E/J with respect to this sequence. As before, we have the exact sequence

$$0 \longrightarrow H^j(E/(J + (e_n, \ldots, e_{n-i+3}, e_{n-i+1})), e_{n-i+2}) \longrightarrow (E/\,(J + (e_n, \ldots, e_{n-i+1})))_j$$
$$\stackrel{\cdot e_{n-i+2}}{\longrightarrow} (E/(J + (e_n, \ldots, e_{n-i+3}, e_{n-i+1})))_{j+1} \longrightarrow (E/(J + (e_n, \ldots, e_{n-i+1})))_{j+1} \longrightarrow 0$$

which leads to

$$\begin{aligned}\alpha_{i,j}(\mathbf{e}; E/J) = &\dim_K (E/(J + (e_n, \ldots, e_{n-i+1})))_j \\ &- \dim_K (E/(J + (e_n, \ldots, e_{n-i+3}, e_{n-i+1})))_{j+1} \\ &+ \dim_K (E/(J + (e_n, \ldots, e_{n-i+1})))_{j+1}).\end{aligned}$$

Our aim is to show that $\alpha_{i,j}(E/J) > \alpha_{i,j}(\mathbf{e}; E/J)$. We therefore need to show that

$$\dim_K (E/(J + (e_{n-i+1}, \ldots, e_{n-i+3}, e_{n-i+1})))_{j+1} > \dim_K (E/(J + (e_n, \ldots, e_{n-i+3}, e_{n-i+2})))_{j+1}.$$

Let $u = e_{l_1} \cdots e_{l_{j+1}} \in E_{j+1}$ with $l_1 < \ldots < l_{j+1}$. If $l_{j+1} > n - i + 1$, it already holds that $u \in (e_n, \ldots, e_{n-i+3}, e_{n-i+2})_{j+1}$. If $l_{j+1} \leq n - i + 1$, we have $u \in J$. Thus, $u \in (J + (e_n, \ldots, e_{n-i+3}, e_{n-i+2}))_{j+1}$ in any case and therefore

$$\dim_K (E/(J + (e_n, \ldots, e_{n-i+3}, e_{n-i+2})))_{j+1} = 0.$$

Consider now $u' = e_{n-i+2} e_{n-i} \cdots e_{n-i-j+1} \in E_{j+1}$. By definition, it holds that $u' \notin J$ and $u' \notin (e_n, \ldots, e_{n-i+3}, e_{n-i+1})$. Since J is a monomial ideal this implies $u' \notin (J + (e_n, \ldots, e_{n-i+3}, e_{n-i+1}))_{j+1}$. We therefore get

$$\dim_K (E/(J + (e_{n-i+1}, \ldots, e_{n-i+3}, e_{n-i+1})))_{j+1} > 0.$$

This finally shows

$$\alpha_{i,j}(E/J) > \alpha_{i,j}(\mathbf{e}; E/J)$$

and we conclude that the answer to Question 4.4.14 is negative. We also compute the annihilator numbers one step before w.r.t. this two sequences, i.e., we compute $\alpha_{i-1,j}(E/J)$ and $\alpha_{i-1,j}(\mathbf{e};E/J)$ to show that this numbers are related to each other the other way round, i.e., we have

$$\alpha_{i-1,j}(E/J) < \alpha_{i-1,j}(\mathbf{e};E/J). \tag{2}$$

This suggests that, to have a chance to become smaller than the generic numbers, the annihilator numbers with respect to **e** have to become “worse”, i.e., greater, in some other position. Similar to the i-th annihilator numbers of E/J we can compute the $(i-1)$-st annihilator numbers using the exact sequence. We therefore get

$$\begin{aligned}\alpha_{i-1,j}(E/J) = &\dim_K\left(E/(J+(e_n,\dots,e_{n-i+2}))\right)_j\\ &-\dim_K\left(E/(J+(e_n,\dots,e_{n-i+3}))\right)_{j+1}\\ &+\dim_K\left(E/(J+(e_n,\dots,e_{n-i+2}))\right)_{j+1})\end{aligned}$$

for the $(i-1)$-st generic annihilator of E/J in degree j. In the same way, we obtain

$$\begin{aligned}\alpha_{i-1,j}(\mathbf{e};E/J) = &\dim_K\left(E/(J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))\right)_j\\ &-\dim_K\left(E/(J+(e_n,\dots,e_{n-i+3}))\right)_{j+1}\\ &+\dim_K\left(E/(J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))\right)_{j+1})\end{aligned}$$

for the $(i-1)$-st exterior annihilator number of E/J in degree j with respect to the sequence **e**. Since J is generated by monomials of degree strictly larger than j it holds that

$$\dim_K\left(E/(J+(e_n,\dots,e_{n-i+2}))\right)_j = \dim_K\left(E/(e_n,\dots,e_{n-i+2})\right)_j$$

and

$$\dim_K\left(E/(J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))\right)_j = \dim_K\left(E/(e_n,\dots,e_{n-i+3},e_{n-i+1})\right)_j.$$

Since

$$\dim_K\left(E/(J+(e_n,\dots,e_{n-i+2}))\right)_j = \dim_K\left(E/(J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))\right)_j,$$

in order to show (2) we need to prove that

$$\dim_K\left(E/(J+(e_n,\dots,e_{n-i+2}))\right)_{j+1} < \dim_K\left(E/(J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))\right)_{j+1}.$$

This follows from

$$(J+(e_n,\dots,e_{n-i+2}))_{j+1} \supsetneq (J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))_{j+1}. \tag{3}$$

To show (3) let $u = e_{l_1}\cdots e_{l_{j+1}} \in (J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))_{j+1}$ with $l_1 < \ldots < l_{j+1}$. If $l_{j+1} \le n-i+1$ it follows that $u \in J_{j+1}$. If $l_{j+1} > n-i+1$ it already holds that $u \in (e_n,\dots,e_{n-i+2})$ and thus

$$u \in (J+(e_n,\dots,e_{n-i+2}))_{j+1}.$$

Since

$$e_{n-i+2}e_{n-i}\cdots e_{n-i-j+1} \in (J+(e_n,\dots,e_{n-i+2}))_{j+1}$$

but

$$e_{n-i+2}e_{n-i}\cdots e_{n-i-j+1} \notin (J+(e_n,\dots,e_{n-i+3},e_{n-i+1}))_{j+1}$$

we obtain (3).

After slight modifications to Example 4.4.15 we also get a counterexample of the question whether the generic annihilator numbers over a polynomial ring are the minimal ones among all the annihilator numbers with respect to a sequence. See [**39**, Example 4.5] for more details.

Example 4.4.15 in particular shows that changing the order of the elements of a sequence might change the annihilator numbers. However, when taking the sequence from a certain non-empty Zariski-open set the order of the elements does not matter.

Theorem 4.4.16. *Let $J \subset E$ be a graded ideal and $|K| = \infty$. If K has enough algebraically independent elements over its base field, then there exists a non-empty Zariski-open set $V \subseteq GL_n(K)$ such that*

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(\gamma(e_{\sigma(1)}), \dots, \gamma(e_{\sigma(n)}); E/J)$$

for all $\gamma \in V$ and all $\sigma \in S_n$, where S_n denotes the symmetric group on $[n]$.

PROOF. If K has enough algebraically independent elements over its base field, then the subset $U \subseteq GL_n(K)$ consisting of matrices with algebraically independent entries is non-empty and Zariski-open. Furthermore, it is invariant under row permutations, column permutations and also under taking inverses. Let U' be a non-empty Zariski-open set of linear transformations that can be used to compute the generic initial ideal. We show that $U \cap U' = V$ is as required.

First of all V is non-empty and Zariski-open since it is the intersection of two such sets. Consider $\gamma \in V$, $\sigma \in S_n$ and set $\gamma_i = \gamma(e_{n-i+1})$. The proof of Theorem 4.4.2 shows that it is enough to prove

$$\dim_K \left(E/\left(\text{gin}(J) + (e_n, \dots, e_{n-i+1})\right)\right)_j = \dim_K \left(E/\left(J + (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(i)})\right)\right)_j$$

for $i = 1, \dots, n$ and $j \in \mathbb{Z}$.

Since $\text{gin}(J) = \text{in}(\gamma^{-1}(J))$ it holds that

$$\dim_K \left(E/\left(\text{gin}(J) + (e_n, \dots, e_{n-i+1})\right)\right)_j = \dim_K \left(E/\left(J + (\gamma_1, \dots, \gamma_i)\right)\right)_j.$$

The permutation lemma on page 288 in [**7**] (it is only formulated for monomial ideals, but the proof is valid also for arbitrary graded ideals) implies that

$$\dim_K \left(E/\left(J + (\gamma_1, \dots, \gamma_i)\right)\right)_j = \dim_K \left(E/\left(J + (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(i)})\right)\right)_j.$$

Both equations together conclude the proof. □

Open problem 4.4.17. *Is it possible to drop the assumption on K in the above Theorem 4.4.16?*

CHAPTER 5

Properties of graded E-modules

We study several module-theoretic properties in this chapter. If possible, also the dual notion is formulated and investigated. An important property is having a linear injective resolution, which is studied in the first section and corresponds to the Cohen-Macaulay property of a module over a polynomial ring. Most of the results in this section appeared in [**40**]. We note some results on face rings of Gorenstein simplicial complexes in Section 5.2. Furthermore, we study componentwise linear and componentwise injective linear modules in Section 5.3 and modules with linear or pure decomposable quotients in Section 5.4. The last section, Section 5.5, is devoted to strongly decomposable modules, a more combinatorial property.

5.1. Modules with linear injective resolutions

Modules with linear injective resolutions behave nicer in many cases than arbitrary modules. Recall from Section 2.2 that a module $M \in \mathscr{M}$ has a d-linear injective resolution if its minimal graded injective resolution is of the form

$$0 \longrightarrow M \longrightarrow E(n-d)^{\mu_0} \longrightarrow E(n-d+1)^{\mu_1} \longrightarrow \dots$$

or, equivalently, if $\mu_{i,j-i}(M) = 0$ for $j \neq d$.

The following example shows that it is in some sense the exterior counterpart to the notion of a Cohen-Macaulay module.

Example 5.1.1. Let Δ be a simplicial complex on $[n]$. Then Δ is Cohen-Macaulay if and only if the face ideal J_{Δ^*} of the Alexander dual Δ^* has a linear projective resolution. This is the Eagon-Reiner theorem [**17**, Theorem 3].

It is equivalent to say that the face ring $K\{\Delta\} = E/J_\Delta$ has a linear injective resolution as it is the dual of J_{Δ^*} by Lemma 1.2.1.

This equivalence could be derived directly from Corollary 1.2.2 and Proposition 1.2.3, see also [**2**, Corollary 7.6].

This is even true in a more general context. Römer proves in [**53**, Corollary 3.7] that a squarefree S-module N is Cohen-Macaulay if and only if the corresponding sqaurefree E-module N_E has a linear injective resolution.

Lemma 5.1.2. *Let $J \subset E$ be a graded ideal and v be E/J-regular Then E/J has a d-linear injective resolution over E if and only if $E/(J+(v))$ has a $(d-1)$-linear injective resolution over $E/(v)$.*

PROOF. Proposition 4.1.15 shows that E/J has a d-linear injective resolution over E if and only if $v(E/J)$ has one over $E/(v)$. The latter module is isomorphic to the $E/(v)$-module

$$v(E/J) = \big(J+(v)\big)/J \cong \big(E/(v)\big)/\big(J+(v)/(v)\big)(-1)$$

where the isomorphism is induced by the homomorphism

$$E/(v) \to \big(J+(v)/J\big)(+1),\ a+(v) \mapsto av+J.$$

Thus $v(E/J)$ has a d-linear injective resolution if and only if $\big(E/(v)\big)/\big(J+(v)/(v)\big)$ has a $(d-1)$-linear injective resolution. □

Recall from Section 1.1 that $H(M,t) = \sum_{i\in\mathbb{Z}} \dim_K M_i t^i$ denotes the Hilbert series of a graded E-module M. Analogously to the well-known Hilbert-Serre theorem (see, e.g., [**11**, Proposition 4.4.1]) we have the following result.

Theorem 5.1.3. *Let $|K| = \infty$ and $0 \neq J \subset E$ be a graded ideal with $\operatorname{depth}_E(E/J) = s$. Let E/J have a linear injective resolution. Then there exists a polynomial $Q(t) \in \mathbb{Z}[t]$ with non-negative coefficients such that the Hilbert series of E/J has the form*

$$H(E/J,t) = Q(t)\cdot(1+t)^s \text{ with } Q(-1)\neq 0.$$

Note that it is not possible to generalise the equation in this form to the case of arbitrary quotient rings. The ideal $(e_{12}, e_{13}, e_{14}, e_{234})$ provides a counterexample.

PROOF. Let $M = E/J$. First of all we show that if v is M-regular, then

$$H(M,t) = (1+t)H(M/vM,t). \tag{4}$$

We have the exact sequence

$$0 \longrightarrow vM \longrightarrow M \longrightarrow M/vM \longrightarrow 0$$

which implies

$$H(vM,t) = H(M,t) - H(M/vM,t). \tag{5}$$

As v is M-regular the sequence

$$0 \longrightarrow vM(-1) \longrightarrow M(-1) \xrightarrow{\cdot v} M \longrightarrow M/vM \longrightarrow 0$$

is exact and gives

$$(1-t)H(M,t) = H(M/vM,t) - tH(vM,t). \tag{6}$$

Equations (5) and (6) together show (4).

Thus if $v_1,\dots,v_s$ is a maximal E/J-regular sequence, we obtain inductively

$$H(E/J,t) = (1+t)^s H(E/J+(v_1,\dots,v_s),t).$$

The Hilbert series of $E/(J+(v_1,\dots,v_s))$ is a polynomial with non-negative coefficients and $\operatorname{depth}_E(E/(J+(v_1,\dots,v_s))) = 0$. We claim that the polynomial $1+t$ does not divide $H(E/(J+(v_1,\dots,v_s)),t)$.

To this end we may assume that $\operatorname{depth}_E(E/J) = 0$. The Hilbert series and the depth of E/J and $E/\operatorname{gin}(J)$ coincide and by Corollary 3.2.4 $E/\operatorname{gin}(J)$ has a linear injective resolution. So we may assume in addition that J is strongly stable. Then we know the

Betti numbers of J. Proving that $1+t$ does not divide the Hilbert series of E/J is the same as showing this for J as the Hilbert series of E is $(1+t)^n$.

Let $m_{k,j}(J) = |\{u \in G(J) : \max(u) = k, \deg(u) = j\}|$. Computing the Hilbert series of J via the minimal graded free resolution of J gives

$$\begin{aligned}
H(J,t) &= \sum_{i\geq 0}(-1)^i H(\bigoplus\nolimits_{j\in\mathbb{Z}} E(-j)^{\beta_{i,j}(J)},t) \\
&= \sum_{i\geq 0}(-1)^i \sum_{j\in\mathbb{Z}} t^j \beta_{i,j}(J)(1+t)^n \\
&= \sum_{i\geq 0}(-1)^i \sum_{j\in\mathbb{Z}} t^{i+j} \beta_{i,i+j}(J)(1+t)^n \\
&= \sum_{i\geq 0}(-1)^i \sum_{j\in\mathbb{Z}} t^{i+j}(1+t)^n \sum_{u\in G(J)_j} \binom{\max(u)+i-1}{\max(u)-1} \\
&= \sum_{i\geq 0}(-1)^i \sum_{k=1}^{n}\sum_{j=1}^{k} m_{k,j}(J)\binom{k+i-1}{k-1} t^{i+j}(1+t)^n \\
&= \sum_{k=1}^{n}\sum_{j=1}^{k} m_{k,j}(J)t^j(1+t)^n \sum_{i\geq 0}\binom{k+i-1}{k-1}(-1)^i t^i \\
&= \sum_{k=1}^{n}\sum_{j=1}^{k} m_{k,j}(J)t^j(1+t)^n \frac{1}{(1+t)^k} \\
&= \sum_{k=1}^{n}\sum_{j=1}^{k} m_{k,j}(J)t^j(1+t)^{n-k},
\end{aligned}$$

where we used the formula for the Betti numbers of stable ideals from Lemma 3.1.2. All coefficients appearing in the last sum are non-negative hence no term can be cancelled by another. The formula $n - \operatorname{cx}_E(E/J) = \operatorname{depth}_E(E/J) = 0$ and Lemma 3.1.4 imply that $m_{n,j}(J) \neq 0$ for some j. Let $u = e_F e_n \in G(J)$. We have $e_F e_i \in J$ for all $i = 1,\dots,n$ because J is stable. The dual of E/J is $(E/J)^* \cong 0 :_E J$, which is generated by all monomials e_F with $e_{F^c} \notin J$ (cf. Section 1.1). But then $e_{(F\cup\{i\})^c} = e_{F^c\setminus\{i\}} \notin (E/J)^*$ for all $i \notin F$. As $e_F \notin J$ (otherwise $e_F e_n$ would not be a minimal generator), the complement e_{F^c} is in $(E/J)^*$ and even a minimal generator. The ideal $(E/J)^*$ has an $(n-d)$-linear projective resolution, in particular it is generated in degree $n-d$, so $|F| = n - |F^c| = n-(n-d) = d$. Thus we have seen that every minimal generator $u \in G(J)$ with $n \in \operatorname{supp}(u)$ has degree $d+1$. Hence $m_{n,j}(J) = 0$ for $j \neq d+1$ and $m_{n,d+1}(J) \neq 0$. So there is exactly one summand in $H(J,t)$ that is not divisible by $1+t$ whence $1+t$ does not divide $H(J,t)$. □

As an application of this theorem we obtain the following corollary.

Corollary 5.1.4. *Let $|K| = \infty$ and Δ be a Cohen-Macaulay simplicial complex of dimension ≥ 0. The depth of the face ring $K\{\Delta\}$ is 0 if and only if $\tilde{\chi}(\Delta) \neq 0$.*

PROOF. By Theorem 5.1.3 the depth of $K\{\Delta\}$ is zero if and only if $H(K\{\Delta\},-1) \neq 0$. Then the assertion follows from

$$H(K\{\Delta\},-1) = \sum_{i=0}^{d}(-1)^i \dim_K K\{\Delta\}_i = \sum_{i=0}^{d}(-1)^i f_{i-1}(\Delta) = -\tilde{\chi}(\Delta).$$

□

Remark 5.1.5. If $M \in \mathscr{M}$ has a d-linear injective resolution, then its graded Bass numbers are determined by its Hilbert series. The graded Bass numbers are the coefficients in the Hilbert series of the S-module $\operatorname{Ext}_E^*(K,M)$, cf. Section 2.3. Using that $H(M,\frac{1}{t}) = t^n H(M^*,t)$, which follows from Lemma 1.1.1, one obtains

$$H(\operatorname{Ext}_E^*(K,M),t) = \frac{(-1)^{n-d}}{t^{n-d}(1-t)^n} H(M^*,-t).$$

For quotient rings with linear injective resolution there is a nice formula for the regularity.

Theorem 5.1.6. *Let $|K| = \infty$, $0 \neq J \subset E$ be a graded ideal and E/J have a d-linear injective resolution. Then*

$$\operatorname{reg}_E(E/J) + \operatorname{depth}_E(E/J) = d.$$

PROOF. First assume that $\operatorname{depth}_E(E/J) = 0$. Then $d = d(E/J)$ is an upper bound for $\operatorname{reg}_E(E/J)$ and we want to show that both numbers are equal. Theorem 3.2.2 says that $\operatorname{reg}_E(E/J) = \operatorname{reg}_E(E/\operatorname{gin}(J))$. Since $\operatorname{depth}_E(E/\operatorname{gin}(J)) = \operatorname{depth}_E(E/J) = 0$ by Theorem 4.2.1, we may assume in addition that J is strongly stable.

In this situation we have already seen in the proof of Theorem 5.1.3 that there exists a minimal generator of J of degree $d+1$ which implies $\operatorname{reg}_E(E/J) = \operatorname{reg}_E J - 1 \geq d+1-1 = d$. This proves the case $\operatorname{depth}_E(E/J) = 0$.

Now suppose $\operatorname{depth}_E(E/J) = s$. Reducing E/J modulo a maximal regular sequence $v_1,\dots,v_s$ does not change the regularity by Proposition 4.1.9, but $E/J+(v_1,\dots,v_s)$ has a $(d-s)$-linear injective resolution over $E/(v_1,\dots,v_s)$ by Lemma 5.1.2. Then the considerations for the case depth zero yield $\operatorname{reg}_E(E/J+(v_1,\dots,v_s)) = d-s$ and so

$$\operatorname{reg}_E(E/J) = \operatorname{reg}_E(E/J+(v_1,\dots,v_s)) = d-s = d - \operatorname{depth}_E(E/J).$$

□

Remark 5.1.7. Let $|K| = \infty$ and $0 \neq J \subset E$ so that E/J has a d-linear injective resolution. By Theorem 4.1.2 we have $\operatorname{cx}_E(E/J) = n - \operatorname{depth}_E(E/J)$. As $d \leq n$ this proves that

$$\operatorname{reg}_E(E/J) \leq \operatorname{cx}_E(E/J).$$

This inequality is even true for general quotient rings E/J. For arbitrary graded E-modules there is no such relation between the regularity and the complexity since the first one is changed by shifting while the other is invariant.

If E/J has a d-linear injective resolution, then $(E/J)^*$ has an $(n-d)$-linear projective resolution, whence $\operatorname{reg}_E(E/J)^* = n-d$. Therefore it holds that

$$\operatorname{reg}_E(E/J) + \operatorname{reg}_E(E/J)^* = d - \operatorname{depth}_E(E/J) + n - d = \operatorname{cx}_E(E/J).$$

For a graded ideal $J \subset E$ Eisenbud, Popescu and Yuzvinsky characterise in [**20**] the case when both J has a linear projective and E/J has a linear injective resolution over E. In their proof they use the Bernstein-Gel'fand-Gel'fand correspondence. We present a (partly) more direct proof using generic initial ideals.

Theorem 5.1.8. [**20**, Theorem 3.4] *Let $|K| = \infty$ and $0 \neq J \subset E$ be a graded ideal. Then J and $(E/J)^*$ have linear projective resolutions if and only if J reduces to a power of the maximal ideal modulo some (respectively any) maximal E/J-regular sequence of linear forms of E.*

PROOF. At first we show that it is enough to consider the case $\operatorname{depth} E/J = 0$. Note that the ideal J has a linear projective resolution over E if and only if the ideal $J + (v_1, \dots, v_s)/(v_1, \dots, v_s)$ has a linear projective resolution over $E/(v_1, \dots, v_s)$ by Proposition 4.1.9. Furthermore, by Lemma 5.1.2, E/J has a linear injective resolution over E if and only if $E/(J + (v_1, \dots, v_s))$ has one over $E/(v_1, \dots, v_s)$. All in all we may indeed assume that $\operatorname{depth}_E(E/J) = 0$.

The t-th power of the maximal ideal $\mathfrak{m} = (e_1, \dots, e_n)$ has a t-linear projective resolution because it is strongly stable and generated in one degree (cf. Lemma 3.1.2). For the same reason $(E/\mathfrak{m}^t)^* \cong 0 :_E \mathfrak{m}^t = \mathfrak{m}^{n-t+1}$ has a linear projective resolution. Hence the "if" direction is proved.

