# Integral equation approach to reflection and transmission of a plane TE-wave at a (linear/nonlinear) dielectric film with spatially varying permittivity 

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## Chapter 1

## Introduction and statement of the problem

Many problems of optics involve the study of the optical response of a dielectric film with a specific permittivity. For constant permittivity the problem is of particular interest in linear optics [1], [2]. In the case of arbitrary varying field intensity independent permittivity traditionally the transfer matrix approach [3] is used discretizing the film by a number of plane parallel dielectric slabs of infinitesimal thickness with constant permittivity. The optical response of each slab is described by a $2 \times 2$ matrix and the net response of the film is obtained through matrix multiplication. Furthermore the Green's function method [4], the invariant embedding approach [5], and the wave splitting theory [6] are known techniques in this respect. Recently, an iterative approach, based on a pair of coupled differential equations generated from Maxwell's equations was proposed by Menon et al [7].

Within a transfer matrix method Sedrakian et al [8], [9] reduced the problem of finding the reflection and transmission amplitudes for arbitrary polarized plane waves to a set of first-order differential equations for the scattering amplitudes.

In nonlinear optics, the Kerr-like nonlinear dielectric film has been the focus of a number of studies [10]-[15]. With respect to the nonlinear Fabry-Perot system, the present problem has been approached under special conditions by several authors: Marburger and Felber [16] simplified the analysis by imposing boundary conditions which suppose the nonlinear slab is separated from the linear media by perfect mirrors. Danikaert et al. [17] treated the
steady-state response of a nonlinear Fabry-Perot resonator including nonlinear absorbtion and oblique incidence for transverse-electric and transversemagnetic polarized fields. Haeltermann et al. [18] and Vitrant [19] presented a unified nonlinear theory for transverse effects of Fabry-Perot resonators simplifying numerical calculations and providing a good understanding of optical bistability. Yasumoto et al. [20] investigated the characteristics of the directional coupler with Kerr type nonlinear gap layer by the orthogonal collocation method, and analyzed the nonlinear grating couplers by using the singular perturbation technique [21].

A considerable amount of interest in recent years has centered on the nonlinear media with the saturation of the nonlinear refractive index [22]-[25].

To the best of my knowledge, there exists no general solution to Maxwell's equations for space dependent and nonlinear (with respect to the field dependence) permittivities. This is the reason to study the following problem.


Figure 1.1: Configuration considered in this work. A plane wave is incident to a linear slab (situated between two linear media) to be reflected and transmitted (including total reflection at $y=0$ ).

Referring to Figure 1.1 the reflection and transmission of an electromagnetic plane wave at a dielectric film between two linear semi-infinite media (substrate and cladding) is considered. All media are assumed to be homogeneous
in $x$ - and $z$ - direction, isotropic, and non-magnetic. The permittivity of the film is assumed to be characterized by a function $\varepsilon_{f}(y)+\varepsilon_{N L}(E)$, where $\varepsilon_{N L}(E)$ denotes the nonlinear part of the permittivity function.

A plane wave of frequency $\omega_{0}$ and intensity $E_{0}^{2}$, with electric vector $\mathbf{E}_{\mathbf{0}}$ parallel to the $z$-axis (TE) is incident on the film of thickness $d$. Since the geometry is independent of the $z$-coordinate and because of the supposed TE-polarization the fields are parallel to the $z$-axis $\left(\mathbf{E}=\mathbf{E}_{z}\right)$. The problem is to find the solutions of Maxwell's equations, neglecting higher harmonics, for various permittivity functions $\varepsilon_{f}(y)$ (linear, nonlinear, real-valued, complex-valued) subject to the boundary conditions (continuity of $\mathbf{E}_{z}$ and $\partial \mathbf{E}_{z} / \partial y$ at interfaces $y \equiv 0$ and $\left.y \equiv d\right)$.

## Chapter 2

## Description of the method illustrated at a lossless dielectric film

### 2.1 Transmission and reflection at a linear lossless dielectric film

### 2.1.1 Reduction of the problem to a Volterra integral equation

To elucidate the method the case of a linear dielectric film with a real-valued permittivity is considered in this chapter [26].

The permittivity is modelled by

$$
\varepsilon(y)=\left\{\begin{array}{l}
\varepsilon_{c}, y>d  \tag{2.1.1}\\
\varepsilon_{f}(y), 0<y<d \\
\varepsilon_{s}, y<0
\end{array}\right.
$$

with real constants $\varepsilon_{c}, \varepsilon_{s}$ and with $\varepsilon_{f}(y)$ as a real continuously differentiable function of $y$ on $[0, d]$.

Due to the requirement of the translational invariance in $x$-direction and partly satisfying the boundary conditions the fields tentatively are written
as ( $\hat{z}$ denotes the unit vector in $z$-direction)

$$
\mathbf{E}(x, y, t)=\left\{\begin{array}{cl}
\hat{z} \frac{1}{2}\left[E_{0} e^{i\left(p x-q_{c} \cdot(y-d)-\omega_{0} t\right)}+\right. &  \tag{2.1.2}\\
\left.E_{r} e^{i\left(p x+q_{c} \cdot(y-d)-\omega_{0} t\right)}+c . c .\right], & y>d, \\
\hat{z} \frac{1}{2}\left[E(y) e^{i\left(p x+\vartheta(y)-\omega_{0} t\right)}+c . c .\right], & 0<y<d, \\
\hat{z} \frac{1}{2}\left[E_{3} e^{i\left(p x-q_{s} y-\omega_{0} t\right)}+c . c .\right], & y<0,
\end{array}\right.
$$

where $E(y), p=\sqrt{\varepsilon_{c}} k_{0} \sin \varphi, q_{c}$, and $\vartheta(y)$ are real and $E_{r}=\left|E_{r}\right| \exp \left(i \delta_{r}\right)$ and $E_{3}=\left|E_{3}\right| \exp \left(i \delta_{t}\right)$ are independent of $y$. The parameter $q_{s}$ is assumed to be real (transmission case) or purely imaginary $q_{s}$ (total reflection case) in the following.

By inserting (2.1.2) and (2.1.1) into Maxwell's equations the linear Helmholtz equations, valid in each of the three media $(j=s, f, c)$, read

$$
\begin{equation*}
\frac{\partial^{2} \tilde{E}_{j}(x, y)}{\partial x^{2}}+\frac{\partial^{2} \tilde{E}_{j}(x, y)}{\partial y^{2}}+k_{0}^{2} \varepsilon_{j} \tilde{E}_{j}(x, y)=0, \quad j=s, f, c \tag{2.1.3}
\end{equation*}
$$

where $k_{0}^{2}=\omega_{0}^{2} / c^{2}$ and $\tilde{E}_{j}(x, y)$ denotes the time-independent part of $\mathbf{E}(x, y, t)$. It should be noted that the assumed TE-polarization of the incident plane wave and the form of the permittivity function $\varepsilon_{f}(y)$ in middle layer are essential for deriving Helmholtz equation (2.1.3) from Maxwell's equations.

Scaling $x, y, z, p, q_{c}, q_{s}$ by the wavelength $\lambda_{0}$ and $\varepsilon$ by $\varepsilon_{0}$, respectively, equations (2.1.3) can be written as

$$
\begin{equation*}
\frac{\partial^{2} \tilde{E}_{j}(x, y)}{\partial x^{2}}+\frac{\partial^{2} \tilde{E}_{j}(x, y)}{\partial y^{2}}+4 \pi^{2} \varepsilon_{j} \tilde{E}_{j}(x, y)=0, \quad j=s, f, c \tag{2.1.4}
\end{equation*}
$$

where the same symbols have been used for unscaled and scaled quantities. Using ansatz (2.1.2) in equation (2.1.4) one gets for the semi-infinite media

$$
\begin{equation*}
q_{j}^{2}=4 \pi^{2} \varepsilon_{j}-p^{2}, \quad j=s, c \tag{2.1.5}
\end{equation*}
$$

For the film $(j=f)$, it follows from equation (2.1.4), by separating real and imaginary parts,

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}-E(y)\left(\frac{d \vartheta(y)}{d y}\right)^{2}+\left[4 \pi^{2} \varepsilon_{f}(y)-p^{2}\right] E(y)=0 \tag{2.1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
E(y) \frac{d^{2} \vartheta(y)}{d y^{2}}+2 \frac{d \vartheta(y)}{d y} \frac{d E(y)}{d y}=0 . \tag{2.1.7}
\end{equation*}
$$

Equation (2.1.7) can be integrated leading to

$$
\begin{equation*}
E^{2}(y) \frac{d \vartheta(y)}{d y}=c_{1} \tag{2.1.8}
\end{equation*}
$$

where $c_{1}$ is a constant that has to be determined by means of the boundary conditions. Insertion of $d \vartheta / d y$ into Eq.(2.1.6) yields

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}+q_{f}^{2}(y) E(y)-\frac{c_{1}^{2}}{E^{3}(y)}=0 \tag{2.1.9}
\end{equation*}
$$

with

$$
\begin{equation*}
q_{f}^{2}(y)=4 \pi^{2} \varepsilon_{f}(y)-p^{2} \tag{2.1.10}
\end{equation*}
$$

As will be shown below, real $q_{s}$ (transmission) implies $c_{1} \neq 0$, pure imaginary $q_{s}$ (total reflection) implies $c_{1}=0$. Introducing $I(y)=E^{2}(y)$, equation (2.1.9) reads

$$
\begin{equation*}
-\frac{1}{4} I^{-\frac{3}{2}}(y)\left(\frac{d I(y)}{d y}\right)^{2}+\frac{1}{2} I^{-\frac{1}{2}}(y) \frac{d^{2} I(y)}{d y^{2}}+q_{f}^{2}(y) I^{\frac{1}{2}}(y)-c_{1}^{2} I^{-\frac{3}{2}}(y)=0 . \tag{2.1.11}
\end{equation*}
$$

Multiplying equation (2.1.11) by $4 I^{\frac{3}{2}}(y)$ one obtains

$$
\begin{equation*}
\left(\frac{d I(y)}{d y}\right)^{2}-2 I(y) \frac{d^{2} I(y)}{d y^{2}}-4 q_{f}^{2}(y) I^{2}(y)+c_{1}^{2}=0 \tag{2.1.12}
\end{equation*}
$$

Differentiating the equation (2.1.12) with respect to $y$ leads to

$$
\begin{array}{r}
-2 \frac{d I(y)}{d y} \frac{d^{2} I(y)}{d y^{2}}+2 \frac{d I(y)}{d y} \frac{d^{2} I(y)}{d y^{2}}+ \\
2 I(y) \frac{d^{3} I(y)}{d y^{3}}+4 \frac{d\left(q_{f}^{2}(y)\right)}{d y} I^{2}(y)+8 q_{f}^{2}(y) I(y) \frac{d I(y)}{d y}=0 \tag{2.1.13}
\end{array}
$$

hence

$$
\begin{equation*}
\frac{d^{3} I(y)}{d y^{3}}+2 \frac{d\left(q_{f}^{2}(y)\right)}{d y} I(y)+4 q_{f}^{2}(y) I(y) \frac{d I(y)}{d y}=0 \tag{2.1.14}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\frac{d^{3} I(y)}{d y^{3}}+4 \frac{d\left(q_{f}^{2}(y) I(y)\right)}{d y}=2 \frac{d\left(q_{f}^{2}(y)\right)}{d y} I(y) . \tag{2.1.15}
\end{equation*}
$$

Representing $\varepsilon_{f}(y)$ in the form $\varepsilon_{f}(y)=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y)$, where $\varepsilon_{f}^{0}$ is a constant, equation (2.1.15) can be integrated with respect to $I(y)$, to yield

$$
\begin{equation*}
\frac{d^{2} I(y)}{d y^{2}}+4 \kappa^{2} I(y)=-16 \pi^{2} \widetilde{\varepsilon}_{f}(y) I(y)+8 \pi^{2} \int_{0}^{y} \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau+c_{2} \tag{2.1.16}
\end{equation*}
$$

where $\kappa^{2}=4 \pi^{2} \varepsilon_{f}^{0}-p^{2}$ and $c_{2}$ denotes another constant of integration. The homogeneous equation $d^{2} I(y) / d y^{2}+4 \kappa^{2} I(y)=0$ has the general solution

$$
\begin{equation*}
\tilde{I}_{0}(y)=A \cos (2 \kappa y)+B \sin (2 \kappa y), \tag{2.1.17}
\end{equation*}
$$

so that the solution of equation (2.1.16) reads [27]

$$
\begin{align*}
& I(y)=\tilde{I}_{0}(y)+\int_{0}^{y} d t \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left(c_{2}+\right. \\
& \left.8 \pi^{2} \int_{0}^{t} d \tau \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau)-16 \pi^{2} \widetilde{\varepsilon}_{f}(t) I(t)\right) \tag{2.1.18}
\end{align*}
$$

where the constant $c_{2}$ must be determined by the boundary conditions. The Volterra equation (2.1.18) is equivalent to equation (2.1.3) for $0<y<d$. According to equation (2.1.18) $I(y)$ and $\tilde{I}_{0}(y)$ satisfy the boundary conditions at $y=0$. Evaluating the first integral on the right hand side, equation (2.1.18) reads [cf. Appendix A]

$$
\begin{equation*}
I(y)=\tilde{I}_{0}(y)+\frac{c_{2}}{2 \kappa^{2}} \sin ^{2}(\kappa y)+\int_{0}^{y} K(y, t) I(t) d t \tag{2.1.19}
\end{equation*}
$$

with

$$
\begin{equation*}
K(y, t)=-8 \pi^{2} \frac{\sin 2 \kappa(y-t)}{\kappa} \widetilde{\varepsilon}_{f}(t)+4 \pi^{2} \frac{\sin ^{2} \kappa(y-t)}{\kappa^{2}} \frac{d \widetilde{\varepsilon}_{f}(t)}{d t} . \tag{2.1.20}
\end{equation*}
$$

If $\widetilde{\varepsilon}_{f}(y) \equiv 0, I(y)$ from equation (2.1.19) is identical to the exact solution of the linear problem [cf. Appendix B].

The solution of equation (2.1.19) can be represented as a uniformly convergent series of iterations [cf. Appendix C]:

$$
\begin{gather*}
I(y)=\sum_{j=0}^{\infty} I_{j}(y)  \tag{2.1.21}\\
I_{j}(y)=\int_{0}^{y} K(y, t) I_{j-1}(t) d t, \quad j=1,2, \ldots \tag{2.1.22}
\end{gather*}
$$

where

$$
\begin{equation*}
I_{0}(y)=\tilde{I}_{0}(y)+\frac{c_{2}}{2 \kappa^{2}} \sin ^{2}(\kappa y) \tag{2.1.23}
\end{equation*}
$$

With the solution $I(y)$, determined by equation (2.1.21), the phase function $\vartheta(y)$ is given, according to equation (2.1.8), by

$$
\begin{equation*}
\vartheta(y)=\vartheta(d)+c_{1} \int_{d}^{y} \frac{d \tau}{I(\tau)} \tag{2.1.24}
\end{equation*}
$$

### 2.1.2 Boundary conditions and associated relations

To evaluate $I(y)$ and $\vartheta(y)$ according to equations (2.1.19), (2.1.24), respectively, it is necessary to determine the integration constants $c_{1}, c_{2}$ by means of the boundary conditions $(\mathbf{E}(y)$ and $d \mathbf{E}(y) / d y$ must be continuous at $y=0$ and $y=d$ ). Hence

$$
\begin{gather*}
E(0)=E_{3} e^{-i \vartheta(0)}  \tag{2.1.25}\\
\left.\frac{d E(y)}{d y}\right|_{y=0}+\left.i \frac{d \vartheta(y)}{d y}\right|_{y=0} E(0)=-i q_{s} E(0)  \tag{2.1.26}\\
E_{0}+E_{r}=E(d) e^{i \vartheta(d)}  \tag{2.1.27}\\
2 E_{0} e^{-i \vartheta(d)}=\left.\frac{i}{q_{c}} \frac{d E(y)}{d y}\right|_{y=d}+E(d)\left(1-\left.\frac{1}{q_{c}} \frac{d \vartheta(y)}{d y}\right|_{y=d}\right) . \tag{2.1.28}
\end{gather*}
$$

In general, subject to the restriction of real $\varepsilon_{f}(y)$, the cases $q_{s}^{2}>0$ (transmission) and $q_{s}^{2} \leq 0$ (total reflection) must be discriminated. In the first case equation (2.1.26) implies

$$
\begin{gather*}
\left.\frac{d E(y)}{d y}\right|_{y=0}=0  \tag{2.1.29}\\
\left.\frac{d \vartheta(y)}{d y}\right|_{y=0}=-q_{s} \tag{2.1.30}
\end{gather*}
$$

in the second case $\left(q_{s}=i \tilde{q}_{s}\right)$

$$
\begin{gather*}
\left.\frac{d E(y)}{d y}\right|_{y=0}=\tilde{q}_{s} E(0),  \tag{2.1.31}\\
\left.\frac{d \vartheta(y)}{d y}\right|_{y=0}=0 \tag{2.1.32}
\end{gather*}
$$

Thus, according to equations (2.1.8), (2.1.25) and (2.1.30) the constant $c_{1}$ is given by

$$
\begin{equation*}
c_{1}=-q_{s}\left|E_{3}\right|^{2}=-q_{s} I(0) \tag{2.1.33}
\end{equation*}
$$

in the transmission case, and by

$$
\begin{equation*}
c_{1}=0 \tag{2.1.34}
\end{equation*}
$$

in the total reflection case, according to equation (2.1.32). It should be noted that this result does not depend on the particular (real-valued) permittivity function.

