# On the theory of planar and cylindrical dielectric waveguides with photorefractive nonlinearity

## Dissertation zur Erlangung des Grades eines Doktors der Naturwissenschaften

von

## Andreas Geisler

vorgelegt dem Fachbereich Physik der



Juli 2004

## Contents

1	Intr	troduction 4						
	1.1 Planar waveguide							
		1.1.1 Even modes	6					
		1.1.2 Odd modes	8					
	1.2	Cylindrical waveguide	9					
2	Pla pho	Planar waveguide with photorefractive permittivity 1						
	2.1	Even modes	16					
		2.1.1 The second iteration function $u_2(x)$	19					
		2.1.2 The amplitude $u_2(0)$	22					
		2.1.3 Exact amplitude $\ldots \ldots \ldots$	24					
		2.1.4 Ratio $\eta$ of power flow in the core $\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$	24					
	2.2	Odd modes	29					
		2.2.1 The second iteration function $v_2(x)$	31					
		2.2.2 Ratio $\eta$ of power flow in the core $\ldots \ldots \ldots \ldots \ldots \ldots$	34					
	2.3	Convergence of the iteration sequence $\{u_n\}$	36					
3	Cylindrical waveguide with photorefractive permittivity of the core							
	3.1 Analytical approximations							

		3.1.1	The second iteration function $u_2(x)$	43		
		3.1.2	Ratio $\eta$ of power flow in the core $\ldots \ldots \ldots \ldots$	46		
	3.2 Convergence of the iteration					
4	Sun	nmary	and Outlook	49		
$\mathbf{A}$	A Planar waveguide					
	A.1	Even f	unction	54		
		A.1.1	Linear case, cut off limit	54		
	A.2 Field dependent permittivity, even modes					
		A.2.1	Construction of function $f$	55		
		A.2.2	Dispersion relation, even mode	57		
		A.2.3	Dispersion relation of the first iteration step, even mode	58		
		A.2.4	The amplitude $u_n(0)$	60		
		A.2.5	The second iteration functions amplitude $u_2(0)$	61		
		A.2.6	Exact amplitude	63		
		A.2.7	The second iteration function $u_2(x)$ , even modes	64		
	A.3 Convergence of the iteration					
		A.3.1	Banach fixed point theorem	67		
		A.3.2	Estimation of $\left  \int_{-R}^{R} ds \ G(s, x) \ B(u(s)) \right $ , even modes	68		
		A.3.3	Operator $A$ maps $S$ into itself $\ldots \ldots \ldots \ldots \ldots$	71		

		A.3.4	Contractivity of operator A	72
в	Cyli	indrica	d waveguide	73
	B.1	Cutoff	limit	73
	B.2	Conve	rgence	75
		B.2.1	Estimation of $\ \rho G\ $	75
		B.2.2	Contraction	79

### 1 Introduction

In recent years, the generation of solitary waves in bulk material with photorefractive (i. e. intensity dependent) permittivity has been the subject of intense experimental and theoretical investigations ([1] - [4]).

In particular, planar waveguides of photorefractive media have been studied theoretically ([5] - [8]) and experimentally ([9], [10]).

Langbein et al. [11] considered TE-polarized guided-wave solutions of a slab waveguide with arbitrary intensity-dependent dielectric functions by means of the first integrals of the Helmholtz equations. However, the field solutions are not determined explicitly, but discussed qualitatively by means of phase path diagrams.

To the best of my knowledge, there are no analytical approximations of the electric field in planar and cylindrical waveguides taking into account boundary conditions which lead to a dispersion relation and assuming a core of photorefractive material with external field.

In this work, the propagation of a TE-wave in a symmetric slab waveguide with real field dependent permittivity is investigated, where solutions of the exact Helmholtz equations are presented with the help of a Green function and approximated by an iteration method.

Approximate solutions and power flow are studied explicitly in the case of a photorefractive permittivity with an external field applied to the core of the waveguide. This kind of photorefractive permittivity is of special interest, because the magnitude of the light induced change of the linear permittivity, in this work expressed in terms of a parameter  $\alpha$ , can be varied experimentally very easily by changing the external field. Since  $|\alpha| < 0.05$  [12], the dependence of the power flow on  $\alpha$  is investigated by linearization with respect to  $\alpha$ .

Whereas Snyder et al. [13] consider circular symmetric solutions of Maxwell's equations with azimuthal polarization in a bulk Kerr medium by numerically solving the associated scalar wave equation, Smirnov et al. [14] investigate analytically azimuthal polarized solutions of a cylindrical waveguide, taking into account the associated dispersion relation (further references concerning

the cylindrical waveguide cf. [14]).

In this work, the cylindrical waveguide with a core of photorefractive permittivity with external field instead of a Kerr permittivity is considered, where particular attention is focused to the cutoff radius of the cylindrical waveguide.

### 1.1 Planar waveguide

In this subsection, a planar waveguide with real and constant permittivity

$$\varepsilon_{i} = \begin{cases} \varepsilon_{1} & x \leq -R & (\text{exterior}) \\ \varepsilon_{2} & -R \leq x \leq R & (\text{interior}) \\ \varepsilon_{1} & x \geq R & (\text{exterior}), \end{cases}$$
(1)

is considered ([15], [16], [17]), where for later purposes the ratio  $\eta_0$  of power flow in the core to the total power flow and the associated cutoff approximation is studied.

The electric field in each of the three domains is determined by the wave equations

$$\Delta_{\perp}\vec{E} + \Delta_{\parallel}\vec{E} + \omega^2 \varepsilon_0 \mu_0 \varepsilon_i \vec{E} = 0.$$
 (2)

Since  $\omega^2 = c_0^2 k^2 = \frac{1}{\varepsilon_0 \mu_0} k^2$ , the space coordinates of the wave equation are dimensionless after dividing Eqs. (2) by  $k^2$ :

$$\frac{1}{k^2} \Delta_{\perp} \vec{E} + \frac{1}{k^2} \Delta_{\parallel} \vec{E} + \varepsilon_i \vec{E} = 0.$$
(3)

Assuming transverse electric (TE) polarization and using an ansatz

$$\vec{E}(x,z) = E_0 u(x) e^{ik\gamma z} \vec{e}_y, \qquad (4)$$

where  $E_0$  is a real constant and u(x) is a real function of the transverse coordinate x, Eqs. (3) read

$$u'' + (\varepsilon_i - \gamma^2) u = 0, \quad i = 1, 2$$
 (5)

where the prime denotes the derivative with respect to  $\tilde{x} := kx$ . The effective refraction index  $\gamma$  is dimensionless, as well as  $\tilde{x}$  and  $\tilde{R} := kR$ . For convenience, the tildes are omitted in the following.

Introducing

$$\kappa_1^2 := \gamma^2 - \varepsilon_1 > 0, \qquad \kappa_2^2 := \varepsilon_2 - \gamma^2 > 0 \tag{6}$$

and assuming  $\varepsilon_2$  to be field-independent, the solutions of Eq. (5) are given by exponential and trigonometric functions, respectively. Even and odd field modes have to be discriminated.

### 1.1.1 Even modes

The solution u(x) of Eqs. (5) reads

$$u(x) = \begin{cases} \frac{e^{+\kappa_1 x}}{e^{+\kappa_1 R}} & x \leq -R \\ \frac{\cos(\kappa_2 x)}{\cos(\kappa_2 R)} & |x| \leq R \\ \frac{e^{-\kappa_1 x}}{e^{-\kappa_1 R}} & x \geq R, \end{cases}$$
(7)

so that the tangential component of the field is continuous at |x| = R.

The tangential component of the magnetic field is continuous if u' is continuous at |x| = R, leading to the dispersion relation

$$g := \tan \kappa_2 R - \frac{\kappa_1}{\kappa_2} = 0.$$
(8)

It determines the possible pairs  $\{\gamma, R\}$  (cf. Fig. 1) and thus the solutions (7). The radius R as a function of  $\gamma$  is given by

$$\kappa_2 R_{\nu} = \nu \frac{\pi}{2} + \arctan \frac{\kappa_1}{\kappa_2}, \qquad \nu = 0, 2, 4, \dots,$$
(9)

where  $\nu$  denotes the mode number. Solutions only exist if  $\tan \kappa_2 R > 0$ , hence

$$\nu \frac{\pi}{2} \leq \kappa_2 R_{\nu} \leq (\nu+1) \frac{\pi}{2}, \qquad \nu = 0, 2, 4, \dots$$
 (10)

If  $\varepsilon_2 > \varepsilon_1$  and  $\kappa_1/\kappa \ll 1$  (cutoff approximation), Eq. (8) leads to (see A.1.1)

$$\kappa R_{\nu} = \nu \frac{\pi}{2} + \frac{\kappa_1}{\kappa} + \mathcal{O}(\frac{\kappa_1}{\kappa})^2 \tag{11}$$

where  $\nu = 0, 2, 4, \ldots$  (even values of  $\nu$  belong to even modes) and

$$\kappa := \sqrt{\varepsilon_2 - \varepsilon_1} \,. \tag{12}$$



Figure 1: Solution  $R = R(\gamma)$  of (8) and (16);  $\varepsilon_1 = 1; \varepsilon_2 = 4$ 

From the experimental point of view, the ratio  $\eta_0$  of the power flow in the core to the total power flow is of interest in order to estimate the guidance of waves. This ratio  $\eta_0$  is given by

$$\eta_{0\nu} := \frac{\int_{-R_{\nu}}^{R_{\nu}} dx \, u^2}{\int_{-\infty}^{\infty} dx \, u^2} = 1 - \frac{1}{1 + \kappa_1 R_{\nu}} \frac{\kappa_2^2}{\kappa^2}.$$
 (13)

It can be evaluated for each of the branches of the dispersion relation (cf. Fig. 1). In the cutoff limit  $\kappa_1 \rightarrow 0$ , the power ratio vanishes according to

$$\eta_{0\nu} = \nu \frac{\pi}{2} \frac{\kappa_1}{\kappa} + \left[ 2 - \left(\nu \frac{\pi}{2}\right)^2 \right] \left(\frac{\kappa_1}{\kappa}\right)^2 + \mathcal{O}\left(\frac{\kappa_1}{\kappa}\right)^3.$$
(14)

Thus if  $\gamma = \sqrt{\varepsilon_1} + \tau$  where  $\tau \ll 1$  and  $\nu = 0$  (fundamental mode), then  $\eta_0 = \eta_0(\tau)$  decreases linearly towards zero, whereas  $\eta_0(\tau)$  vanishes according to a square root law for all higher modes.

In contrast, in the far cutoff limit  $\kappa_2 \to 0$ , due to the divergence of R, the ratio  $\eta_0$  tends to 1, thus the total power flow is in the core.

### 1.1.2 Odd modes

In this case the solution u(x) of Eqs. (5) is given by

$$u(x) = \begin{cases} -\frac{e^{+\kappa_1 x}}{e^{+\kappa_1 R}} & x \leq -R \\ \frac{\sin(\kappa_2 x)}{\sin(\kappa_2 R)} & |x| \leq R \\ \frac{e^{-\kappa_1 x}}{e^{-\kappa_1 R}} & x \geq R \end{cases}$$
(15)

so that the tangential component of the field is continuous at |x| = R.

The continuity of u' at |x| = R leads to the dispersion relation

$$g := -\cot \kappa_2 R - \frac{\kappa_1}{\kappa_2} = 0.$$
 (16)

It determines the possible pairs  $\{\gamma, R\}$  (cf. Fig. 1) and thus the solutions (15).

If  $\varepsilon_2 > \varepsilon_1$  and  $\kappa_1/\kappa \ll 1$  (cutoff approximation), Eq. (16) leads to Eq. (11), where  $\nu = 1, 3, \ldots$  and  $\kappa$  is given by (12).

The power flow through the core for odd modes equals the corresponding power flow for even modes, thus  $\eta_0$  is given by (13) and the associated cutoff limit is given by (14) where  $\nu$  is odd.

### 1.2 Cylindrical waveguide

In this subsection, a cylindrical waveguide is considered with a permittivity

$$\varepsilon_{i} = \begin{cases} \varepsilon_{2} & 0 \leq r \leq R \text{ (core)} \\ \varepsilon_{1} & r \geq R \text{ (cladding)} \end{cases}$$
(17)

where  $\varepsilon_1$  and  $\varepsilon_2$  are real and constant with respect to space coordinates ([15], [16], [17]).

Substitution of an  $\vec{e}_{\varphi}$ -polarized wave

$$\vec{E} = E_{\varphi}(r, z) \, \vec{e}_{\varphi} = E_0 \, u(r) \, e^{i\gamma k z} \, \vec{e}_{\varphi} \tag{18}$$

(with u(r) a real function) in the wave equation (3) leads to the Bessel equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)u + k^2\left(\varepsilon_i - \gamma^2\right)u = 0$$
(19)

where  $\gamma k$  is the propagation constant of the wave.

Using the abbreviation (6), (19) can be written as

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}u + \kappa_2^2 u = 0 \qquad (r \le R)$$
(20a)

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}u + (i\kappa_1)^2 u = 0 \qquad (r \ge R)$$
(20b)

where the prime denotes the derivative with respect to  $\tilde{r} = kr$ .  $\tilde{r}$  and  $\tilde{R} := kR$  are dimensionless, and the tildes are omitted in the following considerations for convenience.

Equations (20) are solved by

$$u(r) = \begin{cases} \frac{J_1(\kappa_2 r)}{J_1(\kappa_2 R)} & r \le R \quad \text{(core)} \\ \frac{K_1(\kappa_1 r)}{K_1(\kappa_1 R)} & r \ge R \quad \text{(cladding)} \end{cases}$$
(21)

where  $J_1(r)$  is the Bessel function of the first kind and  $K_1$  is the McDonald function.  $K_1(r)$  decreases exponentially if  $r \to \infty$ , thus u = u(r) vanishes far away from the core (cf. Fig. 2).



Figure 2: u = u(r) according to (21); R = 2;  $\varepsilon_1 = 1$ ;  $\varepsilon_2 = 4$ 

The continuity conditions of the electric and magnetic field have to be taken into account. They can be expressed as

$$u(R-0) = u(R+0), \qquad u'(R-0) = u'(R+0),$$
 (22)

where the first condition is already fulfilled by (21). The second condition is equivalent to the dispersion relation

$$\kappa_2 \frac{J_1'(\kappa_2 R)}{J_1(\kappa_2 R)} = \kappa_1 \frac{K_1'(\kappa_1 R)}{K_1(\kappa_1 R)}.$$
(23)

For later considerations, the linear dispersion function g is introduced:

$$g(\gamma, R) := \kappa_2 R J_1'(\kappa_2 R) K_1(\kappa_1 R) - \kappa_1 R K_1'(\kappa_1 R) J_1(\kappa_2 R)$$
(24a)  
$$= \kappa_2 R J_0(\kappa_2 R) K_1(\kappa_1 R) + \kappa_1 R K_0(\kappa_1 R) J_1(\kappa_2 R)$$
(24b)

Then (23) is equivalent to g = 0 (see [17], [15]). Equation (23) determines pairs  $\{R, \gamma\}$  (cf. Fig. 3).

Replacing the derivatives  $J'_1$  and  $K'_1$  in (23) by  $J_1$  and  $J_0$  resp.  $K_1$  and  $K_0$  and taking into account the asymptotic behavior

$$K_m(x) \sim \frac{(m-1)!}{2} \left(\frac{2}{x}\right)^m \qquad (x \to 0)$$
 (25a)

$$K_0(x) \sim -\ln(\frac{1}{2}\Gamma x)$$
  $(x \to 0)$  (25b)



Figure 3: Solution  $R = R(\gamma)$  of (23);  $\varepsilon_1 = 1; \varepsilon_2 = 4$ 

([16], (37-72 b) and (37-86), p. 715;  $C \approx 0,577$  (Euler constant),  $\Gamma := e^C$ ), (23) reads

$$\kappa_2 R \frac{J_0(\kappa_2 R)}{J_1(\kappa_2 R)} = (\kappa_1 R)^2 \ln \frac{\Gamma \kappa_1 R}{2}$$
(26)

if  $\kappa_1 R \ll 1$ . It is apparent from inspection of (26) that if  $\kappa_1 \to 0$  (cutoff limit), the cutoff radius must satisfy

$$J_0(\kappa R_{\nu}) = 0 \qquad \text{or} \qquad R_{\nu} = \frac{j_{0\nu}}{\kappa}, \qquad (27)$$

where  $\kappa$  is introduced according to (12) and  $j_{0\nu}$  denotes the  $\nu^{th}$  positive zero of  $J_0$ . If  $\nu = 1$ ,  $\varepsilon_1 = 1$ ,  $\varepsilon_2 = 4$ , then  $R_1 \approx 1.4$  (cf. Fig. 3).