It remains to show that if J has a t-linear projective resolution, E/J has a d-linear injective resolution and $\operatorname{depth}_E(E/J) = 0$, then $J = \mathfrak{m}^t$.

In a first step we will see that J may be replaced by its generic initial ideal. If J has a t-linear projective resolution, its regularity is obviously t. Then by Theorem 3.2.2 the regularity of $\operatorname{gin}(J)$ is also t. As $\operatorname{gin}(J)$ is generated in degree $\geq t$ this implies that $\operatorname{gin}(J)$ has a t-linear resolution as well.

Also $E/\operatorname{gin}(J)$ has a d-linear injective resolution as well by Corollary 3.2.4. Finally

$$\operatorname{depth}_E(E/\operatorname{gin}(J)) = \operatorname{depth}_E(E/J) = 0$$

by Theorem 4.2.1. Altogether $\operatorname{gin}(J)$ satisfies the same conditions as J. Assume that $\operatorname{gin}(J) = \mathfrak{m}^t$. The Hilbert series of J and $\operatorname{gin}(J)$ are the same which implies that in this case $J = \mathfrak{m}^t$ as well because $J \subseteq \mathfrak{m}^t = \operatorname{gin}(J)$.

This allows us to replace J by $\operatorname{gin}(J)$ so in the following we assume that J is strongly stable.

In the proof of Theorem 5.1.3 it is proved in the same situation that there exists a minimal generator of J of degree $d+1$. As J is generated in degree t, it follows that $d = t-1$.

Finally, we will see that this equality implies $J = \mathfrak{m}^t$. As E/J has a d-linear injective resolution, the number $d(E/J) = \max\{i : (E/J)_i \neq 0\}$ equals d. Then, by Lemma 3.1.5,

$$\max\{\min(u) : u \in G(J)\} = n - d = n - t + 1.$$

Thus there exists a monomial $u \in G(J)$ of degree t with $\min(u) = n - t + 1$. The only possibility for u is $u = e_{n-t+1} \cdots e_n$. Then every monomial of degree t is in J because J is strongly stable and this implies $J = \mathfrak{m}^t$ since J is generated in degree t. □

As an application we obtain a characterisation of the Cohen-Macaulay non-acyclic simplicial complexes whose face ideal has a linear resolution. To this end we define $\Delta_{m,n} = \{F \subseteq [n] : |F| \leq m\}$.

Corollary 5.1.9. *Let $|K| = \infty$ and Δ be a Cohen-Macaulay simplicial complex with $\tilde{\chi}(\Delta) \neq 0$. The face ideal J_Δ has a linear projective resolution if and only if $\Delta = \Delta_{m,n}$ for some $m \leq n$.*

PROOF. Note that the face ideal of $\Delta_{m,n}$ is $\mathfrak{m}^{m+1}$. Therefore if $\Delta = \Delta_{m,n}$ for some $m \leq n$, it has all the desired properties.

Conversely let Δ be a Cohen-Macaulay simplicial complex with $\tilde{\chi}(\Delta) \neq 0$ whose face ideal has a linear projective resolution. Since Δ is Cohen-Macaulay, the face ring $K\{\Delta\}$ has a linear injective resolution. Then by Theorem 5.1.8 the ideal J_Δ has a linear projective resolution if and only if it reduces to a power of the maximal ideal modulo some maximal $K\{\Delta\}$-regular sequence. But as $\operatorname{depth}_E K\{\Delta\} = 0$ by Corollary 5.1.4, there are no regular elements and hence J_Δ itself must be a power of the maximal ideal, say $\mathfrak{m}^{m+1}$ for some $m \leq n$. Hence, $\Delta = \Delta_{m,n}$. □

5.2. Face rings of Gorenstein complexes

In this section we investigate Gorenstein complexes with exterior algebra methods. A good overview over Gorenstein complexes is given in the book by Bruns and Herzog, [**11**]. A simplicial complex Δ is called *Gorenstein* if its Stanley-Reisner ring $K[\Delta]$ is Gorenstein, i.e., if $K[\Delta](-a)$ is isomorphic to its canonical module $\omega_{K[\Delta]}$, where $a \in \mathbb{Z}$ is the a-invariant of $K[\Delta]$. We define $\operatorname{core}\Delta$ to be $\Delta_{\operatorname{core}[n]}$, the restriction of Δ to $\operatorname{core}[n] = \{i \in [n] : \operatorname{st}_\Delta\{i\} \neq \Delta\}$, where $\operatorname{st}_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}$ for $F \subseteq [n]$. Notice that Δ is the join

$$\Delta = \operatorname{core}\Delta * \Delta_{[n]\setminus\operatorname{core}[n]}$$

and hence

$$K[\operatorname{core}\Delta][x_i : \{i\} \notin \operatorname{core}[n]].$$

It follows that Δ is Gorenstein if and only if $\operatorname{core}\Delta$ is Gorenstein. Therefore it is sufficient to study simplicial complexes that satisfy $\Delta = \operatorname{core}\Delta$. A Gorenstein complex that satisfies this condition is called *Gorenstein**. Then the a-invariant of $K[\Delta]$ is 0 and hence $K[\Delta] \cong \omega_{K[\Delta]}$.

There is the following combinatorial description of the Gorenstein property.

Theorem 5.2.1. [**11**, Theorem 5.6.1] *Let Δ be a simplicial complex with $\Delta = \operatorname{core}\Delta$. Then Δ is Gorenstein if and only if for all $F \in \Delta$ one has*

$$\widetilde{H}_i(\operatorname{lk}_\Delta F; K) \cong \begin{cases} K & \text{if } i = \dim \operatorname{lk}_\Delta F, \\ 0 & \text{if } i < \dim \operatorname{lk}_\Delta F. \end{cases}$$

Notice that this description together with Proposition 1.2.3 implies that a Gorenstein complex is Cohen-Macaulay.

Lemma 5.2.2. *Let Δ be a Gorenstein complex of dimension $d-1$ with $\Delta = \operatorname{core}\Delta$. Then $\tilde{\chi}(\Delta) \neq 0$.*

PROOF. The reduced Euler characteristic can be computed via the reduced simplicial homology of Δ, i.e.,

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i \dim_K \tilde{H}_i(\Delta;K)$$

(see, e.g., [**11**, p. 231]). As Δ is Gorenstein, Theorem 5.2.1 implies $\tilde{H}_i(\Delta;K) \cong K$ if $i = d-1$ and is zero otherwise. Hence

$$\tilde{\chi}(\Delta) = \dim_K \tilde{H}_{d-1}(\Delta;K) = 1 \neq 0.$$

□

Corollary 5.2.3. *Let Δ be a Gorenstein complex of dimension $d-1$ with $s = n - |\operatorname{core}[n]|$. Then* $\operatorname{depth}_E K\{\Delta\} = s$, $\operatorname{cx}_E K\{\Delta\} = n-s$ *and* $\operatorname{reg}_E K\{\Delta\} = d-s$.

PROOF. Observe that

$$K\{\Delta\} = K\{\operatorname{core}\Delta\} \otimes_K K\{\Delta_{[n]\setminus \operatorname{core}[n]}\} \cong K\{\operatorname{core}\Delta\}\{e_i : \{i\} \notin \operatorname{core}\Delta\}$$

and hence $\operatorname{depth}_E K\{\Delta\} = \operatorname{depth}_E K\{\operatorname{core}\Delta\} + n - |\operatorname{core}\Delta|$. Thus the equations follow from Lemma 5.2.2, Corollary 5.1.4, Theorem 4.1.2 and Theorem 5.1.6. □

Similarly we characterise the Gorenstein* complexes with linear projective resolution.

Proposition 5.2.4. *Let Δ be a Gorenstein complex of dimension $d-1$ with $\Delta = \operatorname{core}\Delta$. The face ideal J_Δ has a linear projective resolution if and only if $\Delta = \Delta_{0,n}$ or $\Delta = \Delta_{n-1,n}$.*

PROOF. From Lemma 5.2.2 and Corollary 5.1.9 it follows that J_Δ has an $(m+1)$-linear projective resolution if and only if $\Delta = \Delta_{m,n}$. We show that $\Delta_{m,n}$ is Gorenstein* if and only if $m = 0$ or $m = n-1$. (The complex $\Delta_{n,n}$ (the simplex) is Gorenstein, but not Gorenstein*.)

Following [**11**, Theorem 5.6.2] a simplicial complex Δ with $\Delta = \operatorname{core}\Delta$ is Gorenstein over K if and only if Δ is Cohen-Macaulay over K and $\tilde{\chi}(\operatorname{lk}_\Delta F) = (-1)^{\dim \operatorname{lk}_\Delta F}$ for all $F \in \Delta$. The complex $\Delta_{m,n}$ is Cohen-Macaulay by Corollary 5.1.9. Let $F \in \Delta_{m,n}$. Then $\operatorname{lk}_{\Delta_{m,n}} F = \{G \subseteq [n] \setminus F : |G| \leq m - |F|\}$. Hence $\dim \operatorname{lk}_{\Delta_{m,n}} F = m - |F| - 1$ and the number of i-dimensional faces is $f_i(\operatorname{lk}_{\Delta_{m,n}} F) = \binom{n-|F|}{i+1}$. The reduced Euler characteristic is

$$\tilde{\chi}(\operatorname{lk}_{\Delta_{m,n}} F) = \sum_{i=-1}^{m-|F|-1} (-1)^i \binom{n-|F|}{i+1}.$$

If $m = 0$ then $\Delta_{0,n} = \{\emptyset\}$ and $\tilde{\chi}(\operatorname{lk}_{\Delta_{0,n}} \emptyset) = -1$ and $\dim \operatorname{lk}_{\Delta_{0,n}} \emptyset = -1$. Thus $\Delta_{0,n}$ is Gorenstein*. If $m > 0$ let $F \in \Delta_{m,n}$ with $|F| = m-1$. Then

$$\begin{aligned}\tilde{\chi}(\operatorname{lk}_{\Delta_{m,n}} F) &= \sum_{i=-1}^{m-(m-1)-1} (-1)^i \binom{n-m+1}{i+1} = \sum_{i=-1}^{0} (-1)^i \binom{n-m+1}{i+1} \\ &= -\binom{n-m+1}{0} + \binom{n-m+1}{1} = -1 + n - m + 1 = n - m.\end{aligned}$$

For $1 \leq m \leq n-1$ the number $n-m$ equals 1 only if $m = n-1$ and otherwise it is neither 1 nor -1. Hence if $m \neq 0, n-1$ then $\Delta_{m,n}$ cannot be Gorenstein*. It remains to show that $\Delta_{n-1,n}$ is Gorenstein*. Let $F \in \Delta_{n-1,n}$ be an arbitrary face. Then

$$\begin{aligned}\tilde{\chi}(\mathrm{lk}_{\Delta_{n-1,n}} F) &= \sum_{i=-1}^{n-1-|F|-1} (-1)^i \binom{n-|F|}{i+1} = -\sum_{i=0}^{n-|F|-1} (-1)^i \binom{n-|F|}{i} \\ &= -\left(\sum_{i=0}^{n-|F|} (-1)^i \binom{n-|F|}{i} - (-1)^{n-|F|} \binom{n-|F|}{n-|F|} \right) \\ &= -(0-(-1)^{n-|F|}) = (-1)^{n-|F|} = (-1)^{\dim \mathrm{lk}_{\Delta_{n-1,n}} F}.\end{aligned}$$

All in all we have proved that the complex $\Delta_{m,n}$ is Gorenstein* if and only if $m \in \{0, n-1\}$. Then Corollary 5.1.9 concludes the proof. □

Since the Stanley-Reisner ideal of a simplicial complex has a linear resolution if and only if its face ideal has one, we obtain the following corollary.

Corollary 5.2.5. *Let Δ be a Gorenstein complex of dimension $d-1$ with $\Delta = \mathrm{core}\,\Delta$. The Stanley-Reisner ideal I_Δ has a linear free resolution over S if and only if $\Delta = \Delta_{0,n}$ or $\Delta = \Delta_{n-1,n}$.*

The graded Bass numbers of the face ring of a Gorenstein* simplicial complex depend only on the f-vector of the complex, because the face ring has a linear injective resolution whence the Bass numbers are determined by the Hilbert series. We give a concrete formula for this relation.

Proposition 5.2.6. *Let Δ be a Gorenstein* simplicial complex of dimension $d-1$ with f-vector $(f_{-1}, \dots, f_{d-1})$. Then $K\{\Delta\}$ has a d-linear injective resolution over E and the graded Bass numbers of $K\{\Delta\}$ are*

$$\mu_{i,d-i}(K\{\Delta\}) = \sum_{k=0}^{d} \binom{n-d+i-1}{i-k} f_{d-k-1}$$

and zero otherwise.

PROOF. A Gorenstein complex is Cohen-Macaulay whence $K\{\Delta\}$ has a d-linear injective resolution, see, e.g., Example 5.1.1. $K\{\Delta\}$ has a natural $\mathbb{Z}^n$-grading. Let $a \in \mathbb{Z}^n$ and $a = a_+ - a_-$ where $a_+ \in \mathbb{Z}^n$ is the vector whose ith component is a_i if $a_i > 0$, and is 0 if $a_i \leq 0$, while $a_- = a - a_+$. Aramova and Herzog [**2**, Proposition 6.2] show that $H^i(\mathbf{e}; K\{\Delta\})_a = 0$ if $a_j > 1$ for some j. Moreover,

$$H^i(\mathbf{e}; K\{\Delta\})_a \cong \tilde{H}^{i-|a_-|-1}(\mathrm{lk}_\Delta G_a; K),$$

if $a_j \leq 1$ for all j and $G_a = \{j : a_j > 0\}$ is a face of Δ. As Δ is Gorenstein*, the simplicial homology of the link of each face $F \in \Delta$ satisfies

$$\tilde{H}_i(\mathrm{lk}_\Delta F; K) = \begin{cases} K & i = \dim \mathrm{lk}_\Delta F, \\ 0 & i < \dim \mathrm{lk}_\Delta F, \end{cases}$$

by Theorem 5.2.1. Combining these two results we can compute the Bass numbers of $K\{\Delta\}$. Recall that $\tilde{H}_i(\Delta;K) \cong \tilde{H}^i(\Delta;K)$ as K-vector spaces. We obtain

$$H^i(\mathbf{e};K\{\Delta\})_a \cong \tilde{H}_{i-|a_-|-1}(\mathrm{lk}_\Delta G_a;K) \cong \begin{cases} K & i-|a_-|-1 = \dim \mathrm{lk}_\Delta G_a, G_a \in \Delta, a_j \leq 1 \text{ for all } j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose from now on that $a_j \leq 1$ for all j. Then $G_a \in \Delta$ if and only if $K\{\Delta\}_{a_+} \neq 0$. If G_a is in Δ, then

$$\dim \mathrm{lk}_\Delta G_a = \dim \Delta - |G_a| = d-1-|a_+|.$$

Hence the condition $i-|a_-|-1 = \dim \mathrm{lk}_\Delta G_a$ is the same as

$$i = d-|a_+|+|a_-| = d-|a|$$

and so

$$H^i(\mathbf{e};K\{\Delta\})_a \cong \begin{cases} K & i = d-|a|, K\{\Delta\}_{a_+} \neq 0, a_j \leq 1 \text{for all } j, \\ 0 & \text{otherwise.} \end{cases}$$

The number $\mu_{i,d-i} = \dim_K H^i(\mathbf{e};K\{\Delta\})_{d-i}$ is the number of $a \in \mathbb{Z}^n$ which satisfies the three conditions $i = d-|a|$, $K\{\Delta\}_{a_+} \neq 0$, $a_j \leq 1$ for all j. When $a_j \leq 1$ for all j, then $G_a = \{j : a_j = 1\}$. Suppose $G_a \in \Delta$. Then G_a has at most $d = \dim \Delta + 1$ elements, say, $|G_a| = d-k$ for some $k \in \{0,\dots,d\}$. There are f_{d-k-1} possibilities for G_a. As $|a| = d-i$, there are $-(d-i-(d-k)) = i-k$ times a "-1" to distribute on $n-(d-k)$ positions. This gives

$$\binom{n-d+k+i-k-1}{i-k} = \binom{n-d+i-1}{i-k}$$

possibilities for each G_a. Altogether these are exactly

$$\sum_{k=0}^{d} \binom{n-d+i-1}{i-k} f_{d-k-1}$$

possibilities to choose $a \in \mathbb{Z}^n$, $|a| = d-i$ such that $H^i(\mathbf{e};K\{\Delta\})_a \cong K$. This ist the vector space dimension of $H^i(\mathbf{e};K\{\Delta\})_{d-i} \cong \mathrm{Ext}^i_E(K,K\{\Delta\})_{d-i}$, which is $\mu_{i,d-i}$ by definition. □

We will need the following identity (which is possibly known by other means).

Lemma 5.2.7. *Let $j,k,d,n \in \mathbb{N}$ with $d \leq n$. Then*

$$\sum_{i=k}^{j} (-1)^i \binom{n}{j-i} \binom{n-d+i-1}{i-k} = (-1)^k \binom{d-k}{j-k}.$$

PROOF. We use a threefold induction. The first one, induction (1), is on k. Suppose $k=0$. Then we have to show that

$$\sum_{i=0}^{j} (-1)^i \binom{n}{j-i} \binom{n-d+i-1}{i} = \binom{d}{j}.$$

We use another induction (2) on j. The case $j=0$ is obvious. Now suppose the claim is proved for j. We use an induction (3) on n to show that then it is also true for $j+1$. The case $n=0$ is trivial since $d \leq n$ implies also $d=0$. For the induction step of the induction (3) on n we have to prove that

$$\sum_{i=0}^{j+1}(-1)^i\binom{n+1}{j+1-i}\binom{n+1-d+i-1}{i}=\binom{d}{j+1}$$

for some $d \leq n+1$ and we have the induction (2) hypothesis for j (for all values of n) and the induction (3) hypothesis for values $\leq n$. Then

$$\begin{aligned}
&\sum_{i=0}^{j+1}(-1)^i\binom{n+1}{j+1-i}\binom{n+1-d+i-1}{i}\\
=\;&\sum_{i=0}^{j+1}(-1)^i\left(\binom{n}{j+1-i}+\binom{n}{j-i}\right)\binom{n+1-d+i-1}{i}\\
=\;&\sum_{i=0}^{j+1}(-1)^i\binom{n}{j+1-i}\binom{n+1-d+i-1}{i}+\sum_{i=0}^{j+1}(-1)^i\binom{n}{j-i}\binom{n+1-d+i-1}{i}\\
=\;&\sum_{i=0}^{j+1}(-1)^i\binom{n}{j+1-i}\binom{n-(d-1)+i-1}{i}+\sum_{i=0}^{j}(-1)^i\binom{n}{j-i}\binom{n-(d-1)+i-1}{i}\\
=\;&\binom{d-1}{j+1}+\binom{d-1}{j}\\
=\;&\binom{d}{j+1}
\end{aligned}$$

where we have used the induction hypothesis for $d-1$ instead of d. This proves the claim of the induction (3) on n which proves the claim of the induction (2) on j.

Now we show the induction step of the induction (1) on k:

$$\begin{aligned}
&\sum_{i=k}^{j}(-1)^i\binom{n}{j-i}\binom{n-d+i-1}{i-k}\\
=\;&\sum_{i=k-1}^{j}(-1)^i\binom{n}{j-i}\binom{n-d+i-1}{i-k}\\
=\;&\sum_{i=k-1}^{j}(-1)^i\binom{n}{j-i}\left(\binom{n-(d-1)+i-1}{i-k+1}-\binom{n-d+i-1}{i-k+1}\right)\\
=\;&\sum_{i=k-1}^{j}(-1)^i\binom{n}{j-i}\binom{n-(d-1)+i-1}{i-k+1}-\sum_{i=k-1}^{j}(-1)^i\binom{n}{j-i}\binom{n-d+i-1}{i-k+1}\\
=\;&(-1)^{k-1}\binom{d-1-k+1}{j-k+1}-(-1)^{k-1}\binom{d-k+1}{j-k+1}\\
=\;&(-1)^k\left(\binom{d-k+1}{j-k+1}-\binom{d-k}{j-k+1}\right)\\
=\;&(-1)^k\binom{d-k}{j-k}.
\end{aligned}$$

□

This identity can be used to prove the following identity for the entries of the f-vector of a Gorenstein* simplicial complex.

Proposition 5.2.8. *Let Δ be a Gorenstein* simplicial complex of dimension $d-1$ with f-vector $(f_{-1},\ldots,f_{d-1})$. Then*

$$f_{d-j-1} = \sum_{k=0}^{j} (-1)^k \binom{d-k}{j-k} f_{d-k-1}$$

for $j = 0,\ldots,d$.