Equations (2.1.27), (2.1.28) imply, introducing $I=E^{2}$,

$$
\begin{gather*}
E_{0}^{2}=\frac{1}{4}\left\{\frac{1}{4 q_{c}^{2} I(d)}\left(\left.\frac{d I(y)}{d y}\right|_{y=d}\right)^{2}+I(d)\left(1-\left.\frac{1}{q_{c}} \frac{d \vartheta(y)}{d y}\right|_{y=d}\right)^{2}\right\}  \tag{2.1.35}\\
E_{r}^{2}=\frac{1}{4}\left\{\frac{1}{4 q_{c}^{2} I(d)}\left(\left.\frac{d I(y)}{d y}\right|_{y=d}\right)^{2}+I(d)\left(1+\left.\frac{1}{q_{c}} \frac{d \vartheta(y)}{d y}\right|_{y=d}\right)^{2}\right\}  \tag{2.1.36}\\
\sin \vartheta(d)=-\frac{\left.\frac{d I(y)}{d y}\right|_{y=d}}{4 q_{c} E_{0} \sqrt{I(d)}} \tag{2.1.37}
\end{gather*}
$$

Evaluation of equations (2.1.35), (2.1.36) leads, by using equation (2.1.8), (2.1.33), (2.1.34), to

$$
\left|E_{r}\right|^{2}-E_{0}^{2}=\left.\frac{1}{q_{c}} I(d) \frac{d \vartheta(y)}{d y}\right|_{y=d}=\left\{\begin{array}{cc}
-\frac{q_{s}}{q_{c}}\left|E_{3}\right|^{2}, & q_{s}^{2}>0  \tag{2.1.38}\\
0, & q_{s}^{2} \leq 0
\end{array}\right.
$$

and hence to

$$
R=\left\{\begin{array}{cc}
1-T, & q_{s}^{2}>0  \tag{2.1.39}\\
1, & q_{s}^{2} \leq 0
\end{array}\right.
$$

where $R=\frac{\left|E_{r}\right|^{2}}{E_{0}^{2}}, T=\frac{q_{s}\left|E_{3}\right|^{2}}{q_{c} E_{0}^{2}}$. In the transmission case the incident intensity $E_{0}^{2}$ is related to the transmitted intensity $I(0)$ according to (cf. (2.1.35), (2.1.8))

$$
\begin{equation*}
E_{0}^{2}=\frac{1}{4}\left\{\frac{1}{4 q_{c}^{2} I(d)}\left(\left.\frac{d I(y)}{d y}\right|_{y=d}\right)^{2}+I(d)\left(1+\frac{q_{s} I(0)}{q_{c} I(d)}\right)^{2}\right\} . \tag{2.1.40}
\end{equation*}
$$

In the total reflection case the incident intensity $E_{0}^{2}$ is related to the transmitted intensity $I(0)$ according to (cf. (2.1.35), (2.1.8)), (2.1.32)),

$$
\begin{equation*}
E_{0}^{2}=\frac{1}{4}\left\{\frac{1}{4 q_{c}^{2} I(d)}\left(\left.\frac{d I(y)}{d y}\right|_{y=d}\right)^{2}+I(d)\right\} \tag{2.1.41}
\end{equation*}
$$

The phase shifts on transmission $\delta_{t}$ and on reflection $\delta_{r}$ are determined by equations (2.1.24), (2.1.27) and (2.1.38) according to

$$
\begin{equation*}
\delta_{t}=\vartheta(0)=\vartheta(d)+I(0) q_{s} \int_{0}^{d} \frac{d \tau}{I(\tau)} \tag{2.1.42}
\end{equation*}
$$

$$
\begin{equation*}
\sin \delta_{r}=\frac{E(d)}{\left|E_{r}\right|} \sin \vartheta(d) \tag{2.1.43}
\end{equation*}
$$

which implies, taking into account equation (2.1.37),

$$
\begin{equation*}
\sin \delta_{r}=-\frac{\left.\frac{d I(y)}{d y}\right|_{y=d}}{4 q_{c} E_{0}^{2} \sqrt{1-\frac{q_{s} I(0)}{q_{c} E_{0}^{2}}}} \tag{2.1.44}
\end{equation*}
$$

for the transmission case and

$$
\begin{gather*}
\delta_{t}=\vartheta(d)=\vartheta(0)=\arccos \frac{I(d)}{2 E_{0}}  \tag{2.1.45}\\
\delta_{r}=2 \vartheta(d)=\arccos \frac{I(d)}{E_{0}} \tag{2.1.46}
\end{gather*}
$$

for the total reflection case [cf. Appendix D].
Contrary to the integration constant $c_{1}$ the integration constant $c_{2}$ depends, according to equation (2.1.16), on the permittivity function $\varepsilon_{f}(y)$.

### 2.1.3 Transmission $\left(q_{s}^{2}>0\right)$

## Solutions

Using equations (2.1.9), (2.1.16), (2.1.25) and (2.1.29), the constant of integration $c_{2}$ in equation (2.1.18) is determined by [cf. Appendix E]

$$
\begin{equation*}
c_{2}=2\left|E_{3}\right|^{2}\left(q_{s}^{2}+q_{f}^{2}(0)\right)=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right) \tag{2.1.47}
\end{equation*}
$$

Taking into account equations (2.1.17), (2.1.25) and (2.1.29), $\tilde{I}_{0}(y)$ is given by

$$
\begin{equation*}
\tilde{I}_{0}(y)=I(0) \cos (2 \kappa y) \tag{2.1.48}
\end{equation*}
$$

with $I(0)\left(=\left|E_{3}\right|^{2}\right)$ related to $E_{0}^{2}$ according to equation (2.1.40). Introducing $\widehat{I}(y)=I(y) / I(0)$ and using the relations of the foregoing subsection the normalized intensity $\widehat{I}(y)$ and the phase $\vartheta(y)$ can be written as

$$
\begin{equation*}
\widehat{I}(y)=\cos (2 \kappa y)+\frac{q_{s}^{2}+q_{f}^{2}(0)}{\kappa^{2}} \sin ^{2}(\kappa y)+\int_{0}^{y} K(y, t) \widehat{I}(t) d t \tag{2.1.49}
\end{equation*}
$$



Figure 2.1: (a) Dependence of the field intensity $I(y)$ (first iteration) inside the slab on the transverse coordinate $y$ for $\varepsilon_{c}=\varepsilon_{s}=1, \varepsilon_{f}^{0}=1.5, \varphi=$ $0.35 \pi, E_{0}^{2}=1, d=1.5$. Dashed curve corresponds to the periodic dependence of $\varepsilon_{f}(y)=\varepsilon_{f}^{0}+\delta \cos ^{2} b\left(\frac{y}{d}\right)$ for $\gamma=\frac{1}{30}, b=10$. The solid curve corresponds to the case of constant permittivity $\varepsilon_{f}(y)=\varepsilon_{f}^{0}(\delta=0)$; (b) the difference between the intensities from (a).
with $\kappa^{2}=4 \pi^{2} \varepsilon_{f}^{0}-p^{2}$, where equations (2.1.47), (2.1.48) have been used, and, taking equations (2.1.24), (2.1.33), (2.1.37) into account,

$$
\begin{equation*}
\vartheta(y)=-\arcsin \frac{\left.\frac{d \hat{I}(y)}{d y}\right|_{y=d}}{\sqrt{\left(\left.\frac{d \hat{I}(y)}{d y}\right|_{y=d}\right)^{2}+4\left(q_{c} \widehat{I}(d)+q_{s}\right)^{2}}}+q_{s} \int_{y}^{d} \frac{d \tau}{\widehat{I}(\tau)} d \tau \tag{2.1.50}
\end{equation*}
$$

Equations (2.1.39), (2.1.40) together with equations (2.1.49), (2.1.50) allow the optical response of the linear film to be calculated for arbitrary thickness $d$, arbitrary angles of incidence $\varphi$ and arbitrary permittivity $\varepsilon_{f}(y)$. Equations (2.1.40), (rewritten with normalized intensity $\widehat{I}(y)$ ), and (2.1.49) constitute a generalization of Fresnel's formulae in linear optics [28].

## Numerical results

To illustrate the foregoing analysis a periodic dependence of $\varepsilon_{f}(y)$ such that $\widetilde{\varepsilon}_{f}(y)=\gamma \cos ^{2} b(y / d)$ is assumed.

The first and the second iteration of (2.1.49) lead to expressions for $I(y)$ and $\vartheta(y)$. The corresponding field intensity inside the slab is shown in Figures 2.1-2.3. In Figure 2.4 the phase $\vartheta(y)$ is plotted.


Figure 2.2: Dependence of the field intensity $I(y)$ (second iteration) inside the slab on the transverse coordinate $y$ for the same parameters as in Figure 2.1.


Figure 2.3: The difference between the field intensities after the first and second iteration.

Using equation (2.1.40) the reflectivity $R$ is given by

$$
\begin{equation*}
R=1-\frac{16 q_{c} q_{s} \widehat{I}(d)}{\left(\left.\frac{d \widehat{I}(y)}{d y}\right|_{y=d}\right)^{2}+4\left(q_{c} \widehat{I}(d)+q_{s}\right)^{2}} \tag{2.1.51}
\end{equation*}
$$

Plots of $R$ are presented in Figure 2.5. The character of the obtained dependence of $R$ on the problem's parameters (thickness $d$, angle of incidence $\varphi$ ) agrees in general with the ones obtained for periodic layers [29]. The region, where $R \approx 1$ is analogous to Bragg reflection, well known in the dynamical theory of X-ray reflection [30].


Figure 2.4: Phase $\vartheta(y)$ inside the film, parameters as in Figure 2.1

(a)

(b)

Figure 2.5: (a) Dependence of the reflectivity $R$ on the layer thickness $d$ for the same parameters as in Figure 2.1; (b) dependence of the reflectivity $R$ on the angle of incidence $\varphi$ for the same parameters as in Figure 2.1.

### 2.1.4 Total reflection $\left(q_{s}^{2}<0\right)$

## Solutions

Using equations (2.1.9),(2.1.16),(2.1.25) and (2.1.31), the constant $c_{2}$ in this case is determined by

$$
\begin{equation*}
c_{2}=2 I(0)\left(q_{f}^{2}(0)+\tilde{q}_{s}^{2}\right) . \tag{2.1.52}
\end{equation*}
$$

According to equations (2.1.17), (2.1.25), (2.1.31), $\tilde{I}_{0}(y)$ for the total reflection case is determined by

$$
\begin{equation*}
\tilde{I}_{0}(y)=I(0) \cos (2 \kappa y)+\frac{\tilde{q}_{s} I(0)}{\kappa} \sin (2 \kappa y), \tag{2.1.53}
\end{equation*}
$$

with $I(0)\left(=\left|E_{3}\right|^{2}\right)$ related to $E_{0}^{2}$ according to equation (2.1.41).

Introducing $\widehat{I}(y)=I(y) / I(0)$ and inserting equations (2.1.52), (2.1.53) into equation (2.1.19) one obtains

$$
\begin{array}{r}
\widehat{I}(y)=\cos (2 \kappa y)+\frac{\tilde{q}_{s}}{\kappa} \sin (2 \kappa y)+\frac{\left(q_{f}^{2}(0)+\tilde{q}_{s}^{2}\right)}{\kappa^{2}} \sin ^{2}(\kappa y) \\
+\int_{0}^{y} K(y, t) \widehat{I}(t) d t . \tag{2.1.54}
\end{array}
$$

The phase constant $\vartheta(0) \equiv \vartheta(d)$ is given by equation (2.1.45).


Figure 2.6: (a) Dependence of the field intensity $I(y)$ (first iteration) inside the slab on the transverse coordinate $y$ for $\varepsilon_{c}=2, \varepsilon_{s}=1, \varepsilon_{f}^{0}=3, \varphi=$ $0.35 \pi, E_{0}^{2}=1, d=0.6$. The full curve corresponds to the periodic dependence of $\varepsilon_{f}(y)$ for $\gamma=0.03, b=10$. Dashed curve corresponds to the case of constant permittivity $\varepsilon_{f}(y)=\varepsilon_{f}^{0}(\gamma=0)$; (b) difference between the intensities from (a).

## Numerical results

To illustrate the procedure in this case, again the periodic dependence of $\varepsilon_{f}(y)\left(\widetilde{\varepsilon}_{f}(y)=\gamma \cos ^{2} b\left(\frac{y}{d}\right)\right)$ is assumed. The first and second iterations of (2.1.54) yield $I(y)$ for the total reflection case. The results are presented in Figures 2.6, 2.7. The permittivity constant $\varepsilon_{s}$ in substrate is less then in cladding and film, thus the sinusoidal wave is (internally) reflected off an interface so that total internal reflection occurs. The corresponding plot of the electromagnetic field in three layers, where an evanescent wave is formed in substrate, is shown in Figure 2.8.


Figure 2.7: (a) Dependence of the field intensity $I(y)$ (second iteration) inside the slab on the transverse coordinate $y$ for the same parameters as in Figure 2.6; (b) Difference between the field intensities after the first and second iteration.


Figure 2.8: Dependence of the field intensity $I(y)$ (first iteration) in cladding, film and substrate on the transverse coordinate $y$ for the periodic dependence of $\varepsilon_{f}(y)$ and for the same parameters as in Figure 2.6.

### 2.2 Transmission and reflection at a Kerr-like nonlinear lossless dielectric film

### 2.2.1 Reduction of the problem to a Volterra integral equation

In the following a permittivity according to (2.1.1) is assumed, but with a Kerr-like nonlinearity inside the film

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y)+a E^{2}(y), 0<y<d, \tag{2.2.1}
\end{equation*}
$$

with a real Kerr constant $a$. Using the same arguments as in Section 2.1.1 (cf. equations (2.1.3), (2.1.4), (2.1.5)) one obtains in place of equation (2.1.6)

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}-E(y)\left(\frac{d \vartheta(y)}{d y}\right)^{2}+\left[4 \pi^{2}\left(\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y)+a E^{2}(y)\right)-p^{2}\right] E(y)=0 \tag{2.2.2}
\end{equation*}
$$

In place of equations (2.1.19),(2.1.20) now the intensity is given by

$$
\begin{equation*}
I(y)=\tilde{I}_{0}(y)+\frac{c_{2}}{2 \kappa^{2}} \sin ^{2}(\kappa y)+\int_{0}^{y} K(y, t, I(t)) I(t) d t \tag{2.2.3}
\end{equation*}
$$

with the kernel

$$
\begin{align*}
K(y, t, I(t))=-\frac{\sin 2 \kappa(y-t)}{\kappa} & \left(8 \pi^{2} \widetilde{\varepsilon}_{f}(t)+6 \pi^{2} a I(t)\right)+ \\
& +4 \pi^{2} \frac{\sin ^{2} \kappa(y-t)}{\kappa^{2}} \frac{d \widetilde{\varepsilon}_{f}(t)}{d t} . \tag{2.2.4}
\end{align*}
$$

Compared with (2.1.20) the kernel now depends on $I(t)$ with the consequence that iteration of (2.2.3) leads to a sequence (instead of a series)

$$
\begin{equation*}
I_{j}(y)=I_{0}(y)+\int_{0}^{y} K\left(y, t, I_{j-1}(t)\right) I_{j-1}(t) d t, \quad j=1,2, \ldots \tag{2.2.5}
\end{equation*}
$$

where $I_{0}(y)$ is given by equation (2.1.23). As shown in Appendix F this sequence uniformly converges to the solution $I(y)$ of equation (2.2.3)

$$
\begin{equation*}
I(y)=\lim _{j \rightarrow \infty} I_{j}(y) \tag{2.2.6}
\end{equation*}
$$

The uniform convergence is proved using the Banach Fixed-Point Theorem [cf. Appendix G]. The condition for convergence leads to a constraint for the
parameters of the problem (definitions of $\left\|N_{1}\right\|,\left\|N_{2}\right\|,\left\|I_{0}\right\|$ see Appendix C and F)

$$
\begin{equation*}
\left\|N_{1}\right\|+2 \sqrt{\left\|N_{2}\right\| \cdot\left\|I_{0}\right\|}<1 \tag{2.2.7}
\end{equation*}
$$

It should be noted that inequality (2.2.7) is only a sufficient condition. To evaluate it, the function $\widetilde{\varepsilon}_{f}(y)$ and the nonlinearity must be prescribed (an example how (2.2.7) can be evaluated is given in Appendix H ).