The ratio of the power flow in the core to the power flow in the cladding is given by I(x, D) I(x, D)

$$\frac{P_{co}}{P_{cl}} := \frac{\int_{0}^{\pi} u^{2} r \, dr}{\int_{R}^{\infty} u^{2} r \, dr} = -\frac{1 - \frac{J_{0}(\kappa_{2}R) J_{2}(\kappa_{2}R)}{J_{1}^{2}(\kappa_{2}R)}}{1 - \frac{K_{0}(\kappa_{1}R) K_{2}(\kappa_{1}R)}{K_{1}^{2}(\kappa_{1}R)}}$$

and vanishes in the cutoff limit  $\kappa_1 R \to 0$  according to

$$\frac{P_{co}}{P_{cl}} \rightarrow -\frac{1}{1 + 2\ln\left(\frac{1}{2}\Gamma\kappa_1R\right)} \qquad (\kappa_1 R \ll 1)$$

where the asymptotic behavior (25b) and (25a) of the McDonald functions and  $J_0(\kappa_2 R) \rightarrow J_0(\kappa R_{\nu}) = 0$  has been taken into account. As a consequence, the ratio  $\eta_0$  of the power flow in the core to the total power flow vanishes according to

$$\eta_0 = \frac{P_{co}}{P_{co} + P_{cl}} \rightarrow -\frac{1}{2\ln\left(\frac{1}{2}\Gamma\kappa_1 R\right)} \qquad (\kappa_1 R \ll 1) .$$
 (28)

In 3.1.2, modifications of this behavior due to a photorefractive permittivity are discussed.

## 2 Planar waveguide with photorefractive permittivity

In the following, a permittivity according to

$$\varepsilon_{i} = \begin{cases} \varepsilon_{1} & x \leq -R & \text{(exterior)} \\ \varepsilon_{2} + \alpha f(u^{2}) & -R \leq x \leq R & \text{(core)} \\ \varepsilon_{1} & x \geq R & \text{(exterior)} \end{cases}$$
(29)

is considered, where  $\varepsilon_1$  and  $\varepsilon_2$  are independent on  $u^2$  and the spatial coordinates and f is a real function.

The electric field is determined by the wave equations

$$u'' + \kappa_2^2 u + \alpha B(u) = 0 \qquad (|x| \le R)$$
(30a)

$$u'' - \kappa_1^2 u = 0 \quad (|x| \ge R)$$
 (30b)

where

$$B(u) = u f(u^2) \tag{31}$$

denotes the inhomogeneous part of the linear differential equation.

Introducing the linear differential operator

$$L := \partial_x^2 + \kappa_2^2, \tag{32}$$

Eq. (30a) reads

$$Lu + \alpha B(u) = 0. (33)$$

It can be solved by means of a Green function G defined by

$$LG(s,x) = -\delta(s-x), \qquad -R < x < R \tag{34a}$$

$$G(R, x) = G(-R, x) = 0.$$
 (34b)

The Green function is given by

$$G(s,x) = \begin{cases} \frac{\sin[\kappa_2 (R+x)] \sin[\kappa_2 (R-s)]}{\kappa_2 \sin(2\kappa_2 R)} & s \ge x\\ \frac{\sin[\kappa_2 (R-x)] \sin[\kappa_2 (R+s)]}{\kappa_2 \sin(2\kappa_2 R)} & s \le x \end{cases}$$
(35)

Using the second Green's formula, the solution of (33) is determined by the integral equation (see A.2.1)

$$u(x,R) = \alpha \int_{-R}^{R} ds \ G(s,x) \cdot B(u(s)) + f(x)$$
 (36)

where, taking into account the Dirichlet boundary condition (34b),

$$f(x,R) := \int_{-R}^{R} ds \left( G(s,x)Lu(s) - u(s)LG(s,x) \right)$$
$$= \left[ G u' \right]_{-R}^{R} - \left[ u G' \right]_{-R}^{R}$$
$$= -u(R) \left. \frac{\partial}{\partial s}G(s,x) \right|_{s=R} + u(-R) \left. \frac{\partial}{\partial s}G(s,x) \right|_{s=-R}.$$
(37)

The continuity of u' at x = R,

$$u'(R) = -\kappa_1 , \qquad (38)$$

together with

$$u'(R) = \alpha \left. \frac{\partial}{\partial x} \int_{-R}^{R} ds \; G(s,x) \cdot B(u(s)) \right|_{x=R} + f'(R)$$

following from (36), leads to (see A.2.2)

$$g(\gamma, R) - \alpha F(\gamma, R; u) = 0$$
(39)

where

$$g := -\frac{\kappa_1}{\kappa_2} - \frac{1}{\kappa_2} f'(R),$$
 (40)

and

$$F(\gamma, R; u) := -\frac{1}{\kappa_2} \int_{-R}^{R} ds \frac{\sin[\kappa_2 (R+s)]}{\sin[2\kappa_2 R]} B(u(s)) .$$
 (41)

By iterating Eq. (36) according to

$$u_{n+1}(x) := \alpha \int_{-R}^{R} ds \ G(s,x) \cdot B(u_n(s)) + f(x), \quad n = 1, 2, \dots (42a)$$
  
$$u_1(x) = f(x)$$
(42b)

one obtains a sequence  $\{u_n(x; \gamma^{(n-1)}, R^{(n-1)})\}$  that converges uniformly to the exact solution of Eq. (36) (see 2.3). The associated dispersion relations read

$$g(\gamma^{(n)}, R^{(n)}) - \alpha F(\gamma^{(n)}, R^{(n)}; u_n) = 0.$$
(43)

By investigating the convergence of iteration (42) (see 2.3), it can be shown that solutions  $\gamma^{(n)}$  resp.  $R^{(n)}$  of (43) exist and that  $|\gamma^{(n)} - \gamma| \to 0$  and  $|R^{(n)} - R| \to 0$  as  $n \to \infty$  if  $|\alpha|$  is sufficient small.

Thus the sequence  $\gamma^{(n)}$  resp.  $R^{(n)}$  approximates the solution  $\gamma$  resp. R of the exact dispersion relation (39), leading to a proof of the existence of solutions  $\gamma$  resp. R of the exact relation (39).

By means of the approximate solutions  $\gamma^{(1)}$  and  $R^{(1)}$ , the second iteration function  $u_2(\gamma^{(1)}, R^{(1)}, x)$  and the ratio  $\eta$  of the power flow in the core will be determined (see below 2.1.1 and 2.1.4 resp. 2.2.1 and 2.2.2).

Since  $\gamma^{(1)}$  and  $R^{(1)}$  approximate the exact solution  $\gamma$  resp. R if  $\alpha$  is small, for obvious reasons it is useful to expand  $\gamma^{(1)}$ ,  $R^{(1)}$  and  $u_2(x)$  with respect to  $\alpha$  around  $\alpha = 0$ . With this results, the ratio  $\eta$  of power flow in the core to the total power flow can be investigated by means of an expansion  $\eta = \eta(\alpha)$  for fixed  $\gamma$  resp. R.

The linear approximation of  $\eta = \eta(\alpha)$  will be expressed by functions  $F_1$  and  $E_1$  (see below (45a) and (54a) for even modes and (72a) and (79a) for odd modes) which depend on the nonlinear (photorefractive) part B of the dielectric function.

If approximate relations are plotted, B(u) is assumed to model a photorefractive permittivity with external field oriented parallel to the optical *c*axis which is assumed to be parallel to the *x*-axis:

$$B(u) = -\frac{u}{1+tu^2}$$
(44)

Taking into account (4), ordinary polarized waves are considered.

Since all following considerations refer to  $\gamma^{(1)}$  and  $R^{(1)}$ , the superscript  $^{(1)}$  is omitted for convenience.

### 2.1 Even modes

Even modes are defined by symmetric functions u with u(R) = u(-R) = 1.

Evaluation of Eq. (37) yields (see A.2.1)

$$f(x) = u_1(x) = \frac{\cos \kappa_2 x}{\cos \kappa_2 R}, \qquad (37a)$$

thus (40) and (8) are identical.

Taking into account the symmetry of u, function F according to (41) is given by

$$F(u) = -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \, \frac{\cos s}{\cos(\kappa_2 R)} B(u(s)) \tag{41a}$$

(see A.2.2).

After inserting the start function  $u_1$  according to (37) and function B according to (44) in (41a) (see A.2.3), it is convenient to introduce a function

$$F_1(\kappa_2 x, \kappa_2 R) := -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \ u_1(s) \ B(u_1(s))$$
(45a)

$$= \frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \; \frac{\left(\frac{\cos s}{\cos \kappa_2 R}\right)^2}{1 + t \left(\frac{\cos s}{\cos \kappa_2 R}\right)^2} \tag{45b}$$

$$= \frac{1}{\kappa_2^2} \frac{1}{t} \left( \kappa_2 x - \sqrt{P_0} \arctan[\sqrt{P_0} \tan(\kappa_2 x)] \right) \quad (45c)$$

where

$$P_0 := \frac{\cos^2(\kappa_2 R)}{\cos^2(\kappa_2 R) + t} = \frac{1}{1 + t \, u_1^2(0)}.$$

Inserting (45) and (8) into (43), the first approximation of the dispersion relation reads

$$\tan(\kappa_2 R) - \frac{\kappa_1}{\kappa_2} - \alpha F_1(\kappa_2 R) = 0 \tag{46}$$

where  $F_1(\kappa_2 R) := F_1(\kappa_2 R, \kappa_2 R)$ .

Equation (46) determines triples  $\{R, \gamma, \alpha\}$  represented in Fig. 4. If  $\gamma$  is fixed, then positive  $\alpha$  increases, whereas negative  $\alpha$  decreases the radius  $R = R(\alpha)$ . In contrast, if R is fixed, positive  $\alpha$  decreases  $\gamma = \gamma(\alpha)$ .



Figure 4: Solution  $R = R(\gamma)$  of (46);  $t = 0.1, \alpha = 0, \pm 0.1; \varepsilon_1 = 1; \varepsilon_2 = 4$ 

In the following, solutions of the dispersion relation (46) are analyzed for  $|\alpha| \ll 1$  and fixed  $\gamma$  resp. R by calculating a series of  $R = R(\alpha)$  resp. of  $\gamma = \gamma(\alpha)$  with respect to  $\alpha$  where terms of second order  $\alpha$  are neglected. In 2.1.1, this linear relationships will be used to determine an expansion of function  $u_2(x)$  with respect to  $\alpha$ .

Let  $\gamma$  be fixed and  $|\alpha| \ll 1$ . To determine the dependence of R on  $\alpha$ 

$$R(\alpha) = R_{0\nu} + R_{1\nu}\alpha + \mathcal{O}(\alpha^2)$$

where  $\nu = 0, 2, 4, \ldots$ , function g is expanded as

$$g(R(\alpha)) = g(R_{0\nu}) + \alpha R_{1\nu} \left. \frac{\partial g}{\partial R} \right|_{R=R_{0\nu}} + \mathcal{O}(\alpha^2)$$

and inserted into Eq. (46). After comparing equal powers of  $\alpha$  one obtains

$$\kappa_2 R_{0\nu} = \nu \frac{\pi}{2} + \arctan \frac{\kappa_1}{\kappa_2} \tag{47}$$

and

$$\kappa_2 R_{1\nu} = \frac{F_1(\kappa_2 R_{0\nu})}{\frac{1}{\kappa_2} \frac{\partial g}{\partial R}\Big|_{R=R_{0\nu}}}$$
(48a)

$$= \cos^{2}(\kappa_{2}R_{0\nu}) F_{1}(\kappa_{2}R_{0\nu})$$
(48b)

$$= \frac{1}{t\kappa^2} \left( \nu \frac{\pi}{2} + \arctan \frac{\kappa_1}{\kappa_2} - \sqrt{P_0} \arctan[\sqrt{P_0} \frac{\kappa_1}{\kappa_2}] \right) \quad (48c)$$

where

$$P_0 = P_0(\kappa_2 R_{0\nu}) := \frac{\cos^2(\kappa_2 R_{0\nu})}{\cos^2(\kappa_2 R_{0\nu}) + t} = \frac{1}{1 + t[1 + \frac{\kappa_1^2}{\kappa_2^2}]}.$$

Since  $F_1$  is positive (see Eq. (45 b)),  $R_{1\nu}$  is positive, too, for arbitrary even mode number  $\nu$ .

Let R be fixed and  $|\alpha| \ll 1$ .

The lowest order term  $\gamma_0$  of the expansion

$$\gamma(\alpha) = \gamma_0 + \gamma_1 \alpha + \mathcal{O}(\alpha^2)$$

is determined by  $g(\gamma_0) = 0$  or

$$\tan \overline{\kappa_2}R = \frac{\overline{\kappa_1}}{\overline{\kappa_2}} \tag{49}$$

where

$$\overline{\kappa_1} := \sqrt{\gamma_0^2 - \varepsilon_1}, \qquad (50a)$$

$$\overline{\kappa_2} := \sqrt{\varepsilon_2 - \gamma_0^2}. \tag{50b}$$

Obviously, it is not possible to represent  $\gamma_0$  explicitly. The first order term  $\gamma_1$  is given by

$$\gamma_1 = \frac{F_1(\gamma_0)}{\frac{\partial g}{\partial \gamma}\Big|_{\gamma=\gamma_0}} = -\overline{\kappa_1 \kappa_2} \frac{\cos^2(\overline{\kappa_2}R) F_1(\overline{\kappa_2}R)}{1+\overline{\kappa_1}R} \frac{1}{\gamma_0}.$$
 (51)

Due to the dependence on  $\gamma_0$  it cannot be represented explicitly.

To investigate the cutoff limit, let  $\kappa_1/\kappa \ll 1$  and  $|\alpha| \ll 1$ .

Neglecting all terms of second order, the cutoff radius is given by

$$\kappa R_{\nu} = \nu \cdot \frac{\pi}{2} \left( 1 + \alpha \frac{1}{t\kappa^2} \right) + \frac{\kappa_1}{\kappa} + \mathcal{O}(2) \qquad (\nu = 0, 2, 4, \ldots) \,. \tag{52}$$

Comparing with (11), the change due to photorefractive permittivity depends very sensitive on t if t < 1.

Fig. 4 shows that in the cutoff limit, the radius of the fundamental mode does not depend on  $\alpha$ . This special property of the fundamental mode can also be derived from (52) with  $\nu = 0$ .

### **2.1.1** The second iteration function $u_2(x)$

Starting from (42a), the second iteration function is given by (see A.2.7)

$$u_{2}(x) = \frac{\cos \kappa_{2} x}{\cos \kappa_{2} R} + \alpha \left[ -\cos(\kappa_{2} x) \sin(\kappa_{2} R) F_{1}(\kappa_{2} R) + \cos(\kappa_{2} R) \sin(\kappa_{2} x) F_{1}(\kappa_{2} R, \kappa_{2} x) \right] + \alpha \cos(\kappa_{2} x) \cos(\kappa_{2} R) \left[ E_{1}(\kappa_{2} R) - E_{1}(\kappa_{2} R, \kappa_{2} x) \right] + \mathcal{O}(\alpha^{2})$$
(53)

where function  $F_1$  is given by (45) and  $E_1$  is defined by

$$E_1(\kappa_2 x, \kappa_2 R) := \frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \ u_1'(s) \ B(u_1(s))$$
(54a)

$$= \frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \; \frac{\sin s}{\cos(\kappa_2 R)} \frac{\frac{\cos s}{\cos(\kappa_2 R)}}{1 + t \left(\frac{\cos s}{\cos(\kappa_2 R)}\right)^2} \quad (54b)$$

$$= \frac{1}{2} \frac{1}{\kappa_2^2} \frac{1}{t} \ln \left[ \frac{1 + t \frac{1}{\cos^2(\kappa_2 R)}}{1 + t \frac{\cos^2(\kappa_2 x)}{\cos^2(\kappa_2 R)}} \right]$$
(54c)

with  $E_1(\kappa_2 R) := E_1(\kappa_2 R, \kappa_2 R)$ . Obviously,  $E_1$  is positive.