PROOF. Since $K\{\Delta\}$ has a linear injective resolution, we can compute the Hilbert series of $K\{\Delta\}$, which is $H(K\{\Delta\},t) = \sum_{l=0}^{d} f_{l-1}t^l$, via the graded Bass numbers of $K\{\Delta\}$ and then compare coefficients:

$$\begin{aligned} H(K\{\Delta\},t) &= \sum_{i\geq 0} (-1)^i \mu_{i,d-i} t^{-n+d-i}(1+t)^n \\ &= \sum_{i\geq 0}\sum_{k=0}^{n} (-1)^i \binom{n}{k} \mu_{i,d-i} t^{-n+d-i+n-k} \\ &= \sum_{i\geq 0}\sum_{k=0}^{n} (-1)^i \binom{n}{k} \mu_{i,d-i} t^{d-i-k}. \end{aligned}$$

Comparing the coefficient of t^{d-j} gives

$$f_{d-j-1} = \sum_{i=0}^{j} (-1)^i \binom{n}{j-i} \mu_{i,d-i}.$$

The graded Bass numbers $\mu_{i,d-i}$ are computed in Proposition 5.2.6:

$$\begin{aligned} f_{d-j-1} &= \sum_{i=0}^{j}\sum_{k=0}^{d} (-1)^i \binom{n}{j-i}\binom{n-d+i-1}{i-k} f_{d-k-1} \\ &= \sum_{i=0}^{j}\sum_{k=0}^{j} (-1)^i \binom{n}{j-i}\binom{n-d+i-1}{i-k} f_{d-k-1} \\ &= \sum_{k=0}^{j} \left(\sum_{i=k}^{j} (-1)^i \binom{n}{j-i}\binom{n-d+i-1}{i-k} \right) f_{d-k-1} \\ &= \sum_{k=0}^{j} (-1)^k \binom{d-k}{j-k} f_{d-k-1}. \end{aligned}$$

In the last step we used Lemma 5.2.7. □

Proposition 5.2.9. *Let Δ be a Gorenstein* simplicial complex of dimension $d-1$ with f-vector $(f_{-1},\ldots,f_{d-1})$. Then the Hilbert series of* $\operatorname{Ext}_E^*(K,K\{\Delta\})$ *is*

$$H(\operatorname{Ext}_E^*(K,K\{\Delta\}),t) = \frac{\sum_{j=0}^{d}(-1)^j f_{d-j-1}t^j}{(1-t)^n}.$$

PROOF. As $K\{\Delta\}$ has a d-linear injective resolution, the i-th graded component of $\operatorname{Ext}_E^*(K, K\{\Delta\})$ is

$$\operatorname{Ext}_E^i(K, K\{\Delta\}) = \operatorname{Ext}_E^i(K, K\{\Delta\})_{d-i}$$

whose vector space dimension $\mu_{i,d-i}(K\{\Delta\})$ is computed in Proposition 5.2.6. Thus

$$\begin{aligned}
H(\operatorname{Ext}_E^*(K, K\{\Delta\}), t) &= \sum_{i\geq 0} \mu_{i,d-i} t^i \\
&= \sum_{i\geq 0}\sum_{k=0}^{d} \binom{n-d+i-1}{i-k} f_{d-k-1} t^i \\
&= \sum_{i\geq 0}\sum_{k=0}^{d} \binom{n-d+k+i-k-1}{i-k} f_{d-k-1} t^i \\
&= \sum_{k=0}^{d} f_{d-k-1} t^k \sum_{i\geq 0} \binom{n-d+k+i-k-1}{i-k} t^{i-k} \\
&= \sum_{k=0}^{d} \frac{f_{d-k-1} t^k}{(1-t)^{n-d+k}} \\
&= \frac{1}{(1-t)^n} \sum_{k=0}^{d} f_{d-k-1} t^k (1-t)^{d-k} \\
&= \frac{1}{(1-t)^n} \sum_{k=0}^{d}\sum_{i=0}^{d-k} (-1)^i \binom{d-k}{i} f_{d-k-1} t^{i+k} \\
&= \frac{1}{(1-t)^n} \sum_{j=0}^{d}\sum_{k=0}^{j} (-1)^{j-k} \binom{d-k}{j-k} f_{d-k-1} t^j \\
&= \frac{1}{(1-t)^n} \sum_{j=0}^{d} (-1)^j f_{d-j-1} t^j,
\end{aligned}$$

where we have used Proposition 5.2.8 in the last step. □

Fløystad [**24**] characterises the Gorenstein* property in terms of what he calls enriched cohomology. This is the cohomology of the complex $\mathbf{L}(K\{\Delta\}(-n))$, cf. Section 2.4. The j-th graded piece of the i-th cohomology of this complex is isomorphic to $\operatorname{Ext}_E^{i-j}(K, K\{\Delta\}(-n))_j$ by Proposition 2.4.2. So if Δ is Cohen-Macaulay of dimension $d-1$, i.e., if $K\{\Delta\}$ has a d-linear injective resolution, then the top cohomology is the only non-zero cohomology and is isomorphic to $\operatorname{Ext}_E^*(K, K\{\Delta\})(-n+d)_{-j}$. With this translation Fløystad's result is the following.

Theorem 5.2.10. [**24**, Theorem 3.1] *Let Δ be a Cohen-Macaulay simplicial complex. Then Δ is Gorenstein* if and only if $\operatorname{Ext}_E^*(K, K\{\Delta\})$ is a torsionfree S-module of rank 1. In this case $\operatorname{Ext}_E^*(K, K\{\Delta\})(-n+d)$ identifies as the Stanley-Reisner ideal I_{Δ^*} of the Alexander dual of Δ.*

Remark 5.2.11. It would be nice to know a property of graded E-modules that corresponds to the Gorenstein property. Such modules should have at least a linear injective resolution. In view of the preceding Theorem 5.2.10 one could try to consider modules M (or only quotient rings E/J), that have a linear injective resolution and in addition it holds that $\operatorname{Ext}_E^*(K,M)$ is a torsionfree rank one S-module. But this condition implies that $\operatorname{depth}_E M = 0$, because the complexity of M equals the Krull dimension of $\operatorname{Ext}_E^*(K,M)$, which is n in that case. (To see the equality between these two invariants apply the Hilbert-Serre theorem, e.g., [**11**, Corollary 4.1.8], to the S-module $\operatorname{Ext}_E^*(K,M)$ and use $\operatorname{cx}_E M = \operatorname{cx}_E M^*$, Theorem 4.1.12.) Therefore this condition could be a definition of Gorenstein*. However, we do not know how to expand this condition on modules of arbitrary depth. The striking definition would be that a module is Gorenstein if modulo a maximal regular sequence it is Gorenstein*. But even for depth one it is not clear at the moment that $\operatorname{Ext}_E^*(K,M/vM)$ is torsionfree and of rank one if and only if $\operatorname{Ext}_E^*(K,M/wM)$ is torsionfree and of rank one, for M-regular elements v,w.

Open problem 5.2.12. *Is there an exterior analogue of the Gorenstein property?*

5.3. Componentwise linear and componentwise injective linear modules

In this section we study a generalisation of the notion of linear projective resolution, the *componentwise linearity*. It has been introduced for ideals in a polynomial ring by Herzog and Hibi in [**29**] to generalise Eagon and Reiner's result in Example 5.1.1, that the Stanley-Reisner ideal of a simplicial complex has a linear resolution if and only if the Alexander Dual of the complex is Cohen-Macaulay. Such ideals have been studied in several articles, e.g., [**25**], [**31**], [**55**]. Römer [**52**] generalises this notion and Herzog and Hibi's result to squarefree modules over a polynomial ring.

Definition 5.3.1. A module $M \in \mathscr{M}$ is called *componentwise linear* if the submodule $M_{\langle j\rangle}$ of M, which is generated by all homogeneous elements of degree j belonging to M, has a j-linear resolution for all $j \in \mathbb{Z}$.

A componentwise linear module that is generated in one degree has a linear resolution. On the other hand, a module that has a linear resolution, is componentwise linear. We encountered a class of componentwise linear ideals in Section 3.1.

Example 5.3.2. If $J \subset E$ is a stable ideal, then each $J_{\langle j\rangle}$ is also stable. It follows from Lemma 3.1.2 that a stable ideal has a linear resolution if it is generated in one degree. Hence a stable ideal is componentwise linear.

Proposition 5.3.3. *Let Δ be a simplicial complex. Then the Stanley-Reisner ideal I_Δ is componentwise linear if and only if the face ideal J_Δ is componentwise linear.*

PROOF. Herzog and Hibi also introduce in [**29**] the notion of a squarefree componentwise linear ideal in a polynomial ring, that is, an ideal I, generated by squarefree monomials, such that $I_{[j]}$, the ideal generated by the squarefree monomials of degree j belonging to I, has a linear resolution for all j. They show in [**29**, Proposition 1.5] that I is squarefree componentwise linear if and only if it is componentwise linear. Obviously $(I_\Delta)_{[j]}$ and $(J_\Delta)_{\langle j\rangle}$ correspond to each other under the equivalence of categories. Hence the assertion follows from Corollary 4.3.3. □

In the following we present some of the results in [**29**] and [**31**] adapted to modules over the exterior algebra.

Lemma 5.3.4. *If $M \in \mathscr{M}$ has a linear projective resolution, then also $\mathfrak{m}M$ has a linear projective resolution.*

PROOF. Assume that M has a t-linear resolution. Then $M/\mathfrak{m}M$ is generated in degree t and is isomorphic as an E-module to a direct sum $\bigoplus_{i=1}^{m} K(-t)$, where K is regarded as the E-module $E/\mathfrak{m}$. Therefore the minimal projective resolution of $M/\mathfrak{m}M$ is a direct sum of Cartan complexes with respect to $e_1, \dots, e_n$, each shifted in degree by t. This resolution is t-linear whence $\operatorname{reg}_E M/\mathfrak{m}M = t$. The short exact sequence $0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0$ and Lemma 2.1.2 yield $\operatorname{reg}_E \mathfrak{m}M \leq t+1$ and thus $\mathfrak{m}M$ has a $(t+1)$-linear resolution, as it is generated in degree $t+1$. □

Corollary 5.3.5. *If $M \in \mathscr{M}$ is componentwise linear, then $M_{\leq k}$ is componentwise linear for all $k \in \mathbb{Z}$.*

PROOF. This follows directly from the previous lemma, since

$$(M_{\leq k})_{\langle j \rangle} = \begin{cases} M_{\langle j \rangle} & \text{for } j \leq k, \\ \mathfrak{m}^{j-k} M_{\langle k \rangle} & \text{for } j > k. \end{cases}$$

□

The graded Betti numbers of a componentwise linear module can be determined by the Betti numbers of its components.

Proposition 5.3.6. *Suppose that $M \in \mathscr{M}$ is componentwise linear. Then*

$$\beta_{i,i+j}(M) = \beta_i(M_{\langle j \rangle}) - \beta_i(\mathfrak{m}M_{\langle j-1 \rangle})$$

for all $j \in \mathbb{Z}$.

PROOF. Let s be the highest degree of a generator in a minimal set of generators of M and t be the lowest degree. We proceed by induction on $s \geq t$. Suppose $s = t$, that is, M is generated in degree t. Since M is componentwise linear, it has a t-linear resolution and thus $\beta_{i,i+j}(M) = 0$ for $j \neq t$. The j-th component is then

$$M_{\langle j \rangle} = \begin{cases} \mathfrak{m}^{j-t} M & \text{for } j > t, \\ M & \text{for } j = t, \\ 0 & \text{for } j < t. \end{cases}$$

Thus in this case the assertion is obvious.

Suppose that $s > t$. Then $M_{\leq s-1}$ is componentwise linear by Corollary 5.3.5, whence we can apply the induction hypothesis on it. Therefore, by Corollary 2.3.4, it holds that

$$\begin{aligned} \beta_{i,i+j}(M) = \beta_{i,i+j}(M_{\leq s-1}) = \beta_i((M_{\leq s-1})_{\langle j \rangle}) - \beta_i(\mathfrak{m}(M_{\leq s-1})_{\langle j-1 \rangle}) \\ = \beta_i(M_{\langle j \rangle}) - \beta_i(\mathfrak{m}M_{\langle j-1 \rangle}) \end{aligned}$$

for $j \leq s-1$. Thus the assertion is proved for $j \leq s-1$. Let $j \geq s$. The short exact sequence

$$0 \to M_{\leq s-1} \to M \to M_{\langle s \rangle}/\mathfrak{m}M_{\langle s-1 \rangle} \to 0,$$

which is exact by of the definition of s, gives rise to the long exact sequence

$$\dots \longrightarrow \operatorname{Tor}_i^E(K, M_{\leq s-1})_{i+j} \longrightarrow \operatorname{Tor}_i^E(K,M)_{i+j} \longrightarrow \operatorname{Tor}_i^E(K, M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle})_{i+j} \longrightarrow \operatorname{Tor}_{i-1}^E(K, M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle})_{(i-1)i+(j+1)} \longrightarrow \dots$$

The induction hypothesis yields

$$\beta_{i,i+j}(M_{\leq s-1}) = \beta_i((M_{\leq s-1})_{\langle j\rangle}) - \beta_i(\mathfrak{m}(M_{\leq s-1})_{\langle j-1\rangle}) = 0.$$

Hence, the long exact sequence of Tor-modules entails

$$\operatorname{Tor}_i^E(K,M)_{i+j} \cong \operatorname{Tor}_i^E(K, M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle})_{i+j}. \tag{7}$$

From the short exact sequence

$$0 \longrightarrow \mathfrak{m}M_{\langle s-1\rangle} \longrightarrow M_{\langle s\rangle} \longrightarrow M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle} \longrightarrow 0$$

we obtain the long exact sequence

$$\operatorname{Tor}_{i+1}^E(K, M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle})_{i+1+(j-1)} \to \operatorname{Tor}_i^E(K, \mathfrak{m}M_{\langle s-1\rangle})_{i+j} \to \operatorname{Tor}_i^E(K, M_{\langle s\rangle})_{i+j} \to \operatorname{Tor}_i^E(K, M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle})_{i+j} \to \operatorname{Tor}_{i-1}^E(K, \mathfrak{m}M_{\langle s-1\rangle})_{i-1+(j+1)}$$

The module at the right end vanishes for $j \neq s-1$, since $\mathfrak{m}M_{\langle s-1\rangle}$ has an s-linear resolution. The module at the left end vanishes for $j = s$, since $M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle}$ is generated in degree s. So for $j = s$ we obtain

$$\beta_{i,i+s}(M_{\langle s\rangle}) = \beta_{i,i+s}(\mathfrak{m}M_{\langle s-1\rangle}) + \beta_{i,i+s}(M_{\langle s\rangle}/\mathfrak{m}M_{\langle s-1\rangle}).$$

Taking into account (7) and that $\mathfrak{m}M_{\langle s-1\rangle}$ and $M_{\langle s\rangle}$ have s-linear resolutions, this proves the formula for $j = s$.

For $j > s$ the second, the third and the last Tor-module vanishes because the corresponding modules have s-linear resolutions. Therefore also the fourth one must be zero, that is, using (7),

$$\operatorname{Tor}_i^E(K,M)_{i+j} = 0.$$

This concludes the proof, since $\mathfrak{m}M_{\langle j-1\rangle} = M_{\langle j\rangle}$ for $j > s$. □

The next theorem characterises componentwise linearity of a module M in terms of the regularity of its truncations $M_{\leq k}$.

Theorem 5.3.7. *Let M be a module in $\mathscr{M}$. Then M is componentwise linear if and only if* $\operatorname{reg}_E(M_{\leq k}) \leq k$ *for all $k \in \mathbb{Z}$.*

PROOF. First assume that M is componentwise linear. Then Corollary 5.3.5 and Proposition 5.3.6 yield

$$\beta_{i,i+j}(M_{\leq k}) = \beta_i((M_{\leq k})_{\langle j\rangle}) - \beta_i(\mathfrak{m}(M_{\leq k})_{\langle j-1\rangle}).$$

For $j > k$ it holds that $(M_{\leq k})_{\langle j\rangle} = \mathfrak{m}(M_{\leq k})_{\langle j-1\rangle}$ and thus $\beta_{i,i+j}(M_{\leq k}) = 0$ in this case. Then $\operatorname{reg}_E(M_{\leq k}) \leq k$ follows immediately from the definition of the regularity.

Now suppose that $\operatorname{reg}_E(M_{\leq k}) \leq k$ for all k. We use induction on j to show that $M_{\langle j\rangle}$ as a j-linear resolution. Let $j = t$ be the lowest degree of a generator in a minimal set of generators. Then $M_{\langle t\rangle} = M_{\leq t}$ and thus $\operatorname{reg}_E M_{\langle t\rangle} = t$. Hence, $M_{\langle t\rangle}$ has a t-linear resolution.

For $j > t$ the induction hypothesis and Lemma 5.3.4 imply that $\mathfrak{m}M_{\langle j-1\rangle}$ has a j-linear resolution. From the short exact sequences

$$0 \longrightarrow M_{\leq j-1} \longrightarrow M_{\leq j} \longrightarrow M_{\langle j\rangle}/\mathfrak{m}M_{\langle j-1\rangle} \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{m}M_{\langle j-1\rangle} \longrightarrow M_{\langle j\rangle} \longrightarrow M_{\langle j\rangle}/\mathfrak{m}M_{\langle j-1\rangle} \longrightarrow 0$$

we obtain, using Lemma 2.1.2,

$$\begin{aligned} \operatorname{reg}_E M_{\langle j\rangle} &\leq \max\{\operatorname{reg}\mathfrak{m}M_{\langle j-1\rangle}, \operatorname{reg}_E M_{\langle j\rangle}/\mathfrak{m}M_{\langle j-1\rangle}\} \\ &\leq \max\{j, \max\{\operatorname{reg}_E M_{\leq j-1} - 1, \operatorname{reg}_E M_{\leq j}\} = \max\{j, j-2, j\} = j. \end{aligned}$$

□

The preceding theorem implies in particular that $\operatorname{reg}_E M \leq s$ if s is the highest degree of a generator in a minimal set of generators of a componentwise linear module M. However, this number is always a lower bound for the regularity. Therefore the following corollary holds.

Corollary 5.3.8. *Let $M \in \mathscr{M}$, $M \neq 0$ be componentwise linear. Then*

$$\operatorname{reg}_E M = \max\{j : \beta_{0,j}(M) \neq 0\}.$$

In the following we want to dualise the notion of a componentwise linear module.

Definition 5.3.9. For $t \in \mathbb{Z}$ we define $M^{\langle t\rangle} = M/N$ where N is the biggest submodule of M such that $N_t = 0$.

We call M *componentwise injective linear* if $M^{\langle t\rangle}$ has a linear injective resolution for all $t \in \mathbb{Z}$.

Remark that such a module N always exist, e.g., take the sum over all submodules N' with $N'_t = 0$.

Theorem 5.3.10. *A module $M \in \mathscr{M}$ is componentwise linear if and only if M^* is componentwise injective linear.*

PROOF. We show that $(M_{\langle t\rangle})^* \cong (M^*)^{\langle n-t\rangle}$ for $t \in \mathbb{Z}$. Then we have the following equivalent statements:

(a) M is componentwise linear.
(b) $M_{\langle t\rangle}$ has a linear projective resolution for all $t \in \mathbb{Z}$.
(c) $(M^*)^{\langle n-t\rangle}$ has a linear injective resolution for all $t \in \mathbb{Z}$.
(d) M^* is componentwise injective linear.

To prove the claim we consider the two exact sequences

$$0 \longrightarrow M_{\langle t\rangle} \longrightarrow M \longrightarrow M/M_{\langle t\rangle} \longrightarrow 0$$

and

$$0 \longrightarrow (M/M_{\langle t\rangle})^* \longrightarrow M^* \longrightarrow (M_{\langle t\rangle})^* \longrightarrow 0$$

which imply

$$(M_{\langle t\rangle})^* \cong M^*/(M/M_{\langle t\rangle})^*.$$

So we have to show that $(M/M_{\langle t\rangle})^*$ is the biggest submodule of M^* that is zero in degree $n-t$. First of all, using Lemma 1.1.1 we see

$$(M/M_{\langle t\rangle})^*_{n-t} \cong ((M/M_{\langle t\rangle})_t)^\vee = 0.$$

Now let N be a submodule of M^* with $N_{n-t} = 0$. We show that N is contained in $(M/M_{\langle t\rangle})^*$. Again using Lemma 1.1.1 we have $N^*_t = 0$. From the exact sequence

$$0 \longrightarrow (M^*/N)^* \longrightarrow M \longrightarrow N^* \longrightarrow 0$$

we obtain $M_t = ((M^*/N)^*)_t$ and thus $M_{\langle t\rangle} \subseteq (M^*/N)^*$. Hence $(M^*/N)^*/M_{\langle t\rangle}$ is a submodule of $M/M_{\langle t\rangle}$ and the quotient module is isomorphic to N^*. Dualizing once more gives $N \subseteq (M/M_{\langle t\rangle})^*$. □

Corollary 5.3.11. *If $M \in \mathscr{M}$ has a linear injective resolution then M is componentwise injective linear.*

Corollary 5.3.12. *If $M \in \mathscr{M}$ is componentwise injective linear and its socle lies in degree d, then M has a d-linear injective resolution.*

PROOF. Recall that the socle of a module M is isomorphic to $\operatorname{Ext}^0_E(K,M)$. Thus if the socle lies in degree d, this implies that $\mu_{0,j}(M) = \beta_{0,n-j}(M^*) = 0$ for all $j \neq d$ by Lemma 2.2.3. Therefore M^* is a componentwise linear module that is generated in one degree, whence it has an $(n-d)$-linear projective and M a d-linear injective resolution. □

Example 5.3.13. Let $J \subset E$ be a stable ideal. Then the ideal $(E/J)^*$ is also stable and thus componentwise linear by Example 5.3.2. So E/J is a componentwise injective linear E-module.

There is a characterisation of $d(M) = \max\{i : M_i \neq 0\}$ using Ext-modules:

Theorem 5.3.14. *Let $M \neq 0$ be in $\mathscr{M}$. Then*

$$d(M) = \min\{l \in \mathbb{Z} : \operatorname{Ext}^i_E(K,M)_{j-i} = 0 \textit{ for all } i \textit{ and } j > l\}.$$

PROOF. We only have to consider $i \geq 0$. Let $d = d(M)$ and

$$s = \min\{l \in \mathbb{Z} : \operatorname{Ext}^i_E(K,M)_{j-i} = 0 \text{ for all } i \text{ and } j > l\}.$$

Computing $\operatorname{Ext}^i_E(K,M)$ via the Cartan cocomplex directly shows $d \geq s$.

On the other hand,

$$\operatorname{Ext}^0_E(K,M)_d \cong \operatorname{Hom}_E(K,M)_d \cong (\operatorname{soc} M)_d \neq 0$$

because $0 \neq M_d \subseteq \operatorname{soc} M$. So this implies $d \leq s$. □

As counterpart for $d(M)$ we define $t(M)$ as follows.

Definition 5.3.15. For $M \in \mathscr{M}$, $M \neq 0$ let $t(M)$ denote the number

$$t(M) = \max\{l \in \mathbb{Z} : \operatorname{Ext}^i_E(K,M)_{j-i} = 0 \text{ for all } i \text{ and } j < l\}.$$

Thus the highest strand in a minimal injective resolution of M is $\operatorname{Ext}_E^i(K,M)_{d(M)-i}$ and the lowest is $\operatorname{Ext}_E^i(K,M)_{t(M)-i}$. Both numbers are equal if and only if M has a d-linear injective resolution (and then $d = d(M) = t(M)$).

For a face ring of a simplicial complex componentwise injective linearity is equivalent to a property of Δ, the sequentially Cohen-Macaulayness. A simplicial complex Δ is called *sequentially Cohen-Macaulay* (over K) if its Stanley-Reisner ring $K[\Delta]$ is sequentially Cohen-Macaulay. This is equivalent to say that $\Delta(k)$ is a Cohen-Macaulay complex for every k, where $\Delta(k)$ is the simplicial complex generated by the k-dimensional faces of Δ.

Proposition 5.3.16. *Let Δ be a simplicial complex. Then $K\{\Delta\}$ is componentwise injective linear if and only if Δ is sequentially Cohen-Macaulay.*

PROOF. Herzog, Reiner and Welker proved in [**31**, Theorem 9] that Δ is sequentially Cohen-Macaulay if and only if I_{Δ^*} is componentwise linear. This is the case if and only if $K\{\Delta\}$ is componentwise injective linear by Theorem 5.3.10 and Proposition 5.3.3. □

In the next part we prove a characterisation of componentwise injective linear modules that corresponds to the known characterisation in [**52**] of componentwise linear squarefree modules, which are exactly those whose Alexander dual is sequentially Cohen-Macaulay. For this purpose we need to consider modules M with chains of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$$

such that

(i) M_i/M_{i-1} has a linear injective resolution for $1 \le i \le t$,
(ii) $d(M_1/M_0) < d(M_2/M_1) < \ldots < d(M_t/M_{t-1})$.

We denote such a chain by filtration $(*)$ and further call the numbers $t, d_1, \ldots, d_t$, where $d_i = d(M_i/M_{i-1})$, the *combinatorial type* of the filtration.