### 2.2.2 Transmission $\left(q_{s}^{2}>0\right)$

## Solutions

For real $q_{s}$, instead of equation (2.1.47) one obtains

$$
\begin{equation*}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right) \tag{2.2.8}
\end{equation*}
$$

so that equation (2.2.3) reads, taking into account equation (2.1.48),

$$
\begin{array}{r}
I(y)=I(0) \cos (2 \kappa y)+\frac{\left(q_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right) I(0)}{\kappa^{2}} \sin ^{2}(\kappa y)+ \\
\int_{0}^{y} K(y, t, I(t)) I(t) d t \tag{2.2.9}
\end{array}
$$

with $I(0)\left(=\left|E_{3}\right|^{2}\right)$ related to $E_{0}^{2}$ according to equation (2.1.40).
The phase $\vartheta(y)$ is given by

$$
\begin{equation*}
\vartheta(y)=-\arcsin \frac{\left.\left(\frac{1}{\sqrt{I(y)}} \frac{d I(y)}{d y}\right)\right|_{y=d}}{4 q_{c} E_{0}}+\int_{y}^{d} \frac{q_{s} I(0)}{I(\tau)} d \tau \tag{2.2.10}
\end{equation*}
$$

## Numerical results

A periodic dependence of $\widetilde{\varepsilon}_{f}(y)$ as for the linear case is assumed. The first iteration of equation (2.2.9) is shown in Figure 2.9. For the special case $\gamma=0$ the results of the present method can be compared with the exact analytical solution [31] (cf. Figure 2.10).

By means of a parametric plot the reflectivity $R$ and the phase on reflection $\delta_{r}$ can be evaluated straightforwardly. Results are depicted in Figures 2.11,2.12.


Figure 2.9: Dependence of the field intensity $I(y)$ (first iteration) inside the slab on the transverse coordinate $y$ for $a=0.01$, other parameters are the same as in Figure 2.1.


Figure 2.10: (a) Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ for $\varepsilon_{c}=1, \varepsilon_{s}=1.7, \varepsilon_{f}^{0}=1.3, \varphi=63.5^{\circ}, b=10, \gamma=$ $0, E_{0}^{2}=1, d=1.5, a=0.01$. Solid curve corresponds to the exact solution and dashed to the first iteration of equation (2.2.9); (b) the difference between the curves from (a).


Figure 2.11: Dependence of the reflectivity $R$ on $a E_{0}^{2}$ and on the thickness $d$ for $\varepsilon_{c}=1, \varepsilon_{s}=1.7, \varepsilon_{f}^{0}=1.3, \varphi=63.5^{\circ}, \gamma=0$.


Figure 2.12: Dependence of the phase of reflection $\delta_{r}$ on $a E_{0}^{2}$ and $d$ for the same parameters as in Figure 2.11.

### 2.2.3 Total reflection $\left(q_{s}^{2}<0\right)$

## Solutions

If $q_{s}$ is pure imaginary, the integration constant $c_{2}$ is given by

$$
\begin{equation*}
c_{2}=2 I(0)\left(\widetilde{q}_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right) \tag{2.2.11}
\end{equation*}
$$

so that the solution of equation (2.2.3) reads, taking into account equation (2.1.53),

$$
\begin{align*}
& I(y)=I(0) \cos (2 \kappa y)+\frac{\tilde{q}_{s} I(0)}{\kappa} \sin (2 \kappa y)+ \\
& \begin{array}{r}
\frac{I(0)\left(\widetilde{q}_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right)}{\kappa^{2}} \sin ^{2}(\kappa y) \\
\quad+\int_{0}^{y} K(y, t, I(t)) I(t) d t .
\end{array}
\end{align*}
$$



Figure 2.13: Dependence of the electric field (first iteration) inside the slab on the transverse coordinate $y$ for the same parameters as in Figure 2.6. Solid curve corresponds to the linear case $(a=0)$, dashed curve to the nonlinear case ( $a=0.01$ ).

## Numerical results

Fixing the parameters of the dielectric slab and assuming periodic dependence of $\widetilde{\varepsilon}_{f}(y)\left(\widetilde{\varepsilon}_{f}(y)=\gamma \cos ^{2} b(y / d)\right)$, the numerical results of the total reflection case can be illustrated. The first iteration of equation (2.2.9) is shown in Figure 2.13. Three dimensional picture of the dielectric slab is presented in Figure 2.14. Again the evanescent wave is formed in substrate, associated to total internal reflection.


E(y)

Figure 2.14: Dependence of the electric field (first iteration) in three layers on the transverse coordinates $y$ and $x$ for $\varepsilon_{c}=2, \varepsilon_{s}=1, \varepsilon_{f}^{0}=1.6, \varphi=$ $0.35 \pi, E_{0}^{2}=1, d=0.9, a=0.01, \gamma=0.03, b=0.1$.

### 2.3 General remarks on the method

Before applying the method to more general permittivities (complex-valued and saturating) it seems suitable to outline the ductus of the method on the basis of the foregoing chapters.

The main steps are:
(I) Transformation of Maxwell's equations to Helmholtz equations (2.1.3) by assuming a harmonic time dependence,
(II) transformation of Helmholtz equations (2.1.3) to a Volterra equation (2.1.18) for the intensity $I(y)$ inside the film by using a rather general ansatz (2.1.2) for the field inside the film $\left(E(y) e^{i(p x+\vartheta(y))}\right)$,
(III) iteration of the Volterra equation to obtain an approximate solution for the intensity $I(y)$ inside the film (check of the condition of convergence in the nonlinear case),
(IV) evaluation of the boundary conditions to determine the integration constants $c_{1}$ (cf. equation (2.1.24)), $c_{2}$ (in the Volterra equation) depending on the permittivity $\varepsilon_{f}(y)$, the parameters $\varepsilon_{c}, \varepsilon_{s}$, and the angle of incidence $\varphi$ (transmission or total reflection),
(V) using the approximate solution $I(y)$ to deduce a generalized Fresnel formula (cf. (2.1.40)) depending on $c_{1}, c_{2}$,
(VI) calculation of the phase function $\vartheta(y)$ (cf. (2.1.24)), the phase shifts $\delta_{r}, \delta_{t}$ (cf. (2.1.42), (2.1.44), (2.1.45), (2.1.46)), and the reflectivity $R$ (cf. (2.1.39)) depending on $c_{1}, c_{2}$
(VII) if necessary, further iterations of the Volterra equation to obtain better approximations for $R, \delta_{r}, \delta_{t}$.

Some comments to the foregoing are appropriate:
(i) Subject to the ansatz the transformation according to step (II) seems always possible; it does neither depend on the permittivity nor on the discrimination between the transmission and the total reflection case.

The essential feature of transformation (I) is the transition from a nonlinear differential equation for the intensity $I(y)$ (cf. (2.1.12)) to a linear integrodifferential equation (cf. (2.1.16)) that is equivalent to a Volterra integral equation (cf. (2.1.18)). Apparently, this procedure works for rather general permittivity functions.
(ii) The integration constant $c_{1}$ does not explicitly depend on the permittivity function $\left(c_{1}=-q_{s} I(0)\right.$ if $q_{s}>0, c_{1}=0$ if $\left.q_{s}=i\left|q_{s}\right|\right)$.
(iii) Comparison of the linear and nonlinear case shows that the structure of the Volterra equation is not changed (cf. equations (2.1.19) and (2.2.3)), whereas the kernel $K$ and the integration constant $c_{2}$ are different (cf. equations (2.1.20), (2.2.4), (2.1.47), (2.2.8), (2.1.52), (2.2.11)).
(iv) As will be seen in the following the main steps remain unchanged if more general permittivities are considered. Changes occur for $K$ and $c_{2}$ leading to more complicated solutions $I(y)$ and thus to $\vartheta(y)$ (cf. equations (2.1.24) and (3.1.4).

## Chapter 3

## Transmission and reflection at an absorbing dielectric film

### 3.1 Transmission and reflection at a linear absorbing dielectric film

### 3.1. 1 Reduction of the problem to a Volterra integral equation

In this chapter the (linear) absorbing dielectric film is considered [32], [33]. The permittivity in the Helmholtz equation is modelled by a complex-valued function according to

$$
\varepsilon(y)=\left\{\begin{array}{l}
\varepsilon_{c}, y>d  \tag{3.1.1}\\
\varepsilon_{f}=\varepsilon_{R}(y)+i \varepsilon_{I}(y), 0<y<d \\
\varepsilon_{s}, y<0
\end{array},\right.
$$

with real constants $\varepsilon_{c}, \varepsilon_{s}$ and real valued continuously differentiable functions $\varepsilon_{R}(y), \varepsilon_{I}(y)$.

The fields are written in the same form as for the case with real-valued permittivity, according to equation (2.1.2). Considering the case of normal incidence [34] $(p=0)$, the parameter $q_{s}$ must be real, if negative $\varepsilon_{s}$ is excluded [34].

Following the lines of the previous chapter one obtains in place of (2.1.6),(2.1.7)

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}-E(y)\left(\frac{d \vartheta(y)}{d y}\right)^{2}+\left[4 \pi^{2} \varepsilon_{R}(y)\right] E(y)=0 \tag{3.1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
E(y) \frac{d^{2} \vartheta(y)}{d y^{2}}+2 \frac{d \vartheta(y)}{d y} \frac{d E(y)}{d y}+4 \pi^{2} \varepsilon_{I}(y) E(y)=0 . \tag{3.1.3}
\end{equation*}
$$

Equation (3.1.3) can be integrated leading to (cf. equation (2.1.8))

$$
E^{2}(y) \frac{d \vartheta(y)}{d y}=c_{1}-4 \pi^{2} \int_{0}^{y} \varepsilon_{I}(\tau) E^{2}(\tau) d \tau
$$

so that the phase is given by

$$
\begin{equation*}
\vartheta(y)=\vartheta(d)+c_{1} \int_{d}^{y} \frac{d \tau}{E^{2}(\tau)}-4 \pi^{2} \int_{d}^{y} \frac{d \tau}{E^{2}(\tau)} \int_{0}^{\tau} \varepsilon_{I}(\xi) E^{2}(\xi) d \xi \tag{3.1.4}
\end{equation*}
$$

Insertion of $d \vartheta(y) / d y$ according to equation (3.1.4) into equation (3.1.2) leads to

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}+q_{f_{R}}^{2}(y) E(y)-\frac{\left(c_{1}-4 \pi^{2} \int_{0}^{y} \varepsilon_{I}(t) E^{2}(t) d t\right)^{2}}{E^{3}(y)}=0 \tag{3.1.5}
\end{equation*}
$$

with

$$
\begin{equation*}
q_{f_{R}}^{2}(y)=4 \pi^{2} \varepsilon_{R}(y) . \tag{3.1.6}
\end{equation*}
$$

As for real permittivity, real $q_{s}$ (transmission) implies $c_{1} \neq 0$.
Setting $I(y)=E^{2}(y)$ and representing $\varepsilon_{R}(y)$ in the form $\varepsilon_{R}(y)=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{R}(y)$, where $\varepsilon_{R}^{0}$ is a constant, one obtains from equation (3.1.5)

$$
\begin{align*}
\frac{d^{3} I(y)}{d y^{3}} & +4 \frac{d\left(q_{f_{R}}^{2}(y) I(y)\right)}{d y}=2 \frac{d\left(q_{f_{R}}^{2}(y)\right)}{d y} I(y) \\
& -16 \pi^{2} \varepsilon_{I}(y)\left(c_{1}-4 \pi_{0}^{2} \int_{0}^{y} \varepsilon_{I}(t) I(t) d t\right) . \tag{3.1.7}
\end{align*}
$$

Equation (3.1.7) can be integrated to yield

$$
\begin{align*}
& \frac{d^{2} I(y)}{d y^{2}}+4 \kappa^{2} I(y)=-16 \pi^{2} \widetilde{\varepsilon}_{R}(y) I(y)+8 \pi^{2} \int_{0}^{y} \frac{d \widetilde{\varepsilon}_{R}(t)}{d t} I(t) d t \\
+ & 64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(t)\left(\int_{0}^{t} \varepsilon_{I}(z) I(z) d z\right) d t-16 \pi^{2} c_{1} \int_{0}^{y} \varepsilon_{I}(t) d t+c_{2}, \tag{3.1.8}
\end{align*}
$$

where $\kappa^{2}=4 \pi^{2} \varepsilon_{R}^{0}$ and $c_{2}$ a constant of integration. The homogeneous equation $d^{2} I(y) / d y^{2}+4 \kappa^{2} I(y)=0$ has the solution

$$
\begin{equation*}
\tilde{I}_{0}(y)=A \cos (2 \kappa y)+B \sin (2 \kappa y) \tag{3.1.9}
\end{equation*}
$$

so that the general solution of equation (3.1.8) reads

$$
\begin{array}{r}
I(y)=\tilde{I}_{0}(y)+\int_{0}^{y} d t \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left(c_{2}-16 \pi^{2} \widetilde{\varepsilon}_{R}(t) I(t)\right. \\
+8 \pi^{2} \int_{0}^{t} \frac{d \widetilde{\varepsilon}_{R}(z)}{d z} I(z) d z+ \\
\left.64 \pi^{4} \int_{0}^{t} \varepsilon_{I}(z)\left(\int_{0}^{z} \varepsilon_{I}\left(z^{\prime}\right) I\left(z^{\prime}\right) d z^{\prime}\right) d t-16 \pi^{2} c_{1} \int_{0}^{t} \varepsilon_{I}(z) d z\right), \tag{3.1.10}
\end{array}
$$

where the constant $c_{2}$ must be determined by means of the boundary conditions. Calculating some of the integrals on the righthand side of equation (3.1.10) one obtains

$$
\begin{array}{r}
I(y)=I_{0}(y)-16 \pi^{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \widetilde{\varepsilon}_{R}(t) I(t) d t \\
+8 \pi^{2} \int_{0}^{y} \frac{\sin ^{2} \kappa(y-t)}{2 \kappa^{2}} \frac{d \widetilde{\varepsilon}_{R}(t)}{d t} I(t) d t \\
+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{z}^{t} \varepsilon_{I}(\tau) d \tau d t d z, \tag{3.1.11}
\end{array}
$$

where

$$
\begin{array}{r}
I_{0}(y)=\tilde{I}_{0}(y)+c_{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} d t \\
-16 \pi^{2} c_{1} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \varepsilon_{I}(z) d z d t \tag{3.1.12}
\end{array}
$$

The integration constants $c_{1}, c_{2}$ are determined analogously to the real case but now taking into account equations (3.1.5), (3.1.8). Hence

$$
\begin{align*}
& c_{1}=-q_{s} I(0), \\
& c_{2}=2 I(0)\left(q_{s}^{2}+q_{f_{R}}^{2}(0)\right) . \tag{3.1.13}
\end{align*}
$$