Fig. 5 compares the field solution  $u_2 = u_2(x)$  according to (53) with the numerical solution of (30a) where  $\alpha = +0.1$ . The agreement is satisfactory.

Let  $\gamma$  be fixed and  $R = R_{0\nu} + \alpha R_{1\nu} + \mathcal{O}(\alpha^2)$ . The second iterate solution  $u_2(x)$  can be expanded as

$$u_2(x,\kappa_2 R) = \frac{\cos \kappa_2 x}{\cos \kappa_2 R_{0\nu}} + \alpha H(\kappa_2 x,\kappa_2 R_{0\nu}) + \mathcal{O}(\alpha^2)$$
(55a)



Figure 5:  $u_2 = u_2(x)$  according to (53) (solid line) and numerical solution of (30a) (dashed line); t = 0.1,  $\varepsilon_1 = 1$ ;  $\varepsilon_2 = 4$ ;  $\alpha = +0.1$ ;  $\gamma = 1.669...$ 

where

$$H(\kappa_{2}x, \kappa_{2}R_{0\nu}) := \cos(\kappa_{2}R_{0\nu}) \left( \sin(\kappa_{2}x) F_{1}(\kappa_{2}x, \kappa_{2}R_{0\nu}) + \cos(\kappa_{2}x) \left[ E_{1}(\kappa_{2}R_{0\nu}) - E_{1}(\kappa_{2}x, \kappa_{2}R_{0\nu}) \right] \right).$$
(55b)

To investigate the sign of H, first it can be stated that  $F_1(\kappa_2 x, \kappa_2 R) \ge 0$ according to (45b). Furthermore, according to (54c)

$$E_1(\kappa_2 R_{0\nu}) - E_1(\kappa_2 x, \kappa_2 R_{0\nu}) = \frac{1}{\kappa_2^2} \frac{1}{2t} \ln\left(\frac{1 + t\cos^2(\kappa_2 x)/\cos^2(\kappa_2 R_{0\nu})}{1 + t}\right) \ge 0.$$

If the fundamental mode  $\nu = 0$  is considered, than  $\kappa_2 x$  is restricted to the interval  $0 \le \kappa_2 x \le \kappa_2 R_{0\nu} < \pi/2$ , thus  $\sin \kappa_2 x > 0$  and  $\cos \kappa_2 x > 0$ . Finally, taking into account  $\cos(\kappa_2 R_{0\nu}) = (-1)^{\nu/2} \kappa_2/\kappa$ , one obtains

$$H(\kappa_2 x, \kappa_2 R_{0\nu}) \ge 0$$
  $(\nu = 0).$  (56)

Therefore if  $\gamma$  is fixed, then positive  $\alpha$  increases both the amplitude and the half width of  $u_2(x)$  (see Fig. 6).



Figure 6:  $u_2 = u_2(x)$  according to (53); t = 0.1,  $\alpha = 0, \pm 0.1$ ;  $\gamma = 1.669...$ ,  $\varepsilon_1 = 1$ ;  $\varepsilon_2 = 4$ 

Let R fixed and  $\gamma = \gamma_0 + \alpha \gamma_1 + \mathcal{O}(\alpha^2)$ . Taking into account

$$\kappa_2 - \overline{\kappa_2} = -\frac{\gamma_0 \gamma_1}{\overline{\kappa_2}} \alpha + \mathcal{O}(\alpha^2),$$
(57a)

$$\kappa_1 - \overline{\kappa_1} = \frac{\gamma_0 \gamma_1}{\overline{\kappa_1}} \alpha + \mathcal{O}(\alpha^2),$$
(57b)

the expansion of  $u_2(x)$  with respect to  $\alpha$  around  $\alpha = 0$  is given by

$$u_2(\kappa_2 x) = \frac{\cos \overline{\kappa_2} x}{\cos \overline{\kappa_2} R} + \alpha H(\overline{\kappa_2} x, \overline{\kappa_2} R) + \mathcal{O}(\alpha^2)$$
(58a)

where

$$H(\overline{\kappa_2}x,\overline{\kappa_2}R) := \cos \overline{\kappa_2}x \cos \overline{\kappa_2}R \left( T(\overline{\kappa_2}R) - T(\overline{\kappa_2}x,\overline{\kappa_2}R) \right) + \sin \overline{\kappa_2}x \sin \overline{\kappa_2}R \frac{\overline{\kappa_2}R F_1(\overline{\kappa_2}x,\overline{\kappa_2}R) - \overline{\kappa_2}x F_1(\overline{\kappa_2}R)}{1 + \overline{\kappa_1}R}$$
(58b)

and

$$T(\overline{\kappa_2}x, \overline{\kappa_2}R) := E_1(\overline{\kappa_2}x, \overline{\kappa_2}R) - \frac{\tan\overline{\kappa_2}x F_1(\overline{\kappa_2}x, \overline{\kappa_2}R)}{1 + \overline{\kappa_1}R}.$$
 (58c)

If  $\varepsilon_1 = 1, \varepsilon_2 = 4$  and R = 0.8, H is negative, thus the amplitude is decreasing with increasing  $\alpha$  (cf. Fig. 7).

Functions  $u_2(\alpha = 0) = u_1$  in Fig. 6 and 7 refer to  $\gamma = 1.669...$  resp. R = 0.8 and are identical since  $\{R, \gamma\} = \{0.8, 1.669...\}$  is a solution of the linear



Figure 7:  $u_2 = u_2(x)$  according to (53); t = 0.1,  $\alpha = 0$ ,  $\pm 0.1$ ; R = 0.8,  $\varepsilon_1 = 1$ ;  $\varepsilon_2 = 4$ 

dispersion relation (8) (cf. Fig. 1).  $u_2$  depends more sensitive on  $\alpha$  if  $\gamma$  is fixed.

#### **2.1.2** The amplitude $u_2(0)$

A special feature of even modes is the amplitude u(0). The linear approximation of the second iteration's amplitude  $u_2(0)$  is obtained by straightforward evaluation of (55a) and (58a), leading to the following formulas (59) and (61).

On the other hand, with the help of (30a) and the boundary conditions (64a) and (64b), in 2.1.3 an implicit equation that determines the exact amplitude  $u_0 := u(0)$  is derived. Next, this implicit equation can be approximated for fixed  $\gamma$  resp. R in order to find an expansion of  $u_0$  with respect to  $\alpha$  around  $\alpha = 0$ . Neglecting terms of second order, the result is identical to (59) resp. (61).

Let  $\gamma$  fixed and  $|\alpha| \ll 1$ .

It is apparent from inspection of (55a) and (55b) that

$$u_2(0) = \frac{1}{\cos(\kappa_2 R_{0\nu})} \left( 1 + \alpha \, \cos^2(\kappa_2 R_{0\nu}) \, E_1(\kappa_2 R_{0\nu}) \right) + \mathcal{O}(\alpha^2) \quad (59a)$$

$$= (-1)^{\nu/2} \frac{\kappa}{\kappa_2} \left( 1 + \frac{\alpha}{2t\kappa^2} \ln\left[1 + \frac{t}{1+t}\frac{\kappa_1^2}{\kappa_2^2}\right] \right) + \mathcal{O}(\alpha^2).$$
(59b)

The even mode number  $\nu$  determines the sign of the amplitude, and the square of the amplitude does not depend on  $\nu$ :

$$u_2^2(0) = \frac{1}{\cos^2(\kappa_2 R_{0\nu})} + 2\alpha \ E_1(\kappa_2 R_{0\nu}) + \mathcal{O}(\alpha^2)$$
(60a)

$$= 1 + \frac{\kappa_1^2}{\kappa_2^2} + \frac{\alpha}{t\kappa_2^2} \ln\left[1 + \frac{t}{1+t}\frac{\kappa_1^2}{\kappa_2^2}\right] + \mathcal{O}(\alpha^2)$$
(60b)

The amplitude increases with increasing  $\alpha$  in accordance with Fig. 6.

In the cutoff limit  $\kappa_1 \to 0$ ,  $E_1(\kappa_2 R_{0\nu})$  vanishes and therefore in this limit, the amplitude is independent on  $\alpha$ .

In contrast, according to (60b), in the limit  $\kappa_2 \to 0$  the change of the amplitude due to  $\alpha$  is remarkable.

Let R be fixed and  $|\alpha| \ll 1$ . If x = 0, then  $T(0, \kappa_2 R) = 0$  and (58a) and (58b) imply

$$u_2(0) = \frac{1}{\cos(\overline{\kappa_2}R)} + \alpha \cos(\overline{\kappa_2}R) T(\overline{\kappa_2}R).$$
(61)

The sign of  $T(\overline{\kappa_2}R)$  depends on  $\overline{\kappa_2} = \sqrt{\varepsilon_2 - \gamma_0^2}$ . Numerical investigation shows that T < 0 if  $\gamma_0 < 1.9$ . Then positive  $\alpha$  produces a decrease of the amplitude  $u_2(0)$ , whereas negative  $\alpha$  increases the amplitude (see Fig. 7). This dependence of the amplitude on  $\alpha$  referring to fixed R is in strong contrast to the situation for fixed  $\gamma$ .

In the cutoff limit  $\kappa_1 \to 0$  where  $\kappa_1/\kappa_2 = \mathcal{O}(\delta) = \kappa_1/\kappa$ , T is given by

$$T = -\frac{1}{2} \frac{1}{\kappa^2} \frac{1}{1+t} \delta^2 + \mathcal{O}(\delta^4)$$
 (62)

and turns out to be negative.

Since T is vanishing if  $\delta \to 0$ , in agreement to the situation for fixed  $\gamma$ , the amplitude is independent on  $\alpha$  in the cutoff limit.

#### 2.1.3 Exact amplitude

Multiplying (30a) with 2u' and integrating with respect to x yields

$$\left[ (u')^2 + \kappa_2^2 u^2 \right]_0^x + 2\alpha \int_0^x ds \ u'(s) B(u(s)) = 0.$$
 (63)

Taking into account the boundary conditions

$$u(0) = u_0, \qquad u(\kappa_2 R) = 1$$
 (64a)

and

$$u'(0) = 0, \qquad u'(\kappa_2 R) = -\kappa_1$$
 (64b)

and inserting function B according to (44), one obtains (see A.2.6)

$$u_0^2 = 1 + \frac{\kappa_1^2}{\kappa_2^2} + 2\frac{\alpha}{\kappa_2^2} \int_0^{\kappa_2 R} ds \ u'(s) \ B(u(s))$$
(65a)

$$= 1 + \frac{\kappa_1^2}{\kappa_2^2} + \frac{\alpha}{t\kappa_2^2} \ln \frac{1+t\,u_0^2}{1+t}.$$
 (65b)

A comparison of  $u_0(\alpha)$ , generated on one hand by a contour plot of (65b) (solid line), and on the other hand determined by  $u_2(0)$  according to (53) where  $\gamma$  is fixed (pointed line), is shown in Fig. 8. The agreement is satisfactory.

Let  $\gamma$  be fixed and  $|\alpha| \ll 1$ . Approximating R by  $R_{\nu 0}$  and u by  $u_1 = \cos(\kappa_2 x) / \cos(\kappa_2 R_{\nu 0})$  in (65a), the linear approximation of the amplitude given by (60a) and (60b) is reproduced.

Let R be fixed and  $|\alpha| \ll 1$ .

Then  $\gamma$  depends on  $\alpha$  according to  $\gamma = \gamma_0 + \alpha \gamma_1 + \mathcal{O}(\alpha^2)$ . In contrast to the situation of fixed  $\gamma$ , the term  $1 + \kappa_1^2/\kappa_2^2$  in (65a) has to be considered as a function of  $\alpha$  using (57a), (57b) and (51).

Taking into account only terms of zero and first order in  $\alpha$ , (61) is reproduced.

### **2.1.4** Ratio $\eta$ of power flow in the core

The ratio  $\eta$  of power flow through the core is given by

$$\eta := \frac{p_{int}}{p_{int} + p_{ext}} \tag{66}$$



Figure 8:  $u_0 = u_0(\alpha)$  according to (65b) (solid line) and (53) (pointed line);  $t = 0.1, \gamma = 1.5, \epsilon_1 = 1; \epsilon_2 = 4$ 

where

$$\begin{array}{lll} p_{int} & := & \int_0^R dx \; u_2^2(x) & (\text{even modes}) \,, \qquad p_{int} \; := \; \int_0^R dx \; v_2^2(x) & (\text{odd modes}) \,, \\ p_{ext} & := & \int_R^\infty dx \; u_2^2(x) \; = \; \frac{1}{2\kappa_1} \,. \end{array}$$

Let  $\alpha$  be fixed.

Fig. 9 shows that  $\eta = \eta(\gamma)$  increases linearly if  $\kappa_1/\kappa \ll 1$ . Furthermore,  $\eta$  increases with positive  $\alpha$ , whereas negative  $\alpha$  diminishes the ratio.

Let  $\gamma$  be fixed and  $R = R_{0\nu} + \alpha R_{1\nu} + \mathcal{O}(\alpha^2)$ . Using (55a), the expansion of  $p_{int}$  with respect to  $\alpha$  is given by

$$p_{int} = p_{0\nu} + \alpha p_{1\nu} + \alpha R_{1\nu} + \mathcal{O}(\alpha^2)$$



Figure 9:  $\eta = \eta(\gamma)$  according to (66);  $\alpha = 0, \pm 0.5, t = 0.1, \varepsilon_1 = 1; \varepsilon_2 = 4$ 

where

$$p_{0\nu} := \int_0^{R_{0\nu}} \left( \frac{\cos \kappa_2 x}{\cos \kappa_2 R_{0\nu}} \right)^2 dx , \qquad (67a)$$

$$p_{1\nu} := 2 \int_0^{R_{0\nu}} \frac{\cos \kappa_2 x}{\cos \kappa_2 R_{0\nu}} H(\kappa_2 x, \kappa_2 R_{0\nu}) dx .$$
 (67b)

Since  $R_{1\nu} \ge 0$  according to (48b), together with (56) one obtains

$$p_{1\nu} + R_{1\nu} \ge 0$$
 if  $\nu = 0$ . (68)

The ratio of power flow through the core is given by

$$\eta(\alpha) = \eta_{0\nu} + \alpha \eta_{0\nu} (1 - \eta_{0\nu}) \frac{p_{1\nu} + R_{1\nu}}{p_{0\nu}} + \mathcal{O}(\alpha^2) , \qquad (69)$$

where

$$\eta_{0\nu} := \frac{2p_{0\nu}}{2p_{0\nu} + \frac{1}{\kappa_1}}$$

is given by (13).

Since  $0 \le \eta_0 \le 1$  and  $p_{1\nu} + R_{1\nu} \ge 0$  if  $\nu = 0$  (fundamental mode), positive  $\alpha$  produces an increase of  $\eta(\alpha)$  in the core.

The agreement of  $\eta$  according to (66) (dotted line in Fig. 10;  $\alpha_i = -0.1 + i \cdot 0.01$ ,  $i = 0, \ldots, 20$ ) with the linear approximation according to (69) (solid



Figure 10:  $\eta = \eta(\alpha)$  according to (66) (dotted line) and (69) (solid line);  $\gamma = 1.5, t = 0.1, \epsilon_1 = 1; \epsilon_2 = 4$ 

line in Fig. 10) is satisfactory if  $|\alpha| < 0.1$  (cf. Fig. 10).