Lemma 5.3.17. *Let $M \in \mathscr{M}$ have a filtration $(*)$ of combinatorial type $t, d_1, \ldots, d_t$.*

(i) *M_i has the filtration $(*)$ $M_0 \subset M_1 \subset \ldots \subset M_i$ of combinatorial type $i, d_1, \ldots, d_i$.*
(ii) *M/M_{i-1} has the filtration $(*)$ $M_i/M_i \subset \ldots \subset M_t/M_i$ of combinatorial type $t - i, d_{i+1}, \ldots, d_t$.*

PROOF. The first statement is evident. For the second just note that

$$d(M_j/M_i/M_{j-1}/M_i) = d(M_j/M_{j-1}).$$

□

The combinatorial type of a filtration $(*)$ is determined by M.

Theorem 5.3.18. *Let $M \in \mathscr{M}$ have a filtration $(*)$ of combinatorial type $t, d_1, \ldots, d_t$. Then*

(i) $\operatorname{Ext}_E^i(K,M)_{j-i} = 0$ *for* $j \notin \{d_1, \ldots, d_t\}$,
(ii) $\operatorname{Ext}_E^i(K,M)_{d_j-i} \cong \operatorname{Ext}_E^i(K, M_j/M_{j-1})_{d_j-i}$ *for* $j = 1, \ldots, t$.

PROOF. We proceed by an induction on t, the length of the filtration. If $t = 0$ then $M = 0$, so the claim is trivial. For $t = 1$ the only possible filtration is $0 \subset M$. In this case $M \cong M_1/M_0$ has a d_1-linear injective resolution and thus $\operatorname{Ext}_E^i(K,M)_{j-i} = 0$ for $j \neq d_1$. Furthermore $\operatorname{Ext}_E^i(K,M) \cong \operatorname{Ext}_E^i(K, M_1/M_0)$, so (ii) is obvious in this case.

Now assume $t > 1$ and consider the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0.$$

The module $M_1 \cong M_1/M_0$ has a d_1-linear injective resolution whence $\operatorname{Ext}^i_E(K,M_1)_{j-i} = 0$ for $j \neq d_1$. On the other side M/M_1 has the filtration $(*)$

$$0 = M_1/M_1 \subset \ldots \subset M_t/M_1 = M/M_1$$

of length $t-1$. By induction hypothesis we have $\operatorname{Ext}^i_E(K,M/M_1)_{j-i} = 0$ for $j \neq d_2,\ldots,d_t$ and $\operatorname{Ext}^i_E(K,M/M_1)_{d_j-i} \cong \operatorname{Ext}^i_E(K,M_j/M_{j-1})_{d_j-i}$ for $j = 2,\ldots,t$. The short exact sequence induces the long exact sequence

$$\operatorname{Ext}^i_E(K,M_1)_{j-i} \longrightarrow \operatorname{Ext}^i_E(K,M)_{j-i} \longrightarrow \operatorname{Ext}^i_E(K,M/M_1)_{j-i} \longrightarrow \operatorname{Ext}^{i+1}_E(K,M_1)_{(j+1)-(i+1)}$$

The first displayed module is zero for $j \neq d_1$, the third one for $j \neq d_2,\ldots,d_t$ and the last one for $j \neq d_1 - 1$. From this we see immediately that the second module, $\operatorname{Ext}^i_E(K,M)_{j-i}$, is zero for $j \neq d_1,d_2,\ldots,d_t$ which proves (i).

If $j = d_1$ we have the short exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_E(K,M_1)_{d_1-i} \longrightarrow \operatorname{Ext}^i_E(K,M)_{d_1-i} \longrightarrow 0.$$

hence $\operatorname{Ext}^i_E(K,M_1)_{d_1-i} \cong \operatorname{Ext}^i_E(K,M)_{d_1-i}$.

For $j \in \{d_2,\ldots,d_t\}$ the condition $d_1 < d_2$ implies $d_j + 1 \neq d_1$ such that we have the exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_E(K,M)_{d_j-i} \longrightarrow \operatorname{Ext}^i_E(K,M/M_1)_{d_j-i} \longrightarrow 0.$$

From this follows

$$\operatorname{Ext}^i_E(K,M)_{d_j-i} \cong \operatorname{Ext}^i_E(K,M/M_1)_{d_j-i} \cong \operatorname{Ext}^i_E(K,M_j/M_{j-1})_{d_j-i}.$$

□

An immediate consequence is:

Corollary 5.3.19. *Let $M \in \mathscr{M}$, $M \neq 0$ have a filtration $(*)$ of combinatorial type $t,d_1,\ldots,d_t$. Then $t(M) = d_1$ and $d(M) = d_t$.*

We will need some more properties of such filtrations.

Lemma 5.3.20. *Let M have a filtration $(*)$ of combinatorial type $t,d_1,\ldots,d_t$. Then $d(M_i) = d(M_i/M_{i-1})$ for $i = 1,\ldots,t$.*

PROOF. This follows from an application of Theorem 5.3.18 to the filtration

$$0 = M_0 \subset \ldots \subset M_i$$

of M_i, using the characterisation

$$d(M) = \min\{l \in \mathbb{Z} : \operatorname{Ext}^i_E(E/\mathfrak{m},M)_{j-i} = 0 \text{ for all } i \text{ and } j > l\}$$

from Theorem 5.3.14. □

Theorem 5.3.21. *A filtration* $(*)$ *is uniquely determined by* M.

PROOF. Let M have a filtration

$$0 = M_0 \subset \ldots \subset M_t = M$$

of combinatorial type $t, d_1, \ldots, d_t$. We show by induction on t that this filtration is the unique filtration $(*)$ of M.

The case $t = 0$ is trivial. If $t = 1$, the unique possibility for a filtration of length 1 is

$$0 = M_0 \subset M_1 = M.$$

Since, by Theorem 5.3.18, the length of a filtration $(*)$ is unique, this is also the only possibility for a filtration $(*)$ of M.

Now assume that $t > 1$ and consider the exact sequence

$$0 \longrightarrow M_{t-1} \longrightarrow M \longrightarrow M/M_{t-1} \longrightarrow 0.$$

The module M_{t-1} has the filtration $(*)$

$$0 = M_0 \subset \ldots \subset M_{t-1}$$

of length $t-1$. By induction hypothesis this filtration is unique. We claim that

$$M/M_{t-1} \cong (M^*_{\langle n-d_t \rangle})^*.$$

Then M/M_{t-1} is independent of the filtration because the right side of the equation depends only on M. (Recall that d_t is independent of the filtration by Theorem 5.3.18). This means thats in every filtration $(*)$ of M the second last term is equal to M_{t-1}. Since the filtration of M_{t-1} is unique, the whole filtration is unique.

It remains to proof the claim. We show the equivalent formulation

$$(M/M_{t-1})^* \cong M^*_{\langle n-d_t \rangle}.$$

Consider the exact sequence

$$0 \longrightarrow (M/M_{t-1})^* \longrightarrow M^* \longrightarrow (M_{t-1})^* \longrightarrow 0.$$

The module M/M_{t-1} has a d_t-linear injective resolution so that $(M/M_{t-1})^*$ has an $(n-d_t)$-linear projective resolution. In particular, $(M/M_{t-1})^*$ is generated in degree $n-d_t$. This already gives the inclusion

$$(M/M_{t-1})^* \subseteq M^*_{\langle n-d_t \rangle}.$$

Combining Lemma 5.3.20 with $d(M) = n - \min\{i \in \mathbb{Z} : (M^*)_i \neq 0\}$ (this follows from Lemma 1.1.1) we see

$$n - d_{t-1} = \min\{i : (M^*_{t-1})_i \neq 0\}.$$

So $d_{t-1} < d_t$ implies

$$(M^*_{t-1})_{n-d_t} = 0.$$

Thus the $(n-d_t)$-th homogeneous component of M^* maps to zero and is henceforth contained in $(M/M_{t-1})^*$. From this equality follows. □

So long we have investigated some properties of these filtrations $(*)$ but have not touched the question of the existence of such filtrations. This question can be answered completely.

Theorem 5.3.22. *M has a filtration (∗) if and only if M is componentwise injective linear.*

PROOF. Assume that M has a filtration $(*)$

$$0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$$

of combinatorial type $t, d_1, \ldots, d_t$. We will show that M is componentwise injective linear For this purpose we use an induction on t. The case $t = 0$ is obvious. If $t = 1$ we have that M has a linear injective resolution which implies that M is componentwise injective linear.

Now assume $t > 1$. By the induction hypothesis M_{t-1} is componentwise injective linear. The module M/M_{t-1} has a d_t-linear injective resolution. In the proof of the uniqueness theorem, Theorem 5.3.21, we have seen that

$$(M/M_{t-1})^* = (M/M_{t-1})^*_{\langle n-d_t\rangle} \cong M^*_{\langle n-d_t\rangle}.$$

Furthermore we know that $n - d_t = \min\{i : (M^*)_i \neq 0\}$. So the exact sequence

$$0 \longrightarrow (M/M_{t-1})^* \longrightarrow M^* \longrightarrow (M_{t-1})^* \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow (M/M_{t-1})^*_{\langle i\rangle} \longrightarrow M^*_{\langle i\rangle} \longrightarrow (M_{t-1})^*_{\langle i\rangle} \longrightarrow 0$$

for all $i \in \mathbb{Z}$ (see [**52**, Lemma 4.4] for a proof of the corresponding statement over a polynomial ring). The first and the third module in this sequence have linear projective resolutions which, by considering the corresponding long exact Tor-sequence, implies that $M^*_{\langle i\rangle}$ has a linear projective resolution as well. This means that M^* is componentwise linear or, equivalent, that M is componentwise injective linear.

Now assume that M is componentwise injective linear. Set $d = d(M)$. We show by induction on $d \geq t(M)$ that M has a filtration $(*)$. If $d = t(M)$, the least possible, M has a linear injective resolution and thus $0 \subseteq M$ is a filtration $(*)$.

For $d > t(M)$ it holds that $M^* \neq M^*_{\langle n-d\rangle}$ because otherwise M^* has a linear projective and M a linear injective resolution.

Since $d(M) = n - \min\{i : (M^*)_i \neq 0\}$, we have that $d((M^*/M^*_{\langle n-d\rangle})^*) < d$ and thus we can apply the induction hypothesis on $(M^*/M^*_{\langle n-d\rangle})^*$, i.e., there exists a filtration $(*)$

$$0 = M_0 \subseteq \ldots \subseteq M_{t-1} = (M^*/M^*_{\langle n-d\rangle})^* \subseteq M$$

for some t. It remains to show that $M/(M^*/M^*_{\langle n-d\rangle})^*$ has a d-linear injective resolution. But this is clear since $M/(M^*/M^*_{\langle n-d\rangle})^* \cong (M^*_{\langle n-d\rangle}) \cong M^{\langle d\rangle}$ and M is componentwise injective linear. □

5.4. Linear quotients and pure decomposable quotients

Monomial ideals with linear quotients have been introduced by Herzog and Takayama [**33**] to study resolutions that arise as iterated mapping cones. Over the exterior algebra one can see them as modules that are generated by iteratively pure decomposable elements.

Definition 5.4.1. Let M be a module in $\mathscr{M}$ with homogeneous generators $m_1, \dots, m_t$. Then M is said to have *linear quotients* with respect to this system of generators if $(m_1, \dots, m_{i-1}) :_E m_i$ is an ideal in E generated by linear forms for $i = 1, \dots, t$. We say that M has linear quotients if there exists a homogeneous system of generators such that M has linear quotients with respect to this system.

Note that in contrast to the definition of linear quotients over the polynomial ring it is important to start with $i = 1$, i.e., $0 :_E m_1$ has to be generated by linear forms or equivalently, m_1 has to be a pure decomposable element.

Example 5.4.2. The definition depends on the order of the generators. For example the ideal $J = (e_1, e_{12} + e_{34})$ has linear quotients w.r.t. this order of the generators, because

$$0 :_E e_1 = (e_1), \quad \text{and} \quad (e_1) :_E (e_{12} + e_{34}) = (e_1, e_3, e_4).$$

But $e_1e_2 + e_3e_4$ is not pure decomposable and hence J has not linear quotients with respect to the reversed order on the generators.

The regularity of a module with linear quotients is the smallest possible, just as for componentwise linear modules.

Theorem 5.4.3. *Let $M \in \mathscr{M}$, $M \neq 0$, have linear quotients w.r.t. homogeneous generators $m_1, \dots, m_t$. Then the regularity of M is*

$$\operatorname{reg}_E(M) = \max\{\deg m_1, \dots, \deg m_t\}.$$

PROOF. It is clear that

$$\operatorname{reg}_E(M) \geq \max\{\deg m_1, \dots, \deg m_t\}$$

by the definition of the regularity.

Let M_i be the module generated by the first i elements $m_1, \dots, m_i$, for $i = 1, \dots, t$. We prove the reverse inequality inductively for each of the M_i. The first module M_1 is generated by a pure decomposable element. Hence, Proposition 4.1.17 implies $\operatorname{reg}_E(M_1) = \deg m_1$.

The sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow (E/(M_{i-1} :_E m_i))(-\deg m_i) \longrightarrow 0$$

is exact. Since $M_{i-1} :_E m_i$ is generated by linear forms, the Cartan complex with respect to this linear forms, shifted in degree by $-\deg m_i$, is a $(\deg m_i)$-linear projective resolution of it. In particular, the regularity of the last module is $\deg m_i$. Thus by induction hypothesis and Lemma 2.1.2 we conclude

$$\begin{aligned} \operatorname{reg}_E M_i &\leq \max\{\operatorname{reg}_E(M_{i-1}), \operatorname{reg}_E(E/(M_{i-1} :_E m_i)(-\deg m_i))\} \\ &= \max\{\max\{\deg m_1, \dots, \deg m_{i-1}\}, \deg m_i\} \\ &= \max\{\deg m_1, \dots, \deg m_i\}. \end{aligned}$$

□

Corollary 5.4.4. *If $M \in \mathscr{M}$ has linear quotients and is generated in one degree, then M has a linear resolution.*

For graded ideals the above corollary also follows from the next theorem, whose proof is quite analogous to the proof of the corresponding statement over a polynomial ring of Sharifan and Varbaro in [**55**, Corollary 2.4].

Theorem 5.4.5. *Let $J \subset E$ be a graded ideal and $f_1, \dots, f_t$ be a minimal homogeneous system of generators of J. If J has linear quotients with respect to $f_1, \dots, f_t$, then J is a componentwise linear ideal.*

PROOF. We proceed by an induction on t. For $t = 1$ the ideal $J = (f_1)$ is generated by a pure decomposable element whence it has a linear resolution. In particular, it is componentwise linear. For the induction step let $d = \deg f_t$ and

$$I = (f_1, \dots, f_{t-1}).$$

By the induction hypothesis I is componentwise linear. Obviously $J_{\langle j \rangle} = I_{\langle j \rangle}$ for $j < d$ hence $J_{\langle j \rangle}$ has a j-linear resolution for $j < d$. As next step we show that $J_{\langle d \rangle}$ has a linear resolution. To this end we show that $I : f_t = I_{\langle d \rangle} : f_t$. Since $I_{\langle d \rangle} \subseteq I$ it follows immediately that

$$I_{\langle d \rangle} : f_t \subseteq I : f_t.$$

As J has linear quotients, the ideal $I : f_t$ is generated by linear forms $v_1, \dots, v_m$. In particular, it holds that $v_i f_t \in I$ for $i = 1, \dots, m$, i.e., there exists $g_{ij} \in E$ such that

$$v_i f_t = \sum_{j=1}^{t-1} g_{ij} f_j.$$

The assumption that $f_1, \dots, f_t, f$ is a minimal system of generators for J implies that

$$\deg g_{ij} \geq 1$$

for all $g_{ij} \neq 0$. As the degree of $v_i f_t$ is $d+1$, the degree of f_j is $\leq d$ for all j with $g_{ij} \neq 0$. Hence $v_i f_t \in I_{\leq d}$ and thus $v_i \in I_{\langle d \rangle} : f_t$ because $(I_{\leq d})_{d+1} = (I_{\langle d \rangle})_{d+1}$. This proves

$$I : f_t = I_{\langle d \rangle} : f_t.$$

Using $J_{\langle d \rangle} = I_{\langle d \rangle} + (f_t)$ we obtain

$$J_{\langle d \rangle} / I_{\langle d \rangle} \cong E / (I_{\langle d \rangle} : f_t)(-d) \cong E / (I : f_t)(-d) \cong E / (v_1, \dots, v_m)(-d).$$

Thus the sequence

$$0 \longrightarrow I_{\langle d \rangle} \longrightarrow J_{\langle d \rangle} \longrightarrow E/(v_1, \dots, v_m)(-d) \longrightarrow 0$$

is exact. The first module in this sequence has a d-linear resolution by induction hypothesis, for the third module the Cartan complex shifted in degrees by $-d$ is a d-linear resolution. Then Lemma 2.1.2 implies that $J_{\langle d \rangle}$ has a d-linear resolution as well.

It remains to show that $J_{\langle j \rangle}$ has a j-linear resolution for $j > d$. Let r be the regularity of J, i.e.,

$$r = \max\{\deg f_1, \dots, \deg f_t\}$$

by Theorem 5.4.3. If $d \geq r$, i.e., if f_t is the generator with the highest degree, then $J_{\langle d+j\rangle} = \mathfrak{m}^j J_{\langle d\rangle}$ for all $j \geq 1$. Since $J_{\langle d\rangle}$ has a linear resolution, also $\mathfrak{m}^j J_{\langle d\rangle}$ has a linear resolution, by Lemma 5.3.4.

For $d \leq r$ we proceed by a second induction on $r-d$. Recall that the colon ideal $I : f_t$ is generated by the linear forms $v_1, \dots, v_m$. If $m = n$ they generate the maximal ideal $\mathfrak{m}$ and it holds that

$$J_{\langle d+j\rangle} = I_{\langle d+j\rangle}. \tag{8}$$

To see the non-trivial inclusion in (8) take $g \in J_{\langle d+j\rangle}$. Then g has a representation $g = g' + hf_t$ with $g' \in I_{\langle d+j\rangle}$ and $h \in E$, $\deg h = j \geq 1$. Thus $hf_t \in \mathfrak{m} f_t \subseteq I$ and $g \in I_{\langle d+j\rangle}$. Therefore we assume that $m < n$ and complement $v_1, \dots, v_m$ with $v_{m+1}, \dots, v_n$ to a basis of E_1. Let

$$g_i = v_i f_t \qquad \text{for } \ i = m+1, \dots, n.$$

Then we will see that

$$(I + (g_{m+1}, \dots, g_n))_{\langle d+j\rangle} = J_{\langle d+j\rangle} \tag{9}$$

holds for all $j \geq 1$. Since $g_i = v_i f_t \in (f_t)$ we have that $(I + (g_{m+1}, \dots, g_n)) \subseteq J = I + (f_t)$. On the other hand let h be in $J_{\langle d+j\rangle}$, $h = h' + h'' f_t$ with $h' \in I$, $h'' \in \mathfrak{m} = (v_1, \dots, v_m, v_{m+1}, \dots, v_n)$. Since $I : f_t = (v_1, \dots, v_m)$ we can assume that $h'' \in (v_{m+1}, \dots, v_n)$ and thus that $h \in I + (g_{m+1}, \dots, g_n)$.

Therefore by (9) it is enough to show that $I + (g_{m+1}, \dots, g_n)$ is componentwise linear. To this end we show that we can apply the induction hypothesis (of the induction on $r-d$) to $I + (g_{m+1}, \dots, g_n)$.

At first we will see that $I + (g_{m+1}, \dots, g_n)$ has linear quotients, more precisely that

$$(I + (g_{m+1}, \dots, g_{m+i-1})) : g_{m+i} = (v_1, \dots, v_m, v_{m+1}, \dots, v_{m+i})$$

for all $i = 1, \dots, n-m$. The inclusion "$\supseteq$" immediately follows from $g_{m+i} = v_{m+i} f_t$ and $I : f_t = (v_1, \dots, v_m)$. So let $h \in (I + (g_{m+1}, \dots, g_{m+i-1})) : g_{m+i}$. Then there exists $a_k, b_k \in E$ such that

$$v_{m+i} f_t h = \sum_{k=1}^{t-1} a_k f_k + \sum_{k=m+1}^{m+i-1} b_k v_k f_t.$$

Thus

$$v_{m+i} h - \sum_{k=m+1}^{m+i-1} b_k v_k \in I : f_t = (v_1, \dots, v_m)$$

and hence

$$v_{m+i} h \in (v_1, \dots, v_m, v_{m+1}, \dots, v_{m+i-1}).$$

Since v_{m+i} is not contained in the linear ideal $(v_1, \dots, v_m, v_{m+1}, \dots, v_{m+i-1})$, it is regular on it. So it follows that

$$h \in (v_1, \dots, v_m, v_{m+1}, \dots, v_{m+i-1}) + (v_{m+i}).$$

We further claim that $f_1,\dots,f_{t-1},g_{m+1},\dots,g_{m+i}$ is a minimal system of generators for $I+(g_{m+1},\dots,g_{m+i})$ for all i. Suppose that there exists $a_k,b_k \in E$ such that

$$f_l = \sum_{k=1,k\neq l}^{t-1} a_k f_k + \sum_{k=1}^{i} b_{m+k} g_{m+k} = \sum_{k=1,k\neq l}^{t-1} a_k f_k + \left(\sum_{k=1}^{i} b_{m+k} v_{m+k}\right) f_t$$

for some $1 \le l < t$. Then $f_l \in (f_1,\dots,\widehat{f_l},\dots,f_t)$, a contradiction to the assumption that $f_1,\dots,f_t$ is a minimal system of generators of J.

Now suppose that there exists $a_k,b_k \in E$ such that

$$g_{m+l} = \sum_{k=1}^{t-1} a_k f_k + \sum_{k=1,k\neq l}^{i} b_{m+k} g_{m+k}$$

for some $1 \le l \le i$. The degree of $g_{m+l} = v_{m+l} f_t$ equals the degree of the $g_{m+k} = v_{m+k} f_t$, hence the coefficients b_{m+k} are elements in K and

$$(v_{m+l} - \sum_{k=1,k\neq l}^{i} b_{m+k} v_{m+k}) f_t = \sum_{k=1}^{t-1} a_k f_k \in I.$$

Therefore $(v_{m+l} - \sum_{k=1,k\neq l}^{i} b_{m+k} v_{m+k}) \in I : f_t = (v_1,\dots,v_m)$, again a contradiction. This proves the claim that $f_1,\dots,f_{t-1},g_{m+1},\dots,g_{m+i}$ is a minimal system of generators for $I+(g_{m+1},\dots,g_{m+i})$ for all i.

Finally we note that

$$\max\{\deg f_1,\dots,\deg f_{t-1},\deg g_{m+1},\dots,\deg g_{m+i}\} = \max\{r,d+1\}$$

and $r-(d+1) = (r-d)-1$. So by induction hypothesis $I+(g_{m+1},\dots,g_{m+i})$ is componentwise linear and this concludes the proof. □

Example 5.4.6. The hypothesis in Theorem 5.4.5 that the generating system is minimal cannot be removed. For example the ideal $J=(e_{12},e_{34})$ does not have a linear resolution, and as it is generated in one degree, this means that J is not componentwise linear. But J has linear quotients with respect to the non-minimal homogeneous generating system $e_{12},e_{123}+e_{234},e_{34}$.