### 3.1.2 Reflectance, transmittance, absorptance and phase shifts

Conservation of energy requires that the absorptance $A$ of the film be expressed by reflectance $R$ and transmittance $T$ according to [36]

$$
\begin{equation*}
A=1-R-T, \tag{3.1.14}
\end{equation*}
$$

with

$$
\begin{gather*}
T=\frac{q_{s}}{q_{c}} \frac{E^{2}(0)}{E_{0}^{2}}  \tag{3.1.15}\\
R=\frac{\left|E_{r}\right|^{2}}{E_{0}^{2}} \tag{3.1.16}
\end{gather*}
$$

The intensities $E_{0}^{2}$ and $E_{r}^{2}$ of the incident and reflected waves are related to the transmitted intensity $I(0)$ according to (cf. (2.1.35), (2.1.36), (3.1.4))

$$
\begin{align*}
& E_{0}^{2}=\frac{1}{4}\left\{\frac{\left(\left.\frac{d I(y)}{d y}\right|_{y=d}\right)^{2}}{4 q_{c}^{2} I(d)}+I(d)\left(1+\frac{q_{s} I(0)+4 \pi^{2} \int_{0}^{d} \varepsilon_{I}(\tau) I(\tau) d \tau}{q_{c} I(d)}\right)^{2}\right\}  \tag{3.1.17}\\
& E_{r}^{2}=\frac{1}{4}\left\{\frac{\left(\left.\frac{d I(y)}{d y}\right|_{y=d}\right)^{2}}{4 q_{c}^{2} I(d)}+I(d)\left(1-\frac{q_{s} I(0)+4 \pi^{2} \int_{0}^{d} \varepsilon_{I}(\tau) I(\tau) d \tau}{q_{c} I(d)}\right)^{2}\right\} \tag{3.1.18}
\end{align*}
$$

Equation (3.1.4) implies

$$
\begin{equation*}
\left.E^{2}(d) \frac{d \vartheta(y)}{d y}\right|_{y=d}=-q_{s} E^{2}(0)-4 \pi^{2} \int_{0}^{d} \varepsilon_{I}(\tau) E^{2}(\tau) d \tau \tag{3.1.19}
\end{equation*}
$$

By insertion of equations (2.1.35) and (2.1.36) into equations (3.1.15) and (3.1.16), using (3.1.19), the absorptance $A$ can be written as

$$
\begin{equation*}
A=\frac{4 \pi^{2}}{q_{c} E_{0}^{2}} \int_{0}^{d} \varepsilon_{I}(\tau) E^{2}(\tau) d \tau \tag{3.1.20}
\end{equation*}
$$

The phase shift on reflection $\delta_{r}$ is determined by equations (2.1.27), (2.1.28), (3.1.14) and (3.1.16). Evaluation yields

$$
\begin{equation*}
\sin \delta_{r}=-\frac{\left.E(d) \frac{d E(y)}{d y}\right|_{y=d}}{2 q_{c} E_{0}^{2} \sqrt{1-T-A}} \tag{3.1.21}
\end{equation*}
$$

The phase on transmission $\delta_{t}$, is equal to $\vartheta(0)$ and can be obtained by integrating equation (3.1.19) using equation (2.1.37). The result is

$$
\begin{equation*}
\delta_{t}=\vartheta(0)=\int_{0}^{d} \frac{q_{s} E^{2}(0)+q_{c} E_{0}^{2} \widetilde{\mathbf{A}}(\tau)}{E^{2}(\tau)} d \tau+\arcsin \left(-\frac{\left.\frac{d I(y)}{d y}\right|_{y=d}}{4 q_{c} E_{0} \sqrt{I(d)}}\right) \tag{3.1.22}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{\mathbf{A}}(y):=\frac{4 \pi^{2}}{q_{c} E_{0}^{2}} \int_{0}^{y} \varepsilon_{I}(\tau) E^{2}(\tau) d \tau \tag{3.1.23}
\end{equation*}
$$

denotes the absorptance of a film of thickness $y$. The phase shift on transmission can be also obtained according to equation (3.1.4).

### 3.1.3 Solutions

Introducing $\widehat{I}(y)=I(y) / I(0)$ and using the relations of the foregoing subsection the normalized intensity $\widehat{I}(y)$ and the phase $\vartheta(y)$ can be written as

$$
\begin{array}{r}
\widehat{I}(y)=\cos (2 \kappa y)+\frac{q_{s}^{2}+q_{f}^{2}(0)}{\kappa^{2}} \sin ^{2}(\kappa y) \\
+16 \pi^{2} q_{s} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \varepsilon_{I}(z) d z d t \\
-16 \pi^{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \widetilde{\varepsilon}_{R}(t) \widehat{I}(t) d t \\
+8 \pi^{2} \int_{0}^{y} \frac{\sin ^{2} \kappa(y-t)}{2 \kappa^{2}} \frac{d \widetilde{\varepsilon}_{R}(t)}{d t} \widehat{I}(t) d t \\
+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) \widehat{I}(z) \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{z}^{t} \varepsilon_{I}(\tau) d \tau d t d z, \tag{3.1.24}
\end{array}
$$

with $\kappa^{2}=4 \pi^{2} \varepsilon_{f}^{0}$, where equations (2.1.48), (2.1.33), (2.1.47), (3.1.11), (3.1.12) have been used, and, taking equations (3.1.19), (2.1.33), (2.1.37) into account,

$$
\begin{equation*}
\vartheta(y)=\int_{y}^{d} \frac{q_{s} E^{2}(0)+q_{c} E_{0}^{2} \widetilde{\mathbf{A}}(\tau)}{E^{2}(\tau)} d \tau+\arcsin \left(-\frac{\left.\frac{d I(y)}{d y}\right|_{y=d}}{4 q_{c} E_{0} \sqrt{I(d)}}\right) \tag{3.1.25}
\end{equation*}
$$

Equations (3.1.17),(3.1.20) together with equations (3.1.24), (3.1.25) allow the optical response of the linear film to be calculated for arbitrary thickness $d$, arbitrary angles of incidence $\varphi$ and rather arbitrary complex-valued permittivity $\varepsilon_{f}(y)$. Equations (3.1.17), with the normalized intensity $\widehat{I}(y)$, and (3.1.24) constitute a generalization of Fresnel's formulae in linear optics [28].

## Numerical results

Again a periodic dependence of $\varepsilon_{f}(y)\left(\widetilde{\varepsilon}_{f}(y)=\gamma \cos ^{2} b(y / d)\right)$, and, for simplicity, $\varepsilon_{I}=$ const is assumed. Expressions for $I(y)$ and $\vartheta(y)$ can be obtained evaluating the first iteration of (3.1.24). The corresponding field intensity inside the slab is shown in Figure 3.1. In Figure 3.2 the phase $\vartheta(y)$ is plotted. The field intensity after the first iteration can be compared with the exact numerical solution of the system of differential equations (3.1.2), (3.1.3). The corresponding plots are shown in Figure 3.3. Plots of $A$ and $R$ are presented in Figure 3.4. Using a parametric plot routine the absorptance $A$ and the reflectivity $R$ can be evaluated straightforwardly. Results are shown in Figure 3.5.


Figure 3.1: Dependence of the field intensity $I(y)$ (first iteration of equation (3.1.24)) inside the slab on the transverse coordinate $y$ for $\varepsilon_{c}=1, \varepsilon_{s}=$ $1.7, \varepsilon_{f}^{0}=3.5, \varepsilon_{I}=0.1, E_{0}=1, d=1, \gamma=0.03, b=0.1$.


Figure 3.2: Phase $\vartheta(y)$ according to equation (3.1.25), parameters as in figure 3.1.


Figure 3.3: (a) Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ for the same parameters as in figure 3.1. Solid curve corresponds to the first iteration of equation (3.1.24) and dashed curve to the numerical solution of the system of differential equations (3.1.2), (3.1.3); (b) the difference between the curves from (a).


Figure 3.4: (a) Dependence of absorptance $A$ on the layer thickness $d$; (b) Dependence of reflectance $R$ on the layer thickness $d$. Parameters are as in figure 3.1.


Figure 3.5: (a) Dependence of absorptance $A$ on the layer thickness $d$ and $E_{0}^{2}$; (b) Dependence of reflectance $R$ on the layer thickness $d$ and $E_{0}^{2}$. Parameters are as in figure 3.1.

### 3.2 Transmission and reflection at an absorbing Kerr-like nonlinear dielectric film

### 3.2.1 Reduction of the problem to a Volterra integral equation

Again the transmission case ( $q_{s}$ real) is considered with a nonlinearity of the permittivity according to

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{R}(y)+i \varepsilon_{I}(y)+a E^{2}(y), 0<y<d, \tag{3.2.1}
\end{equation*}
$$

with a real constant $a$. Using the same arguments as in Section 3.1.1 one obtains in place of equation (3.1.2)

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}-E(y)\left(\frac{d \vartheta(y)}{d y}\right)^{2}+\left[4 \pi^{2}\left(\left(\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{R}(y)\right)+a E^{2}(y)\right)\right] E(y)=0 \tag{3.2.2}
\end{equation*}
$$

Equation (3.1.11) reads in this case

$$
\begin{align*}
& I(y)=I_{0}(y)+\int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left(-16 \pi^{2} \widetilde{\varepsilon}_{R}(t) I(t)-12 \pi^{2} a I^{2}(y)\right) d t \\
&+8 \pi^{2} \int_{0}^{y} \frac{\sin ^{2} \kappa(y-t)}{2 \kappa^{2}} \frac{d \widetilde{\varepsilon}_{R}(t)}{d t} I(t) d t \\
&+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) d z \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left\{\int_{z}^{t} \varepsilon_{I}(\tau) d \tau\right\} d t \tag{3.2.3}
\end{align*}
$$

where $I_{0}(y)$ is given by equation (3.1.12). As in section 2.2 .1 the solution of the nonlinear integral equation (3.2.3) can be represented as a limit of the uniformly convergent sequence $I_{j}(y)$ [cf. Appendix I]

$$
\begin{gather*}
I(y)=\lim _{j \rightarrow \infty} I_{j}(y)  \tag{3.2.4}\\
I_{j}(y)=I_{0}(y)+\int_{0}^{y} K\left(y, t, I_{j-1}(t)\right) I_{j-1}(t) d t, \quad j=1,2, \ldots \tag{3.2.5}
\end{gather*}
$$

where $I_{0}(y)$ is given by equation (3.1.12) and where $K$ can be written similar to (2.2.4), taking into account (3.2.3) (cf. equation (3.2.7)). The uniform convergence is again proved using the Banach Fixed-Point Theorem. The condition for the convergence leads to a constraint for the parameters of the problem (definitions of $\left\|N_{*}\right\|,\left\|N_{2}\right\|,\left\|I_{0 c}\right\|$ see in Appendix I)

$$
\begin{equation*}
\left\|N_{*}\right\|+2 \sqrt{\left\|N_{2}\right\|\left\|I_{0 c}\right\|}<1 \tag{3.2.6}
\end{equation*}
$$

### 3.2.2 Solutions

For real $q_{s}$, the constants of integration $c_{1}$ and $c_{2}$ are given by the equations (2.1.33) and (2.2.8) respectively, so that equation (3.2.3) reads, taking into account equation (2.1.48),

$$
\begin{align*}
I(y)= & I(0) \cos (2 \kappa y)+\frac{\left(q_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right) I(0)}{\kappa^{2}} \sin ^{2}(\kappa y) \\
& +16 \pi^{2} q_{s} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \varepsilon_{I}(z) d z d t \\
& +\int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left(-16 \pi^{2} \widetilde{\varepsilon}_{R}(t) I(t)-12 \pi^{2} a I^{2}(y)\right) d t \\
& +8 \pi^{2} \int_{0}^{y} \frac{\sin ^{2} \kappa(y-t)}{2 \kappa^{2}} \frac{d \widetilde{\varepsilon}_{R}(t)}{d t} I(t) d t \\
+ & 64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) d z \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left\{\int_{z}^{t} \varepsilon_{I}(\tau) d \tau\right\} d t \tag{3.2.7}
\end{align*}
$$

with $I(0)\left(=\left|E_{3}\right|^{2}\right)$ related to $E_{0}^{2}$ according to equation (2.1.40). The phase $\vartheta(y)$ is given by (3.1.25).

## Numerical results

Assuming the same periodic dependence of $\widetilde{\varepsilon}_{f}(y)$, as in the linear case the first iteration of (3.2.7) leads to expressions for the field intensity $I(y)$ and the phase $\vartheta(y)$ of the dielectric film, which are plotted in Figures 3.6-3.7. Plots of the field intensity $I(y)$ after first iteration and of the numerical solution of the system of differential equations (3.2.2), (3.1.3) are shown in Figure 3.8. Illustrations of absorptance $A$ and reflectivity $R$ are presented in Figures 3.9 and 3.10. For larger $E_{0}^{2}$ the dependence of $R$ on $E_{0}^{2}$ is shown in Figure 3.11.


Figure 3.6: Dependence of the field intensity $I(y)$ (first iteration of equation (3.2.7)) inside the slab on the transverse coordinate $y$ for $a=0.01$. The other parameters are as in figure 3.1.


Figure 3.7: Phase $\vartheta(y)$ according to equation (3.1.25), parameters as in figure $3.1, a=0.01$.


Figure 3.8: (a) Dependence of the field intensity $I(y)$ inside an absorbing Kerr-like nonlinear dielectric slab on the transverse coordinate $y$ for the same parameters as in figure 3.6. Solid curve corresponds to the first iteration of equation (3.2.7) and dashed curve to the numerical solution of the system of differential equations (3.2.2), (3.1.3); (b) the difference between the curves from (a).


Figure 3.9: Dependence of the absorptance $A$ on $E_{0}^{2}$ and $d$ for the same parameters as in Figure 3.6.


Figure 3.10: Dependence of the reflectivity $R$ on $E_{0}^{2}$ and $d$ for the same parameters as in Figure 3.6.


Figure 3.11: Dependence of the reflectivity $R$ on $a E_{0}^{2}$ and $d$ for the same parameters as in Figure 3.6.