Let R be fixed and  $|\alpha| \ll 1$ .

By means of (58a), the normalized power flow through the core is determined by

$$p_{int} = p_0 + \alpha p_1 + \mathcal{O}(\alpha^2)$$

where

$$p_0 := \int_0^R \frac{\cos^2 \overline{\kappa_2} x}{\cos^2 \overline{\kappa_2} R} \, dx \tag{70a}$$

$$p_1 := 2 \int_0^R \frac{\cos(\overline{\kappa_2}x)}{\cos(\overline{\kappa_2}R)} H(\overline{\kappa_2}x, \overline{\kappa_2}R) dx.$$
 (70b)

Finally, the ratio of power flow in the core in relation to the total power flow is given by

$$\eta(\alpha) = \eta_0 + \alpha \eta_0 (1 - \eta_0) \left[ \frac{p_1}{p_0} + \frac{\gamma_0 \gamma_1}{\overline{\kappa_1}^2} \right] + \mathcal{O}(\alpha^2).$$
(71)

Numerical investigation shows that  $p_1$  is negative if  $\varepsilon_1 = 1, \varepsilon_2 = 4$  and  $\gamma_0 < 1.7$ . Since  $\gamma_1$  according to (51) is negative, too, the power ratio  $\eta$  decreases if  $\alpha$  is positive (see Fig. 11).



Figure 11:  $\eta = \eta(\alpha)$  according to (66) (dotted line) and (71) (solid line);  $R := 0.5304..., t = 0.1, \ \varepsilon_1 = 1; \varepsilon_2 = 4$ 

Similar to Fig. 10, the agreement of  $\eta$  according to (66) (dotted line in Fig. 11) with the linear approximation according to (71) (solid line in Fig. 11) is satisfactory if  $|\alpha| < 0.1$  (cf. Fig. 11).

The selected pair  $\{R, \gamma\} = \{0.5304..., 1.5\}$  (cf. Fig. 10 resp. 11) is a solution of the linear dispersion relation (8) (cf. Fig. 1).

To summarize, in order to estimate the amplitude resp. the ratio  $\eta$  of power flow as a function of  $\alpha$  if  $|\alpha| \ll 1$  and if a photorefractive permittivity with external electric field is considered, due to the boundary conditions of the waveguide two cases have to be distinguished:

If the propagation constant  $k\gamma$  is fixed and  $\nu = 0$  (fundamental mode), it is shown analytically that an increasing  $\alpha$  leads to a rise of the amplitude as well as the ratio  $\eta$ . However, if R is fixed, supported by numerical investigations for sufficiently small  $\overline{\kappa_1}$  it is shown that an increasing  $\alpha$  leads to a decrease of the amplitude and ratio  $\eta$ .

Experimental results [9] indicate that the assumption of fixed  $\gamma$  may be crucial for a correct description of spatial solitons in a planar waveguide.

### 2.2 Odd modes

Odd modes are defined by odd functions v.

First, evaluation of (37) yields

$$f(x) = v_1(x) = \frac{\sin \kappa_2 x}{\sin \kappa_2 R}.$$
 (37b)

(41) can be simplified by using v(-x) = -v(x) and is given by

$$F(v) = -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \, \frac{\sin(\kappa_2 s)}{\sin(\kappa_2 R)} B(v(s)) \,. \tag{41b}$$

Substitution of (37b) and function B according to (44) into (41b) leads to

$$F_1 := -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \, v_1(s) \, B(v_1)$$
(72a)

$$= \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \quad \frac{v_1^2(s)}{1 + tv_1^2(s)}$$
(72b)

$$= \frac{1}{\kappa_2^2} \frac{1}{t} \left( \kappa_2 R - \frac{1}{\sqrt{P_0}} \arctan[\sqrt{P_0} \tan(\kappa_2 R)] \right)$$
(72c)

where

$$P_0 := \frac{\sin^2(\kappa_2 R) + t}{\sin^2(\kappa_2 R)}.$$

Using g according to (16) and  $F_1$  according to (72c), the dispersion relation can be written as

$$g - \alpha F_1(v_1) = 0.$$
 (73)

(73) determines triples  $\{R, \gamma, \alpha\}$ .

Let  $\gamma$  be fixed and  $|\alpha| \ll 1$ . Following the lines of 2.1 leads to

$$\kappa_2 R_{0\nu} = \nu \frac{\pi}{2} + \frac{\pi}{2} - \arctan \frac{\kappa_2}{\kappa_1}$$
(74)

and

$$\kappa_2 R_{1\nu} = \frac{F_1(R_{0\nu})}{\left.\frac{1}{\kappa_2} \frac{\partial g}{\partial R}\right|_{R=R_{0\nu}}}$$
(75a)

$$= \sin^{2}(\kappa_{2}R_{0\nu}) F_{1}(\kappa_{2}R_{0\nu})$$
(75b)  
$$= \frac{1}{t\kappa^{2}} \left(\nu\frac{\pi}{2} + \frac{\pi}{2} - \arctan\frac{\kappa_{2}}{\kappa_{1}} + \frac{1}{\sqrt{P_{0}}} \arctan[\sqrt{P_{0}}\frac{\kappa_{2}}{\kappa_{1}}]\right)$$
(75c)

where

$$P_0 := (1+t)\frac{\kappa^2}{\kappa_2^2} + t\frac{\kappa_1^2}{\kappa_2^2}.$$

As for even modes, since  $R_{1\nu} \ge 0$ , positive  $\alpha$  increases, whereas negative  $\alpha$  decreases the radius  $R = R(\alpha)$ .

Let R be fixed and  $|\alpha| \ll 1$ .

Similar to 2.1, the coefficient  $\gamma_0$  of lowest order is determined by  $g(\gamma_0) = 0$ , and  $\gamma_1$  is given by

$$\gamma_1 = \frac{F_1(\gamma_0)}{\frac{\partial g}{\partial \gamma}\Big|_{\gamma=\gamma_0}} = -\overline{\kappa_1 \kappa_2} \frac{\sin^2(\overline{\kappa_2}R) F_1(\overline{\kappa_2}R)}{1+\overline{\kappa_1}R} \frac{1}{\gamma_0}$$
(76)

where  $\overline{\kappa_1}$  and  $\overline{\kappa_2}$  are defined by (50a) and (50b).

As for even modes,  $\gamma_0$  and  $\gamma_1$  can't be represented explicitly.  $\gamma_1$  again is negative since  $F_1$  according to (72b) is positive.

Let  $\kappa_1/\kappa \ll 1$  (cutoff limit) and  $|\alpha| \ll 1$ .

Neglecting all terms of second order, the cutoff radius is given by

$$\kappa R_{\nu} = \nu \cdot \frac{\pi}{2} \left( 1 + \alpha \frac{1}{t\kappa^2} \right) + \alpha \frac{1}{t\kappa^2} \frac{1}{\sqrt{1+t}} \frac{\pi}{2} + \frac{\kappa_1}{\kappa} + \mathcal{O}(2) \quad (\nu = 1, 3, 5, \ldots) .$$
(77)

#### **2.2.1** The second iteration function $v_2(x)$

The second iteration function  $v_2$  follows from (42) and is written as

$$v_{2}(x) - \frac{\sin(\kappa_{2}x)}{\sin(\kappa_{2}R)}$$

$$= \alpha \left[ \sin(\kappa_{2}x) \cos(\kappa_{2}R) F_{1}(\kappa_{2}R) - \sin(\kappa_{2}R) \cos(\kappa_{2}x) F_{1}(\kappa_{2}x, \kappa_{2}R) \right]$$

$$+ \alpha \sin(\kappa_{2}x) \sin(\kappa_{2}R) \left[ E_{1}(\kappa_{2}R) - E_{1}(\kappa_{2}x, \kappa_{2}R) \right] + \mathcal{O}(\alpha^{2})$$
(78)

where

$$E_1(\kappa_2 x, \kappa_2 R) := \frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \, v_1'(s) \, B(v_1(s))$$
(79a)

$$= -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \quad \frac{\cos s}{\sin(\kappa_2 R)} \frac{\frac{\sin s}{\sin(\kappa_2 R)}}{1 + t \left(\frac{\sin s}{\sin(\kappa_2 R)}\right)^2} \quad (79b)$$

$$= -\frac{1}{\kappa_2^2} \frac{1}{2t} \ln\left(1 + t \frac{\sin^2(\kappa_2 x)}{\sin^2(\kappa_2 R)}\right).$$
 (79c)

A special property of odd functions v is the slope v'(0). By integration of (30a) and application of the boundary conditions

 $v(0) = 0, \qquad v(\kappa_2 R) = 1, \qquad v'(\kappa_2 R) = -\kappa_1$  (80)

one obtains without approximation

$$(v'(0))^2 = \kappa^2 + 2\alpha \int_0^{\kappa_2 R} ds \, v'(s) \, B(v(s))$$
(81a)

$$= \kappa^2 - \frac{\alpha}{t} \ln(1+t) , \qquad (81b)$$

which neither depends on R nor on  $\gamma$ .

The expansion of  $v'_2(0)$  with respect to  $\alpha$  around  $\alpha = 0$  for fixed  $\gamma$  resp. fixed R reproduces (81b) if terms of second order are neglected.

In Fig. 12,  $v_2(x)$  according to (78) is compared with a numerical solution of (30a) where  $\alpha = +0.01$ . The agreement is satisfactory.

Let  $\gamma$  be fixed and  $|\alpha| \ll 1$ . Then

$$v_2(x) = \frac{\sin(\kappa_2 x)}{\sin(\kappa_2 R_{0\nu})} + \alpha H(\kappa_2 x, \kappa_2 R) + \mathcal{O}(\alpha^2)$$
(82)



Figure 12:  $v_2 = v_2(x)$  according to (78) (solid line) and numerical solution of (30a) (dashed line); t = 0.1,  $\varepsilon_1 = 1$ ;  $\varepsilon_2 = 4$ ;  $\alpha = +0.01$ ;  $\gamma = 1.5$ 

where

$$H(\kappa_{2}x,\kappa_{2}R_{0\nu}) := \\ \sin(\kappa_{2}R_{0\nu}) \left( -\cos(\kappa_{2}x) F_{1}(\kappa_{2}x,\kappa_{2}R_{0\nu}) + \sin(\kappa_{2}x) \left[ E_{1}(\kappa_{2}R_{0\nu}) - E_{1}(\kappa_{2}x,\kappa_{2}R_{0\nu}) \right] \right).$$
(83)

If  $\gamma = 1.5$ , the dependence of  $v_2(x)$  on  $\alpha$  is shown in Fig. 13 where  $x \in [-R, R]$ .

Let R be fixed and  $|\alpha| \ll 1$ . Then

$$v_2(x) = \frac{\sin(\overline{\kappa_2}x)}{\sin(\overline{\kappa_2}R)} + \alpha H(\overline{\kappa_2}x, \overline{\kappa_2}R) + \mathcal{O}(\alpha^2)$$
(84)

where

$$H(\overline{\kappa_2}x,\overline{\kappa_2}R) := \sin(\overline{\kappa_2}x)\sin(\overline{\kappa_2}R)\left(T(\overline{\kappa_2}R) - T(\overline{\kappa_2}x) + \left[E_1(\overline{\kappa_2}R) - E_1(\overline{\kappa_2}x)\right]\right) + \cos(\overline{\kappa_2}R)\cos(\overline{\kappa_2}x)\left(\overline{\kappa_2}R \frac{F_1(\overline{\kappa_2}x)}{1 + \overline{\kappa_1}R} - \overline{\kappa_2}x\frac{F_1(\overline{\kappa_2}R)}{1 + \overline{\kappa_1}R}\right)$$
(85a)

with

$$T(\overline{\kappa_2}x, \overline{\kappa_2}R) := E_1(\overline{\kappa_2}x, \overline{\kappa_2}R) + \frac{\cot(\overline{\kappa_2}x)F_1(\overline{\kappa_2}x, \overline{\kappa_2}R)}{1+\overline{\kappa_1}R} .$$
(85b)



Figure 13:  $v_2 = v_2(x)$  according to (78);  $x \in [-R, R]$ ;  $t = 0.1, \alpha = 0, \pm 0.01$ ;  $\gamma = 1.5, \varepsilon_1 = 1; \varepsilon_2 = 4; \nu = 1$ 

If R = 1.7178..., the dependence of  $v_2(x)$  on  $\alpha$  is shown in Fig. 14 where  $x \in [-R, R]$ .



Figure 14:  $v_2 = v_2(x)$  according to (78); t = 0.1,  $\alpha = 0, \pm 0.01$ ; R = 1.7178,  $\varepsilon_1 = 1$ ;  $\varepsilon_2 = 4$ ;  $\nu = 1$ 

 $\gamma = 1.5$  and R = 1.7178... are related by the linear dispersion relation (16), thus the graphs of functions  $v_2(x, \alpha = 0)$  in Fig. 13 and 14 coincide.

In Fig. 13 and 14, though v'(0) depends on  $\alpha$  according to (81b), all curves are indistinguishable in the vicinity of the origin due to the smallness of  $\alpha$ .

In contrast to Fig. 6 resp. Fig. 7 where  $\alpha = \pm 0.1$  and  $\nu = 0$ , in Fig. 13

and 14  $\alpha$  has been chosen to be much smaller: Investigating higher modes numerically, it is necessary to choose  $\alpha$  sufficiently small in order to satisfy the condition of convergence.

### **2.2.2** Ratio $\eta$ of power flow in the core

Let  $\gamma$  be fixed and  $|\alpha| \ll 1$ . Then the ratio  $\eta$  of power flow is given by (69) where

$$p_{0\nu} = \int_0^{R_{0\nu}} \left(\frac{\sin\kappa_2 x}{\sin\kappa_2 R_{0\nu}}\right)^2 dx, \qquad (86a)$$

$$p_{1\nu} = 2 \int_0^{R_{0\nu}} \frac{\sin \kappa_2 x}{\sin \kappa_2 R_{0\nu}} H(\kappa_2 x, \kappa_2 R_{0\nu}) dx$$
(86b)

and H is given by (83).

The agreement of  $\eta$  according to (66) (dotted line in Fig. 15;  $\alpha_i = -0.01 + i0.0005$ ,  $i = 0, \ldots, 40$ ) and the linear approximation (69) where  $p_{0\nu}$  resp.  $p_{1\nu}$  is given by (86a) resp. (86b) and  $R_{1\nu}$  is given by (75) (solid line in Fig. 15) is, compared to Fig. 10, restricted to a smaller range of  $|\alpha|$  (see Fig. 15).



Figure 15:  $\eta = \eta(\alpha)$  according to (66) (dotted line) and (69) (solid line);  $\gamma := 1.5, t = 0.1, \epsilon_1 = 1, \epsilon_2 = 4;$ 

Let R be fixed and  $|\alpha| \ll 1$ . Then the ratio  $\eta$  of power flow is given by (71)

where

$$p_0 = \int_0^R \frac{\sin^2 \overline{\kappa_2} x}{\sin^2 \overline{\kappa_2} R} \, dx \,, \tag{87a}$$

$$p_1 = 2 \int_0^R \frac{\sin \overline{\kappa_2} x}{\sin \overline{\kappa_2} R} H(\overline{\kappa_2} x, \overline{\kappa_2} R) dx$$
(87b)

and H is given by (85a).

Numerical investigation shows that  $p_1$  is negative if  $\varepsilon_2 = 4$  and  $\gamma_0 < 1.7$ . Since  $\gamma_1$  according to (76) is negative, too, the power ratio  $\eta$  according to (71) diminishes if  $\alpha$  increases (see Fig. 16).

Similar to Fig. 15, the agreement of  $\eta$  according to (66) (dotted line in Fig. 16) and the linear approximation (71) where  $p_0$  resp.  $p_1$  is given by (87a) resp. (87b) and  $\gamma_1$  is given by (76) (solid line in Fig. 16) is, compared to Fig. 11, restricted to a smaller range of  $|\alpha|$  (see Fig. 16).