The definition of linear quotients for a module over E is exactly the same as for modules over a polynomial ring. For simplicial complexes these notions coincide. This is not surprising since these notions correspond to a combinatorial property of Δ, as we will see later on in this section.

Proposition 5.4.7. *Let Δ be a simplicial complex. The face ideal J_Δ of Δ has linear quotients if and only if the Stanley-Reisner ideal I_Δ of Δ has linear quotients.*

PROOF. Both ideals are generated by the respective monomials corresponding to the minimal non-faces $F_1,\dots,F_m$ of Δ. Thus the colon ideals appearing in the definition are monomial ideals, too. Then $(x_{F_1},\dots,x_{F_{i-1}}) : x_{F_i}$ is generated by $\{x_j : j \in A\}$ if and only if $(e_{F_1},\dots,e_{F_{i-1}}) : e_{F_i}$ is generated by $\{e_j : j \in A \cup F_i\}$. □

In the following we introduce a notion that turns out to be the dual notion to linear quotients. Recall from Section 4.1 that an element in E is called pure decomposable if it is a product of linear forms. An ideal is called pure decomposable if it is generated by pure decomposable elements.

Definition 5.4.8. Let $J = J_1 \cap \ldots \cap J_m \subset E$ be a graded ideal and let the ideals J_i be graded. We say that E/J has *pure decomposable quotients* with respect to this decomposition if

(i) $\dim_K \operatorname{soc}(E/J_i) = 1$ for $i = 1, \ldots, m$;
(ii) the quotient $\bigcap_{j=1}^{i-1} J_j / \bigcap_{j=1}^{i} J_j$ is isomorphic to an ideal that is generated by a pure decomposable element, shifted in degrees by $n - d_i$, $d_i = d(E/J_i)$, for $i = 1, \ldots, m$.

We say that E/J has *pure decomposable quotients* if there exists a decomposition such that E/J has pure decomposable quotients with respect to this decomposition.

The second condition in the definition implies that the i-th quotient $\bigcap_{j=1}^{i-1} J_j / \bigcap_{j=1}^{i} J_j$ has an $(n-d_i)$-linear injective resolutions.

Observe that a decomposition into an intersection of ideals always exists. Suppose that $f_1, \ldots, f_m$ is a homogeneous system of generators of $0 :_E J$, then take $J_i = 0 :_E f_i$ since $J = 0 :_E (0 :_E J) = 0 :_E \sum_{i=1}^{m} (f_i) = \bigcap_{i=1}^{m} 0 :_E (f_i)$.

Example 5.4.9. The definition depends on the order of the ideals. For example consider the ideal $J = (e_1e_3, e_1e_4)$ with the decomposition

$$J = J_1 \cap J_2, \quad J_1 = (e_1), \quad J_2 = (e_{13}, e_{14}, e_{23}, e_{24}, e_{12} - e_{34}).$$

We have $\operatorname{soc}(E/J_1) = (e_{234}) + (e_1)/(e_1)$ and $\operatorname{soc}(E/J_2) = (e_{34}) + J_2/J_2$. Thus the socles are one-dimensional K-vector spaces. They lie in degree $d_1 = d(E/J_1) = 3$ and $d_2 = d(E/J_2) = 2$ respectively. In this order of the decomposition the first quotient is

$$E/(e_1) \cong (e_1)(+1)$$

and the second is

$$(e_1)/(e_{13}, e_{14}) \cong (e_{134})(+2).$$

(The isomorphisms are induced by the multiplication with e_1 resp. e_{34}.) Therefore E/J has pure decomposable quotients with respect to the decomposition $J_1 \cap J_2$. But if one reverses the order of the decomposition, the first quotient is

$$E/(e_{13}, e_{14}, e_{23}, e_{24}, e_{12} - e_{34})$$

and this cannot be isomorphic to a pure decomposable principal ideal. Otherwise the annihilator of E/J_2, which is J_2, would be the annihilator of this ideal. But the annihilator of a pure decomposable principal ideal is a linear ideal.

The argument used in the above example is true in general, so we state that the first ideal in a decomposition that gives pure decomposable quotients must be a linear ideal.

On the level of monomial ideals, or equivalently, exterior face rings, this algebraic property corresponds to a well-known combinatorial property of simplicial complexes.

We recall one formulation of this property from, e.g., [**11**, Definition 5.1.11], which is perhaps not the most frequently used one, but the one we need in the proof. A simplicial complex Δ is called *shellable* if there exists an order $F_1, \ldots, F_m$ of the facets of Δ in such a way that the set $\{F : F \in \langle F_1, \ldots, F_i \rangle, F \notin \langle F_1, \ldots, F_{i-1} \rangle\}$ has a unique minimal element for all i, $2 \leq i \leq m$. Here $\langle F_1,\ldots,F_i\rangle$ denotes the simplicial complex whose facets are $F_1,\ldots,F_i$. Such an order of the facets is called a *shelling*.

Theorem 5.4.10. *Let Δ be a simplicial complex with facets $F_1,\ldots,F_m$ and let J_i be the ideal generated by the variables e_j with $j \notin F_i$ for $i = 1,\ldots,m$. Then $J_\Delta = J_1 \cap \ldots \cap J_m$ and $K\{\Delta\}$ has pure decomposable quotients with respect to this decomposition if and only if $F_1,\ldots,F_m$ is a shelling of Δ.*

PROOF. J_Δ is the ideal generated by the monomials corresponding to the (minimal) non-faces of Δ, i.e., a monomial e_F is contained in J_Δ if and only if $F \not\subseteq F_i$, or equivalently $e_F \in J_i$, for $i = 1,\ldots,m$.

First assume that $F_1,\ldots,F_m$ is a shelling of Δ. The first condition in Definition 5.4.8 is easy to check:

$$\operatorname{soc}(E/J_i) = \operatorname{soc}(E/(e_j : j \notin F_i)) = K\overline{e_{F_i}}.$$

This also shows $d_i = d(E/J_i) = |F_i|$. Let $F_{\min}$ denote the minimal element of the set $\{F : F \in \langle F_1,\ldots,F_i\rangle, F \notin \langle F_1,\ldots,F_{i-1}\rangle\}$ which exists by the definition of a shelling. We show that

$$\bigcap_{j=1}^{i-1} J_j \Big/ \bigcap_{j=1}^{i} J_j \cong (e_{F_{\min}\cup F_i^c})(n-d_i), \tag{10}$$

where $F_i^c = [n] \setminus F_i$. To this end we consider the following map

$$\varphi : \bigcap_{j=1}^{i-1} J_j \xrightarrow{\cdot e_{F_i^c}} (e_{F_i^c})(n-d_i).$$

It is a graded homomorphism because $\deg e_{F_i^c} = n - |F_i| = n - d_i$. The kernel is

$$\operatorname{Ker}\varphi = \bigcap_{j=1}^{i-1} J_j \cap (0 :_E e_{F_i^c}) = \bigcap_{j=1}^{i-1} J_j \cap J_i = \bigcap_{j=1}^{i} J_j.$$

We claim that the image of φ is $(e_{F_{\min}\cup F_i^c})(n-d_i)$. The set $F_{\min}$ is the minimal element of the set $\{F : F \subseteq F_i, F \not\subseteq F_j \text{ for all } j < i\}$. Thus $e_{F_{\min}}$ is contained in J_j for all $j < i$, i.e., $e_{F_{\min}} \in \bigcap_{j=1}^{i-1} J_j$, and $e_{F_{\min}\cup F_i^c}$ is in the image of φ. An arbitrary monomial e_G is an element in $\bigcap_{j=1}^{i-1} J_j$ if and only if $G \not\subseteq F_j$ for all $j = 1,\ldots,i-1$. If $G \cap F_i^c \neq \emptyset$ then e_G is in the kernel of φ. Otherwise $G \subseteq F_i$ and hence $G \in \{F : F \subseteq F_i, F \not\subseteq F_j \text{ for all } j < i\}$. Since $F_{\min}$ is the unique minimal element of this set it follows that $F_{\min} \subseteq G$. Therefore the image of φ is contained in $(e_{F_{\min}\cup F_i^c})$. All together φ induces an isomorphism as in (10).

Now assume that $K\{\Delta\}$ has pure decomposable quotients with respect to $J_\Delta = J_1 \cap \ldots \cap J_m$. Then there exists linear forms v_{i_j}, $i = 1,\ldots,m$, $j = 1,\ldots,s_i$ such that

$$\bigcap_{j=1}^{i-1} J_j \Big/ \bigcap_{j=1}^{i} J_j \cong (v_{i_1}\cdots v_{i_{s_i}})(n-d_i),$$

with $d_i = d(E/J_i) = |F_i|$ as above. Since the J_j are monomial ideals, these quotients are $\mathbb{Z}^n$-graded. Thus also their annihilators are $\mathbb{Z}^n$-graded, i.e., monomial ideals. From the above isomorphism follows

$$\operatorname{ann}_E\left(\bigcap_{j=1}^{i-1} J_j \Big/ \bigcap_{j=1}^{i} J_j\right) = \operatorname{ann}_E\left((v_{i_1}\cdots v_{i_{s_i}})(n-d_i)\right) = (v_{i_1},\ldots,v_{i_{s_i}}),$$

whence we conclude that the v_{i_j} are variables, say $v_{i_j} = e_{i_j}$.

Set $S_i = \{i_1,\ldots,i_{s_i}\}$. We show that $S_i \cap F_i$ is the unique minimal element of the set

$$\mathscr{F}_i = \{F : F \subseteq F_i, F \not\subseteq F_j \text{ for all } j < i\}$$

for $i = 1,\ldots,m$. Let F be an element of $\mathscr{F}_i$ and suppose there exists an element $l \in S_i \cap F_i$, $l \notin F$. Then $e_l\overline{e_F} = 0$ and thus, since $e_l e_F \neq 0$, $F \cup \{l\} \not\subseteq F_i$. This is a contradiction and so every element of $\mathscr{F}_i$ indeed contains $S_i \cap F_i$. The complement of F_i is contained in S_i: suppose there exists an element $l \in F_i^c \setminus S_i$. Since $F_i \not\subseteq F_j$ for $j < i$, the monomial e_{F_i} is in $\bigcap_{j=1}^{i-1} J_j$ and its residue class is not zero in $\bigcap_{j=1}^{i-1} J_j \big/ \bigcap_{j=1}^{i} J_j$. Also $e_l\overline{e_{F_i}} \neq 0$ because $l \notin S_i$ and $l \notin F_i$. But this means that $F_i \cup \{l\} \subseteq F_i$, obviously a contradiction. Hence

$$|S_i \cap F_i| = |S_i \setminus F_i^c| = |S_i| - (n - |F_i|) = s_i - n + d_i.$$

The degree of $e_{i_1}\cdots e_{i_{s_i}}$ in $(e_{i_1}\cdots e_{i_{s_i}})(n-d_i)$ also is $s_i - n + d_i$. Therefore there exists a non-zero element of this degree in $\bigcap_{j=1}^{i-1} J_j \big/ \bigcap_{j=1}^{i} J_j$ and it can be chosen as the residue class of a monomial e_F with $F \in \mathscr{F}_i$, $|F| = s_i - n + d_i$. We have already seen that $S_i \cap F_i \subseteq F$. Since these sets have the same cardinality, they are equal. In particular, it holds that $S_i \cap F_i \in \mathscr{F}_i$. □

The dual property to pure decomposable quotients are linear quotients.

Theorem 5.4.11. *Let J be a graded ideal in E. Then E/J has pure decomposable quotients if and only if $(E/J)^*$ has linear quotients.*

PROOF. Every system of generators $f_1,\ldots,f_t$ of $(E/J)^* \cong 0 :_E J$ corresponds to a decomposition $J = J_1 \cap \ldots \cap J_t$ with $\dim_K \operatorname{soc}(E/J_i) = 1$ via $J_i = 0 :_E (f_i)$ for $i = 1,\ldots,t$ and vice versa. The reason for this are the equalities

$$J = 0 :_E (0 :_E J)) = 0 : \sum_{i=1}^{t}(f_i) = \bigcap_{i=1}^{t} 0 :_E (f_i) = \bigcap_{i=1}^{t} J_i$$

and

$$\dim_K \operatorname{soc}(E/J_i) = \mu_0(E/J_i) = \beta_0((E/J_i)^*) = \beta_0((f_i)) = 1,$$

using Lemma 2.2.3 and the fact that $\operatorname{soc}(E/J_i) \cong \operatorname{Hom}_E(K, E/J_i) \cong \operatorname{Ext}_E^0(K, E/J_i)$. If one also takes into account the grading one obtains that the socle lies in degree $n - \deg f_i$, i.e., $d_i = d(E/J_i) = n - \deg f_i$.

The sequence

$$0 \longrightarrow (E/(f_1,\ldots,f_{i-1}) : f_i)(-\deg f_i) \xrightarrow{\cdot f_i} E/(f_1,\ldots,f_{i-1}) \longrightarrow E/(f_1,\ldots,f_i) \longrightarrow 0$$

is exact. Thus also the dualized sequence

$$0 \longrightarrow 0 :_E (f_1,\ldots,f_i) \longrightarrow 0 :_E (f_1,\ldots,f_{i-1}) \longrightarrow (0 :_E ((f_1,\ldots,f_{i-1}) : f_i))(+\deg f_i) \longrightarrow 0$$

is exact.

Using $0 :_E (f_1,\dots,f_i) = \bigcap_{j=1}^{i} J_j$, $\deg f_i = n - d_i$ and the fact that the annihilator of a linear ideal is a pure decomposable principal ideal and vice versa we conclude that $(f_1,\dots,f_t)$ has linear quotients if and only if $E/J_1 \cap \dots \cap J_t$ has pure decomposable quotients. □

Theorems 5.4.10, 5.4.11 and Proposition 5.4.7 together form another proof of the well-known fact that a simplicial complex Δ is shellable if and only if the face ideal of the Alexander dual I_{Δ^*} has linear quotients.

Corollary 5.4.12. *Let J be a graded ideal in E. If E/J has pure decomposable quotients with respect to a decomposition $J = J_1 \cap \dots \cap J_m$ that satisfies $\bigcap_{j=1, j\neq i}^{m} J_j \not\subseteq J_i$, then E/J is componentwise injective linear. If in addition the socle of E/J lies in one degree, then E/J has a linear injective resolution.*

PROOF. This follows from Theorems 5.3.10, 5.4.11 and 5.4.5. The assumption on the decomposition guarantees that one obtains a minimal system of generators for $0 :_E J$. □

Open problem 5.4.13. *Is there a direct proof of Corollary 5.4.12 (without use of duality)?*

5.5. Strongly pure decomposable ideals

Pure decomposable ideals are generated by elements whose annihilators are generated by linear forms. In this section we investigate a sharpening of this condition: in addition, these annihilators should have nice intersections.

Definition 5.5.1. An ideal J in the exterior algebra E is called *strongly pure decomposable* if J has a system of pure decomposable generators $f_1,\dots,f_t$ such that the annihilators $\operatorname{ann}_E f_i = (v : v \in V_i)$ satisfy the condition $V_i \cap (V_1 + \dots + \hat{V_i} + \dots + V_t) = \{0\}$ for every $i = 1,\dots,t$.

For simplicity we often omit the pure and say strongly decomposable ideal. The definition depends on the chosen generating system as the following example shows.

Example 5.5.2. Let $J = (f_1, f_2)$ in $E = K\langle e_1,\dots,e_6\rangle$ with $f_1 = e_{123}$, $f_2 = e_{456}$. Then $V_1 = \langle e_1,e_2,e_3\rangle$ and $V_2 = \langle e_4,e_5,e_6\rangle$. Therefore this generating system obviously satisfies the conditions in Definition 5.5.1. But the (minimal) generating system $g_1 = f_1 + f_2$, $g_2 = f_2$ does not, since the annihilator of g_1 is generated by the monomials e_ie_j with $i \in \{1,2,3\}$, and $j \in \{4,5,6\}$ and not by linear forms.

The following observation will be useful in some proofs.

Remark 5.5.3. Let $J = (f_1,\dots,f_t)$ be as in Definition 5.5.1. Then $V_1 \oplus \dots \oplus V_t$ is a direct sum of vector spaces. Therefore we can choose bases of the V_i and complement the union of all the basis elements to a basis of E_1. Thus we can assume that after a coordinate transformation the annihilators are generated by variables. Then Lemma 4.1.16 shows that the f_i become monomials under this coordinate transformation.

To check whether a monomial ideal is strongly decomposable is easy.

Proposition 5.5.4. *Let $J = (e_{F_1}, \ldots, e_{F_t})$ be a monomial ideal in E. Then J is strongly decomposable (with respect to this generating system) if and only if $F_1, \ldots, F_t$ are pairwise disjoint.*

PROOF. A monomial is obviously pure decomposable. With $V_i = \langle e_j : j \in F_i \rangle$ the condition $V_i \cap (V_1 + \ldots + \hat{V}_i + \ldots + V_t) = \{0\}$ is equivalent to $F_i \cap (F_1 \cup \ldots \cup \hat{F}_i \cup \ldots \cup F_t) = \emptyset$. It is satisfied for all $i = 1, \ldots, t$ if and only if $F_i \cap F_j = \emptyset$ for all $j \neq i$. □

A monomial ideal in a polynomial ring is a *complete intersection* if and only if the support of the monomials are pairwise disjoint. In view of the preceding proposition the notion of strongly decomposable could be the exterior analogue of a complete intersection. We will see that there are even more parallels, but nevertheless the behaviour of strongly decomposable is not completely satisfactory. An arbitrary ideal in a polynomial ring is a *complete intersection* if and only if it is generated by a regular sequence. This is also true for strongly decomposable ideals, at least if there is a notion of a regular sequence, i.e., if it is generated by linear forms.

Proposition 5.5.5. *Let $J = (v_1, \ldots, v_t)$ with linear forms $v_1, \ldots, v_t \in E_1$. Then J is strongly decomposable if and only if $v_1, \ldots, v_t$ is a regular sequence on E.*

PROOF. The annihilator of a linear form is the ideal generated by the linear form. Therefore we have that $V_i = \langle v_i \rangle$ for $i = 1, \ldots, t$. Then the condition $V_i \cap (V_1 + \ldots + \hat{V}_i + \ldots + V_t) = \{0\}$ is equivalent to $v_i \notin V_1 + \ldots + \hat{V}_i + \ldots + V_t$. It is satisfied for all i if and only if $v_1, \ldots, v_t$ are linearly independent, which is equivalent to say that $v_1, \ldots, v_t$ is a regular sequence on E. □

The generating systems that satisfy the conditions of the definition of strongly decomposable are minimal systems of generators.

Proposition 5.5.6. *If $J = (f_1, \ldots, f_t)$ is strongly decomposable, then $f_1, \ldots, f_t$ is a minimal system of generators of J.*

PROOF. As explained in Remark 5.5.3 we can assume that after a coordinate change the f_i are monomials since a coordinate transformation does not change the minimality property.

Following Proposition 5.5.4 the monomials have pairwise disjoint support and are therefore a minimal set of generators. □

Strongly decomposable ideals are a class of modules over the exterior algebra with a linear injective resolution.

Theorem 5.5.7. *If $J = (f_1, \ldots, f_t)$ is strongly decomposable, then E/J has an $(n-t)$-linear injective resolution.*

PROOF. Again we may assume that the f_i are monomials as in Remark 5.5.3 since a coordinate transformation does not touch the property of having a minimal injective resolution. Therefore J has the form $J = J_\Delta$ for some simplicial complex Δ. As stated in Proposition 5.5.4 this implies that I_Δ is a complete intersection. In particular, Δ is Cohen-Macaulay and thus E/J_Δ has a linear injective resolution. The dual $0 : J$ of E/J is generated in degree t so the resolution must be $(n-t)$-linear. □

Note that, in contrast to the property strongly decomposable, having a linear injective resolution does not depend on the choice of generators.

Lemma 5.5.8. *Let* $J = (e_{F_1}, \dots, e_{F_t})$ *be a strongly decomposable monomial ideal in* E *and* $[n] \setminus (F_1 \cup \dots \cup F_t) = \{i_1, \dots, i_s\}$. *Then* $e_{i_1}, \dots, e_{i_s}$ *is a maximal regular sequence on* E/J. *In particular,* $\operatorname{depth}_E(E/J) = n - |F_1 \cup \dots \cup F_t|$ *and, if* $|K| = \infty$, $\operatorname{cx}_E(E/J) = |F_1 \cup \dots \cup F_t|$.

PROOF. It is easy to see that $e_{i_1}, \dots, e_{i_s}$ is a regular sequence. Therefore we assume that $F_1 \cup \dots \cup F_t = [n]$ and show that in this situation $\operatorname{depth}_E(E/J) = 0$. The ideal J has a decomposition $J = J_1 \cap \dots \cap J_t$ with $J_i = (e_j : j \in F_i)$. If $t = 1$ then $J = J_1 = (e_1, \dots, e_n)$ has depth 0. For $t > 1$ there is the exact sequence

$$0 \longrightarrow J \longrightarrow (J_1 \cap \dots \cap J_{t-1}) \oplus J_t \longrightarrow (J_1 \cap \dots \cap J_{t-1}) + J_t \longrightarrow 0.$$

The F_i are pairwise disjoint and thus $F_1 \cup \dots \cup F_{t-1} \neq [n]$. By the induction hypothesis the variables that generate J_t are a maximal regular sequence on $J_1 \cap \dots \cap J_{t-1}$. Hence, $(J_1 \cap \dots \cap J_{t-1}) + J_t$ has depth 0. There exist regular elements on $J_1 \cap \dots \cap J_{t-1}$ and on J_t. Since this is an open condition by Proposition 4.1.7, there even exists an element that is regular on $(J_1 \cap \dots \cap J_{t-1}) \oplus J_t$. If there were a regular element on J, this and Lemma 4.1.11 would imply that there exists a regular element on $(J_1 \cap \dots \cap J_{t-1}) + J_t$ which is not possible. We conclude that $\operatorname{depth}_E J = 0$. □

Corollary 5.5.9. *Let* $|K| = \infty$ *and* $J = (f_1, \dots, f_t)$ *be strongly decomposable. Then*

(i) $\operatorname{cx}_E(E/J) = \sum_{i=1}^t |\deg f_i| \geq t$.
(ii) $\operatorname{depth}_E(E/J) = n - \sum_{i=1}^t |\deg f_i| \leq n - t$.
(iii) $\operatorname{reg}_E(E/J) = \sum_{i=1}^t |\deg f_i| - t$.

Equality holds in (i) and (ii) if and only if $\deg f_i = 1$ *for* $i = 1, \dots, t$.

PROOF. (i) This follows after a change of coordinates from Lemma 5.5.8 and Proposition 5.5.4.