## Chapter 4

## Transmission and reflection at a lossless dielectric film with a saturating permittivity

### 4.1 Reduction of the problem to a Volterra integral equation

The permittivity of the film is assumed to be modelled according to

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y)+\frac{a E^{2}(y)}{1+\operatorname{ar} E^{2}(y)}, 0<y<d \tag{4.1.1}
\end{equation*}
$$

with real constants $\varepsilon_{f}^{0}, a, r$ and a real-valued continuously differentiable function $\widetilde{\varepsilon}_{f}(y)$. Using the same arguments as in Section 3.1 equation (3.1.2) reads

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}-E(y)\left(\frac{d \vartheta(y)}{d y}\right)^{2}+\left[4 \pi^{2}\left(\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y)+\frac{a E^{2}(y)}{1+\operatorname{ar} E^{2}(y)}\right)-p^{2}\right] E(y)=0 \tag{4.1.2}
\end{equation*}
$$

In place of equations (2.1.19), (2.1.20) one obtains

$$
\begin{align*}
& I(y)= I_{0}(y)+ \\
&+\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau \\
&-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{f}(\tau) I(\tau) d \tau \\
&-\frac{8 \pi^{2}}{r \kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) d \tau \\
&-\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{1}{1+\operatorname{ar} I(\tau)} d \tau  \tag{4.1.3}\\
&+ \frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+\operatorname{ar} I(\tau)) d \tau
\end{align*}
$$

with

$$
\begin{equation*}
I_{0}(y)=\tilde{I}_{0}(y)+\left(c_{2}-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}} \tag{4.1.4}
\end{equation*}
$$

where $\tilde{I}_{0}(y)$ is given by equation (2.1.17). Following the same lines of calculation as for the linear case, the constant of integration $c_{2}$ in equation (4.1.4) is determined by [cf. Appendix E]

$$
\begin{array}{r}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+\operatorname{arI}(0)} \\
+\frac{8 \pi^{2}}{a r^{2}}\left(2+2 \operatorname{ar} I(0)+\frac{1}{1+\operatorname{arI} I(0)}-\ln (1+\operatorname{arI}(0))\right) \tag{4.1.5}
\end{array}
$$

for the transmission case and

$$
\begin{array}{r}
c_{2}=2 I(0)\left(\tilde{q}_{s}^{2}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+\operatorname{arI} I(0)} \\
+\frac{8 \pi^{2}}{a r^{2}}\left(2+2 \operatorname{ar} I(0)+\frac{1}{1+\operatorname{arI}(0)}-\ln (1+\operatorname{arI}(0))\right) \tag{4.1.6}
\end{array}
$$

for the total reflection case.
Similar to the forgoing section 3.2.1 the solution of the nonlinear integral equation (4.1.3) can be represented as a limit of the uniformly convergent sequence $I_{j}(y)$, according to equations (3.2.4),(3.2.5), where $K$ can be written similar to (2.2.4), taking into account (4.1.3) (cf. equation (4.2.1)) [cf. Appendix J]

The uniform convergence can be proved and the condition for convergence implies (definitions of $\left\|N_{1}\right\|,\left\|N_{2}\right\|,\left\|N_{3}\right\|$ and $\left\|N_{4}\right\|$ see in Appendix J)

$$
\begin{equation*}
\left\|N_{1}\right\|+\left\|N_{2}\right\|+\frac{\left\|N_{3}\right\|+(1+a)\left\|N_{4}\right\|}{r}<1 \tag{4.1.7}
\end{equation*}
$$

### 4.2 Transmission $\left(q_{s}^{2}>0\right)$

### 4.2.1 Solutions

Using equation (2.1.48) for $\tilde{I}_{0}(y)$ for the case of real $q_{s}$ and taking into account the constant of integration $c_{2}$, which is given by equation (4.1.5), equation (4.1.3) yields

$$
\begin{array}{r}
I(y)=I(0) \cos (2 \kappa y)+ \\
+\frac{8 \pi^{2}}{a r^{2}}\left(2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+\operatorname{ar} I(0)}\right. \\
\left.\left.+\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y)+\frac{1}{1+\operatorname{arI}(0)}-\ln (1+\operatorname{ar} I(0))\right)-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}} \\
-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \\
d \tau \\
\widetilde{\varepsilon}_{f}(\tau) I(\tau) d \tau \\
-\frac{8 \pi^{2}}{r \kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) \\
-\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{1}{1+\operatorname{arI} I(\tau)} d \tau \\
+\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+\operatorname{arI(\tau ))d\tau ,(4.2.1)}
\end{array}
$$

with $I(0)\left(=\left|E_{3}\right|^{2}\right)$ related to $E_{0}^{2}$ according to equation (2.1.40). The phase $\vartheta(y)$ is given by (2.2.10).

### 4.2.2 Numerical results

Again a periodic dependence of $\varepsilon_{f}(y)\left(\widetilde{\varepsilon}_{f}(y)=\gamma \cos ^{2} b(y / d)\right)$ is assumed. The first iteration of (4.2.1) leads to expressions for $I(y)$ and $\vartheta(y)$. The corresponding plots of the field intensity are shown in Figure 4.1. The results after the first iteration can be compared to the numerical solution of the system of differential equations (4.1.2), (2.1.7). The corresponding plots are in Figure 4.2. In Figure 4.3 the phase $\vartheta(y)$ is plotted. A plot of the reflectivity $R$ is presented in Figure 4.4.


Figure 4.1: Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ and $E_{0}^{2}$ for $a=0.01, r=1000, \varepsilon_{c}=1, \varepsilon_{s}=1.7, \varepsilon_{f}^{0}=$ $3.5, \varphi=1.107, d=1, \gamma=0.033, b=0.1$.


Figure 4.2: Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ for $\left|E_{3}\right|^{2}=0.1$. Other parameters are as in figure 4.1. Solid curve corresponds to the first iteration of equation (4.2.1) and dashed curve to the numerical solution of the system of differential equations (4.1.2), (2.1.7).


Figure 4.3: Phase function $\vartheta\left(y, E_{0}^{2}\right)$ according to equation (2.2.10) inside the slab. Parameters as in figure 4.1.


Figure 4.4: Dependence of the reflectivity $R$ on the layer thickness $d$ and on the incident wave intensity $E_{0}^{2}$ Parameters as in figure 4.1.

### 4.3 Total reflection $\left(q_{s}^{2}<0\right)$

### 4.3.1 Solutions

Using equation (2.1.53) for $\tilde{I}_{0}(y)$ for the case of pure imaginary $q_{s}$ and taking into account the constant of integration $c_{2}$, which is given by equation (4.1.5), equation (4.1.3) yields

$$
\begin{array}{r}
I(y)=I(0) \cos (2 \kappa y)+\frac{\tilde{q}_{s} I(0)}{\kappa} \sin (2 \kappa y)+ \\
\left.\left.+\frac{8 \pi^{2}}{a r^{2}}\left(2+2 \operatorname{ar} I(0)+\frac{1}{1+\operatorname{arI} I(0)}-\ln (1+\operatorname{ar} I(0))\right)-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+\operatorname{arI(0)}} \\
+\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau \\
-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{f}(\tau) I(\tau) d \tau \\
\\
-\frac{8 \pi^{2}}{r \kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) d \tau \\
-\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{1}{1+\operatorname{ar} I(\tau)} d \tau  \tag{4.3.1}\\
+\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+\operatorname{arI(\tau ))d\tau }
\end{array}
$$

with $I(0)\left(=\left|E_{3}\right|^{2}\right)$ related to $E_{0}^{2}$ according to equation (2.1.41). The phase constant $\vartheta(0) \equiv \vartheta(d)$ is given by equation (2.1.45).

### 4.3.2 Numerical results

The numerical results are shown for a periodic dependence of $\varepsilon_{f}(y)\left(\widetilde{\varepsilon}_{f}(y)=\right.$ $\left.\gamma \cos ^{2} b(y / d)\right)$. The field intensity inside the slab after the first iteration of (4.3.1) is shown in Figure 4.5. The results after the first iteration can be compared to the numerical solution of the system of differential equations (4.1.2), (2.1.7). The corresponding plots are in Figure 4.6.


Figure 4.5: Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ and $E_{0}^{2}$ for $a=0.01, r=1000, \varepsilon_{c}=1.7, \varepsilon_{s}=1, \varepsilon_{f}^{0}=$ $3.5, \varphi=1.107, d=1, \gamma=0.033, b=0.1$.


Figure 4.6: Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ for $\left|E_{3}\right|^{2}=0.1$. The other parameters are as in figure 4.5. Solid curve corresponds to the first iteration of equation (4.3.1) and dashed curve to the numerical solution of the system of differential equations (4.1.2), (2.1.7).

## Chapter 5

## Transmission and reflection at an absorbing dielectric film with a saturating permittivity

### 5.1 Reduction of the problem to a Volterra integral equation

For absorbing media the permittivity in the Helmholtz equation is modelled by

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{R}(y)+i \varepsilon_{I}(y)+\frac{a E^{2}(y)}{1+\operatorname{ar} E^{2}(y)}, 0<y<d \tag{5.1.1}
\end{equation*}
$$

with real constants $\varepsilon_{f}^{0}, a, r$ and real-valued continuously differentiable functions $\widetilde{\varepsilon}_{R}(y), \varepsilon_{I}(y)$. Using the same arguments as in Section 3.1.1 equation (3.1.2) is replaced by

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}-E(y)\left(\frac{d \vartheta(y)}{d y}\right)^{2}+\left[4 \pi^{2}\left(\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{R}(y)+\frac{a E^{2}(y)}{1+\operatorname{ar} E^{2}(y)}\right)\right] E(y)=0 \tag{5.1.2}
\end{equation*}
$$

In place of equation (4.1.3) one obtains

$$
\begin{array}{r}
I(y)=I_{0}(y)+\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d \widetilde{\varepsilon}_{R}(\tau)}{d \tau} I(\tau) d \tau \\
-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{R}(\tau) I(\tau) d \tau \\
-\frac{8 \pi^{2}}{r \kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) d \tau \\
-\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{1}{1+\operatorname{arI}(\tau)} d \tau \\
\left.+\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+\operatorname{arI} I \tau)\right) d \tau \\
+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) d z \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left\{\int_{z}^{t} \varepsilon_{I}(\tau) d \tau\right\} d t, \tag{5.1.3}
\end{array}
$$

where $I_{0}(y)$ is given by

$$
\begin{array}{r}
I_{0}(y)=\tilde{I}_{0}(y)+\left(c_{2}-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}} \\
-16 \pi^{2} c_{1} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \varepsilon_{I}(z) d z d t . \tag{5.1.4}
\end{array}
$$

The constant of integration $c_{2}$ in equation (5.1.4) is determined by equation (4.1.5).

Following the arguments of the forgoing section the solution of the nonlinear integral equation (5.1.3) can be also represented as a limit of the uniformly convergent sequence $I_{j}(y)$ [cf. Appendix K]. The uniform convergence is proved and the condition for convergence now yields (definitions of $\left\|N_{1}\right\|,\left\|N_{2}\right\|,\left\|N_{3}\right\|,\left\|N_{4}\right\|$ and $\left\|N_{c}\right\|$ see in Appendix K)

$$
\begin{equation*}
\left\|N_{1}\right\|+\left\|N_{2}\right\|+\left\|N_{c}\right\|+\frac{\left\|N_{3}\right\|+(1+a)\left\|N_{4}\right\|}{r}<1 . \tag{5.1.5}
\end{equation*}
$$

### 5.1.1 Solutions for the transmission case

Using equation (2.1.48) for $\tilde{I}_{0}(y)$ for of real $q_{s}$ and taking into account the constant of integration $c_{2}$, which is given by equation (4.1.5), equation (4.1.3)
reads

$$
\begin{array}{r}
I(y)=I(0) \cos (2 \kappa y)+\left(2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+a r I(0)}\right. \\
\left.+\frac{8 \pi^{2}}{a r^{2}}\left(2+2 a r I(0)+\frac{1}{1+a r I(0)}-\ln (1+\operatorname{ar} I(0))\right)-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}} \\
-16 \pi^{2} c_{1} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \varepsilon_{I}(z) d z d t \\
+\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d_{f}(\tau)}{d \tau} I(\tau) d \tau \\
-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{f}(\tau) I(\tau) d \tau \\
-\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{8 \pi^{2}}{r \kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) d \tau \\
+\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+a r I(\tau)) d \tau \\
+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) d z \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left\{\int_{z}^{t} \varepsilon_{I}(\tau) d \tau\right\} d t
\end{array}
$$

with $I(0)\left(=\left|E_{3}\right|^{2}\right)$ related to $E_{0}^{2}$ according to equation (2.1.40). The phase $\vartheta(y)$ is given by (2.2.10).

### 5.1.2 Numerical results

For illustration again a periodic dependence of $\varepsilon_{f}(y)$ is assumed $\left(\widetilde{\varepsilon}_{f}(y)=\right.$ $\left.\gamma \cos ^{2} b(y / d)\right)$. The first iteration of (5.1.6) leads to expressions for $I(y)$ and $\vartheta(y)$. The corresponding field intensity inside the slab is given in Figure 5.1. The results after the first iteration can be compared to the numerical solution of the system of differential equations (5.1.2), (3.1.3) for the fixed parameter $\left|E_{3}\right|^{2}$. The illustration is shown in Figure 5.2. In Figure 5.3 the phase $\vartheta\left(y, E_{0}^{2}\right)$ is plotted. Pictures of absorptanse $A$ and of reflectivity $R$ are presented in Figures 5.4-5.5. Again, $\varepsilon_{I}=$ const is assumed for simplicity.


Figure 5.1: Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ and $E_{0}^{2}$ for $a=0.01, r=1000, \varepsilon_{I}=0.1, \varepsilon_{c}=1, \varepsilon_{s}=$ $1.7, \varepsilon_{f}^{0}=0, \varphi=1.107, d=1, \gamma=0.033, b=0.1$.


Figure 5.2: Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate $y$ for $\left|E_{3}\right|^{2}=0.1$. The other parameters are as in figure 5.1. Solid curve corresponds to the first iteration of equation (4.2.1) and dashed curve to the numerical solution of the system of differential equations (4.1.2), (3.1.3).


Figure 5.3: Phase function $\vartheta\left(y, E_{0}^{2}\right)$ according to equation (3.1.25) inside the slab. Parameters as in figure 5.1.


Figure 5.4: Absorptsance $A$ depending on the layer thickness $d$ and on the incident wave intensity $E_{0}^{2}$ for the same parameters as in figure 5.1.


Figure 5.5: Reflectivity $R$ depending on the layer thickness $d$ and on the incident wave intensity $E_{0}^{2}$ for the same parameters as in figure 5.1.

## Chapter 6

## Summary and outlook

A rather general iterative approach was presented to solve the Helmholtz equation for a dielectric film with various permittivities. The solutions for the linear case and for the nonlinear case have been expressed in terms of a uniformly convergent series and a uniformly convergent sequence of iterations of the Volterra equation, respectively. The main emphasis was on the derivation of the relationship between the Helmholtz equation (with a specific permittivity) and the associated Volterra integral equation.

The following integral equations were obtained:
(a) If the permittivity of the film is given by

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y) \tag{6.0.1}
\end{equation*}
$$

where $\varepsilon_{f}^{0}$ is a real constant and $\widetilde{\varepsilon}_{f}(y)$ is a real-valued continuously differentiable function, the intensity $I(y)$ is determined according to equations (2.1.19), (2.1.20)

$$
\begin{equation*}
I(y)=\tilde{I}_{0}(y)+\frac{c_{2}}{2 \kappa^{2}} \sin ^{2}(\kappa y)+\int_{0}^{y} K(y, t) I(t) d t \tag{6.0.2}
\end{equation*}
$$

with

$$
\begin{equation*}
K(y, t)=-8 \pi^{2} \frac{\sin 2 \kappa(y-t)}{\kappa} \widetilde{\varepsilon}_{f}(t)+4 \pi^{2} \frac{\sin ^{2} \kappa(y-t)}{\kappa^{2}} \frac{d \widetilde{\varepsilon}_{f}(t)}{d t} \tag{6.0.3}
\end{equation*}
$$

and the integration constant

$$
\begin{equation*}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right) \tag{6.0.4}
\end{equation*}
$$

for the case of real $q_{s}$ (transmission), and

$$
\begin{equation*}
c_{2}=2 I(0)\left(\tilde{q}_{s}^{2}+q_{f}^{2}(0)\right) \tag{6.0.5}
\end{equation*}
$$

for the case of pure imaginary $q_{s}=i \tilde{q}_{s}$ (total reflection);
(b) if the permittivity of the film is modelled by

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{R}(y)+i \varepsilon_{I}(y), \tag{6.0.6}
\end{equation*}
$$

where $\widetilde{\varepsilon}_{R}(y), \varepsilon_{I}(y)$ are real-valued continuously differentiable functions, the intensity $I(y)$ is determined according to equations (3.1.11), (3.1.12) (if $p=$ $0)$

$$
\begin{array}{r}
I(y)=I_{0}(y)-8 \pi^{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{\kappa} \widetilde{\varepsilon}_{R}(t) I(t) d t \\
+4 \pi^{2} \int_{0}^{y} \frac{\sin ^{2} \kappa(y-t)}{\kappa^{2}} \frac{d \widetilde{\varepsilon}_{R}(t)}{d t} I(t) d t \\
+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{z}^{t} \varepsilon_{I}(\tau) d \tau d t d z \tag{6.0.7}
\end{array}
$$

where

$$
\begin{array}{r}
I_{0}(y)=\tilde{I}_{0}(y)+c_{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} d t \\
-16 \pi^{2} c_{1} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \varepsilon_{I}(z) d z d t \tag{6.0.8}
\end{array}
$$

with

$$
\begin{equation*}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right) ; \tag{6.0.9}
\end{equation*}
$$

(c) if the permittivity of the film is given by

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{R}^{0}+\widetilde{\varepsilon}_{R}(y)+i \varepsilon_{I}(y)+a E^{2}(y) \tag{6.0.10}
\end{equation*}
$$

where $a$ is a real constant, the intensity $I(y)$ is determined according to equation (3.2.3) (if $p=0$ )

$$
\begin{array}{r}
I(y)=I_{0}(y)-8 \pi^{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{\kappa} \widetilde{\varepsilon}_{R}(t) I(t) d t \\
+4 \pi^{2} \int_{0}^{y} \frac{\sin ^{2} \kappa(y-t)}{\kappa^{2}} \frac{d \widetilde{\varepsilon}_{R}(t)}{d t} I(t) d t \\
+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) d z \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left\{\int_{z}^{t} \varepsilon_{I}(\tau) d \tau\right\} d t \\
-6 \pi^{2} a \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{\kappa} I^{2}(y) d t \tag{6.0.11}
\end{array}
$$

where $I_{0}(y)$ is given according to equation (6.0.8) and the integration constant $c_{2}$ for the transmission case is given by

$$
\begin{equation*}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right) ; \tag{6.0.12}
\end{equation*}
$$