Figure 16:  $\eta = \eta(\alpha)$  according to (66) (dotted line) and (71) (solid line);  $R := 1.7178..., t = 0.1, \ \varepsilon_1 = 1; \varepsilon_2 = 4; \ \nu = 1$ 

To summarize, similar to even modes it is shown that if the propagation constant  $k\gamma$  is fixed, an increasing  $\alpha$  leads to a rise of the ratio  $\eta$ . However, if R is fixed, an increasing  $\alpha$  leads to a decrease of  $\eta$ .
## **2.3** Convergence of the iteration sequence $\{u_n\}$

The convergence of the iteration (42) is proved for even modes using the Banach fixed point theorem (see A.3.1); the investigation of odd modes is analogous.

Consider the operator

$$F: \begin{cases} S \to F(S) \\ u \mapsto F(u) = \alpha \int_{-R}^{R} ds \ G(s, x) \ B(u(s)) + f(x) \end{cases}$$
(88)

S is set of functions in C[-R, R] which will be specified below.

Then (42a) is equivalent to  $u_{n+1} = F(u_n)$ .

To check the convergence of operator F with the norm

$$||F|| := \max_{x \in [-R,R]} |F(u(x))|, \qquad (89)$$

let  $\kappa_2 R = \nu \cdot \pi/2 + \delta$ ,  $\nu = 0, 2, 4, \dots$  and  $|\delta| < \pi/2$ . Then the estimation

$$\left\| \int_{-R}^{R} ds \ G(s,x) \ B(u(s)) \right\| \leq C_{\nu} \frac{1}{2\sqrt{t}}$$
(90)

holds where

$$C_{\nu} := \frac{1}{\kappa_2^2} \left( 1 + \frac{1}{\cos \delta} \right) \left( \nu + 1 \right) \tag{91}$$

(see A.3.2) and consequently

$$\|F(u)\| \leq |\alpha| \quad \left\| \int_{-R}^{R} ds \ G(s,x) \ B(u) \right\| + \|f(x)\|$$
  
$$\leq |\alpha| \frac{1}{2\sqrt{t}} C_{\nu} + \frac{1}{\cos \delta}.$$
(92)

In A.3, the estimation

$$||F(u) - F(v)|| \leq |\alpha| C_{\nu} ||u - v||$$
(93)

is proved.

Choose  $\alpha$  sufficiently small and  $\kappa_2$  sufficiently large so that

$$|\alpha| C_{\nu} < 1. \tag{94}$$

Then according to (93), F is a contraction.

Furthermore, choose  $S_t := \{ u \in C[-R, R] | ||u|| < \frac{1}{2\sqrt{t}} + \frac{1}{\cos \delta} \}.$ 

Then according to (92)

$$\|F(u)\| \leq \frac{1}{2\sqrt{t}} + \frac{1}{\cos\delta} \tag{95}$$

or

$$F(S_t) \subseteq S_t$$
,

i. e. F maps S into itself and finally, from the Banach fix point theorem, it follows the convergence of the sequence  $(u_n)_{n \in \mathbb{N}}$ .

## 3 Cylindrical waveguide with photorefractive permittivity of the core

In the following, a permittivity according to

$$\varepsilon = \begin{cases} \varepsilon_2 - \alpha \frac{1}{1 + tu^2} & r \le R \quad \text{(core)} \\ \varepsilon_1 & x \ge R \quad \text{(cladding)} \end{cases}$$
(96)

is considered, where  $\varepsilon_1$  and  $\varepsilon_2$  are independent on  $u^2$  and the spatial coordinates and B is given by (44).

The electric field is determined by the wave equations

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}u + \kappa_2^2 u + \alpha B(u) = 0 \qquad (r \le R) \qquad (97a)$$

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}u - \kappa_1^2 u = 0 \quad (r \ge R) \quad (97b)$$

and the boundary conditions (22).

After multiplication with r and introducing the linear differential operator

$$L := r \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \kappa_2^2 - \frac{1}{r^2} \right)$$
$$= \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + r \kappa_2^2 - \frac{1}{r} \qquad (r \le R) \quad , \tag{98}$$

Eq. (97a) reads

$$Lu + \alpha r B(u) = 0. (99)$$

The solution of (99) can be expressed in terms of a Green function defined by

$$LG := -\delta(\rho - r), \qquad (100a)$$

$$G(r,0) := 0 =: G'(r,R),$$
 (100b)

with [14]

$$\rho(r) = \int \frac{\pi}{2} \left[ \frac{J_1(\kappa_2 \rho) J_1(\kappa_2 r)}{J_1'(\kappa_2 R)} Y_1'(\kappa_2 R) - J_1(\kappa_2 \rho) Y_1(\kappa_2 r) \right] \qquad \rho < r \le R$$

$$G(\rho, r) = \begin{cases} \frac{\pi}{2} \left[ \frac{J_1(\kappa_2 \rho) J_1(\kappa_2 r)}{J_1'(\kappa_2 R)} Y_1'(\kappa_2 R) - J_1(\kappa_2 r) Y_1(\kappa_2 \rho) \right] & r < \rho \le R \end{cases}$$
(101)

where  $Y_1$  denotes the Bessel function of the second kind. Applying the second Green's formula, the solution of (99) is given by

$$u(r) = \begin{cases} \alpha \int_{0}^{R} d\rho \, \rho \, G(\rho, r) \, B(u(\rho)) + R \, G(R, r) \, u'(R-0) & r \le R \\ \frac{K_{1}(\kappa_{1} r)}{K_{1}(\kappa_{1} R)} & r \ge R \,. \end{cases}$$
(102)

If  $\rho = R$ , (101) leads to

$$G(r,R) = G(R,r) = \frac{J_1(\kappa_2 r)}{J_1'(\kappa_2 R)} \frac{1}{\kappa_2 R}.$$
 (103)

Taking into account (103) and the boundary conditions (22) together with

$$u(R+0) = 1,$$
 (104a)

$$u'(R+0) = \kappa_1 \frac{K'_1(\kappa_1 R)}{K_1(\kappa_1 R)},$$
 (104b)

one obtains

$$u(r) = \alpha \int_{0}^{R} d\rho \,\rho \,G(\rho, r) \,B(u(\rho)) + \frac{\kappa_{1}}{\kappa_{2}} \frac{J_{1}(\kappa_{2}r)}{J_{1}'(\kappa_{2}R)} \frac{K_{1}'(\kappa_{1}R)}{K_{1}(\kappa_{1}R)} \qquad (r \le R)$$
(105)

and

$$1 = \alpha \int_0^R d\rho \, \rho \, G(\rho, R) \, B(u(\rho)) \, + \, R \, \frac{1}{\kappa_2 R} \, \frac{J_1(\kappa_2 R)}{J_1'(\kappa_2 R)} \, \kappa_1 \, \frac{K_1'(\kappa_1 R)}{K_1(\kappa_1 R)}$$

Multiplication by  $(\kappa_2 R) J'_1(\kappa_2 R) K_1(\kappa_1 R)$  yields the dispersion relation

$$g(\gamma, R) - \alpha F(\gamma, R; u) = 0, \qquad (106)$$

where g is given by (24) and

$$F(\gamma, R; u) := K_1(\kappa_1 R) \int_0^R dr \, r \, B(u(r)) \, J_1(\kappa_2 r)$$
 (107a)

$$= -K_1(\kappa_1 R) \int_0^R dr \, r \, \frac{u}{1+tu^2} J_1(\kappa_2 r) \,. \tag{107b}$$

By iterating (105) according to

$$u_{n+1}(r) = \alpha \int_{0}^{R} d\rho \,\rho \,G(\rho, r) \,B(u_{n}(\rho)) + \frac{\kappa_{1}}{\kappa_{2}} \frac{K_{1}'(\kappa_{1}R)}{K_{1}(\kappa_{1}R)} \frac{J_{1}(\kappa_{2}r)}{J_{1}'(\kappa_{2}R)} \,(108a)$$
$$u_{1} := \frac{J_{1}(\kappa_{2}r)}{J_{1}(\kappa_{2}R)}$$
(108b)

one obtains a sequence  $\{u_n(r; \gamma^{(n-1)}, R^{(n-1)})\}$ . It can be shown that  $\{u_n\}$  converges uniformly to the exact solution (105) (see 3.2).

The associated dispersion relation is formally equal to the dispersion relation (43) of the planar waveguide. If  $\alpha = 0$ , the linear dispersion relation g = 0 holds and by means of (24*a*) it is easy to verify that  $u_n = u_1$ .

The sequence  $\gamma^{(n)}$  resp.  $R^{(n)}$  of solutions of the approximate dispersion relations  $g(\gamma^{(n)}, R^{(n)}) - \alpha F(\gamma^{(n)}, R^{(n)}; u_n) = 0$  converges to the exact solution  $\gamma$  resp. R of the dispersion relation (106), thus the existence of solutions of (106) is proved (see 3.2 and [14]).

Substitution of  $u_1$  into (106) leads to first the approximate dispersion relation

$$g(\gamma^{(1)}, R^{(1)}) - \alpha F(\gamma^{(1)}, R^{(1)}; u_1) = 0.$$
(109)

If  $\alpha$  is fixed, it determines pairs  $\{\gamma^{(1)}, R^{(1)}\}$  represented by Fig. 17 where three modes  $R^{(1)} = R^{(1)}(\gamma^{(1)})$  are depicted, each of them consisting of three parts referring to  $\alpha = 0$ ,  $\alpha = +0.1$  and  $\alpha = -0.1$ .

In order to study the dependence of the relation  $R^{(1)} = R^{(1)}(\gamma^{(1)})$  on  $\alpha$  analytically, it is useful to study the cutoff region  $\gamma^{(1)} \approx \sqrt{\varepsilon_1}$ .

For convenience, in the following the superscript <sup>(1)</sup> is omitted.



Figure 17: Solution  $R = R(\gamma)$  of (110); t = 0.1,  $\alpha = 0, \pm 0.1$ ;  $\varepsilon_1 = 1$ ;  $\varepsilon_2 = 4$ 

## 3.1 Analytical approximations

Dividing (109) by  $K_1(\kappa_1 R) J_1(\kappa_2 R)$ , one obtains

$$a(\kappa_2 R) + b(\kappa_1 R) + \frac{\alpha}{\kappa_2^2} F_1(\kappa_2 R) = 0$$
 (110)

where

$$a(\kappa_2 R) := \kappa_2 R \frac{J_0(\kappa_2 R)}{J_1(\kappa_2 R)}, \qquad (111a)$$

$$b(\kappa_1 R) := \kappa_1 R \frac{K_0(\kappa_1 R)}{K_1(\kappa_1 R)}$$
(111b)

and

$$F_1(\kappa_2 R) := -\int_0^{\kappa_2 R} dr \, r \, u_1(r) \, B(u_1(r))$$
(112a)

$$= \int_{0}^{\kappa_{2}R} dr \, r \, \frac{u_{1}^{2}(r)}{1 + tu_{1}^{2}(r)} \tag{112b}$$

$$= \int_{0}^{\kappa_{2}R} dr \, r \, \frac{\left(\frac{J_{1}(r)}{J_{1}(\kappa_{2}R)}\right)^{2}}{1 + t \left(\frac{J_{1}(r)}{J_{1}(\kappa_{2}R)}\right)^{2}}$$
(112c)

Introducing

$$\delta := \frac{\kappa_1}{\kappa} \ll 1$$

and

$$R := R_{\nu}(1 + \chi + \mathcal{O}(\chi^2)) \quad \text{where} \quad \chi \ll 1 \,, \tag{113}$$

Equation (110) reads

$$\chi - \frac{1}{2}\delta^2 + \delta^2 \ln[\frac{\Gamma}{2}j_{0m}\delta] - \alpha \frac{F_1(j_{0\nu})}{(\kappa j_{0\nu})^2} = 0$$
(114)

(see B.1) if terms of second order  $\alpha$  and  $\chi$  are neglected.

If  $\kappa_1 = 0$  resp.  $\delta = 0$ , combination of (113) and (114) leads to

$$\kappa R_{\nu}(\alpha) = j_{0\nu} \left( 1 + \frac{\alpha}{\kappa^2} \frac{F_1(j_{0\nu})}{j_{0\nu}^2} \right) + \mathcal{O}(\alpha^2) + \mathcal{O}(\chi^2).$$
(115)

The dependence of  $R_{\nu} = R_{\nu}(\alpha)$  on  $\alpha$  is presented in Fig. 18 by a plot of (115) (dashed straight line) and a ContourPlot of (110).

Let R be fixed.

Then  $\chi$  is fixed, too, and (114) determines  $\alpha = \alpha(\delta)$ .

If  $\chi = 0$  and  $\delta < 0.7698$ , then

$$\alpha(\delta) = \frac{(\kappa j_{0\nu})^2}{F_1(j_{0\nu})} \left( -\frac{1}{2} \delta^2 + \delta^2 \ln[\frac{\Gamma}{2} j_{0\nu} \delta] \right) < 0$$

since  $F_1 > 0$  according to (112b).



Figure 18:  $R_{\nu} = R_{\nu}(\alpha)$  according to (115) (dashed straight line) and as a solution of (110) (solid line);  $\nu = 1$ ;  $\gamma = 1.0001$ ;  $\varepsilon_1 = 1, \varepsilon_2 = 4$ ; t = 1;

If a radius larger than the cutoff radius is considered, then  $\chi > 0$  has to be added in the bracket, and in the limit  $\delta \to 0$  the associated  $\alpha$  is first positive, then  $\alpha$  again becomes negative with increasing  $\delta$ . This relation is sketched in Fig. 19 by a ContourPlot  $\alpha = \alpha(\gamma)$  of (110).

#### **3.1.1** The second iteration function $u_2(x)$

The second iteration function follows from (108a) and is given by

$$u_{2}(r) = \begin{cases} \alpha \int_{0}^{R} d\rho \, \rho \, G(\rho, r) \, B(u_{1}(\rho)) + \frac{\kappa_{1}}{\kappa_{2}} \frac{K_{1}'(\kappa_{1}R)}{K_{1}(\kappa_{1}R)} \frac{J_{1}(\kappa_{2}r)}{J_{1}'(\kappa_{2}R)} & r \leq R \\ \frac{K_{1}(\kappa_{1}r)}{K_{1}(\kappa_{1}R)} & r \geq R \,. \end{cases}$$
(116)

The integrals in (116) can't be solved analytically; the second iteration function  $u_2$  is plotted with the plot-routine (cf. Fig. 20) for fixed  $\alpha$ ,  $\gamma$  and R, and compared with the numerical solution of (97). Except for the origin, the agreement is satisfactory.



Figure 19: Solution  $\alpha = \alpha(\gamma)$  of (110);  $R = R_{\nu}(1 + \chi)$ ;  $\chi = 0.01$ ; t = 0.1;  $\varepsilon_1 = 1, \varepsilon_2 = 4$ ;  $\nu = 1$ 



Figure 20:  $u_2 = u_2(r)$  according to (116) and numerical solution of (97);  $t = 0.1, \varepsilon_1 = 1; \varepsilon_2 = 4; \alpha = +0.1; R = 2, \gamma = 1.3697...$ 

Let  $\gamma$  be fixed.

With increasing  $\alpha$ , the radius  $R = R(\alpha)$ , determined by (110), increases as well as the amplitude of function  $u_2$ , whereas negative  $\alpha$  leads to the opposite effect (see Fig. 21).



Figure 21:  $u_2 = u_2(r)$  according to (116);  $\gamma = 1.394...; \alpha = -0.1, 0, +0.1; \epsilon_1 = 1, \epsilon_2 = 4$ 

Let R be fixed.

 $\gamma = \gamma(\alpha)$  is determined by (110). In contrast to the situation of fixed  $\gamma$ , positive  $\alpha$  now decreases the amplitude, whereas negative  $\alpha$  produces a higher amplitude (see Fig. 22).

 $\gamma = 1.394...$  and R = 2 are related by the linear dispersion relation (23), thus the graphs of functions  $u_2(r, \alpha = 0)$  in Fig. 21 and 22 coincide.