(ii) This follows immediately from (i) and the formula

$$\operatorname{depth}_E(E/J) + \operatorname{cx}_E(E/J) = n.$$

(iii) As E/J has an $(n-t)$-linear injective resolution, the regularity is, by Theorem 5.1.6,

$$\operatorname{reg}_E(E/J) = n - t - \operatorname{depth}_E(E/J) = n - t - (n - \sum_{i=1}^t |\deg f_i|) = \sum_{i=1}^t |\deg f_i| - t.$$

□

This shows that such generating systems are not a good generalisation of regular sequences as we had hoped, because the depth does not necessarily diminish by one when dividing out an element of such a sequence. (In fact the depth diminishes exactly if it is a regular element in the known sense.)

Nevertheless the property is maintained under reduction modulo regular sequences.

Lemma 5.5.10. *Let J be a strongly decomposable ideal and $v \in E_1$ a regular element on E/J. Then $J+(v)/(v)$ as an ideal in $E/(v)$ is strongly decomposable as well.*

PROOF. We may assume that $J = (e_{F_1}, \dots, e_{F_t})$ with $F_i \cap F_j = \emptyset$ by Remark 5.5.3 because the property of being a regular element is independent from coordinates. As v is regular on E/J it is not in the vector space spanned by the variables e_j with $j \in F_i$ for some i (cf. Lemma 5.5.8). Thus we can assume as well that $v = e_s$ for some $s \in [n]$.

Then the residue classes $\overline{e_{F_i}}$, $i = 1, \dots, t$ generate $J+(e_s)/(e_s)$. They are monomials in the exterior algebra $E/(e_s)$ with pairwise disjoint supports and thus $J+(e_s)/(e_s)$ is strongly decomposable in $E/(e_s)$. □

But strongly decomposability is not carried over from the initial ideal to the ideal, simply because pure decomposability is not.

Example 5.5.11. Let $J = (e_{12} + e_{34})$. Then $\operatorname{in} J = (e_{12})$ with respect to the revlex order. This ideal is strongly decomposable, but J is not because $e_{12} + e_{34}$ is not a pure decomposable element.

Open problem 5.5.12.

(i) *Is there a proof of Theorem 5.5.7 without use of the fact that a complete intersection ideal in the polynomial ring is Cohen-Macaulay? Open problem 2.1.4 could be helpful.*
(ii) *Is there an exterior analogue of the complete intersection property?*
(iii) *Is there a generalisation of the notion regular for elements of higher degree?*

CHAPTER 6

Orlik-Solomon algebras

In this chapter we apply results of the preceding chapters on Orlik-Solomon algebras. These are quotient rings over the exterior algebra defined by a certain combinatorial structure. They have linear injective resolutions and their defining ideal is pure decomposable.

Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be an essential central affine hyperplane arrangement in $\mathbb{C}^m$, X its complement. We refer to the book by Orlik and Terao [**48**] for background on hyperplane arrangements. We choose linear forms $\alpha_i \in (\mathbb{C}^m)^*$ such that $\operatorname{Ker}\alpha_i = H_i$ for $i = 1, \dots, n$. It is well-known that the singular cohomology $H^\bullet(X;K)$ of X with coefficients in K is isomorphic to E/J where J is the *Orlik-Solomon ideal* of X which is generated by all

$$\partial e_S = \sum_{i=1}^{t} (-1)^{i-1} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_i}} \wedge \cdots \wedge e_{j_t} \text{ for } S = \{j_1, \dots, j_t\} \subseteq [n]$$

where $\{H_{j_1}, \dots, H_{j_t}\}$ is a dependent set of hyperplanes of $\mathscr{A}$, i.e., $\alpha_{j_1}, \dots, \alpha_{j_t}$ are linearly dependent. The algebra E/J is also known as the *Orlik-Solomon algebra* of X. Note that the definition of E/J does only depend on the matroid of $\mathscr{A}$ on $[n]$. Therefore in this chapter we study more generally the Orlik-Solomon algebra of a matroid. Most of the results in this chapter appeared in [**40**].

6.1. Orlik-Solomon algebra of a matroid

For the convenience of the reader we collect all necessary matroid notions that are used in this section. They can be found in introductory books on matroids, as for example [**49**] or [**61**]. Notice that from now on the letter M always denotes a matroid, and not a module.

Let M be a non-empty *matroid* over $[n]$, i.e., M is a collection $\mathscr{I}$ of subsets of $[n]$, called *independent sets*, satisfying the following conditions:

(i) $\emptyset \in \mathscr{I}$.
(ii) If $A \in \mathscr{I}$ and $B \subseteq A$, then $B \in \mathscr{I}$.
(iii) If $A, B \in \mathscr{I}$ and $|A| < |B|$, then there exists an element $i \in B \setminus A$ such that $A \cup \{i\} \in \mathscr{I}$.

The subsets of $[n]$ that are not in $\mathscr{I}$ are called *dependent*, minimal dependent sets are called *circuits*. The cardinality of maximal independent sets (called *bases*) is constant and denoted by $r(M)$, the *rank* of M.

Equivalently one can define a matroid via its set $\mathscr{C}$ of circuits, which has to satisfy the following three conditions:

(i) $\emptyset \notin \mathscr{C}$.
(ii) If C_1 and C_2 are members of $\mathscr{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(iii) If C_1 and C_2 are distinct members of $\mathscr{C}$ and $i \in C_1 \cap C_2$, then there is a member C_3 of $\mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{i\}$.

Condition (iii) is called the *circuit elimination axiom*.

On E exists a derivation $\partial : E \to E$ of degree -1 which maps e_i to 1 and obeys the Leibniz rule

$$\partial(ab) = (\partial a)b + (-1)^{\deg a} a(\partial b)$$

for homogeneous $a \in E$ and all $b \in E$. One easily checks

$$\partial e_S = (e_{i_1} - e_{i_0}) \cdots (e_{i_m} - e_{i_0}) = \sum_{j=0}^{m} (-1)^j e_{S \setminus \{i_j\}}$$

for $S = \{i_0, \ldots, i_m\}$. The *Orlik-Solomon ideal* of M is the ideal

$$J(M) = (\partial e_S : S \text{ dependent }) = (\partial e_C : C \text{ circuit }).$$

If there is no danger of confusion we simply write J for $J(M)$. The quotient ring E/J is called the *Orlik-Solomon algebra* of M.

Orlik-Solomon algebras of matroids are generalisations of Orlik-Solomon algebras of central hyperplane arrangements, since not every matroid is defined by a hyperplane arrangement.

Example 6.1.1. The matroid on $[9]$ whose circuits are

$$\{\{1,2,3\},\{1,6,8\},\{1,5,7\},\{2,4,7\},\{2,6,9\},\{3,4,8\},\{3,5,9\},\{4,5,6\}\}$$

is called the *non-Pappus matroid*. Matroids defined by hyperplane arrangements are representable matroids (over $\mathbb{C}$). Oxley showed in [**49**, Proposition 6.1.11] that the non-Pappus matroid is not representable over any field and hence cannot be defined by a hyperplane arrangement.

A circuit whose minimal element (respective to a chosen order on $[n]$) is deleted is called a *broken circuit*. A set that does not contain any broken circuit is called *nbc*. Björner proved in [**6**, Theorem 7.10.2] that the set of all monomials corresponding to nbc-sets is a K-linear basis of E/J.

A *loop* is a subset $\{i\}$ that is dependent. If M has a loop $\{i\}$, then $\partial e_i = 1$ is in J and thus E/J is zero. Quite often it is enough to consider the case that M is *simple*, i.e., M has no loops and no non-trivial parallel classes. A *parallel class* is a maximal subset such that any two distinct members i, j are parallel, i.e., $\{i, j\}$ is a circuit, and no member is a loop.

Note that if M has no loops, a monomial e_S is contained in J if and only if the set S is dependent (see for example [**6**, Lemma 7.10.1]).

Example 6.1.2. The simplest matroids are the *uniform matroids* $U_{m,n}$ with $m \leq n$. They are matroids over $[n]$ such that all subsets of $[n]$ of cardinality $\leq m$ are independent. The rank of $U_{m,n}$ is obviously m and the circuits of $U_{m,n}$ are all subsets of $[n]$ of cardinality $m+1$. Thus the Orlik-Solomon ideal $J_{m,n} := J(U_{m,n})$ of $U_{m,n}$ is the ideal $J_{m,n} = (\partial e_A : A \subset [n], |A| = m+1)$. The relation

$$\partial e_S = \sum_{j=0}^{k} (-1)^j \partial e_{S \setminus \{i_j\} \cup \{1\}}$$

for $S = \{i_0, \ldots, i_k\} \subseteq [n]$ with $1 \notin S$ is easily verified by a simple computation. Then we can rewrite the Orlik-Solomon ideal as

$$J_{m,n} = (\partial e_A : A \subset [n], |A| = m+1, 1 \in A).$$

The rank of a subset $X \subseteq [n]$ is the rank of the matroid $M|X$ which results from restricting M on X. Then the closure operator cl is defined as

$$\mathrm{cl}(X) = \{i \in [n] : r(X \cup \{i\}) = r(X)\}$$

for $X \subseteq [n]$. If $\mathrm{cl}(X) = X$, then X is called a *flat* (or a closed set). The by inclusion partially ordered set L of all flats of M is a graded lattice. On L we consider the *Möbius function* which can be defined recursively by

$$\mu(X,X) = 1 \qquad \text{and} \qquad \mu(X,Z) = -\sum_{X \leq Y < Z} \mu(X,Y) \text{ if } X < Z$$

and the *characteristic polynomial*

$$p(L;t) = \sum_{X \in L} \mu(\emptyset, X) t^{r(M)-r(X)}.$$

The *beta-invariant* $\beta(M)$ of a matroid M was introduced by Crapo in [**13**] as

$$\beta(M) = (-1)^{r(M)} \sum_{S \subseteq [n]} (-1)^{|S|} r(S) = (-1)^{r(M)} \sum_{X \in L} \mu(\emptyset, X) r(X).$$

The Möbius function, the characteristic polynomial and the beta-invariant are considered in detail in [**63**].

The *direct sum* of two matroids M_1 and M_2 on disjoint ground sets V_1 and V_2 is the matroid $M_1 \oplus M_2$ on the ground set $V_1 \cup V_2$ whose independent sets are the unions of an independent set of M_1 and an independent set of M_2. The circuits of $M_1 \oplus M_2$ are those of M_1 and those of M_2. Let $E_i = K\langle e_j : j \in V_i \rangle$ for $i = 1, 2$ be the exterior algebras corresponding to the ground sets, and $E = E_1 \otimes_K E_2$. Then the Hilbert series of the Orlik-Solomon algebra is multiplicative on direct sums, i.e.,

$$H(E/J(M_1 \oplus M_2), t) = H(E_1/J(M_1), t) \cdot H(E_2/J(M_2), t).$$

This can be proved using the fact that the set of all nbc-sets of cardinality k is a K-basis of $(E/J)_k$ and that the nbc-sets of $M_1 \oplus M_2$ are the unions of an nbc-set of M_1 and an nbc-set of M_2.

On a matroid M exists the equivalence relation:

$$x \sim y \Leftrightarrow x = y \text{ or there is a circuit } C \text{ which contains both } x \text{ and } y.$$

The equivalence classes of this relation are called the *connected components* or, more briefly, *components* of M. They are disjoint subsets of the ground set and each circuit contains only elements of one component. If $T_1, \ldots, T_k$ are the components of M then $M = M|T_1 \oplus \cdots \oplus M|T_k$. In abuse of notation we also call the matroids $M|T_i$ the components of M. The matroid M is called *connected* if it has only one connected component.

The Orlik-Solomon algebra has a linear injective resolution, which was first observed by Eisenbud, Popescu and Yuzvinsky in [**20**] for Orlik-Solomon algebras defined by hyperplane arrangements, although their proof works for arbitrary Orlik-Solomon algebras as well. For the convenience of the reader we present a compact proof.

Theorem 6.1.3. [**20**, Theorem 1.1] *Let $l = r(M)$ be the rank of the matroid M. Then the Orlik-Solomon algebra E/J of M has an l-linear injective resolution.*

PROOF. Let Γ be the simplicial complex whose faces are the nbc-sets of M. The face ideal of Γ is the ideal

$$J_\Gamma = (e_A : A \notin \Gamma) = (e_A : A \text{ is a broken circuit}).$$

This ideal is the initial ideal $\text{in}(J)$ of J (this is implicitly contained in the proof of [**14**, Theorem 3.3]).

By [**6**, Theorem 7.4.3] the complex Γ is shellable and hence $K\{\Gamma\}$ has pure decomposable quotients by Theorem 5.4.10. We show that the socle of $K\{\Gamma\}$ is $K\{\Gamma\}_l$. Then it follows from Corollary 5.4.12 that $K\{\Gamma\} = E/\text{in}(J)$ has an l-linear injective resolution. Corollary 3.2.3 finally implies that also E/J has an l-linear injective resolution.

It remains to show that $\text{soc} K\{\Gamma\} = K\{\Gamma\}_l$. As l is the rank of the matroid, there is no independet set with more than l elements. In particular, there are no nbc-sets with more than l elements, as nbc-sets are independent. This shows $K\{\Gamma\}_{l+1} = 0$ and thus $K\{\Gamma\}_l$ lies in the socle of $K\{\Gamma\}$. On the other hand, let $0 \neq \overline{e_A} \in K\{\Gamma\}$ be of degree $< l$, in other words, A is an nbc-set with less than l elements. Every nbc-set is contained in an nbc-base (see, e.g., [**6**, Proposition 7.4.2(ii)]), and hence there exists an element $i \in [n] \setminus A$ such that $A \cup \{i\}$ is nbc. Equivalently, $\overline{e_i e_A} \neq 0$ in $K\{\Gamma\}$ and thus $\overline{e_A} \notin \text{soc} K\{\Gamma\}$. All in all this shows $\text{soc} K\{\Gamma\} = K\{\Gamma\}_l$. Alternatively one could use that a pure shellable complex is Cohen-Macaulay. □

Björner's result [**6**, Theorem 7.4.3] that is used in the above proof also says that the quotient ring modulo the ideal generated by the monomials corresponding to circuits (which is the face ring of a matroid complex, cf. Section 6.5) has a linear injective resolution. Therefore one could ask if it is true in general that $E/\partial J$ has a linear injective resolution if E/J has one, for an arbitrary graded ideal J. But the answer is negative:

Example 6.1.4. Let $J = (e_{123}, e_{234}, e_{245}, e_{456}) \subseteq K\langle e_1, \dots, e_6\rangle$. The simplicial complex whose face ideal is J is clearly shellable, hence Cohen-Macaulay. Therefore E/J has a linear injective resolution. We use the computer algebra system Macaulay 2 [**26**] to compute the annihilator of $\partial J = (\partial e_{123}, \partial e_{234}, \partial e_{245}, \partial e_{456})$. It is the ideal

$$0 : \partial J = (\partial e_{23} \partial e_{45}, \partial e_{23} \partial e_{256}, \partial e_{13} \partial e_{456} - \partial e_{12} \partial e_{256}, \partial e_{13} \partial e_{345}),$$

which is not generated in one degree. Thus $E/\partial J$ cannot have a linear injective resolution.

6.2. Depth of Orlik-Solomon algebras

We want to determine the depth of the Orlik-Solomon algebra. We are able to find at least one E/J-regular element if M has no loops. If M has a loop, then $E/J = 0$ and $\text{depth}_E(E/J) = 0$, so we can restrict our attention to loopless matroids.

Proposition 6.2.1. *If the matroid M has no loops, then the variable e_i is E/J-regular for all $i \in [n]$. In particular,* $\text{depth}_E(E/J) \geq 1$.

PROOF. By Theorem 2.4.5 and Theorem 6.1.3 it is enough to show that the annihilator of e_i in E/J and the ideal $(\overline{e_i}) = e_i(E/J)$ in E/J coincide in degree l.

Every set of cardinality $l+1$ is dependent and therefore every monomial of degree $l+1$ is contained in J whence $(E/J)_{l+1}=0$. So every element in E/J of degree l is annihilated by e_i.

Now let T be an independent set of cardinality l that does not contain i. Then $T\cup\{i\}$ is dependent and thus $\partial e_{T\cup\{i\}}\in J$. Arrange $T\cup\{i\}$ such that i is the first element. Then in E/J there is the relation

$$\overline{e_T}=\overline{e_T}-\overline{\partial e_{T\cup\{i\}}}=\overline{e_T}-\overline{e_T+(\dots)e_i}=(\dots)\overline{e_i}.$$

So the residue class of every monomial of degree l is in the ideal generated by $\overline{e_i}$, which shows that the annihilator and the ideal $(\overline{e_i})$ coincide in degree l. This shows that e_i is E/J-regular and thus the depth of E/J is at least 1. □

The matroids M whose corresponding depth is exactly 1 can be characterised by their *beta-invariant* $\beta(M)$.

Theorem 6.2.2. *If $|K|=\infty$ and M has no loops, then the depth of the Orlik-Solomon algebra E/J equals 1 if and only if $\beta(M)\neq 0$.*

PROOF. Theorem 5.1.3 shows that the depth of E/J is the maximal number s such that the Hilbert series can be written as $H(E/J,t)=(1+t)^sQ(t)$ for some $Q(t)\in\mathbb{Z}[t]$ with $Q(-1)\neq 0$.

Björner proved in [**6**, Corollary 7.10.3] that

$$H(E/J,t)=(-t)^{r(M)}p(L;-\frac{1}{t}).$$

Replacing the characteristic polynomial $p(L;-\frac{1}{t})$ by its definition gives

$$H(E/J,t)=\sum_{X\in L}\mu(\emptyset,X)(-1)^{r(X)}t^{r(X)}.$$

Observe that $H(E/J,-1)=0$ because $1+t$ divides $H(E/J,t)$ at least once since e_i is regular on E/J by the preceding lemma. Thus the Taylor expansion of $H(E/J,t)$ at -1 is

$$\begin{aligned}H(E/J,t)&=\left(\sum_{X\in L}\mu(\emptyset,X)(-1)^{r(X)}r(X)(-1)^{r(X)-1}\right)(1+t)+(1+t)^2(\dots)\\&=\left(-\sum_{X\in L}\mu(\emptyset,X)r(X)\right)(1+t)+(1+t)^2(\dots)\\&=(-1)^{r(M)-1}\beta(M)(1+t)+(1+t)^2(\dots).\end{aligned}$$

Now one sees that $H(E/J,t)$ can be divided twice by $1+t$ if and only if $\beta(M)=0$. □

Crapo [**13**, Theorem II] proves that M is connected if and only if $\beta(M)\neq 0$ (see also Welsh [**61**, Chapter 5.2]). Thus the above result says that if M is connected, the depth of E/J equals the number of components of M. This is true in general.

Theorem 6.2.3. *Let $|K|=\infty$ and M be a loopless matroid with k components and J its Orlik-Solomon ideal. Then* $\operatorname{depth}_E(E/J)=k$.

PROOF. Let $M_1,\dots,M_k$ be the matroids on the components of M, i.e., $M = M_1 \oplus \dots \oplus M_k$ and let $J_i = J(M_i)$ be the corresponding Orlik-Solomon ideals. Theorem 5.1.3 and Theorem 6.2.2 imply that their Hilbert series can be written as

$$H(K\langle e_j : j \in T_i\rangle/J_i,t) = Q_i(t)\cdot(1+t)$$

such that $Q_i(-1) \neq 0$. The Hilbert series is multiplicative on direct sums, thus

$$H(E/J,t) = \prod_{i=1}^{k}(Q_i(t)\cdot(1+t)) = Q(t)\cdot(1+t)^k$$

with $Q(-1) \neq 0$ and so $\operatorname{depth}_E(E/J) = k$. □

For Orlik-Solomon algebras of hyperplane arrangements this result has been proved by Eisenbud, Popescu and Yuzvinsky. In [**20**, Corollary 2.3] they state that the codimension of the singular variety (i.e., the set of all non-regular elements on the Orlik-Solomon algebra) of the arrangement is the number of central factors in an irreducible decomposition of the arrangement. This codimension is exactly the depth of the Orlik-Solomon algebra as Aramova, Avramov and Herzog show in [**1**, Theorem 3.1].

Remark 6.2.4. Let M be a loopless matroid with components $T_1,\dots,T_k$ and $M_i = M|T_i$. A "canonical" maximal regular sequence on E/J can be found as follows. For every component T_j choose an element $i_j \in T_j$. Then $e_{i_1},\dots,e_{i_k}$ is an E/J-regular sequence: As $E/(J+(e_{i_1},\dots,e_{i_{j-1}}))$ has an $(l-j+1)$-linear injective resolution over $E/(e_{i_1},\dots,e_{i_{j-1}})$ by Lemma 5.1.2, it is enough to prove that e_{i_j} is regular on $E/(J+(e_{i_1},\dots,e_{i_{j-1}}))$ in degree $l-j+1$. Let A be an independent subset of $[n]\setminus\{i_1,\dots,i_{j-1}\}$ with $|A| = l-j+1$. Then $A = S_1\cup\dots\cup S_k$ with $S_i \subseteq T_i$. The rank of M is the sum of the ranks of the M_i, i.e., $l = r(M_1)+\dots+r(M_k)$. So at most $j-1$ of the S_i are not bases of their matroid, which means that there exists a $t \in \{1,\dots,j\}$ such that $S_t \cup \{i_t\}$ is dependent in M_t. Then $A\cup\{i_t\}$ is dependent in M. The same trick as in the proof of Proposition 6.2.1 shows that $e_A \in J+(e_{i_1},\dots,e_{i_j})$.

Remark 6.2.5. Deletion and contraction are standard matroid operations. Let M be a matroid on $[n]$. For a subset $X \subseteq [n]$ the *deletion* of X from M is the matroid $M\setminus X$, whose independent sets are the independent sets of M having no common element with X. In other words, it is the restriction of M on $[n]\setminus X$. The *contraction* of X from M is the matroid M/X, whose circuits are the minimal non-empty members of $\{C\setminus X : C \text{ is a circuit of } M\}$. Both matroids are matroids on the ground set $[n]\setminus X$.

In general, there seems to be no relation between the depth of the Orlik-Solomon algebra of a matroid and the depth of the Orlik-Solomon algebra of its deletion or contraction. To see this we look at the connected components. It is known that every connected component of $M\setminus X$ or M/X is contained in a connected component of M. Let $M = M_1\oplus\dots\oplus M_k$ with $M_i = M|T_i$ be the decomposition of M into its connected components. Then

$$M\setminus X = M_1\setminus(T_1\cap X)\oplus\dots\oplus M_k\setminus(T_k\cap X) \text{ and } M/X = M_1/(T_1\cap X)\oplus\dots\oplus M_k/(T_k\cap X)$$

are decompositions of $M\setminus X$ and M/X, but $M_i\setminus(T_i\cap X)$ and $M_i/(T_i\cap X)$ are not necessarily connected. Thus the number of connected components may diminish or grow under these operations.