(d) if $\widetilde{\varepsilon}_{I}(y)=0$ in (c), then the following permittivity can be considered

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y)+a E^{2}(y) \tag{6.0.13}
\end{equation*}
$$

and, the intensity $I(y)$ is determined according to equations (2.2.3), (2.2.4)

$$
\begin{equation*}
I(y)=\tilde{I}_{0}(y)+\frac{c_{2}}{2 \kappa^{2}} \sin ^{2}(\kappa y)+\int_{0}^{y} K(y, t, I(t)) I(t) d t, \tag{6.0.14}
\end{equation*}
$$

with the kernel

$$
\begin{gather*}
K(y, t, I(t))=-8 \pi^{2} \frac{\sin 2 \kappa(y-t)}{\kappa} \widetilde{\varepsilon}_{f}(t)+4 \pi^{2} \frac{\sin ^{2} \kappa(y-t)}{\kappa^{2}} \frac{d \widetilde{\varepsilon}_{f}(t)}{d t}  \tag{6.0.15}\\
\\
-\frac{\sin 2 \kappa(y-t)}{\kappa} 6 \pi^{2} a I(t),
\end{gather*}
$$

where the integration constant $c_{2}$ is given by

$$
\begin{equation*}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right) \tag{6.0.16}
\end{equation*}
$$

for the case of real $q_{s}$ (transmission) and

$$
\begin{equation*}
c_{2}=2 I(0)\left(\widetilde{q}_{s}^{2}+q_{f}^{2}(0)+2 \pi^{2} a I(0)\right), \tag{6.0.17}
\end{equation*}
$$

for the case of pure imaginary $q_{s}$ (total reflection);
(e) if the nonlinearity of the permittivity of the film is saturating and if $\widetilde{\varepsilon}_{I}(y)=0$ in $(\mathrm{c})$, then $\varepsilon_{f}$ is given by

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{f}(y)+\frac{a E^{2}(y)}{1+\operatorname{ar} E^{2}(y)} \tag{6.0.18}
\end{equation*}
$$

The intensity $I(y)$ is determined according to equations (4.1.3), (4.1.4)

$$
\begin{array}{r}
I(y)=I_{0}(y)-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{f}(\tau) I(\tau) \\
+\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau \\
\quad-\frac{8 \pi^{2}}{r \kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) d \tau \\
\\
\quad-\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{1}{1+\operatorname{ar} I(\tau)} d \tau  \tag{6.0.19}\\
+\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+\operatorname{ar} I(\tau)) d \tau
\end{array}
$$

where $I_{0}(y)$ is given according to equation

$$
\begin{equation*}
I_{0}(y)=\tilde{I}_{0}(y)+\left(c_{2}-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}} \tag{6.0.20}
\end{equation*}
$$

The constant of integration $c_{2}$ in equation (6.0.20) is determined by

$$
\begin{array}{r}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+\operatorname{arI} I(0)} \\
+\frac{8 \pi^{2}}{a r^{2}}\left(2+2 \operatorname{ar} I(0)+\frac{1}{1+\operatorname{arI}(0)}-\ln (1+\operatorname{ar} I(0))\right) \tag{6.0.21}
\end{array}
$$

for the transmission case and

$$
\begin{array}{r}
c_{2}=2 I(0)\left(\tilde{q}_{s}^{2}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+\operatorname{arI} I(0)} \\
+\frac{8 \pi^{2}}{a r^{2}}\left(2+2 \operatorname{ar} I(0)+\frac{1}{1+\operatorname{arI}(0)}-\ln (1+\operatorname{ar} I(0))\right) \tag{6.0.22}
\end{array}
$$

for the total reflection case;
$(f) \quad$ if the permittivity of the film is modelled by

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}^{0}+\widetilde{\varepsilon}_{R}(y)+i \varepsilon_{I}(y)+\frac{a E^{2}(y)}{1+\operatorname{ar} E^{2}(y)}, \tag{6.0.23}
\end{equation*}
$$

the intensity $I(y)$ is determined according to equations (4.1.3), (4.1.4) (if $p=0$ )

$$
\begin{array}{r}
I(y)=I_{0}(y)-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{R}(\tau) I(\tau) \\
+\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d \widetilde{\varepsilon}_{R}(\tau)}{d \tau} I(\tau) d \tau \\
-\frac{8 \pi^{2}}{r \kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) d \tau \\
-\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{1}{1+\operatorname{ar} I(\tau)} d \tau \\
+\frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+\operatorname{ar} I(\tau)) d \tau \\
+64 \pi^{4} \int_{0}^{y} \varepsilon_{I}(z) I(z) d z \int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left\{\int_{z}^{t} \varepsilon_{I}(\tau) d \tau\right\} d t, \tag{6.0.24}
\end{array}
$$

where $I_{0}(y)$ is given according to equation

$$
\begin{array}{r}
I_{0}(y)=\tilde{I}_{0}(y)+\left(c_{2}-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}} \\
-16 \pi^{2} c_{1} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \varepsilon_{I}(z) d z d t . \tag{6.0.25}
\end{array}
$$

The constant of integration $c_{2}$ for the transmission case in equation (6.0.25) is equal to

$$
\begin{array}{r}
c_{2}=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right)-\frac{8 \pi^{2} I^{2}(0)}{1+\operatorname{arI} I(0)} \\
+\frac{8 \pi^{2}}{a r^{2}}\left(2+2 \operatorname{ar} I(0)+\frac{1}{1+\operatorname{arI}(0)}-\ln (1+\operatorname{arI}(0))\right) . \tag{6.0.26}
\end{array}
$$

If the intensity $I(y)$ has been determined the phase $\vartheta(y)$ is given by (cf. equation (3.1.4))

$$
\begin{equation*}
\vartheta(y)=\vartheta(d)+c_{1} \int_{d}^{y} \frac{d \tau}{E^{2}(\tau)}-4 \pi^{2} \int_{d}^{y} \frac{d \tau}{E^{2}(\tau)} \int_{0}^{\tau} \varepsilon_{I}(\xi) E^{2}(\xi) d \xi \tag{6.0.27}
\end{equation*}
$$

where (cf. equation (2.1.37))

$$
\begin{equation*}
\sin \vartheta(d)=-\frac{\left.\frac{d I(y)}{d y}\right|_{y=d}}{4 q_{c} E_{0} \sqrt{I(d)}}, \tag{6.0.28}
\end{equation*}
$$

and $c_{1}$ is general to $-q_{s} I(0)$ and to zero for the transmission case and the total reflection case, respectively.

As shown above, the agreement between the approximate analytical solutions and the exact numerical solutions is satisfactory. Thus it seems that the method proposed can serve as a means to optimize certain parameters (material and/or geometric) for particular purposes.

Furthermore it seems that the method can be applied to a rather large class of different (real, complex, linear, nonlinear) permittivity functions. For a example for the film exhibiting the local Kerr-like nonlinearity the constant $a$ was assumed to be real in the foregoing, but it can also be complex [cf. Appendix L].

This leads to the question where this method is not applicable.
First, as outlined above, certain parameters must be small in order to use the first iteration of the Volterra equation as a good approximation.

Second, TE-polarization and the dependence $\varepsilon=\varepsilon\left(y,|E(y)|^{2}\right)$ is essential; otherwise $\nabla \cdot \vec{E} \neq 0$ so that Helmholtz equation (2.1.4) is not valid.

Third, it is not clear yet how reflection and transmission of a plane TM-wave can be treated since a system of two coupled Helmholtz equations must be considered [37], [38].

Finally it should be remarked, that the approach can be applied to a wide field of further investigations concerning the great number of different parameter combinations which all apply to interesting theoretical and practical situations.

## Appendix A

## On the evaluation of the integral in (2.1.18)

The second integral from (2.1.18) reads

$$
\begin{align*}
A & =8 \pi^{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} d t \int_{0}^{t} \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau \\
= & 8 \pi^{2} \int_{0}^{y} \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau \int_{\tau}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} d t \\
& =8 \pi^{2} \int_{0}^{y} \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau\left(\frac{\sin ^{2} \kappa(y-\tau)}{2 \kappa^{2}}\right) \\
& =8 \pi^{2} \int_{0}^{y} \frac{\sin ^{2} \kappa(y-\tau)}{2 \kappa^{2}} \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau \tag{A.0.1}
\end{align*}
$$

For $\widetilde{\varepsilon}_{I}(y)=0\left(\widetilde{\varepsilon}_{R}(y)=\widetilde{\varepsilon}_{f}(y)\right)$ equation (2.1.18) is a special case of equation (3.1.11). Thus the relation

$$
\begin{align*}
8 \pi^{2} \frac{\sin ^{2} \kappa(y-t)}{2 \kappa^{2}}=\frac{4 \pi^{2}}{\kappa^{2}} & \left(\frac{1-\cos 2 \kappa(y-t)}{2}\right)= \\
& -2 \pi^{2} \frac{\cos 2 \kappa(y-t)-1}{\kappa^{2}} \tag{A.0.2}
\end{align*}
$$

holds, and, equations (3.1.11) and (2.1.19) - (2.1.20) are consistent (if $\widetilde{\varepsilon}_{I}(y)=$ $0)$.

## Appendix B

## On the evaluation of the limiting case $\widetilde{\varepsilon}(y) \equiv 0$ for the linear dielectric lossless film

If $\widetilde{\varepsilon}_{f}(y) \equiv 0$, equations (2.1.19), (2.1.20) yield

$$
\begin{equation*}
I(y)=\left|E_{3}\right|^{2} \cos (2 \kappa y)+\frac{\left|E_{3}\right|^{2}\left(q_{s}^{2}+q_{f}^{2}(0)\right)}{\kappa^{2}} \sin ^{2}(\kappa y) . \tag{B.0.1}
\end{equation*}
$$

After some algebra one obtains

$$
\begin{equation*}
I(y)=E^{2}(y)=\frac{\left|E_{3}\right|^{2}}{q_{s}^{2}}\left(q_{s}^{2} \sin ^{2} q_{f} y+q_{f}^{2} \cos ^{2} q_{f} y\right) \tag{B.0.2}
\end{equation*}
$$

hence the result is consistent with equation (37), Section 5 "The linear case" from [31].

## Appendix C

## Proof of the uniform convergence of series (2.1.21) by induction

Denoting the norm by $\left\|I_{0}\right\|=\max _{0 \leq y \leq d}\left|I_{0}(y)\right|$ and $\|K\|=\max _{0 \leq y, t \leq d}|K(y, t)|$ the iterations $I_{j}(y)$ can be estimated according to

$$
\begin{equation*}
\left|I_{j}(y)\right| \leq\left\|I_{0}\right\|\|K\|^{j} \frac{y^{j}}{j!} . \tag{C.0.1}
\end{equation*}
$$

For $j=1$ equation (2.1.22) implies

$$
\begin{equation*}
\left|I_{1}(y)\right| \leq\left\|I_{0}\right\| \int_{0}^{y}|K(y, t)| d t \leq\left\|I_{0}\right\|\|K\| y \tag{C.0.2}
\end{equation*}
$$

Assuming that (C.0.1) holds one obtains

$$
\begin{equation*}
\left|I_{j+1}(y)\right| \leq\left\|I_{0}\right\|\|K\|^{j+1} \int_{0}^{y} \frac{t^{j}}{j!} d t, \tag{C.0.3}
\end{equation*}
$$

which yields the required estimate (C.0.1). Thus (C.0.1) is valid for all $j$, leading to

$$
\begin{equation*}
|I(y)| \leq \sum_{j=0}^{\infty}\left|I_{j}(y)\right| \leq\left\|I_{0}\right\| \sum_{j=0}^{\infty} \frac{(\|K\| y)^{j}}{j!}=\left\|I_{0}\right\| e^{y\|K\|} \tag{C.0.4}
\end{equation*}
$$

Hence series (2.1.21) converges uniformly on $[0, d]$ according to the Weierstrass uniform convergence criterion.

## Appendix D

## On the evaluation of the phase on reflection $\delta_{r}$ for the total reflection case

According to the formula $E_{r}=\left|E_{r}\right| \exp \left(i \delta_{r}\right)$, the phase on reflection $\delta_{r}$ for the total reflection case is given by

$$
\begin{equation*}
\delta_{r}=2 \vartheta(d), \tag{D.0.1}
\end{equation*}
$$

where equations (2.1.27), (2.1.39) were used.
Actually, according to $\left|E_{r}^{2}\right|=\left|E_{0}^{2}\right|$,

$$
\begin{equation*}
E_{r}=E_{0} e^{i \delta_{r}} \tag{D.0.2}
\end{equation*}
$$

and thus

$$
\begin{equation*}
E_{0}+E_{0} e^{i \delta_{r}}=E(d) e^{i \vartheta(d)} \tag{D.0.3}
\end{equation*}
$$

This equation implies

$$
\left[\begin{array}{l}
E_{0}+E_{0} \cos \delta_{r}=E(d) \cos \vartheta(d),  \tag{D.0.4}\\
E_{0} \sin \delta_{r}=E(d) \sin \vartheta(d),
\end{array}\right.
$$

hence

$$
\begin{equation*}
\tan \vartheta(d)=\frac{E_{0} \sin \delta_{r}}{E_{0}+E_{0} \cos \delta_{r}}=\frac{\sin \delta_{r}}{1+\cos \delta_{r}}=\tan \frac{\delta_{r}}{2}, \tag{D.0.5}
\end{equation*}
$$

and thus equation (D.0.1), disregarding the periodicity of tan.

## Appendix E

## On the evaluation of the constant of integration $c_{2}$

The constant of integration $c_{2}$ depends on the permittivity function inside the film $\varepsilon_{f}(y)$ and on the parameter $q_{s}$. It can be calculated for every case independently, but always in the same manner, that can be exemplified by the case of the lossless linear dielectric film.

Considering equation (2.1.16) at $y=0$, the constant of integration $c_{2}$ is determined by

$$
\begin{equation*}
c_{2}=\left.\frac{d^{2} I(y)}{d y^{2}}\right|_{y=0}+4 q_{f}^{2}(0) I(0) \tag{E.0.1}
\end{equation*}
$$

According to equation (2.1.9), the second derivative of the field $E(y)$ at $y=0$ is given by

$$
\begin{equation*}
\left.\frac{d^{2} E(y)}{d y^{2}}\right|_{y=0}=\frac{c_{1}^{2}}{E^{3}(0)}-4 q_{f}^{2}(0) E^{2}(0) \tag{E.0.2}
\end{equation*}
$$

In case of real $q_{s}$ (transmission) equation (E.0.2) yields, taking into account equations (2.1.25), (2.1.29),

$$
\begin{equation*}
c_{2}=2\left|E_{3}\right|^{2}\left(q_{s}^{2}+q_{f}^{2}(0)\right)=2 I(0)\left(q_{s}^{2}+q_{f}^{2}(0)\right) \tag{E.0.3}
\end{equation*}
$$

For the pure imaginary $q_{s}$ (total reflection) the constant of integration $c_{2}$ is determined by

$$
\begin{equation*}
c_{2}=2\left|E_{3}\right|^{2}\left(\tilde{q}_{s}^{2}+q_{f}^{2}(0)\right)=2 I(0)\left(\tilde{q}_{s}^{2}+q_{f}^{2}(0)\right) \tag{E.0.4}
\end{equation*}
$$

where equations $(2.1 .31)$, (2.1.34) have been used.