Figure 22:  $u_2 = u_2(r)$  according to (116); R = 2;  $\alpha = -0.1, 0, +0.1$ ;  $\varepsilon_1 = 1, \varepsilon_2 = 4$ 

#### **3.1.2** Ratio $\eta$ of power flow in the core

By means of the second iteration function  $u_2$ , the ratio of power flow in the core to the total power flow

$$\eta = \frac{P_{co}}{P} = \frac{\int_0^R dr \; r u_2^2(r)}{\int_0^\infty dr \; r u_2^2(r)}$$
(117)

can be investigated.

Let  $R = R_{\nu}(1 + \chi)$  be fixed where  $\chi \ll 1$  and  $\nu = 1$ .

Let further  $\delta \ll 1$  (cutoff limit). Then  $\delta$  depends on  $\alpha$  according to (114).

Fig. 23 depicts  $\eta = \eta(\alpha)$ . If  $\chi = 0$  and  $\alpha \to 0$ , the cylindrical waveguide with constant permittivity is reproduced and  $\eta$  vanishes in consistence with (28). If R deviates from the cutoff radius by  $\chi \ll 1$ , then  $\eta$  converges towards a positive constant if  $\alpha \to 0$  (cf. Fig. 23). The sensitive dependence of  $\eta$  on  $\alpha$ if  $R \gtrsim R_c$  may be of interest for application by optical switches.



Figure 23:  $\eta = \eta(\alpha)$  according to (117);  $R = R_c = 1.3884...$  and R = 1.39; t = 0.1;  $\varepsilon_1 = 1, \varepsilon_2 = 4$ 

### 3.2 Convergence of the iteration

After introduction of the operator

$$F: \begin{cases} S \to F(S) \\ u \mapsto F(u) := \alpha \int_0^R d\rho \ \rho \ G(\rho, r) \ \frac{u}{1+tu^2} + f \end{cases}$$
(118)

where

$$f := \frac{\kappa_1}{\kappa_2} \frac{K_1'(\kappa_1 R)}{J_1'(\kappa_2 R)} \frac{J_1(\kappa_2 r)}{K_1(\kappa_1 R)},$$

iteration (108a) can be written as

$$u_{n+1} = F(u_n), \qquad n = 1, 2, \dots$$

Similar to 2.3, the convergence of the iteration (108a) with respect to the norm (89) is proved by application of the Banach fixed point theorem.

Let

$$j_{01} \leq \kappa_2 R \leq j_{11}$$
 (119)

Then the estimation

$$\|\rho G(\rho, r)\| \leq \frac{c_1}{\kappa_2^2}$$
 (120)

holds where

$$c_1 := \frac{1}{|J_1'(j_{01})|} \frac{1}{j_{01}} \int_0^{j_{01}} dt \ t \ J_1(t) \approx 2.83214$$

(see B.2.1).

Equation (119) implies that

$$\frac{\kappa_1}{\kappa_2} j_{01} \leq \kappa_1 R \leq \frac{\kappa_1}{\kappa_2} j_{11} \tag{121}$$

and leads to the estimation

$$\|f\| = \frac{\kappa_1}{\kappa_2} j_{11} \frac{K_1'(\frac{\kappa_1}{\kappa_2} j_{11})}{K_1(\frac{\kappa_1}{\kappa_2} j_{11})} \frac{J_1(j_{11}')}{j_{01} J_1'(j_{01})} =: c(\kappa_1, \kappa_2).$$

These preliminary investigations can be summarized by the estimation

$$||F(u)|| \leq |\alpha| \frac{1}{2\sqrt{t}} \frac{c_1}{\kappa_2^2} + c(\kappa_1, \kappa_2).$$
 (122)

Further analytical considerations yield the estimation

$$||F(u_2) - F(u_1)|| \leq |\alpha| ||\rho G|| ||u_2 - u_1|| \leq |\alpha| \frac{c_1}{\kappa_2^2} ||u_2 - u_1||$$
(123)

(see B.2.2).

Choose  $|\alpha|$  sufficient small and  $\kappa_2$  sufficient large so that

$$|\alpha|\frac{c_1}{\kappa_2^2} < 1. \tag{124}$$

Then F is a contraction.

Furthermore, choose  $S_t := \{ u \in C[0, R] \mid ||u|| \le \frac{1}{2\sqrt{t}} + c(\kappa_1, \kappa_2) \}$ 

Then F maps  $S_t$  onto itself, and from the Banach fixed point theorem follows the convergence of the iteration (108a).

## 4 Summary and Outlook

The planar and cylindrical waveguide with linear cladding and a core of diameter 2R with real, field dependent permittivity modelled by (29) are considered, where even and odd modes are investigated. It is assumed that the field independent part  $\varepsilon_2$  of the permittivity of the core is larger than the constant permittivity  $\varepsilon_1$  of the cladding.

The strength of the additional field dependent permittivity in the core is expressed by a nonlinearity parameter  $\alpha$ .

Assuming a wave with suitable polarization and a propagation constant  $k_0\gamma$ , Maxwell's equations for the electric field lead to the linear, inhomogeneous differential equation (33) whose solution is approximated by means of a Green function and an iteration method. Referring to a photorefractive permittivity with external field, the approximate solution is compared with the numerical solution; furthermore, the amplitude of even modes in the planar waveguide is compared with the analytically determined amplitude. In both cases, the agreement is satisfactory.

If  $\alpha$ ,  $\gamma$  and R are chosen according to (94) resp. (124) (both conditions can, in principle, be satisfied for arbitrary R and  $\gamma$  if  $|\alpha|$  is chosen sufficiently small), the conditions of convergence of the iteration method are given for a photorefractive permittivity with external field.

In contrast to numerical investigations, analytical considerations by means of the iteration method allow to study the dependence of the ratio  $\eta$  of power flow in the core to the total power flow on the parameters  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\gamma$ , R and  $\alpha$ .

The boundary conditions give rise to a dispersion relation (39) relating the half diameter R of the waveguide, the propagation constant  $k_0\gamma$  of the solitary wave and the nonlinear parameter  $\alpha$ . By means of the iteration method, the change of the linear dispersion relation due to the field dependent permittivity (29) is described with the help of function  $F_1$  ((45a) resp. (72a) for even resp. odd modes).

Approximate solutions of  $\gamma$  and R are investigated with respect to  $\alpha$  for photorefractive permittivity. If  $\gamma$  is fixed, R enlarges with increasing  $\alpha$  for all modes of the planar waveguide as well as the odd modes of the cylindrical waveguide.

The amplitude of the solution of (30a) resp. (33) is presented by an implicit equation (65a). The change of the amplitude due to the field dependent permittivity (29) is described by function  $E_1$  ((54a) resp. (79a) for even resp. odd modes).

The ratio  $\eta$  of the power flow in the core to the total power flow is linearized with respect to  $\alpha$  with the help of functions  $F_1$  and  $E_1$  for even and odd modes of the planar waveguide with field dependent permittivity (29).

If R is fixed and a photorefractive permittivity with external field is considered, an increasing  $\alpha$  leads to a decrease of  $\eta$  for even as well as for odd modes. However, if the propagation constant  $k_0\gamma$  is fixed, an increasing  $\alpha$  leads to an increase of  $\eta$ .

As an outlook, it seems that the approach outlined in this work can be applied to more general permittivities (higher order, saturating, complex valued). Furthermore, the restriction of a linear cladding seems not to be essential. Crucial for the approach are the assumption of TE-waves and the possibility of finding an explicit expression for the Green function.

## References

- Valley, G., Segev, M., Crosignani, B., Yariv, A., Fejer, A., Bashaw, M., Dark and bright photovoltaic spatial solitons, Physical Review A 50 (1994), 4457-4460
- [2] Christodoulides, D. N., Carvalho, M. I., Bright, dark and gray spatial soliton states in photorefractive media J. Opt. Soc. Am. B 12 (1995), 1628-1633
- [3] Segev, M., Shih, M., Valley, G., Photorefractive screening solitons of high and low intensity, J. Opt. Soc. Am. B 13 (1996), 706-718
- [4] Geisler, A., Homann, F., Schmidt, H.-J., General properties and analytical approximations of photorefractive solitons, accepted in Optics Communications
- [5] Boardman, A., Egan, P.,
   Optically nonlinear waves in thin films,
   IEEE J. Quantum Electron., Vol. QE-22, pp.319 324, 1986
- [6] Mihalache, D., Bertolotti, M., Sibilia, C., Nonlinear wave propagation in planar structures, Progress in Optics XXVII, Amsterdam 1989
- [7] Ponath, H., Stegeman, G., Nonlinear Surface Electromagnetic Phenomena, Amsterdam 1991
- [8] Schürmann, H. W., Serov, V., Shestopalov, Y., TE-polarized waves guided by a lossless nonlinear three-layer structure, Phys. Rev. E 58 (1998), 1040-1050
- [9] Kip, D., Wesner, M., Shandarov, V., Moretti, P., Observation of bright spatial photorefractive solitons in a planar strontium barium niobate waveguide, Optics Letters 23 (1998) 921-923
- [10] Wesner, M., Herden, C., Kip, D., Krätzig, E., Moretti, P., Photorefractive steady state solitons up to telecommunication wavelengths in planar SBN waveguides, Optics Communications 188 (2001) 69-76

- [11] Langbein, U., Lederer, F., Ponath, H., Generalized dispersion relations for nonlinear slab-guided waves, Optics Comm. 53 (1985) 417-420
- [12] Luennemann, M., Hartwig, U., Panotopoulos, G., Buse, K., Electrooptic Properties of Lithium Niobate Crystals for Extremely High External Electric Fields, Appl. Phys. B 76, 403 - 406 (2003).
- [13] Snyder, A., Polodian, L., Mitchell, D.,
   Stable black self-guided beams of circular symmetry in a bulk Kerr medium
   Opt. Lett 17 (1992), 789 - 791
- [14] Smirnov, Y., Schürmann, H. W., Shestopalov, Y., Integral Equation approach for the Propagation of TE-Waves in a Nonlinear Dielectric Cylindrical Waveguide, Journal of Nonlinear Mathematical Physics 11, 256-268, (2004)
- [15] Marcuse, D., Light Transmission Optics, New York 1982
- [16] Snyder, A., Love, J., Optical Waveguide Theory, New York 1983
- [17] Jackson, J. D., Classical Electrodynamics, New York 1975
- [18] Collin, R. E., Field Theory of Guided Waves, New York 1991

## Acknowledgement

I am very grateful to Prof. Dr. Hans Werner Schürmann for giving me the opportunity to work on a PhD project in physics at the university of Osnabrück and for supporting my work. Without his motivations, I would not have finished my thesis.

Furthermore, I would like to thank Prof. Dr. Juri Gennadjevich Smirnov and Prof. Dr. Valery Segeevich Serov for giving me valuable advices.

I would like to give heartfelt thanks to Dipl. Math. Lena Svetogorova for supporting my work by advices for numerical calculations and to Dr. Monika Wesner who gave me many advices concerning experiments.

The support by lectures of the Graduate College 695 "Nonlinearities of optical materials" is gratefully acknowledged.

## A Planar waveguide

## A.1 Even function

#### A.1.1 Linear case, cut off limit

The cutoff limit is defined by  $\gamma \to \sqrt{\varepsilon_1}$  or  $\kappa_1 \to 0$ . Then

$$\delta \ := \ \frac{\kappa_1}{\kappa} \ \ll \ 1$$

is an appropriate parameter for an expansion of g. Then

$$\kappa_2^2 = \varepsilon_2 - \gamma^2 = (\varepsilon_2 - \varepsilon_1) - (\gamma^2 - \varepsilon_1) = \kappa^2 - \kappa_1^2 = \kappa^2 (1 - \delta^2)$$

and

$$\kappa_2 = \kappa \sqrt{1 - \delta^2} = \kappa \left( 1 - \frac{1}{2} \delta^2 + \mathcal{O}(\delta^4) \right).$$

Next  $R = R(\gamma)$  is expanded in a series with respect to  $\delta$ :

$$R(\delta) := R_0 + R_1 \delta + \mathcal{O}(\delta^2)$$

It follows immediately

$$\kappa_2 R = \kappa R_0 + \kappa R_1 \delta + \mathcal{O}(2) \, .$$

Inserting

$$\tan \kappa_2 R = \tan \kappa R_0 + \frac{1}{\cos^2(\kappa R_0)} \kappa R_1 \delta + \mathcal{O}(2)$$

and

$$\frac{\kappa_1}{\kappa_2} = \frac{\kappa_1}{\kappa} \frac{\kappa}{\kappa_2} = \delta \frac{1}{\sqrt{1-\delta^2}} = \delta + \mathcal{O}(3)$$

in the linear dispersion relation

$$\tan \kappa_2 R = \frac{\kappa_1}{\kappa_2} \tag{8}$$

leads to

$$\tan \kappa R_0 + \frac{1}{\cos^2(\kappa R_0)} \kappa R_1 \delta = \delta + \mathcal{O}(3),$$

and after comparing equal orders in  $\delta$  it follows

$$\tan \kappa R_0 = 0 \qquad \Longleftrightarrow \qquad \kappa R_0 = z \cdot \pi, \quad (z = 0, 1, 2, \ldots)$$

 $\kappa R_1 = 1.$ 

Therefore if  $\kappa_1/\kappa \ll 1$ ,  $R = R(\kappa_1)$  can be approximated by

$$\kappa R = z \cdot \pi + \delta + \mathcal{O}(\delta^2) = z \cdot \pi + \frac{\kappa_1}{\kappa} + \mathcal{O}(2).$$

The width  $R = R(\gamma)$  of the core in the cutoff limit for both, even and odd modes, can be described by

$$\kappa R = \nu \cdot \frac{\pi}{2} + \delta + \mathcal{O}(\delta^2), \qquad \nu = 0, 1, 2, \dots; \quad \delta := \frac{\kappa_1}{\kappa}$$
(11)

where even values of  $\nu$  belong to even TE modes, while odd values of  $\nu$  belong to odd modes (cf. Marcuse, Light transmission optics, p. 327.)

Additionally, Fig. 24 shows the ContourPlot of function g in the region of small  $\kappa_1$   $(1 \leq \gamma \leq 1.1$  where  $\varepsilon_1 = 1$ ) together with the function  $R(\gamma) = \kappa_1/\kappa^2$  for the fundamental mode ( $\nu=0$ ).



Figure 24:  $R = R(\gamma); \ \varepsilon_1 = 1; \varepsilon_2 = 4; \kappa_1 \to 0 \ (1 \le \gamma \le 1.1)$ 

### A.2 Field dependent permittivity, even modes

#### A.2.1 Construction of function f

Function f is determined by following considerations:

and

First, with the help of  $Lu = -\alpha B(u)$  and  $LG = -\delta(s - x)$ , we have

$$\begin{split} \int_{-R}^{R} ds \, \left( \, G(s,x)Lu \, - \, uLG(s,x) \right) &= \int_{-R}^{R} ds \, \left( \, - \, G(s,x) \, \alpha B(u) \, + \, u(s)\delta(s-x) \right) \\ &= u(x) \quad - \quad \alpha \, \int_{-R}^{R} ds \, G(s,x)B(u) \\ &= f(x) \, . \end{split}$$

On the other hand, taking into account  $L := \partial_s^2 + \kappa_2^2$ , the relation

$$\begin{aligned} \int_{-R}^{R} ds \left( vLu - uLv \right) &= \int_{-R}^{R} ds \left( v \, u'' - u \, v'' \right) \\ &= \left[ v \, u' \right]_{-R}^{R} - \int_{-R}^{R} ds \, v'u' \quad - \quad \left( \left[ u \, v' \right]_{-R}^{R} - \int_{-R}^{R} ds \, u'v' \right) \\ &= \left[ v \, u' \right]_{-R}^{R} - \left[ u \, v' \right]_{-R}^{R} \end{aligned}$$

is valid.