At least, there is an upper and a lower bound for it. The upper bound is (as for every matroid) the cardinality of the ground set, i.e., $n-|X|$. An example where this bound is attained is the deletion of a subset X with x elements of the uniform matroid $U_{n-x,n}$,

$$U_{n-x,n} \setminus X \cong U_{n-x,n-x},$$

or the contraction of the uniform matroid $U_{x,n}$,

$$U_{x,n}/X \cong U_{0,n-x}.$$

The lower bound is the number of connected components T_i of M such that $T_i \cap X \neq T_i$. This is the number of the non-empty matroids in the decomposition of $M \setminus X$ and M/X as above. It is attained if $M_i \setminus (T_i \cap X)$ (or $M_i/(T_i \cap X)$) is connected for all i. For example if X is a *separator* of M, i.e., a union of components of M, then this bound is always attained. For instance if $X = T_1 \cup \ldots \cup T_s$ for some $s \leq k$, then the number of connected components of $M \setminus X = M/X$ is $k-s$, because $T_i \setminus (T_i \cap X) \neq \emptyset$ if and only if $i > s$.

6.3. On resolutions of Orlik-Solomon algebras

As we know now the depth, we can compute the regularity of the Orlik-Solomon algebra as well.

Corollary 6.3.1. *Let $|K| = \infty$ and M be a loopless matroid of rank l with k components. The regularity of its Orlik-Solomon algebra is*

$$\operatorname{reg}_E(E/J) = l-k.$$

PROOF. This is just an application of Theorem 5.1.6. □

Example 6.3.2. We consider the uniform matroids $U_{m,n}$ and their Orlik-Solomon ideals $J_{m,n}$.

If $m = 0$ then every non-empty set is dependent. The circuits are all sets with one element, in particular they are loops. Thus $U_{0,n}$ has rank 0 and n components $U_{0,1}$. The Orlik-Solomon ideal is $J_{0,n} = E$.

If $m = n$ then every set is independent. There are no circuits hence $J_{n,n} = 0$. The rank of $U_{n,n}$ is n and it has n components $U_{1,1}$. Thus $\operatorname{depth}_E(E/J) = n$ and $\operatorname{cx}_E(E/J) = 0$. The regularity is $\operatorname{reg}_E(E/J) = n-n = 0$.

If $m \neq 0, n$ then $U_{m,n}$ is connected. Thus $\operatorname{depth}_E(E/J) = 1$ and $\operatorname{cx}_E(E/J) = n-1$. The rank is m hence the regularity is $\operatorname{reg}_E(E/J) = m-1$.

We say that an E-module has *linear relations* if it is generated in one degree and the first syzygy module is generated in degree one. Thus a linear projective resolution implies linear relations.

Theorem 6.3.3. *Let M be a simple matroid and have no singleton components. If the Orlik-Solomon ideal J has linear relations then M is connected.*

PROOF. As M is simple there exists no circuits with one or two elements, so J is generated in degree $m \geq 2$. Suppose $J = (\partial e_{C_i} : i = 1,\ldots,r)$ where $C_1,\ldots,C_r$ are circuits of M of cardinality $m+1$. Let $f_1,\ldots,f_r$ be the free generators of $\bigoplus_{i=1}^{r} E(-m)$ such that

f_i is mapped to ∂e_{C_i} in the minimal graded free resolution of J. Then the assumption says that the kernel of this map,

$$U = \{\sum_{i=1}^{r} a_i f_i : a_i \in E, \sum_{i=1}^{r} a_i \partial e_{C_i} = 0\},$$

is generated by elements $g_k = \sum_{i=1}^{r} v_{ik} f_i$ with $v_{ik} \in E_1$. We may assume that the generators g_k are minimal in the sense that no sum $\sum_{i \in I'} v_{ik} f_i$ with $I' \subsetneq \{1, \dots, r\}$ is in U.

Under this conditions we claim that for each k the elements of the circuits C_i with $v_{ik} \neq 0$ are in the same component of M.

Fix $k \in \{1, \dots, r\}$. The support of a linear form $v = \sum_{j=1}^{n} \alpha_j e_j$ with $\alpha_j \in K$ is the set $\operatorname{supp}(v) = \{j : \alpha_j \neq 0\}$. The monomials in the expression $\sum_{i=1}^{r} v_{ik} \partial e_{C_i} = 0$ have the form $e_j e_{C_i \setminus \{l\}}$ with $l \in C_i$ and $j \in \operatorname{supp}(v_{ik})$. Because of the structure of ∂e_{C_i} the monomials $e_j e_{C_i \setminus \{l\}}$ cannot be zero for all $l \in C_i$. If it is not zero, then there exists another C_p, $q \in C_p$ and $t \in \operatorname{supp}(v_{pk})$ such that

$$\{j\} \cup C_i \setminus \{l\} = \{t\} \cup C_p \setminus \{q\}.$$

As C_i and C_p have at least three elements, it follows that their intersection is not empty. This means that their elements are both in the same component of M. Then the minimality of g_k implies that all elements of circuits C_i with $v_{ik} \neq 0$ belong to the same component, because otherwise one could split the sum g_k into two sums $\sum_{i \in I_1} v_{ik} \partial e_{C_i} = 0$ and $\sum_{i \in I_2} v_{ik} \partial e_{C_i} = 0$, where I_1 corresponds to one component and I_2 to the other(s). This shows the claim that for each k the elements of the circuits C_i with $v_{ik} \neq 0$ are in the same component of M.

As the next step we show that, given $i \in \{1, \dots, r\}$, every $j \in \operatorname{supp}(v_{ik})$ must belong to some circuit C_p with $v_{pk} \neq 0$. Suppose that the converse is true, that is, $j \notin C_p$ for all p with $v_{pk} \neq 0$. Then

$$e_j \sum_{\{p : j \in \operatorname{supp}(v_{pk})\}} \alpha_{pjk} \partial e_{C_p} = 0$$

with $v_{pk} = \sum_{j=1}^{n} \alpha_{pjk} e_j$. This implies $\sum_{\{p : j \in \operatorname{supp}(v_{pk})\}} \alpha_{pjk} \partial e_{C_p} \in (e_j)$ and thus this sum must equal zero since $j \notin C_p$ for all these p. But this is not possible as this is a relation of degree 0 and U is generated by relations of degree 1. Therefore every $j \in \operatorname{supp}(v_{ik})$ must belong to some circuit C_p with $v_{pk} \neq 0$. From this and the first claim we conclude that all indices in the support of v_{ik} belong to the same component of M as the elements of the circuits C_i.

If M is not connected and has no singleton components, there exists at least two components and thus two circuits C_l and C_j whose intersection is empty. There is a trivial relation of degree $2m$ between the generators corresponding to these two circuits, namely $\partial e_{C_l} f_j \pm \partial e_{C_j} f_l$. This relation has a representation

$$\partial e_{C_l} f_j \pm \partial e_{C_j} f_l = \sum_k h_k g_k = \sum_k \sum_{i=1}^{r} h_k v_{ik} f_i$$

where $h_k \in E_{m-1}$. Then

$$\partial e_{C_l} = \sum_k h_k v_{jk}$$

since the f_i are free generators. Each monomial in the sum on the right-hand side has a variable whose index is in the support of v_{jk}. As shown above this support is contained in the same component of M as C_j. Thus C_l contains elements of the component of C_j which implies that both circuits belong to the same component, a contradiction to the choice of C_l and C_j. □

Finally we classify all Orlik-Solomon ideals with linear projective resolutions. Joining or removing “superfluous” variables has no effect on the linearity of J. This operation can be expressed using the direct sum of matroids. A singleton $\{i\}$ is a component of a (loopless) matroid M if and only if it is contained in no circuit, or equivalently, is contained in each base. In this case i is called a *coloop*. The matroid on $\{i\}$ is $U_{1,1}$ if i is a coloop, so we can write $M = M' \oplus U_{1,1}$ with $M' = M|_{[n]\setminus\{i\}}$. Let $E' = E/(e_i)$ and J' the Orlik-Solomon ideal of M' in E'. Then e_i is E/J-regular and $J = J'E$ has a linear projective resolution if and only if J' has one. By iterating this procedure we can split up M in the direct sum of a matroid M' which has no singleton components and a copy of $U_{f,f}$ where f is the number of coloops of M (note that $U_{f-1,f-1} \oplus U_{1,1} = U_{f,f}$). Then $J(M')$ has a linear resolution if and only if $J(M)$ has one.

Theorem 6.3.4. *Let $|K| = \infty$ and M be a matroid on $[n]$. The Orlik-Solomon ideal J of M has an m-linear projective resolution if and only if M satisfies one of the following three conditions:*

(i) *M has a loop and $m = 0$.*
(ii) *M has no loops, but non-trivial parallel classes, $m = 1$ and $M = U_{1,n_1} \oplus \cdots \oplus U_{1,n_k} \oplus U_{f,f}$ for some $k, f \geq 0$.*
(iii) *M is simple and $M = U_{m,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n$.*

PROOF. First of all we will see that if M satisfies one of the three conditions then J has a linear projective resolution:

(i) If M has a loop $\{i\}$, then $\partial e_i = 1 \in J$ so J is the whole ring E which has a linear resolution.

If M satisfies (ii) then the circuits of M are the circuits of the U_{1,n_i}. Thus all circuits of M have cardinality two which means that J is generated by linear forms $v_1, \dots, v_s$ and has the Cartan complex $C_{\bullet}(v_1, \dots, v_s; E)$ as a linear resolution.

(iii) Following the remark preceding this theorem we may assume that M has no singleton components, so we have $M = U_{m,n}$. If $m = 0$ or $m = n$ then the Orlik-Solomon ideal is E or zero and has a linear resolution. By Example 6.3.2 the matroid $U_{m,n}$ is connected if $m \neq 0, n$. Hence it follows from Theorem 6.2.2 that $\operatorname{depth}_E(E/J) = 1$. By Proposition 6.2.1 the variable e_1 is E/J-regular. The Orlik-Solomon ideal

$$J = J_{m,n} = (\partial e_A : |A| = m+1, 1 \in A)$$

of $U_{m,n}$ was computed in Example 6.1.2. So $J + (e_1) = (e_A : |A| = m) + (e_1)$ and thus J reduces modulo e_1 to the m-th power of the maximal ideal in the exterior algebra $E/(e_1)$ and hence has a linear projective resolution by Theorem 5.1.8.

Now let J have an m-linear projective resolution. If M has a loop, then this is a circuit of cardinality one whence $m = 0$. Thus M satisfies (i).

Now we consider the case that M is simple. As above we assume that M has no singleton components. So we have to show that $M = U_{m,n}$. Theorem 6.3.3 implies that M is connected. Then $\operatorname{depth}_E(E/J) = 1$ by Theorem 6.2.2 and e_1 is a maximal regular sequence on E/J by Proposition 6.2.1. Reducing J modulo (e_1) gives the m-th power of the maximal ideal of the exterior algebra $E/(e_1)$ by Theorem 5.1.8 .

Let $A \subseteq [n]$ with $1 \in A$, $|A| = m+1$ and let $A' = A \setminus \{1\}$. The degree of the residue class of $e_{A'}$ in $E/(e_1)$ is m and so $\overline{e_{A'}} \in J + (e_1)/(e_1)$. Thus there exists a representation

$$e_{A'} = f + ge_1 \qquad f \in J, g \in E.$$

Then

$$e_A = \pm e_{A'} e_1 = \pm f e_1 \in J$$

which is the case if and only if A is dependent. So every subset of cardinality $m+1$ containing 1 is dependent. An analogous argument for $i > 1$ shows that every subset of cardinality $m+1$ is dependent. No subset of cardinality $\leq m$ is dependent because $J_j = 0$ for $j < m$. Thus we conclude $M = U_{m,n}$.

Finally we assume that M has no loops or singleton components, but non-trivial parallel classes. Then there exists at least one circuit with two elements. As J is generated in degree m this implies $m = 1$. Let $J_1, \dots, J_k$ be the Orlik-Solomon ideals of the components $M_1, \dots, M_k$ of M, i.e., $J = J_1 + \ldots + J_k$. Each J_j is generated by linear forms, because no ∂e_C with C of one component can be represented by elements ∂e_{C_i} with C_i of other components. Ideals generated by linear forms have the Cartan complex with respect to these linear forms as minimal graded free resolution and this is a linear resolution. Thus J_j has a linear resolution. It is the Orlik-Solomon ideal of the connected loopless matroid M_j. Following the argumentation in the preceding paragraph for simple matroids this implies $M_j = U_{1,n_j}$ with n_j the cardinality of the j-th component of M and $M = \bigoplus_{j=1}^k U_{1,n_j}$. □

Since the powers of the maximal ideal of E are strongly stable, their minimal resolution and especially their Betti numbers are known from Lemma 3.1.2. Also Eisenbud, Fløystad and Schreyer give in [**19**, Section 5] an explicit description of the minimal graded free resolution of the power of the maximal ideal using Schur functors. Their result gave the hint how a “nicer” formula of the Betti numbers could look like.

Proposition 6.3.5. *The graded Betti numbers of* $\mathfrak{m}^t$ *are*

$$\beta_{i,i+t}(\mathfrak{m}^t) = \binom{n+i}{t+i}\binom{t+i-1}{i} \text{ and } \beta_{i,i+j}(\mathfrak{m}^t) = 0 \text{ for } j \neq t.$$

PROOF. There are $\binom{k-1}{t-1}$ monomials of degree t whose highest supporting variable is e_k, i.e., $m_k(\mathfrak{m}^t) = |\{u \in G(\mathfrak{m}^t) : \max(u) = k\}| = \binom{k-1}{t-1}$. Hence by Lemma 3.1.2 we obtain

$$\beta_{i,i+t}(\mathfrak{m}^t) = \sum_{k=t}^{n} m_k(\mathfrak{m}^t)\binom{k+i-1}{k-1} = \sum_{k=t}^{n}\binom{k-1}{t-1}\binom{k+i-1}{k-1}.$$

That this sums equals $\binom{n+i}{t+i}\binom{t+i-1}{i}$ can be seen by an induction on n, where the induction step from n to $n+1$ is the following:

$$\begin{aligned}&\sum_{k=t}^{n+1}\binom{k-1}{t-1}\binom{k+i-1}{k-1}\\&=\binom{n+i}{t+i}\binom{t+i-1}{i}+\binom{n}{t-1}\binom{n+i}{n}\\&=\left(\binom{n+i+1}{t+i}-\binom{n+i}{t+i-1}\right)\binom{t+i-1}{i}+\binom{n}{t-1}\binom{n+i}{n}\\&=\binom{n+i+1}{t+i}\binom{t+i-1}{i},\end{aligned}$$

where we used that

$$\binom{n+i}{t+i-1}\binom{t+i-1}{i}=\binom{n}{t-1}\binom{n+i}{n}$$

which can be verified by a direct computation. □

Now we obtain:

Theorem 6.3.6. *Let $|K|=\infty$ and M be a matroid and $J=J(M)$ be its Orlik-Solomon ideal.*

(i) *If $M=U_{m,n-f}\oplus U_{f,f}$ for some $f\geq 0$, then*

$$\beta_i(J)=\binom{n-f-1+i}{m+i}\binom{m+i-1}{i}.$$

(ii) *If $M=U_{1,n_1}\oplus\cdots\oplus U_{1,n_k}\oplus U_{f,f}$ for some $k,f\geq 0$, $n=f+\sum_{i=1}^{k}n_i$, then*

$$\beta_i(J)=\binom{n-f-k+i}{i+1}.$$

PROOF. Reducing modulo a regular sequence does not change the Betti numbers by Proposition 4.1.9 so the Betti numbers of $\mathfrak{m}^m$ give the Betti numbers of $J_{m,n}$.

(i) The Betti numbers of J are the same as the Betti numbers of the m-th power of the maximal ideal in the exterior algebra $\overline{E}$ on $n-f-1$ variables:

$$\beta_i^E(J)=\beta_i^{\overline{E}}(\overline{\mathfrak{m}}^m)=\binom{n-f-1+i}{m+i}\binom{m+i-1}{i}.$$

(ii) In this case J reduces to the maximal ideal in the exterior algebra on $n-f-k$ variables because for each component U_{1,n_i} one reduces modulo one variable as in Remark 6.2.4. □

Remark 6.3.7. (a) It would be interesting to characterise the matroids whose Orlik-Solomon ideals are componentwise linear. Of course the matroids from Theorem 6.3.4 are, and these are all such ideals that are generated in one degree. We give a necessary condition for a matroid to have a componentwise linear Orlik-Solomon ideal. By Corollary 5.3.8 the regularity of the ideal is then the maximal degree of its generators. On the other hand, Corollary 6.3.1 shows that the regularity of the Orlik-Solomon algebra is the rank of the matroid minus its number of connected components. Therefore a matroid M whose Orlik-Solomon ideal is componentwise linear must have a subset $\mathscr{C}'$ of its circuits such that $J(M) = (\partial e_C : C \in \mathscr{C}')$ and

$$\operatorname{rank}(M) - \#\text{connected components of } M + 1 = \max\{|C| - 1 : C \in \mathscr{C}'\}.$$

So if M is connected, then the rank must equal $\max\{|C| - 1 : C \in \mathscr{C}'\}$. This is not always satisfied, e.g., the ideal of the matroid on $[5]$ with circuits $123, 345, 1245$ has a generating system induced by the first two circuits. Also, this is not a sufficient condition. The $(9_3)_2$ matroid (see [**14**, Example 4.6]) has rank 3 and is connected. Using Macaulay 2 [**26**], we compute that its Orlik-Solomon ideal J is minimally generated by the ∂e_C with

$$C \in \{278, 179, 467, 368, 269, 458, 359, 134, 125, 5789, 4789\}.$$

Hence, it satisfies the above condition, but it is not componentwise linear, because $J_{\langle 2 \rangle}$ does not have a linear resolution.

(b) Another interesting question is for which matroids the Orlik-Solomon ideal has linear quotients. If the ideal is generated in one degree, then by Theorem 5.4.3 these are exactly the ones with a linear projective resolution, determined in Theorem 6.3.4. Let M be a simple matroid without singleton components whose Orlik-Solomon ideal has linear quotients w.r.t. a generating system consisting of elements of the type ∂e_C for circuits C of M. Then M must be connected: Suppose the first m generators $C_1, \ldots, C_m$ belong to one component, and the $(m+1)$-th belongs to another. Then the ideal

$$(\partial e_{C_1}, \ldots, \partial e_{C_m}) : \partial e_{C_{m+1}}$$

must be generated by linear forms. A similar way of argumentation as in the proof of Theorem 6.3.3 shows that this is not possible. The regularity of the ideal is the same as in (a), by Theorem 5.4.3. Hence, also the condition $\operatorname{rank}(M) = \max\{|C| - 1 : C \in \mathscr{C}'\}$ for a generating set $\mathscr{C}'$ must be satisfied. Again this is not sufficient; the $(9_3)_2$ matroid does not have linear quotients w.r.t. the above generating system (or any permutation of it), because otherwise it would be componentwise linear by Theorem 5.4.5. The matroid on $[5]$ with the circuits $1234, 2345, 1245, 135$ has linear quotients w.r.t. this generating set. Note that it is not minimal.

(c) To prove that the Orlik-Solomon algebra has a linear injective resolution, one uses that the initial ideal of the Orlik-Solomon ideal is the face ideal of a shellable complex, the complex Γ whose faces are the nbc-sets. By Theorem 5.4.10 the face ring of this complex has pure decomposable quotients w.r.t. a decomposition induced by the shelling. So one could wonder if the same is true for the Orlik-Solomon algebra itself. But the answer is negative, at least for the so induced decomposition: Consider the matroid $AG(3,2)'$ (see, e.g., [**49**]), whose circuits are

$$1234, 1256, 1278, 1357, 1368, 1458, 1467, 2358, 2367, 2468, 3456, 3478, 5678$$

and

$$12457, 23457, 24567, 24578.$$

Let $T_1,\ldots,T_{22}$ be the lexicographically ordered nbc-bases of $AG(3,2)'$. The decomposition of J that corresponds to the canonical generating set of $\mathrm{in}(J)=J_\Gamma$ is given by

$$J=\bigcap_{i=1}^{22} J_i,$$

where J_i is defined as follows (cf. [**14**]): Let $T_i=\{j_1,\ldots,j_l\}$, $l=\operatorname{rank} AG(3,2)'=4$ with $j_1<\ldots<j_l$. Then J_i is generated by the linear forms $e_x-e_{j_k}$ such that k is maximal with $x\in \mathrm{cl}\{j_l,\ldots,j_k\}$, $x\notin T_i$. With Macaulay 2 [**26**] we compute that the fifth quotient is not isomorphic to a principal pure decomposable ideal. The first five nbc-bases are

$$T_1=1235,\ T_2=1236,\ T_3=1237,\ T_4=1238,\ T_5=1245,$$

and the corresponding ideals are

$$\begin{aligned}
&J_1=(e_8-e_2,e_7-e_1,e_6-e_1,e_4-e_1), \quad J_2=(e_7-e_2,e_8-e_1,e_5-e_1,e_4-e_1),\\
&J_3=(e_6-e_2,e_8-e_1,e_5-e_1,e_4-e_1), \quad J_4=(e_5-e_2,e_7-e_1,e_6-e_1,e_4-e_1),\\
&J_5=(e_8-e_1,e_7-e_1,e_6-e_1,e_3-e_1).
\end{aligned}$$

The annihilator of the fifth quotient is the ideal

$$\operatorname{ann}_E \bigcap_{i=1}^{4} J_i\Big/\bigcap_{i=1}^{5} J_i=(e_8-e_1,e_7-e_1,e_6-e_1,e_3-e_1,\partial e_{458},\partial e_{248}),$$

which is not generated by linear forms. Hence, $\bigcap_{i=1}^{4} J_i\big/\bigcap_{i=1}^{5} J_i$ is not isomorphic to a principal pure decomposable ideal.

Open problem 6.3.8.

(i) *Is there a characterisation of the matroids whose Orlik-Solomon ideal is componentwise linear?*

(ii) *Is there a characterisation of the matroids whose Orlik-Solomon ideal has linear quotients?*

6.4. Strongly decomposable Orlik-Solomon ideals

In this section we analyse which Orlik-Solomon ideals are strongly decomposable. In abuse of notation we will also call the matroids strongly decomposable whose Orlik-Solomon ideals are so. Therefore throughout this section let J be the Orlik-Solomon ideal of a matroid.

Recall from Section 5.5 that J is called strongly (pure) decomposable if it has a system of pure decomposable generators $f_1,\ldots,f_t$ such that the annihilators $\operatorname{ann}_E f_i=(v: v\in V_i)$ satisfy the condition $V_i\cap(V_1+\ldots+\hat{V}_i+\ldots+V_t)=\{0\}$ for every $i=1,\ldots,t$.

One generating system of J is the set $\{\partial e_C : C \text{ is a circuit }\}$. But this is not necessarily minimal. Often it is enough to consider only a subset of the set of circuits of the matroid. The annihilator of ∂e_C is generated by the elements e_i-e_j with $i,j\in C$. Therefore every Orlik-Solomon ideal is pure decomposable. Thus for being strongly decomposable J should have a generating system whose circuits do not intersect “too much”. Obviously if two of the circuits have at least two common elements, then J cannot be strongly decomposable w.r.t. a generating set that contains these two circuits.