## Appendix $\mathbf{F}$

## Proof of the uniform convergence of sequence (2.2.6)

The nonlinear operator $F$ is considered:

$$
\begin{equation*}
F(I):=I_{0}(y)+N_{1} I+N_{2} I^{2}, \tag{F.0.1}
\end{equation*}
$$

where $N_{1}$ and $N_{2}$ are linear bounded integral operators in the Banach space $C[o, d]$ and $I_{0}(y)$ is given by equation (2.1.23). Let

$$
\begin{equation*}
N_{1} \psi:=\int_{0}^{y} K_{1} \psi(t) d t, \quad N_{2} \varphi:=\int_{0}^{y} K_{2} \varphi(t) d t, \tag{F.0.2}
\end{equation*}
$$

with

$$
\begin{align*}
K_{1} & =\left(-\frac{\sin 2 \kappa(y-t)}{\kappa} 8 \pi^{2} \widetilde{\varepsilon}_{f}(t)-2 \pi^{2} \frac{\cos 2 \kappa(y-t)-1}{\kappa^{2}} \frac{d \widetilde{\varepsilon}_{f}(t)}{d t}\right) \\
K_{2} & =\left(-\frac{6 \pi^{2} a}{\kappa} \sin 2 \kappa(y-t)\right) \tag{F.0.3}
\end{align*}
$$

The norms $\left\|N_{1}\right\|,\left\|N_{2}\right\|$ are defined by

$$
\begin{equation*}
\left\|N_{1}\right\|=\max _{0 \leq y \leq d} \int_{0}^{y}\left|K_{1}\right| d t, \quad\left\|N_{2}\right\|=\max _{0 \leq y \leq d} \int_{0}^{y}\left|K_{2}\right| d t . \tag{F.0.4}
\end{equation*}
$$

Then equation (2.2.3) can be rewritten in operator form

$$
\begin{equation*}
I(y)=F(I)(y) . \tag{F.0.5}
\end{equation*}
$$

In order to prove that the equation (F.0.5) under certain assumptions has only one solution the following quadratic equation is considered

$$
\begin{equation*}
z=\left\|I_{0}\right\|+\left\|N_{1}\right\| z+\left\|N_{2}\right\| z^{2} \tag{F.0.6}
\end{equation*}
$$

where $\left\|I_{0}\right\|$ is defined in Appendix A. This equation has two positive roots if and only if the following conditions are satisfied:

$$
\begin{equation*}
\left(\left\|N_{1}\right\|-1\right)^{2}-4\left\|N_{2}\right\|\left\|I_{0}\right\|>0, \quad\left\|N_{1}\right\|<1 \tag{F.0.7}
\end{equation*}
$$

These inequalities imply

$$
\begin{equation*}
\left\|N_{1}\right\|+2 \sqrt{\left\|N_{2}\right\| \cdot\left\|I_{0}\right\|}<1 \tag{F.0.8}
\end{equation*}
$$

Let $r$ and $R$ be the smallest and the largest root of equation (F.0.6), respectively. In order to satisfy the conditions of the Banach Fixed-Point Theorem [cf. Appendix G], [39] it must be checked whether operator $F$ maps the ball $S_{R}(0)=\{y \in C[0, d]:\|y\|<R\}\left(\right.$ and $\left.S_{r}(0)\right)$ to itself. If $I(y) \in S_{R}(0)$ then

$$
\begin{align*}
& \|F(I)\| \leq\left\|I_{0}\right\|+\left\|N_{1}\right\|\|I\|+\left\|N_{2}\right\|\|I\|^{2}  \tag{F.0.9}\\
& <\left\|I_{0}\right\|+\left\|N_{1}\right\| R+\left\|N_{2}\right\| R^{2}=R .
\end{align*}
$$

Thus $F(I) \in S_{R}(0)$. Hence equation (F.0.5) has at least one solution inside $S_{R}(0) . F$ is contractive [39] in $S_{r}(0)$, because, if $I_{1}, I_{2} \in S_{r}(0)$, then

$$
\begin{aligned}
& \left\|F\left(I_{1}\right)-F\left(I_{2}\right)\right\|=\left\|N_{1}\left(I_{1}-I_{2}\right)+N_{2}\left(I_{1}^{2}-I_{2}^{2}\right)\right\| \\
& \leq\left\|N_{1}\right\|\left\|I_{1}-I_{2}\right\|+\left\|N_{2}\right\|\left\|I_{1}-I_{2}\right\|\left\|I_{1}+I_{2}\right\| \\
& \leq\left\|N_{1}\right\|\left\|I_{1}-I_{2}\right\|+2 r\left\|N_{2}\right\|\left\|I_{1}-I_{2}\right\| \\
& =\left(\left\|N_{1}\right\|+2 r\left\|N_{2}\right\|\right)\left\|I_{1}-I_{2}\right\| .
\end{aligned}
$$

Thus the inequality

$$
\begin{equation*}
\left\|N_{1}\right\|+2 r\left\|N_{2}\right\|<1 \tag{F.0.10}
\end{equation*}
$$

holds and thus the contraction of $F$. Inequality (F.0.10) is satisfied if (F.0.8) holds. Hence one can conclude [39] that the iteration procedure (2.2.5), (2.2.6) converges uniformly on $[0, d]$ inside the ball $S_{r}(0)$.

The set of the balls, where the iteration procedure converges uniformly, can be increased. Let us consider $\rho$ from the interval $r \leq \rho \leq R$. Following the foregoing arguments it must be checked whether operator $F$ maps the ball $S_{\rho}(0)$ to itself. If $I(y) \in S_{\rho}(0)$ then

$$
\begin{align*}
& \|F(I)\| \leq\left\|I_{0}\right\|+\left\|N_{1}\right\|\|I\|+\left\|N_{2}\right\|\|I\|^{2}  \tag{F.0.11}\\
& <\left\|I_{0}\right\|+\left\|N_{1}\right\| \rho+\left\|N_{2}\right\| \rho^{2} \leq \rho .
\end{align*}
$$

This inequality holds if $r \leq \rho \leq R$. Thus for $\rho$ from the interval

$$
\begin{align*}
& \frac{1-\left\|N_{1}\right\|-\sqrt{\left(1-\left\|N_{1}\right\|\right)^{2}-4\left\|N_{2}\right\| \cdot\left\|I_{0}\right\|}}{2\left\|N_{2}\right\|} \leq \rho \\
& \leq \frac{1-\left\|N_{1}\right\|+\sqrt{\left(1-\left\|N_{1}\right\|\right)^{2}-4\left\|N_{2}\right\| \cdot\left\|I_{0}\right\|}}{2\left\|N_{2}\right\|} \tag{F.0.12}
\end{align*}
$$

$F(I) \in S_{\rho}(0)$. Hence equation (F.0.5) has at least one solution inside $S_{\rho}(0)$. According to inequality (F.0.10), contraction of $F$ holds for $\rho$ from the following interval

$$
\begin{equation*}
\frac{1-\left\|N_{1}\right\|-\sqrt{\left(1-\left\|N_{1}\right\|\right)^{2}-4\left\|N_{2}\right\| \cdot\left\|I_{0}\right\|}}{2\left\|N_{2}\right\|} \leq \rho \leq \frac{1-\left\|N_{1}\right\|}{2\left\|N_{2}\right\|} . \tag{F.0.13}
\end{equation*}
$$

## Appendix G

## The Banach Fixed-Point Theorem

The Banach fixed-point theorem represents a fundamental convergence theorem for a broad class of iterations methods.

The operator equation

$$
\begin{equation*}
u=A u, \quad u \in M \tag{G.0.1}
\end{equation*}
$$

can be solved by means of the following iteration method

$$
\begin{equation*}
u_{n+1}=A u_{n}, \quad n=0,1, \ldots, \tag{G.0.2}
\end{equation*}
$$

where $u_{0} \in M$. Each solution of (G.0.1) is called a fixed point of the operator A.

Theorem (The fixed-point theorem of Banach). We assume that:
(a) $M$ is a closed nonempty set in the Banach space $X$ over $\mathbb{K}$, and
(b) the operator $A: M \rightarrow M$ is $k$-contractive, i.e., by definition,

$$
\begin{equation*}
\|A u-A v\| \leq k\|u-v\| \tag{G.0.3}
\end{equation*}
$$

for all $u, v \in M$, and fixed $k, 0 \leq k<1$.
Then, the following hold true:
(i) Existence and uniqueness.

The original equation (G.0.1) has exactly one solution $u$, i.e., the operator $A$ has exactly one fixed point $u$ on the set $M$.
(ii) Convergence of the iteration method.

For each given $u_{0} \in M$, the sequence $\left(u_{n}\right)$ constructed by (G.0.2) converges to the unique solution of equation (G.0.1).
(iii) Error estimates.

For all $n=0,1, \ldots$ there is the so-called a priori error estimate

$$
\begin{equation*}
\left\|u_{n}-u\right\| \leq k^{n}(1-k)^{-1}\left\|u_{1}-u_{0}\right\|, \tag{G.0.4}
\end{equation*}
$$

and the so-called a posteriori error estimate

$$
\begin{equation*}
\left\|u_{n+1}-u\right\| \leq k(1-k)^{-1}\left\|u_{n+1}-u_{n}\right\| . \tag{G.0.5}
\end{equation*}
$$

(iv) Rate of convergence.

The following is true for all $n=0,1, \ldots$

$$
\begin{equation*}
\left\|u_{n+1}-u\right\| \leq k\left\|u_{n+1}-u_{n}\right\| . \tag{G.0.6}
\end{equation*}
$$

This theorem was proved by Banach in 1920. The Banach fixed-point theorem is also called the contraction principle [39].

## Appendix H

## On the evaluation of inequality (2.2.7)

As an example, function $\widetilde{\varepsilon}_{f}(y)$ in $(2.2 .1)$ and the parameters are chosen according to

$$
\begin{equation*}
\widetilde{\varepsilon}_{f}(y)=\frac{1}{30} \cos ^{2} \frac{y}{10} . \tag{H.0.1}
\end{equation*}
$$

and $d=1, \varphi=1.107, E_{0}=1, \varepsilon_{c}=1, \varepsilon_{s}=1.7, \varepsilon_{f}^{0}=2.3, a=0.01$. Regarding the transmission case, one obtains $c_{2}=61.32, I(0)=0.319$, so that $\left\|I_{0}\right\|=$ 0.32 , according to (2.1.17), (2.1.23). The kernels $K_{1}, K_{2}$, defined by (F.0.3), can be estimated by


Figure H.1: $K_{1}$ from (F.0.3).

$$
\begin{gather*}
K_{1}=\frac{8 \pi^{2}}{30 \kappa} \sin 2 \kappa(y-t) \cos ^{2} 0.1 t+\frac{4 \pi^{2}}{150 \kappa^{2}} \sin ^{2} \kappa(y-t) \sin 0.1 t \cos 0.1 t \\
\leq \frac{8 \pi^{2}}{\kappa} \cos ^{2} 0.1 t+\frac{4 \pi^{2}}{150 \kappa^{2}} \sin ^{2} \kappa(y-t),  \tag{H.0.2}\\
\left|K_{2}\right|=\left|\frac{6 \pi^{2} a}{\kappa} \sin 2 \kappa(y-t)\right| \leq \frac{6 \pi^{2} a}{\kappa} . \tag{H.0.3}
\end{gather*}
$$

As shown in figure H.1, $K_{1}(y, t) \geq 0,0 \leq y, t \leq d$ holds. Hence the norm $\left\|N_{1}\right\|$ (cf. (F.0.4)) can be calculated to yield $\left\|N_{1}\right\|=0.34$. According to (H.0.3), (F.0.4) one obtains $\left\|N_{2}\right\|=0.078$. Inserting $\left\|I_{0}\right\|,\left\|N_{1}\right\|,\left\|N_{2}\right\|$ into (2.2.7) this inequality is fulfilled, so that the sequence of iterate solutions $I_{j}(y)$ of equation (2.2.6) converges uniformly.

## Appendix I

## Proof of the uniform convergence of sequence (3.2.4)

The nonlinear operator $F$ is considered

$$
\begin{equation*}
F(I):=I_{0 c}(y)+N_{1}(I)+N_{2}(I)+N_{3}(I)+N_{4}(I), \tag{I.0.1}
\end{equation*}
$$

where $N_{1}, N_{2}, N_{3}$ and $N_{4}$ are bounded integral operators in the Banach space $C[0, d]$. These operators and $I_{0 c}(y)$ are given by

$$
\begin{array}{r}
N_{1}(I)=-\frac{8 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{R}(\tau) I(\tau) d \tau \\
N_{2}(I)=-\frac{6 a \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I^{2}(\tau) d \tau \\
N_{3}(I)=\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d \widetilde{\varepsilon}_{R}(\tau)}{d \tau} I(\tau) d \tau \\
N_{4}(I)=\frac{24 \pi^{4}}{\kappa} \int_{0}^{y} \widetilde{\varepsilon}_{I}(z) \psi(y, t, z) I(z) d z \\
I_{0 c}(y)=\tilde{I}_{0}(y)+c_{2} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} d t \\
-16 \pi^{2} c_{1} \int_{0}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa} \int_{0}^{t} \widetilde{\varepsilon}_{I}(z) d z d t \tag{I.0.2}
\end{array}
$$

where constant $c_{2}$ is from equation (2.2.8) and

$$
\begin{equation*}
\psi(y, t, z)=\int_{z}^{y} \frac{\sin 2 \kappa(y-t)}{2 \kappa}\left\{\int_{z}^{t} \widetilde{\varepsilon}_{I}(\tau) d \tau\right\} d t \tag{I.0.3}
\end{equation*}
$$

The norms $\left\|N_{1}\right\|,\left\|N_{2}\right\|,\left\|N_{3}\right\|,\left\|N_{4}\right\|,\left\|I_{0 c}\right\|$ are defined by

$$
\begin{array}{r}
\left\|N_{1}\right\|=\frac{8 \pi^{2}}{\kappa^{2}} \max _{0 \leq y \leq d} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| \cdot\left|\widetilde{\varepsilon}_{R}(\tau)\right| d \tau \\
\left\|N_{2}\right\|=\frac{6 a \pi^{2}}{\kappa} \max _{0 \leq y \leq d} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| d \tau \\
\left\|N_{3}\right\|=\frac{4 \pi^{2}}{\kappa^{2}} \max _{0 \leq y \leq d} \int_{0}^{y}\left|\sin ^{2} \kappa(y-\tau)\right| \cdot\left|\frac{d \widetilde{\varepsilon}_{R}(\tau)}{d \tau}\right| d \tau \\
\left\|N_{4}\right\|=\frac{24 \pi^{4}}{\kappa} \max _{0 \leq y \leq d} \int_{0}^{y}\left|\widetilde{\varepsilon}_{I}(z)\right| \cdot|\psi(y, t, z)| d z \\
\left\|I_{0 c}\right\|=\max _{0 \leq y \leq d}\left|I_{0 c}\right| . \tag{I.0.4}
\end{array}
$$

Then equation (3.2.3) can be rewritten in operator form

$$
\begin{equation*}
I(y)=F(I)(y) . \tag{I.0.5}
\end{equation*}
$$

Let us consider $R$ such that $\|I\|=\max _{0 \leq y \leq d} I(y) \leq R$. In order to satisfy the conditions of the Banach Fixed-Point Theorem [39] it must be checked whether operator $F$ maps the ball $S_{R}(0)$ to itself. If $I(y) \in S_{R}(0)$ then

$$
\begin{align*}
&\|F(I)\| \leq\left\|I_{0 c}\right\|+\left\|N_{1}\right\|\|I\|+\left\|N_{2}\right\|\|I\|^{2}+\left\|N_{3}\right\|\|I\|+\left\|N_{4}\right\|\|I\| \\
& \leq\left\|I_{0 c}\right\|+\left\|N_{*}\right\| R+\left\|N_{2}\right\| R^{2} \leq R \tag{I.0.6}
\end{align*}
$$

with

$$
\begin{equation*}
\left\|N_{*}\right\|=\left\|N_{1}\right\|+\left\|N_{3}\right\|+\left\|N_{4}\right\| . \tag{I.0.7}
\end{equation*}
$$

This inequality holds for $R$ from the interval

$$
\begin{array}{r}
\frac{1-\left\|N_{*}\right\|-\sqrt{\left(1-\left\|N_{*}\right\|\right)^{2}-4\left\|N_{2}\right\|\left\|I_{0 c}\right\|}}{2\left\|N_{2}\right\|} \leq R \\
\leq \frac{1-\left\|N_{*}\right\|+\sqrt{\left(1-\left\|N_{*}\right\|\right)^{2}-4\left\|N_{2}\right\|\left\|I_{0 c}\right\|}}{2\left\|N_{2}\right\|} \tag{I.0.8}
\end{array}
$$

and imply

$$
\begin{equation*}
\left\|N_{*}\right\|+2 \sqrt{\left\|N_{2}\right\|\left\|I_{0 c}\right\|}<1 \tag{I.0.9}
\end{equation*}
$$

For this $R F(I) \in S_{R}(0)$. Hence equation (I.0.5) has at least one solution inside $S_{R}(0) . F$ is contractive [39] in $S_{R}(0)$, because, if $I_{1}, I_{2} \in S_{R}(0)$, then

$$
\begin{align*}
& \left\|F\left(I_{1}\right)-F\left(I_{2}\right)\right\|=\left\|N_{*}\left(I_{1}-I_{2}\right)+N_{2}\left(I_{1}^{2}-I_{2}^{2}\right)\right\| \\
& \leq\left\|N_{*}\right\|\left\|I_{1}-I_{2}\right\|+\left\|N_{2}\right\|\left\|I_{1}-I_{2}\right\|\left\|I_{1}+I_{2}\right\|  \tag{I.0.10}\\
& \leq\left\|N_{*}\right\|\left\|I_{1}-I_{2}\right\|+2 R\left\|N_{2}\right\|\left\|I_{1}-I_{2}\right\| \\
& =\left(\left\|N_{*}\right\|+2 R\left\|N_{2}\right\|\right)\left\|I_{1}-I_{2}\right\| .
\end{align*}
$$

Thus the inequality

$$
\begin{equation*}
\left\|N_{*}\right\|+2 R\left\|N_{2}\right\|<1 \tag{I.0.11}
\end{equation*}
$$

holds and thus the contraction of $F$. Inequality (I.0.11) is satisfied if (I.0.9) holds. Hence one can conclude [39] that the iteration procedure (3.2.4), (3.2.5) converges uniformly on $[0, d]$.