With v(s) := G(s, x) and taking into account G(R, x) = G(-R, x) = 0, we find

$$f(x) = \int_{-R}^{R} ds \left( G(s,x)Lu(s) - u(s)LG(s,x) \right)$$
  
$$= -[uv']_{-R}^{R}$$
  
$$= -u(R) \left. \frac{\partial}{\partial s}G(s,x) \right|_{s=R} + u(-R) \left. \frac{\partial}{\partial s}G(s,x) \right|_{s=-R}$$
  
$$= - \left. \frac{\partial}{\partial s}G(s,x) \right|_{s=R} \pm \left. \frac{\partial}{\partial s}G(s,x) \right|_{s=-R}$$

since u(R) = 1 and  $u(-R) = \pm 1$ , where u(R) = u(-R) for even functions and u(-R) = -u(R) for odd functions.

From (35), the derivation of the Greens function with respect to s is given by

$$\frac{\partial}{\partial s}G(s,x) = \begin{cases} -\frac{\sin[\kappa_2 (R+x)] \cos[\kappa_2 (R-s)]}{\sin[2\kappa_2 R]} & x \le s \le R\\ \frac{\sin[\kappa_2 (R-x)] \cos[\kappa_2 (R+s)]}{\sin[2\kappa_2 R]} & x \ge s \ge -R \end{cases}$$

It follows

$$\left. \frac{\partial}{\partial s} G(s,x) \right|_{s=R} = \left. -\frac{\sin[\kappa_2 \left(R+x\right)]}{\sin[2\kappa_2 R]} \right.$$

and

$$\left. \frac{\partial}{\partial s} G(s, x) \right|_{s=-R} = \left. \frac{\sin[\kappa_2 \left( R - x \right)]}{\sin[2\kappa_2 R]} \right.,$$

thus

$$\int_{-R}^{R} ds \left( G(s,x)Lu - uLG(s,x) \right) = -\partial_s G(s,x)|_{s=R} \pm \partial_s G(s,x)|_{s=-R}$$
$$= \frac{\sin[\kappa_2 (R+x)] \pm \sin[\kappa_2 (R-x)]}{\sin[2\kappa_2 R]}$$

For even functions, u(-R) = u(R), and therefore

$$f(x) = \frac{\sin[\kappa_2 (R+x)] + \sin[\kappa_2 (R-x)]}{\sin[2\kappa_2 R]} = \frac{2\sin(\kappa_2 R)\cos(\kappa_2 x)}{2\sin(\kappa_2 R)\cos(\kappa_2 R)} = \frac{\cos(\kappa_2 x)}{\cos(\kappa_2 R)},$$

whereas odd functions require u(-R) = -u(R), and in this case we have

$$f(x) = \frac{\sin[\kappa_2 (R+x)] - \sin[\kappa_2 (R-x)]}{\sin[2\kappa_2 R]} = \frac{2\cos(\kappa_2 R)\sin(\kappa_2 x)}{2\sin(\kappa_2 R)\cos(\kappa_2 R)} = \frac{\sin(\kappa_2 x)}{\sin(\kappa_2 R)}.$$

### A.2.2 Dispersion relation, even mode

A dispersion relation connecting  $\gamma$  and R follows from the continuity of the derivative of u(x) at x = R:

$$-\kappa_{1} = \alpha \frac{\partial}{\partial x} \int_{-R}^{R} ds \ G(s,x) \cdot B(u(s)) \Big|_{x=R} + f'(R)$$
$$= -\alpha \kappa_{2} \int_{-R}^{R} ds \ \frac{\sin[\kappa_{2} (R+s)]}{\kappa_{2} \sin[2\kappa_{2}R]} \cdot B(u(s)) - \kappa_{2} \tan(\kappa_{2}R)$$
$$\kappa_{2} \tan \kappa_{2}R - \kappa_{1} = -\alpha \int_{-R}^{R} ds \ \frac{\sin[\kappa_{2} (R+s)]}{\sin[2\kappa_{2}R]} \cdot B(u(s)) + \frac{1}{\kappa_{2}}$$
$$\tan \kappa_{2}R - \frac{\kappa_{1}}{\kappa_{2}} = -\frac{\alpha}{\kappa_{2}} \int_{-R}^{R} ds \ \frac{\sin[\kappa_{2} (R+s)]}{\sin[2\kappa_{2}R]} \cdot B(u(s))$$

or

$$g - \alpha F(u) = 0$$

where

$$g := \tan \kappa_2 R - \frac{\kappa_1}{\kappa_2},$$
  

$$F(u) := -\frac{1}{\kappa_2} \int_{-R}^{R} ds \; \frac{\sin[\kappa_2 (R+s)]}{\sin[2\kappa_2 R]} \cdot B(u(s)).$$

If we assume that u(s) is a symmetric function, than B(u(s)) is a symmetric function, too, and therefore

$$\int_{-R}^{R} ds \sin(\kappa_2 s) \cdot B(u(s)) = 0.$$

Thus

$$F(u) := -\frac{1}{\kappa_2} \int_{-R}^{R} ds \; \frac{\sin(\kappa_2 R) \, \cos(\kappa_2 s)}{\sin[2\kappa_2 R]} \cdot B(u(s))$$
  
$$= -\frac{1}{2\kappa_2} \int_{-R}^{R} ds \; \frac{\cos(\kappa_2 s)}{\cos(\kappa_2 R)} \cdot B(u(s))$$
  
$$= -\frac{1}{\kappa_2^2} \int_{0}^{\kappa_2 R} ds \; \frac{\cos s}{\cos(\kappa_2 R)} \cdot B(u(s))$$

### A.2.3 Dispersion relation of the first iteration step, even mode

Introducing

$$u_1(s) = \frac{\cos s}{\cos(\kappa_2 R)}$$

and the photorefractive nonlinearity  $B_{Ph}(u) = -u/(1 + tu^2)$ , we have

$$F_{Ph}(u_1) = -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \ u_1(s) \cdot B(u_1(s))$$
$$= \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \ \frac{u_1^2(s)}{1 + tu_1^2(s)}$$
$$= \frac{1}{\kappa_2^2} \frac{1}{t} \int_0^{\kappa_2 R} ds \ \frac{t \ u_1^2}{1 + tu_1^2}$$
$$= \frac{1}{\kappa_2^2} \frac{1}{t} \int_0^{\kappa_2 R} ds \ \left[1 - \frac{1}{1 + tu_1^2}\right]$$

With the help of

$$1 + tu_1^2 = 1 + t \frac{\cos^2 s}{\cos^2(\kappa_2 R)} = \frac{1}{P_0} \cos^2 s \left[ 1 + P_0 \tan^2 s \right]$$

where

$$P_0 := \frac{\cos^2(\kappa_2 R)}{\cos^2(\kappa_2 R) + t}$$

and the substitution:  $z(s) = \sqrt{P_0} \tan s$ ;  $dz = \sqrt{P_0} \frac{1}{\cos^2 s} ds$ , we have

$$\begin{aligned} \int_{a}^{b} ds & \frac{1}{1+tu_{1}^{2}} &= P_{0} \int_{a}^{b} ds \frac{1}{\cos^{2} s} \frac{1}{1+P_{0} \tan^{2} s} \\ &= \sqrt{P_{0}} \int_{z(a)}^{z(b)} dz \frac{1}{1+z^{2}} \\ &= \sqrt{P_{0}} \left[ \arctan z \right]_{z(a)}^{z(b)} \\ &= \sqrt{P_{0}} \left( \arctan \left[ \sqrt{P_{0}} \tan b \right] - \arctan \left[ \sqrt{P_{0}} \tan a \right] \right) \end{aligned}$$

For later purposes, we introduce the function

$$F_{1}(a,b) := -\frac{1}{\kappa_{2}^{2}} \int_{a}^{b} ds \ u_{1}(s) \cdot B(u_{1}(s))$$
  
$$= \frac{1}{\kappa_{2}^{2}} \frac{1}{t} \int_{a}^{b} ds \quad \left[1 - \frac{1}{1 + tu_{1}^{2}}\right]$$
  
$$= \frac{1}{\kappa_{2}^{2}} \frac{1}{t} \left[s - \sqrt{P_{0}} \arctan[\sqrt{P_{0}} \tan s]\right]_{a}^{b}$$

which depends on two arguments referring to the lower and upper boundary of the integral.

Taking into account  $F(u_1) := F_1(0, \kappa_2 R)$ , we finally have

$$F(u_1) = \frac{1}{\kappa_2^2} \frac{1}{t} \left( \kappa_2 R - \sqrt{P_0} \arctan[\sqrt{P_0} \tan(\kappa_2 R)] \right) .$$

Summary:

The first iteration step dispersion relation is given by

$$g - \alpha F_1 = 0 \tag{46}$$

where

$$g = \tan \kappa_2 R - \frac{\kappa_1}{\kappa_2}$$

and

$$F_1(\kappa_2 R) := -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \ u_1(s) \ B(u_1)$$
(45a)

$$= \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \quad \frac{u_1^2(s)}{1 + t u_1^2(s)}$$
(45b)

$$= \frac{1}{\kappa_2^2} \frac{1}{t} \left( \kappa_2 R - \sqrt{P_0} \arctan[\sqrt{P_0} \tan(\kappa_2 R)] \right) \quad (45c)$$

with

$$u_1(s) := \frac{\cos s}{\cos(\kappa_2 R)}, \qquad P_0 := \frac{\cos^2(\kappa_2 R)}{\cos^2(\kappa_2 R) + t} = \frac{1}{1 + t u_1^2(0)}$$

## **A.2.4** The amplitude $u_n(0)$

If x = 0, from (42a) we have

$$u_{n+1}(0) = \alpha \int_{-R}^{R} ds \ G(s,0) \ B(u_n(s)) \ + \ f(0).$$

From (35) we get

$$G(s,0) = \begin{cases} \frac{\sin(\kappa_2 R) \sin[\kappa_2 (R-s)]}{\kappa_2 \sin(2\kappa_2 R)} & s \ge 0\\ \frac{\sin(\kappa_2 R) \sin[\kappa_2 (R+s)]}{\kappa_2 \sin(2\kappa_2 R)} & s \le 0 \end{cases}$$

thus using the axial symmetry,  $u_n(-s) = u_n(s)$ , we have

$$u_{n}(0) - f(0) = \alpha \frac{\sin(\kappa_{2}R)}{\kappa_{2}\sin(2\kappa_{2}R)} \left( \int_{-R}^{0} ds \, \sin[\kappa_{2}\,(R+s)] \, B(u_{n}(s)) + \int_{0}^{R} ds \, \sin[\kappa_{2}\,(R-s)] \, B(u_{n}(s)) \right)$$

$$= \alpha \frac{\sin(\kappa_{2}R)}{\kappa_{2}\sin(2\kappa_{2}R)} \int_{-R}^{R} ds \, \sin[\kappa_{2}\,(R+s)] \, B(u_{n}(s))$$

$$+ \alpha \frac{\sin(\kappa_{2}R)}{\kappa_{2}\sin(2\kappa_{2}R)} \left( -\int_{0}^{R} ds \, \sin[\kappa_{2}\,(R+s)] \, B(u_{n}(s)) + \int_{0}^{R} ds \, \sin[\kappa_{2}\,(R-s)] \, B(u_{n}(s)) \right)$$

$$= -\alpha \, \sin(\kappa_{2}R) \, F(u_{n}) - \alpha \, \frac{\sin(\kappa_{2}R)}{\kappa_{2}\sin(2\kappa_{2}R)} \int_{0}^{R} ds \, 2\cos(\kappa_{2}R) \sin(\kappa_{2}s) \, B(u_{n}(s))$$

$$= -\alpha \, \sin(\kappa_{2}R) \, F(u_{n}) - \frac{\alpha}{\kappa_{2}} \int_{0}^{R} ds \, \sin(\kappa_{2}s) \, B(u_{n}(s))$$

$$= -\alpha \, \sin(\kappa_{2}R) \, F(u_{n}) - \frac{\alpha}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}R} dz \, \sin z \, B(u_{n}(z))$$

where, assuming that  $u_n$  is an even function,

$$F(u_n) := -\frac{1}{\kappa_2} \int_{-R}^{R} ds \, \frac{\sin[\kappa_2 \, (R+s)]}{\sin(2\kappa_2 R)} \, B(u_n(s))$$
$$= -\frac{1}{2\kappa_2} \int_{-R}^{R} ds \, \frac{\cos(\kappa_2 s)}{\cos(\kappa_2 R)} \, B(u_n(s))$$
$$= -\frac{1}{\kappa_2^2} \int_{0}^{\kappa_2 R} ds \, \frac{\cos(\kappa_2 s)}{\cos(\kappa_2 R)} \, B(u_n(s))$$

Summary:

$$u_n(0) - f(0) = -\alpha \, \sin(\kappa_2 R) \, F(u_n) \, - \, \frac{\alpha}{\kappa_2^2} \, \int_0^{\kappa_2 R} dz \, \sin z \, B(u_n(z))$$

### A.2.5 The second iteration functions amplitude $u_2(0)$

The first iteration function is given by  $u_1(z) = \frac{\cos z}{\cos(\kappa_2 R)}$ ; the photorefractive nonlinearity is given by  $B(u) = -\frac{u}{1+tu^2}$ .

Let n = 2. Then, remembering Eqs. (45a) - (45c),

$$F_{1} := F(u_{1}) = -\frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}R} ds \, \frac{\cos(\kappa_{2}s)}{\cos(\kappa_{2}R)} B(u_{1}(s))$$
  
$$= -\frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}R} ds \, u_{1}(s) B(u_{1}(s))$$
  
$$= \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}R} ds \, \frac{u_{1}^{2}(s)}{1 + tu_{1}^{2}(s)}$$
  
$$= \frac{1}{\kappa_{2}^{2}} \frac{1}{t} \left(\kappa_{2}R - \sqrt{P_{0}} \arctan[\sqrt{P_{0}}\tan(\kappa_{2}R)]\right),$$

it is convenient to introduce the function

$$E_{1} := E(u_{1}) := + \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}R} dz \ u_{1}'(z) \ B(u_{1}(z))$$

$$= -\frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}R} ds \ u_{1}'(s) \frac{u_{1}(s)}{1 + tu_{1}^{2}(s)}$$

$$= -\frac{1}{\kappa_{2}^{2}} \frac{1}{2t} \ln \left[ \frac{1 + tu_{1}^{2}(\kappa_{2}R)}{1 + tu_{1}^{2}(0)} \right], \quad u_{1}(\kappa_{2}R) = 1$$

$$= -\frac{1}{\kappa_{2}^{2}} \frac{1}{2t} \ln \left[ \frac{1 + t}{1 + tu_{1}^{2}(0)} \right]$$

$$= \frac{1}{\kappa_{2}^{2}} \frac{1}{2t} \ln \left[ \frac{1 + tu_{1}^{2}(0)}{1 + t} \right]$$

$$= \frac{1}{\kappa_{2}^{2}} \frac{1}{2t} \ln \left[ \frac{1 + tu_{1}^{2}(0)}{1 + t} \right]$$

Since  $\cos^2(\kappa_2 R) < 1$ , the counter of the logarithm's argument is larger than the denominator, therefore  $E_1(u_1) > 0$ .