Proposition 6.4.1. *J is strongly decomposable w.r.t a generating set of circuits if and only if the Orlik-Solomon ideals of all of its connected components are.*

PROOF. Let $T_1,\dots,T_k$ be the ground sets of the components of M and $J = (\partial e_{C_{i,j}})$ where $C_{i,j}$ is the j-th circuit of the i-th component (which is part of the generating set). The vector space corresponding to the annihilator of the (i,j)-th generator is $V_{i,j} = \langle e_x - e_y : x,y \in C_{i,j}\rangle$. Then J is strongly decomposable if and only if

$$V_{h,l} \cap \sum_{i\neq h \text{ or } (h,j)\neq(h,l)} V_{i,j} = \{0\}.$$

Assume that J is strongly decomposable. Then

$$V_{h,l} \cap \sum_{j\neq l} V_{h,j} \subseteq V_{h,l} \cap \sum_{i\neq h \text{ or } (h,j)\neq(h,l)} V_{i,j} = \{0\}$$

for each $h = 1,\dots,k$. Thus the Orlik-Solomon ideal $(\partial e_{C_{h,j}})$ of the h-th component is strongly decomposable.

Now suppose that the ideal of each component is strongly decomposable. Notice that $\sum_j V_{i,j} = \langle e_x - e_y : x,y \in T_i\rangle$ for all $i = 1,\dots,k$ because whenever $|C\cap C'| \geq 1$ for two circuits C,C', then

$$\langle e_x - e_y : x,y \in C\rangle + \langle e_x - e_y : x,y \in C'\rangle = \langle e_x - e_y : x,y \in C\cup C'\rangle.$$

Let v be an element in $V_{h,l} \cap \sum_{i\neq h \text{ or } (h,j)\neq(h,l)} V_{i,j}$. Then there exist elements a_p, b_p which are in the same ground set T_{i_p}, $i_p \neq h$ and c_s, d_s which are in the same circuit C_{h,j_s} (in T_h) and coefficients $\alpha_p, \beta_s \in K$ such that

$$v = \sum_p \alpha_p(e_{a_p} - e_{b_p}) + \sum_s \beta_s(e_{c_s} - e_{d_s}) \in V_{h,l}.$$

The set of variables occuring in the first sum is disjoint to the set of variables that occur in the second sum and in $V_{h,l}$ because they lie in different components. Thus the first sum must be zero. Then

$$v = \sum_s \beta_s(e_{c_s} - e_{d_s}) \in V_{h,l} \cap \sum_{j\neq l} V_{h,j}$$

whence $v = 0$ since the ideal of the h-th component is strongly decomposable by assumption. This concludes the proof. □

Therefore it is enough to look at connected matroids. We start by looking at the number of circuits.

Example 6.4.2. (One circuit) If the matroid has only one circuit then J is strongly decomposable. By the way this is the only case when J is generated by one element.

Example 6.4.3. (Two circuits) A connected matroid cannot have exactly two circuits because of the circuit elimination axiom. Combining Proposition 6.4.1 and Example 6.4.2 one sees that a (disconnected) matroid with exactly two circuits obviously is strongly decomposable.

Example 6.4.4. (Three circuits) If a connected matroid has exactly three circuits C_1, C_2, C_3, then the circuit elimination axiom implies that each circuit is the symmetric difference of the two others, i.e.,

$$C_i = (C_j \cup C_k) \setminus (C_j \cap C_k) \text{ for } \{i,j,k\} = \{1,2,3\},$$

as follows: First of all we see that the intersection of any two of the circuits is not empty. Suppose for the contrary that $C_1 \cap C_2 = \emptyset$ and consider $x \in C_1$. Then x is not contained in C_2. Every $y \in C_2$ is not contained in C_1, but since the matroid is connected, there must exist a circuit that contains both x and y. This must be C_3. Hence $C_2 \subseteq C_3$, a contradiction. If the intersection is not empty, then it is immediate that $C_i \subseteq (C_j \cup C_k) \setminus (C_j \cap C_k)$ by the circuit elimination axiom. Suppose there exists an element $x \in (C_j \cup C_k) \setminus (C_j \cap C_k)$ and $x \notin C_i$, say $x \in C_j$. This means that $x \notin C_k$. But this is a contradiction to $C_j \subseteq (C_i \cup C_k) \setminus (C_i \cap C_k)$.

We distinguish two cases:

(a) The intersection $C_1 \cap C_2$ contains exactly one element.
(b) The intersection $C_1 \cap C_2$ contains more than one element.

(a) We show that in this situation J is strongly decomposable w.r.t. the generating system $J = (\partial e_{C_1}, \partial e_{C_2})$. At first sight, the Orlik-Solomon ideal is generated by all three circuits, but J is not strongly decomposable w.r.t. this generating set. One reason for this is that it is not minimal. It is enough to take the first two circuits, that is, $J = (\partial e_{C_1}, \partial e_{C_2})$. To prove this, one has to do some computations showing that $\partial e_{C_3} \in (\partial e_{C_1}, \partial e_{C_2})$. This is done in Lemma 6.4.5 following this example. Since $V_i = \langle e_x - e_y : x, y \in C_i \rangle$ for $i = 1,2$ and $|C_1 \cap C_2| = 1$ we conclude that $V_1 \cap V_2 = \{0\}$. Thus J is strongly decomposable.

(b) We show that in this situation J is not strongly decomposable w.r.t any generating set. If the intersection $C_1 \cap C_2$ consists of more than one element, say $x, y \in C_1 \cap C_2$, then any generating set containing ∂e_{C_1} and ∂e_{C_2} cannot be strongly decomposable w.r.t. this generating set since $e_x - e_y$ is a nonzero element in the intersection $V_1 \cap V_2$. We show that J is also not strongly decomposable w.r.t. every other system of generators because the number of variables is not big enough. We can assume $n = |C_1 \cup C_2| = |C_1| + |C_2| - |C_1 \cap C_2|$ because $C_3 = C_1 \cup C_2 \setminus C_1 \cap C_2 \subseteq C_1 \cup C_2$ and because we can divide out supernumerous variables (i.e., variables that do not appear in any circuit) and afterwards the ideal would still be strongly decomposable by Lemma 5.5.10.

It is not always the case that ∂e_{C_1} and ∂e_{C_2} are a minimal system of generators of J (we do not know exactly when). But they give a lower bound for the number of elements and degrees of a minimal system of generators. Therefore we look at the ideal generated by them.

If $(\partial e_{C_1}, \partial e_{C_2})$ has a generating system that satisfies the strongly decomposable property, then there exists a coordinate transformation such that this generating set is transformed to monomials with pairwise disjoint support. Thus we would need at least $|C_1| - 1 + |C_2| - 1$ variables. If $|C_1 \cap C_2| > 2$, then

$$n = |C_1| + |C_2| - |C_1 \cap C_2| < |C_1| + |C_2| - 2$$

whence there are not enough variables. If $|C_1 \cap C_2| = 2$, then there are exactly as much variables as needed. But the Orlik-Solomon algebra of a connected matroid has depth one by Theorem 6.2.2 so we can divide out one regular element. This operation does not change the property of being strongly decomposable nor the necessary number of variables but diminishes the number of available variables by one. So there are not enough also for $|C_1 \cap C_2| = 2$.

For readability we relabel the variables in the following lemma.

Lemma 6.4.5. *Let $C_1 = \{1,2,\dots,s-1,s\}$, $C_2 = \{s,s+1,\dots,n-1,n\}$ and $C_3 = \{1,2,\dots,s-1,s+1,\dots,n-1,n\}$ for some natural numbers $s \le n$. Then*

$$\partial e_{C_3} = (-1)^{(n+1)(s+1)} \partial e_{C_2 \setminus \{s\}} \partial e_{C_1} + \partial e_{C_1 \setminus \{s\}} \partial e_{C_2}.$$

In particular, the Orlik-Solomon ideal $(\partial e_{C_1}, \partial e_{C_2}, \partial e_{C_3})$ is generated by ∂e_{C_1} and ∂e_{C_2}.

PROOF. The proof is just a rewriting of the right-hand side of the equation using the definition of ∂, that is

$$\begin{aligned}
&(-1)^{(n+1)(s+1)} \partial e_{C_2 \setminus \{s\}} \partial e_{C_1} + \partial e_{C_1 \setminus \{s\}} \partial e_{C_2} \\
&= (-1)^{(n+1)(s+1)} \sum_{j=s+1}^{n} (-1)^{j-s-1} e_{C_2 \setminus \{s,j\}} \sum_{i=1}^{s} (-1)^{i-1} e_{C_1 \setminus \{i\}} \\
&+ \sum_{i=1}^{s-1} (-1)^{i-1} e_{C_1 \setminus \{s,i\}} \sum_{j=s}^{n} (-1)^{j-s} e_{C_2 \setminus \{j\}} \\
&= \sum_{j=s+1}^{n} \sum_{i=1}^{s} (-1)^{(n+1)(s+1)+j+i-s} e_{C_2 \setminus \{s,j\}} e_{C_1 \setminus \{i\}} \\
&+ \sum_{i=1}^{s-1} \sum_{j=s}^{n} (-1)^{j+i-s-1} e_{C_1 \setminus \{s,i\}} e_{C_2 \setminus \{j\}} \\
&\overset{(*)}{=} \sum_{j=s+1}^{n} \sum_{i=1}^{s} (-1)^{j+i-s} e_{(C_1 \cup C_2) \setminus \{s,j,i\}} + \sum_{i=1}^{s-1} \sum_{j=s}^{n} (-1)^{j+i-s-1} e_{(C_1 \cup C_2) \setminus \{s,j,i\}} \\
&= \sum_{j=s+1}^{n} \sum_{i=1}^{s-1} \left((-1)^{j+i-s} + (-1)^{j+i-s-1} \right) e_{(C_1 \cup C_2) \setminus \{s,j,i\}} \\
&+ \sum_{j=s+1}^{n} (-1)^{j} e_{(C_1 \cup C_2) \setminus \{s,j\}} + \sum_{i=1}^{s-1} (-1)^{i-1} e_{(C_1 \cup C_2) \setminus \{s,i\}} \\
&= 0 + \partial e_{C_3}.
\end{aligned}$$

In the step labelled $(*)$ we used $e_{C_2 \setminus \{s,j\}} e_{C_1 \setminus \{i\}} = (-1)^{(|C_2|-2)(|C_1|-1)} e_{(C_1 \cup C_2) \setminus \{s,j,i\}}$, since $\deg e_{C_2 \setminus \{s,j\}} = |C_2| - 2$, $\deg e_{C_1 \setminus \{i\}} = |C_1| - 1$ and C_1 and C_2 are ordered. Into the last step goes $C_3 = (C_1 \cup C_2) \setminus \{s\}$. □

For the uniform matroids, our standard example class, we are able to characterise completely which of them are strongly decomposable.

Example 6.4.6. Let $|K| = \infty$. We claim that the uniform matroid $U_{m,n}$ is strongly decomposable if and only if $m \in \{0,1,n-1,n\}$.

One possible generating system of $J_{m,n}$ we know is the set $\{\partial e_C : C \subset [n], |C| = m + 1, 1 \in C\}$. But as 1 is an element of each circuit in this set, it is probably not an appropriate generating system for our purpose.

Instead of looking for a better generating system we first construct a necessary condition. Therefore suppose that J has a generating set that satisfy the strongly decomposable property. Following Proposition 5.5.6 this must be a minimal system of generators. The number of elements in a minimal system of generators is determined by the zeroth Betti number which is $\binom{n-1}{m}$ by Theorem 6.3.6.

Hence after a change of coordinates the ideal is generated by $\binom{n-1}{m}$ monomials of degree m with pairwise distinct support. Thus the exterior algebra must have at least $\binom{n-1}{m} \cdot m$ variables. So our necessary condition is

$$\binom{n-1}{m} \cdot m \leq n.$$

The cases in which this is true are characterised in the following lemma, whose proof we postpone to the end of the example.

Lemma 6.4.7. *Let m, n be natural numbers with $m \leq n$. Then $\binom{n-1}{m} \cdot m \leq n$ if and only if $m \in \{0, 1, n-1, n\}$.*

This result says that the uniform matroid $U_{m,n}$ can only be strongly decomposable if $m \in \{0, 1, n-1, n\}$. One easily finds generating sets in this four cases such that $J_{m,n}$ is strongly decomposable w.r.t. these:

(i) If $m = 0$, then $J_{0,n} = (1)$.
(ii) If $m = 1$, then $J_{1,n}$ is generated by linear forms.
(iii) If $m = n-1$, then $J_{n-1,n}$ is generated by one element.
(iv) If $m = n$, then $J_{n,n} = (0)$.

PROOF OF LEMMA 6.4.7. We consider $\binom{n-1}{m} \cdot m$ as a function in m. It is increasing on $[0, \lfloor \frac{n}{2} \rfloor] \cap \mathbb{N}$ and symmetric to the axis $m = \frac{n-1}{2}$.

The claim then follows from the observation that $\binom{n-1}{m} \cdot m \leq n$ for $m = 1, n-1$ and $\binom{n-1}{m} \cdot m \geq n$ for $m = 2, n-2$. □

6.5. Matroid complexes

There is another quotient ring of the exterior algebra related to matroids. The set of independent sets of a matroid M is closed under taking subsets and thus a simplicial complex Δ_M. Therefore one can investigate the face ring $K\{\Delta_M\}$. Björner [**6**, Theorem 7.3.3] proves that Δ_M is shellable. So $K\{\Delta_M\}$ has pure decomposable quotients (cf. Theorem 5.4.10) and in particular a linear injective resolution, because Δ_M is pure and hence its socle lies in one degree.

Theorem 6.5.1. *Let $|K| = \infty$ and M be a matroid with s singleton components. Then*

$$\operatorname{depth}_E K\{\Delta_M\} = s.$$

PROOF. M can be decomposed into a direct sum $M = M' \oplus U_{s,s}$ such that M' is a matroid without singleton components. If $i_1, \ldots, i_s$ are the elements of the singleton components, then $e_{i_1}, \ldots, e_{i_s}$ is a regular sequence on $K\{\Delta_M\}$ and $K\{\Delta_M\}/(e_{i_1}, \ldots, e_{i_s})K\{\Delta_M\}$

is isomorphic to the face ring of $\Delta_{M'}$. Hence we may assume that M has no singleton components. Then we have to show that $\operatorname{depth}_E K\{\Delta_M\} = 0$ which is equivalent to $\tilde{\chi}(\Delta_M) \neq 0$ by Corollary 5.1.4.

Björner [**6**, Proposition 7.4.7] proved that

$$\tilde{\chi}(\Delta_M) = (-1)^{r(M)}\tilde{\mu}(M^*)$$

(observe that he denotes the reduced Euler characteristic by $\chi(\Delta)$). Here $r(M)$ is the rank of M and M^* is its dual matroid, whose bases are the complements of the bases of M. Furthermore $\tilde{\mu}(M^*)$ the Möbius invariant of M^* defined as

$$\tilde{\mu}(M^*) = \begin{cases} |\mu_L(\hat{0},\hat{1})| & \text{if } M^* \text{ is loopless,} \\ 0 & \text{if } M^* \text{ has loops,} \end{cases}$$

where L is the lattice of flats of M^*. By [**51**, Theorem 4] the number $|\mu_L(\hat{0},\hat{1})|$ is not zero.

The loops of M^* are exactly the singleton components (or coloops) of M. Since we have assumed that M has no singleton components, $\tilde{\mu}(M^*)$ is not zero and hence $\tilde{\chi}(\Delta_M) \neq 0$. □

With the depth the homological invariants complexity and regularity can be computed, cf. Theorems 4.1.2 and 5.1.6.

Corollary 6.5.2. *Let $|K| = \infty$ and M be a matroid on $[n]$ of rank l with s singleton components. Then*

(i) $\operatorname{cx}_E K\{\Delta_M\} = n - s$.
(ii) $\operatorname{reg}_E K\{\Delta_M\} = l - s$.

Proposition 6.5.3. *Let $|K| = \infty$ and M be a matroid. The face ideal J_{Δ_M} has an m-linear projective resolution over E if and only if $M = U_{m-1,n-s} \oplus U_{s,s}$.*

PROOF. As in the proof of Theorem 6.5.1 we may assume that M has no singleton components and thus $\operatorname{depth}_E K\{\Delta_M\} = 0$. Then we have to show that J_{Δ_M} has an m-linear projective resolution over E if and only if $M = U_{m-1,n}$. This follows directly from Corollary 5.1.9 and $\Delta_{U_{m-1,n}} = \Delta_{m-1,n}$. □

Corollary 6.5.4. *Let $|K| = \infty$ and M be a matroid. The Stanley-Reisner ideal I_{Δ_M} has an m-linear free resolution over S if and only if $M = U_{m-1,n-s} \oplus U_{s,s}$.*

An easy observation is that any strongly decomposable monomial ideal is the face ideal of a matroid complex.

Corollary 6.5.5. *A monomial ideal $J \subset E$ is strongly decomposable if and only if $J = J_{\Delta_M}$ for a matroid M such that every component of M has at most one circuit.*

PROOF. If J is strongly decomposable, then it is equal to the face ideal of Δ_M, where the circuits of M are the sets corresponding to the generators of J. Since these sets are pairwise disjoint by Proposition 5.5.4, they define indeed a matroid, and every component of this matroid is given by one of these sets.

Conversely let $J = J_{\Delta_M}$ with M as above. The minimal non-faces of Δ_M, which correspond to the generators of J, are the circuits of M. They are pairwise disjoint and thus J is strongly decomposable by Proposition 5.5.4. □

6.6. Examples

In this section we study some examples of matroids with small rank or small number of elements.

Oxley enumerates in [**49**, Table 1.1] all non-isomorphic matroids with three or fewer elements. The only loopless matroids among them are the uniform matroids $U_{1,1}$, $U_{1,2}$, $U_{2,2}$, $U_{1,3}$, $U_{2,3}$ and $U_{3,3}$. Their depth, complexity and regularity were already computed in Example 6.3.2.

Now we turn to matroids defined by central hyperplane arrangements in $\mathbb{C}^l$ with $l \leq 3$. The arrangement is called central if the common intersection of all hyperplanes is not empty. A set of t hyperplanes defines an independent set if and only if their intersection has codimension t. Thus every two hyperplanes in a central arrangement define an independent set and so the matroids defined by central hyperplane arrangements are simple.

In $\mathbb{C}^1$ the only central hyperplane arrangement consists of a single point, thus the underlying matroid is $U_{1,1}$.

In $\mathbb{C}^2$ a central hyperplane arrangement consists of n lines through the origin. The underlying matroid is $U_{2,n}$ if $n \geq 2$ and $U_{1,1}$ if $n = 1$.

In $\mathbb{C}^3$ central hyperplane arrangement define various matroids. One single hyperplane defines a $U_{1,1}$, two hyperplanes a $U_{2,2}$. Three hyperplanes intersecting in a point give a $U_{3,3}$, if their intersection is a line then the underlying matroid is $U_{2,3}$. More generally n hyperplanes through a line define the matroid $U_{2,n}$. Such an arrangement is called a *pencil*. For the first time one obtains a matroid that is not uniform with four hyperplanes taking three hyperplanes intersecting in a line and a fourth in general position, i.e., the intersection of the fourth with every two others is a point. The underlying matroid has two components, one containing the first three hyperplanes and one singleton component for the fourth hyperplane. It is the matroid $U_{2,3} \oplus U_{1,1}$. Such an arrangement is an example for a *near pencil*. For simplicity we define the notions of pencil and near pencil in terms of their underlying matroid.

Definition 6.6.1. A central arrangement of $n \geq 3$ hyperplanes is called

(i) a *pencil* if its underlying matroid is $U_{2,n}$.
(ii) a *near pencil* if its underlying matroid is $U_{2,n-1} \oplus U_{1,1}$.

In abuse of notation we also call the matroid $U_{2,n}$ a pencil and $U_{2,n-1} \oplus U_{1,1}$ a near pencil.

A matroid defined by n hyperplanes in $\mathbb{C}^3$ is a simple matroid of rank 3 unless it is not a pencil which has rank 2. We classify all simple rank 3 matroids by their connectedness. Then we determine their homological invariants depth, complexity and regularity.

It is well-known that a near pencil is the unique reducible central hyperplane arrangement in $\mathbb{C}^3$; we present a homological proof for this fact.

Theorem 6.6.2. *Let M be a simple matroid of rank 3. Then M is connected if and only if it is not a near pencil.*

PROOF. Note that $n \geq 3$ since M has rank 3. If $M = U_{2,n-1} \oplus U_{1,1}$ is a near pencil, it has two components if $n > 3$ and three components if $n = 3$. Thus it is not connected in any case.

Suppose that M has k components with $k > 1$ and let J be its Orlik-Solomon ideal. It is zero if and only if all subsets are independent. Then $r(M) = 3$ implies that $M = U_{3,3} = U_{2,2} \oplus U_{1,1}$ is a near pencil. So from now on we assume $J \neq 0$. Since M is simple, J is generated in degree ≥ 2 and thus $\operatorname{reg}_E J \geq 2$. Theorem 6.2.3 and Corollary 6.3.1 imply that

$$\operatorname{reg}_E(J) = \operatorname{reg}_E(E/J) + 1 = 3 - k + 1 = 4 - k \leq 2.$$

Thus the regularity of J is exactly 2 and $k = 2$. Then J has a 2-linear resolution and we may apply Theorem 6.3.4 which says that $M = U_{2,n-i} \oplus U_{i,i}$ for some $0 \leq i \leq n$. We may assume $2 < n - i$ otherwise M is $U_{3,3}$ and has three components. Then $3 = r(M) = 2 + i$ so $i = 1$ and $M = U_{2,n-1} \oplus U_{1,1}$ is a near pencil. □

In the following table we have collected the homological invariants of all simple matroids of rank 3 which are given in the preceding Theorem 6.6.2, using [**1**, Theorem 3.2], Theorem 6.2.3 and Corollary 6.3.1. It is a generalisation of Proposition 4.6 of Schenck and Suciu in [**54**], even including the special case $n = 3$.

	$\operatorname{depth}_E(E/J)$	$\operatorname{cx}_E(E/J)$	$\operatorname{reg}_E(E/J)$
no near pencil	1	$n-1$	2
near pencil, $n > 3$	2	$n-2$	1
near pencil, $n = 3$	3	0	0

The number of simple rank 3 matroids is, e.g., determined in [**16**]. If $n = 4$ there exist only two simple rank 3 matroids, namely $U_{3,4}$ and $U_{2,3} \oplus U_{1,1}$. If $n = 5$ there exist 4 simple rank 3 matroids, $U_{3,5}$, $U_{2,4} \oplus U_{1,1}$ and two further which cannot be expressed as sum of uniform matroids since they must be connected by Theorem 6.6.2. One is the underlying matroid of an arrangement of five hyperplanes, three intersecting in a line and two in general position to each other and to the first three hyperplanes. The matroid has only one circuit with three elements corresponding to the first three hyperplanes and three circuits with four elements. The arrangement of five hyperplanes defining the second matroid has twice three hyperplanes intersecting in a line. The matroid has two circuits with three elements corresponding to these triples, and one circuit with four elements, not containing the element in the intersection of the other circuits.

Open problem 6.6.3. *Are there more interesting applications to hyperplane arrangements?*

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