## Appendix J

## Proof of the uniform convergence of iteration sequence for (4.1.3)

The nonlinear operator $F$ is considered

$$
\begin{equation*}
F(I):=I_{0 s}(y)+N_{1}(I)+N_{2}(I)+\frac{1}{r} N_{3}(I)+\frac{1}{r^{2}} N_{4}(I)+\frac{1}{r^{2}} N_{5}(I), \tag{J.0.1}
\end{equation*}
$$

where $N_{1}, N_{2}, N_{3}, N_{4}$ and $N_{5}$ are bounded integral operators in the Banach space $C[0, d]$. These operators and $I_{0 s}(y)$ are given by

$$
\begin{array}{r}
N_{1}(I)=\frac{4 \pi^{2}}{\kappa^{2}} \int_{0}^{y} \sin ^{2} \kappa(y-\tau) \frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau} I(\tau) d \tau \\
N_{2}(I)=-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) \widetilde{\varepsilon}_{f}(\tau) I(\tau) d \tau \\
N_{3}(I)=-\frac{8 \pi^{2}}{\kappa} \int_{0}^{y} \sin 2 \kappa(y-\tau) I(\tau) d \tau \\
N_{4}(I)=-\frac{4 \pi^{2}}{\kappa a} \int_{0}^{y} \sin 2 \kappa(y-\tau) \frac{1}{1+\operatorname{arI(\tau )}} d \tau \\
N_{5}(I)=\frac{4 \pi^{2}}{\kappa a} \int_{0}^{y} \sin 2 \kappa(y-\tau) \ln (1+\operatorname{arI(\tau ))d\tau } \\
I_{0 s}(y)=\tilde{I}_{0}(y)+\left(c_{2}-\frac{16 \pi^{2}}{a r^{2}}\right) \frac{\sin ^{2} \kappa y}{2 \kappa^{2}}, \tag{J.0.2}
\end{array}
$$

where constant $c_{2}$ is from equation (4.1.5).

The norms $\left\|N_{1}\right\|,\left\|N_{2}\right\|,\left\|N_{3}\right\|,\left\|N_{4}\right\|,\left\|N_{5}\right\|,\left\|I_{0 s}\right\|$ are defined by

$$
\begin{array}{r}
\left\|N_{1}\right\|=\frac{4 \pi^{2}}{\kappa^{2}} \max _{0 \leq y \leq d} \int_{0}^{y}\left|\sin ^{2} \kappa(y-\tau)\right| \cdot\left|\frac{d \widetilde{\varepsilon}_{f}(\tau)}{d \tau}\right| d \tau \\
\left\|N_{2}\right\|=\frac{8 \pi^{2}}{\kappa} \max _{0 \leq y \leq d} \int_{0}^{y}\left|\sin ^{2} \kappa(y-\tau)\right| \cdot\left|\widetilde{\varepsilon}_{f}(\tau)\right| d \tau \\
\left\|N_{3}\right\|=\frac{8 \pi^{2}}{\kappa} \max _{0 \leq y \leq d} \int_{0}^{y}\left|\sin ^{2} \kappa(y-\tau)\right| d \tau \\
\left\|N_{4}\right\|=\frac{4 \pi^{2}}{\kappa a} \max _{0 \leq y \leq d} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| d \tau \\
\left\|N_{5}\right\|=\frac{4 \pi^{2}}{\kappa a} \max _{0 \leq y \leq d} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| d \tau \\
\left\|I_{0 s}\right\|=\max _{0 \leq y \leq d}\left|I_{0 s}\right| . \tag{J.0.3}
\end{array}
$$

The norms $\left\|N_{4}\right\|,\left\|N_{5}\right\|$ of the operators $N_{4}(I)$ and $N_{5}(I)$, respectively, are equal. In following for the both norms only the determination $\left\|N_{4}\right\|$ will be used.

Equation (4.1.3) can be rewritten in operator form

$$
\begin{equation*}
I(y)=F(I)(y) . \tag{J.0.4}
\end{equation*}
$$

Let us consider $R$ such that $\|I\|=\max _{0 \leq y \leq d} I(y) \leq R$. In order to satisfy the conditions of the Banach Fixed-Point Theorem [39] it must be checked whether operator $F$ maps the ball $S_{R}(0)$ to itself. If $I(y) \in S_{R}(0)$ then

$$
\begin{array}{r}
\|F(I)\| \leq\left\|I_{0 s}\right\|+\left\|N_{1}\right\| \cdot\|I\|+\left\|N_{2}\right\| \cdot\|I\|+\frac{1}{r}\left\|N_{3}\right\| \cdot\|I\| \\
+\frac{1}{r^{2}}\left\|N_{4}\right\| \cdot \frac{1}{1+a r \min _{0 \leq y \leq d} I(y)}+\frac{1}{r^{2}}\left\|N_{4}\right\| \cdot a r\|I\| \\
\leq\left\|I_{0 s}\right\|+\left\|N_{1}\right\| \cdot R+\left\|N_{2}\right\| \cdot R+\frac{1}{r}\left\|N_{3}\right\| \cdot R+\frac{1}{r^{2}}\left\|N_{4}\right\| \\
+\frac{a}{r}\left\|N_{4}\right\| \cdot R<R . \\
\left\|I_{0 s}\right\|+\frac{1}{r^{2}}\left\|N_{4}\right\|+\left(\left\|N_{1}\right\|+\left\|N_{2}\right\|+\frac{1}{r}\left(\left\|N_{3}\right\|+a\left\|N_{4}\right\|\right)\right) \cdot R<R . \tag{J.0.6}
\end{array}
$$

This inequality holds if

$$
\begin{equation*}
\frac{\left\|I_{0 s}\right\|+\frac{1}{r^{2}}\left\|N_{4}\right\|}{1-\left(\left\|N_{1}\right\|+\left\|N_{2}\right\|+\frac{\left\|N_{3}\right\|+a\left\|N_{4}\right\|}{r}\right)}<R \tag{J.0.7}
\end{equation*}
$$

$$
\begin{equation*}
\left\|N_{1}\right\|+\left\|N_{2}\right\|+\frac{\left\|N_{3}\right\|+a\left\|N_{4}\right\|}{r}<1 . \tag{J.0.8}
\end{equation*}
$$

Thus for $R$, which satisfy condition (J.0.7), $F(I) \in S_{\rho}(0)$. Hence equation (F.0.5) has at least one solution inside $S_{R}(0)$.

The iteration procedure for (4.1.3) convergence on $[0, d]$ uniformly, if operator $F$ is contractive [39]. To prove the contraction of $F$ the following difference is considered

$$
\begin{gather*}
F\left(I_{1}\right)-F\left(I_{2}\right)=N_{1}\left(I_{1}-I_{2}\right)+N_{2}\left(I_{1}-I_{2}\right)+N_{3}\left(\frac{1}{r}\left(I_{1}-I_{2}\right)\right)+ \\
N_{4}\left(I_{1}-I_{2}\right)+N_{5}\left(I_{1}-I_{2}\right), \quad \Rightarrow  \tag{J.0.9}\\
\left\|F\left(I_{1}\right)-F\left(I_{2}\right)\right\| \leq\left\|N_{1}\right\|\left\|I_{1}-I_{2}\right\|+\left\|N_{2}\right\|\left\|I_{1}-I_{2}\right\|+ \\
\left\|N_{3}\right\|\left\|\frac{1}{r}\left(I_{1}-I_{2}\right)\right\|+\left\|\frac{1}{r^{2}} N_{4}\left(I_{1}-I_{2}\right)\right\|+\left\|\frac{1}{r^{2}} N_{5}\left(I_{1}-I_{2}\right)\right\| \tag{J.0.10}
\end{gather*}
$$

The following terms are considered to be estimated:

$$
\begin{array}{r}
\left\|\frac{1}{r^{2}} N_{4}\left(I_{1}-I_{2}\right)\right\| \leq \max _{0 \leq y \leq d} \frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| \\
\left|\frac{1}{1+\operatorname{ar} I_{1}(\tau)}-\frac{1}{1+\operatorname{ar} I_{2}(\tau)}\right| d \tau= \\
\max _{0 \leq y \leq d} \frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| \cdot \left\lvert\, \frac{\operatorname{arI_{2}(\tau )-\operatorname {ar}I_{1}(\tau )}}{\left(1+\operatorname{arI_{1}(\tau ))(1+\operatorname {arI_{2}(\tau ))}} \mid d \tau\right.}\right. \\
\leq \frac{1}{r} \max _{0 \leq y \leq d} \frac{4 \pi^{2}}{\kappa} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| d \tau \cdot\left\|I_{1}-I_{2}\right\|, \tag{J.0.11}
\end{array}
$$

hence

$$
\begin{equation*}
\left\|\frac{1}{r^{2}} N_{4}\left(I_{1}-I_{2}\right)\right\| \leq \frac{\left\|N_{4}\right\|}{r} \cdot\left\|I_{1}-I_{2}\right\| \tag{J.0.12}
\end{equation*}
$$

$$
\begin{gather*}
\left\|\frac{1}{r^{2}} N_{5}\left(I_{1}-I_{2}\right)\right\| \leq \max _{0 \leq y \leq d} \frac{4 \pi^{2}}{\kappa a r^{2}} \int_{0}^{y}|\sin 2 \kappa(y-\tau)| \\
\left|\ln \left(1+\operatorname{ar} I_{1}(\tau)\right)-\ln \left(1+\operatorname{ar} I_{2}(\tau)\right)\right| d \tau \tag{J.0.13}
\end{gather*}
$$

Let us consider the difference of $\ln$ in the equation (J.0.13)

$$
\begin{align*}
& \left|\ln \left(1+\operatorname{ar} I_{1}\right)-\ln \left(1+\operatorname{ar} I_{2}\right)\right|=\left|\ln \frac{1+\operatorname{ar} I_{1}}{1+\operatorname{ar} I_{2}}\right| \\
& \quad=\left|\ln \left(1+\frac{\operatorname{ar}\left(I_{1}-I_{2}\right)}{1+\operatorname{ar} I_{2}}\right)\right| \leq \operatorname{ar}\left\|I_{1}-I_{2}\right\| \tag{J.0.14}
\end{align*}
$$

which implies

$$
\begin{equation*}
\left\|\frac{1}{r^{2}} N_{5}\left(I_{1}-I_{2}\right)\right\| \leq \frac{1}{r^{2}} \cdot\left\|N_{4}\right\| \cdot a r \cdot\left\|I_{1}-I_{2}\right\|=\frac{a\left\|N_{4}\right\|}{r}\left\|I_{1}-I_{2}\right\| . \tag{J.0.15}
\end{equation*}
$$

Thus from equation (J.0.10) one obtains

$$
\begin{array}{r}
\left\|F\left(I_{1}\right)-F\left(I_{2}\right)\right\| \leq\left(\left\|N_{1}\right\|+\left\|N_{2}\right\|+\frac{\left\|N_{3}\right\|}{r}+\frac{\left\|N_{4}\right\|}{r}+\frac{a\left\|N_{4}\right\|}{r}\right) \\
\cdot\left\|I_{1}-I_{2}\right\| \\
=\left(\left\|N_{1}\right\|+\left\|N_{2}\right\|+\frac{\left\|N_{3}\right\|+(1+a)\left\|N_{4}\right\|}{r}\right) \cdot\left\|I_{1}-I_{2}\right\| . \tag{J.0.16}
\end{array}
$$

According to inequality (J.0.16), contraction of $F$ holds if the following condition is satisfied

$$
\begin{equation*}
\left\|N_{1}\right\|+\left\|N_{2}\right\|+\frac{\left\|N_{3}\right\|+(1+a)\left\|N_{4}\right\|}{r}<1 . \tag{J.0.17}
\end{equation*}
$$

Hence one can conclude [39] that the iteration procedure for (4.1.3) converges uniformly on $[0, d]$.

## Appendix K

## Proof of the uniform convergence of iteration sequence for (5.1.3)

The proof of the uniform convergence of iteration sequence for (5.1.3) follows the lines of the proof in Appendix E. The additional term, associated to the imaginary part of permittivity function $\widetilde{\varepsilon}_{I}(y)$, must be taken into account. After some algebra, the final condition of uniform convergence reads

$$
\begin{equation*}
\left\|N_{1}\right\|+\left\|N_{2}\right\|+\left\|N_{c}\right\|+\frac{\left\|N_{3}\right\|+(1+a)\left\|N_{4}\right\|}{r}<1 \tag{K.0.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\|N_{c}\right\|=\frac{24 \pi^{4}}{\kappa} \max _{0 \leq y \leq d} \int_{0}^{y}\left|\widetilde{\varepsilon}_{I}(z)\right| \cdot|\psi(y, t, z)| d z \tag{K.0.2}
\end{equation*}
$$

with $\psi(y, t, z)$ from equation (I.0.3).

## Appendix L

## Complex constant of nonlinearity for the Kerr-like nonlinear case

For the Kerr-like nonlinear dielectric film with a complex constant of nonlinearity $a=a_{R}+i a_{I}$ the permittivity in the Helmholtz equation is modelled by a complex-valued function according to

$$
\begin{equation*}
\varepsilon_{f}=\varepsilon_{f}(y)+\left(a_{R}+i a_{I}\right) E^{2}(y), 0<y<d \tag{L.0.1}
\end{equation*}
$$

with real constants $a_{R}, a_{I}$.
Following the lines of section 3.1.1 one obtains in place of equations (3.1.2)), (3.1.3))

$$
\begin{equation*}
\frac{d^{2} E(y)}{d y^{2}}-E(y)\left(\frac{d \vartheta(y)}{d y}\right)^{2}+\left[4 \pi^{2} \varepsilon_{f}(y)+a_{R} E^{2}(y)\right] E(y)=0 \tag{L.0.2}
\end{equation*}
$$

and

$$
\begin{equation*}
E(y) \frac{d^{2} \vartheta(y)}{d y^{2}}+2 \frac{d \vartheta(y)}{d y} \frac{d E(y)}{d y}+4 \pi^{2} a_{I} E^{3}(y)=0 \tag{L.0.3}
\end{equation*}
$$

Equation (L.0.3) can be integrated leading to (cf. equation (3.1.4))

$$
E^{2}(y) \frac{d \vartheta(y)}{d y}=c_{1}-4 \pi^{2} a_{I} \int_{0}^{y} E^{4}(\tau) d \tau
$$

so that the following steps of the method can be done leading to the corresponding Volterra equation.

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