The sign of  $E_1(u_1)$  can more easy determined by the transformation

$$E(u_1) := + \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} dz \ u_1'(z) \ B(u_1(z))$$
  
=  $- \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} dz \ \frac{\sin z}{\cos(\kappa_2 R)} \ B(u_1(z))$   
=  $+ \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} dz \ \frac{\sin z}{\cos(\kappa_2 R)} \frac{\frac{\cos z}{\cos(\kappa_2 R)}}{1 + t\left(\frac{\cos z}{\cos(\kappa_2 R)}\right)^2}.$ 

Inserting the definition of  $E(u_1)$ , we can immediately write

$$u_2(0) - u_1(0) = -\alpha \sin(\kappa_2 R) F_1(\kappa_2 R) + \alpha \cos(\kappa_2 R) E_1(\kappa_2 R).$$
 (125)

$$\begin{split} F_1(\kappa_2 R) &:= -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \; u_1(s) \, B(u_1(s)) \; > \; 0 \\ E_1(\kappa_2 R) &:= \; \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} ds \; u_1'(s) \, B(u_1(s)) \; > \; 0 \end{split}$$

#### A.2.6 Exact amplitude

Multiplying the exact differential equation

$$u'' + \kappa_2^2 u + \alpha B(u) = 0$$

with 2u' and integrating with respect to x with the upper boundary 0 yields

$$\left[(u')^2 + \kappa_2^2 u^2\right]_0^x + 2\alpha \int_0^x ds \ u'(s) \ B(u(s)) = 0 \ .$$

The amplitude is defined by

$$u(0) = u_0, \qquad u'(0) = 0,$$

thus we have

$$(u')^{2} + \kappa_{2}^{2}(u^{2} - u_{0}^{2}) + 2\alpha \int_{0}^{x} ds \ u'(s) \ B(u(s)) = 0 \ .$$

Taking into account the boundary conditions for  $x = \kappa_2 R$ ,

$$u(\kappa_2 R) = 1, \qquad u'(\kappa_2 R) = -\kappa_1,$$

leads to an equation which determines  $u_0$  as a function of  $\alpha$ :

$$\kappa_1^2 + \kappa_2^2 (1 - u_0^2) + 2\alpha \int_0^{\kappa_2 R} ds \, u'(s) \, B(u(s)) = 0 \qquad \left| \frac{1}{\kappa_2^2} \right.$$
$$1 + \frac{\kappa_1^2}{\kappa_2^2} - u_0^2 + 2\frac{\alpha}{\kappa_2^2} \int_0^{\kappa_2 R} ds \, u'(s) \, B(u(s)) = 0$$
$$u_0^2 = \left( 1 + \frac{\kappa_1^2}{\kappa_2^2} \right) + 2\frac{\alpha}{\kappa_2^2} \int_0^{\kappa_2 R} ds \, u'(s) \, B(u(s))$$

The integral can be solved analytically:

$$\int_{0}^{\kappa_{2}R} ds \ u'(s) \ B(u(s)) = -\int_{0}^{\kappa_{2}R} ds \ u'(s) \ \frac{u}{1+tu^{2}}$$
$$= -\int_{u(0)}^{u(\kappa_{2}R)} du \ \frac{u}{1+tu^{2}}$$
$$= \frac{1}{2t} \ln \frac{1+tu^{2}}{1+t}$$

Therefore we have an implicit equation for the exact amplitude  $u_0$ :

$$u_0^2 = 1 + \frac{\kappa_1^2}{\kappa_2^2} + \frac{\alpha}{t\kappa_2^2} \ln \frac{1 + t \, u_0^2}{1 + t} \tag{65b}$$

(65a)

## A.2.7 The second iteration function $u_2(x)$ , even modes

$$u_2(x) := \alpha \int_{-R}^{R} ds \ G(s,x) \ B(u_1(s)) + \frac{\cos(\kappa_2 x)}{\cos(\kappa_2 R)}$$
(42a)

$$= \alpha \int_{-R}^{x} ds \frac{\sin[\kappa_{2} (R-x)] \sin[\kappa_{2} (R+s)]}{\kappa_{2} \sin(2\kappa_{2}R)} B(u_{1}(s)) + \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \\ + \alpha \int_{x}^{R} ds \frac{\sin[\kappa_{2} (R+x)] \sin[\kappa_{2} (R-s)]}{\kappa_{2} \sin(2\kappa_{2}R)} B(u_{1}(s))$$

$$= \alpha \int_{R}^{-x} (-dt) \frac{\sin[\kappa_2 (R-x)] \sin[\kappa_2 (R-t)]}{\kappa_2 \sin(2\kappa_2 R)} B(u_1(-t)) + \frac{\cos(\kappa_2 x)}{\cos(\kappa_2 R)} \\ + \alpha \int_{x}^{R} ds \frac{\sin[\kappa_2 (R+x)] \sin[\kappa_2 (R-s)]}{\kappa_2 \sin(2\kappa_2 R)} B(u_1(s))$$

$$= \alpha \int_{-x}^{R} dt \frac{\sin[\kappa_2 (R-x)] \sin[\kappa_2 (R-t)]}{\kappa_2 \sin(2\kappa_2 R)} B(u_1(t)) + \frac{\cos(\kappa_2 x)}{\cos(\kappa_2 R)} \\ + \alpha \int_{x}^{R} ds \frac{\sin[\kappa_2 (R+x)] \sin[\kappa_2 (R-s)]}{\kappa_2 \sin(2\kappa_2 R)} B(u_1(s))$$

$$= \alpha \frac{\sin \kappa_2 (R-x)}{2\sin(\kappa_2 R)} J[-\kappa_2 x, \kappa_2 R] + \alpha \frac{\sin \kappa_2 (R+x)}{2\sin(\kappa_2 R)} J[\kappa_2 x, \kappa_2 R] + u_1(x)$$

where

$$J[a,b] := \frac{1}{\cos(\kappa_2 R)} \frac{1}{\kappa_2^2} \int_a^b dz \, \sin(\kappa_2 R - z) \, B(u_1(z))$$
  

$$= \frac{1}{\cos(\kappa_2 R)} \frac{1}{\kappa_2^2} \int_a^b dz \, (\sin\kappa_2 R \cos z - \cos\kappa_2 R \sin z) \, B(u_1(z))$$
  

$$= \sin\kappa_2 R \underbrace{\frac{1}{\kappa_2^2} \int_a^b dz \, u_1(z) \, B(u_1(z))}_{-F_1[a,b]} + \cos\kappa_2 R \underbrace{\frac{1}{\kappa_2^2} \int_a^b dz \, u_1'(z) \, B(u_1(z))}_{E_1[a,b]}$$

where

$$B(u_1) = -\frac{u_1}{1+tu_1^2}$$
 and  $u_1(z) := \frac{\cos z}{\cos(\kappa_2 R)}$ ,  $u_1'(z) = -\frac{\sin z}{\cos(\kappa_2 R)}$ 

Further simplification can be achieved by using integrals with zero boundary:

$$u_{2}(x) - u_{1}(x) := \frac{\alpha}{2} \left( \frac{\sin \kappa_{2}(R-x)}{\sin \kappa_{2}R} + \frac{\sin \kappa_{2}(R+x)}{\sin \kappa_{2}R} \right) J[0, \kappa_{2}R] \\ + \frac{\alpha}{2} \frac{\sin \kappa_{2}(R-x)}{\sin(\kappa_{2}R)} J[-\kappa_{2}x, 0] - \frac{\alpha}{2} \frac{\sin \kappa_{2}(R+x)}{\sin(\kappa_{2}R)} J[0, \kappa_{2}x]$$

Due to the symmetry of the integrand of  $F_1$ , according to (45b)

$$F_1[-\kappa_2 x, 0] = F_1[\kappa_2 x, 0],$$

whereas the antisymmetry of the integrand according to (54b) leads to

$$E_1[-\kappa_2 x, 0] = -E_1[\kappa_2 x, 0],$$

 $\operatorname{thus}$ 

$$J[-\kappa_2 x, 0] = -\sin(\kappa_2 R) F_1[-\kappa_2 x, 0] + \cos(\kappa_2 R) E_1[-\kappa_2 x, 0] = -\sin(\kappa_2 R) F_1[0, \kappa_2 x] - \cos(\kappa_2 R) E_1[0, \kappa_2 x]$$

and introducing the abbreviations

$$F_1(\kappa_2 x, \kappa_2 R) := F_1[0, \kappa_2 x], \qquad E_1(\kappa_2 x, \kappa_2 R) := E_1[0, \kappa_2 x],$$

- for convenience, in contrast to the main part of this work, in this calculation  $F_1(\kappa_2 x, \kappa_2 R)$  is abbreviated by  $F_1(\kappa_2 x)$  and  $E_1(\kappa_2 x, \kappa_2 R)$  by  $E_1(\kappa_2 x)$  - we have

$$u_{2}(x) - \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} = \frac{\alpha}{2} 2\cos(\kappa_{2}x) J[0, \kappa_{2}R] + \frac{\alpha}{2} \frac{\sin\kappa_{2}(R-x)}{\sin(\kappa_{2}R)} [-\sin(\kappa_{2}R) F_{1}(\kappa_{2}x) - \cos(\kappa_{2}R) E_{1}(\kappa_{2}x)] - \frac{\alpha}{2} \frac{\sin\kappa_{2}(R+x)}{\sin(\kappa_{2}R)} [-\sin(\kappa_{2}R) F_{1}(\kappa_{2}x) + \cos(\kappa_{2}R) E_{1}(\kappa_{2}x)] = \alpha \cos(\kappa_{2}x) [-\sin(\kappa_{2}R) F_{1}(\kappa_{2}R) + \cos(\kappa_{2}R) E_{1}(\kappa_{2}R)] + \frac{\alpha}{2} \frac{2\cos(\kappa_{2}R) \sin(\kappa_{2}x)}{\sin(\kappa_{2}R)} \sin(\kappa_{2}R) F_{1}(\kappa_{2}x) - \frac{\alpha}{2} \frac{2\sin(\kappa_{2}R) \cos(\kappa_{2}x)}{\sin(\kappa_{2}R)} \cos(\kappa_{2}R) E_{1}(\kappa_{2}x) = \alpha \left[ -\cos(\kappa_{2}x) \sin(\kappa_{2}R) F_{1}(\kappa_{2}R) + \cos(\kappa_{2}R) \sin(\kappa_{2}x) F_{1}(\kappa_{2}x) \right] + \alpha \cos(\kappa_{2}x) \cos(\kappa_{2}R) \left[ E_{1}(\kappa_{2}R) - E_{1}(\kappa_{2}x) \right]$$

$$u_2(x) - \frac{\cos(\kappa_2 x)}{\cos(\kappa_2 R)}$$
  
=  $\alpha \left[ -\cos(\kappa_2 x) \sin(\kappa_2 R) F_1(\kappa_2 R) + \cos(\kappa_2 R) \sin(\kappa_2 x) F_1(\kappa_2 x, \kappa_2 R) \right]$   
+  $\alpha \cos(\kappa_2 x) \cos(\kappa_2 R) \left[ E_1(\kappa_2 R) - E_1(\kappa_2 x, \kappa_2 R) \right]$ 

It follows immediately that

$$u_2(R) = u_1(R) = 1$$

and

$$u_{2}(0) = u_{1}(0) + \alpha \left( -\sin(\kappa_{2}R) F_{1}(\kappa_{2}R) + \cos(\kappa_{2}R) \left[ E_{1}(\kappa_{2}R) - E_{1}(0) \right] \right)$$
  
=  $u_{1}(0) + \alpha \left( -\sin(\kappa_{2}R) F_{1}(\kappa_{2}R) + \cos(\kappa_{2}R) E_{1}(\kappa_{2}R) \right)$ 

where

$$F_{1}(\kappa_{2}x,\kappa_{2}R) = -\frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} dz \ u_{1}(z) B(u_{1}(z))$$
  
$$= +\frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} dz \ \frac{\left(\frac{\cos z}{\cos \kappa_{2}R}\right)^{2}}{1 + t\left(\frac{\cos z}{\cos \kappa_{2}R}\right)^{2}}$$
  
$$= +\frac{1}{\kappa_{2}^{2}} \frac{1}{t} \left(\kappa_{2}x - \sqrt{P_{0}} \arctan[\sqrt{P_{0}}\tan(\kappa_{2}x)]\right)$$

and

$$E_{1}(\kappa_{2}x,\kappa_{2}R) = \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} dz \ u_{1}'(z) B(u_{1}(z))$$
  
$$= -\frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} dz \ u_{1}'(z) \ \frac{u_{1}(z)}{1 + tu_{1}^{2}(z)}$$
  
$$= +\frac{1}{\kappa_{2}^{2}} \frac{1}{2t} \ln \left( \frac{1 + t \frac{1}{\cos^{2}(\kappa_{2}R)}}{1 + t \frac{\cos^{2}(\kappa_{2}R)}{\cos^{2}(\kappa_{2}R)}} \right),$$

and

$$F_1(\kappa_2 R) := F_1(\kappa_2 R, \kappa_2 R), \qquad E_1(\kappa_2 R) := E_1(\kappa_2 R, \kappa_2 R),$$

and

$$E_{1}(\kappa_{2}R) - E_{1}(\kappa_{2}x, \kappa_{2}R) = \frac{1}{2t} \ln\left(\frac{1 + t\frac{1}{\cos^{2}(\kappa_{2}R)}}{1 + t}\right) - \frac{1}{2t} \ln\left(\frac{1 + t\frac{1}{\cos^{2}(\kappa_{2}R)}}{1 + t\frac{\cos^{2}(\kappa_{2}R)}{\cos^{2}(\kappa_{2}R)}}\right)$$
$$= \frac{1}{2t} \ln\left(\frac{1 + t\left(\frac{\cos\kappa_{2}x}{\cos\kappa_{2}R}\right)^{2}}{1 + t}\right)$$

where  $u_1(\kappa_2 R) = \frac{\cos \kappa_2 R}{\cos \kappa_2 R} = 1$  was used. Since  $0 \le \kappa_2 x \le \kappa_2 R$  it follows  $\cos(\kappa_2 x) \ge \cos(\kappa_2 R)$ , and for this reason  $E_1(\kappa_2 R) - E_1(\kappa_2 x, \kappa_2 R) \ge 0$ .

### A.3 Convergence of the iteration

#### A.3.1 Banach fixed point theorem

A solution of the operator equation

$$u = Au, \qquad u \in M, \tag{126a}$$

by means of the iteration method

$$u_{n+1} = Au_n, \qquad n = 0, 1, ...,$$
 (126b)

(where  $u_0 \in M$ ) is called a *fixed point* of the operator A.

Let

(a) M be a nonempty set in the Banach space X over  $\mathbb{K}$ , and

(b) the operator  $A: M \to M$  be k-contractive, i.e., by definition,

$$||Au - Av|| \le k ||u - v||$$
 for all  $u, v \in M$ , and fixed  $k, 0 \le k < 1$ .

Then the following hold true:

(i) Existence and uniqueness.

The original equation (126a) has exactly one solution u, i.e., the operator A has exactly one fixed point u on the set M.

(ii) Convergence of the iteration method.

For each given  $u_0 \in M$ , the sequence  $(u_n)$  constructed by (126b) converges to the unique solution of equation (126a).

# A.3.2 Estimation of $\left| \int_{-R}^{R} ds \ G(s,x) \ B(u(s)) \right|$ , even modes

Assuming the symmetry of function u, for even functions u it can be shown (see A.2.7) that

$$\int_{-R}^{R} ds \ G(s, x) \ B(u(s))$$

$$= \left[ -\cos(\kappa_2 x) \ \sin(\kappa_2 R) \ \tilde{F}_1(\kappa_2 R) + \cos(\kappa_2 R) \ \sin(\kappa_2 x) \ \tilde{F}_1(\kappa_2 x) \right]$$

$$+ \cos(\kappa_2 x) \ \cos(\kappa_2 R) \ \left[ \ \tilde{E}_1(\kappa_2 R) - \tilde{E}_1(\kappa_2 x) \right]$$

$$= \cos(\kappa_2 R) \ \left[ \ \sin(\kappa_2 x) \tilde{F}_1(\kappa_2 x) - \cos(\kappa_2 x) \tilde{E}_1(\kappa_2 x) \right]$$

$$- \frac{\cos(\kappa_2 R)}{\cos(\kappa_2 R)} \ \cos(\kappa_2 R) \ \left[ \ \sin(\kappa_2 R) \ \tilde{F}_1(\kappa_2 R) - \cos(\kappa_2 R) \ \tilde{E}_1(\kappa_2 R) \right]$$

where

$$\tilde{F}_1(\kappa_2 x) := -\frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \ u_1(s) B(u(s)) = +\frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \ \frac{\cos s}{\cos \kappa_2 R} \frac{u(s)}{1 + tu^2(s)}$$

and

$$\tilde{E}_1(\kappa_2 x) := \frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \ u_1'(s) B(u(s)) = + \frac{1}{\kappa_2^2} \int_0^{\kappa_2 x} ds \ \frac{\sin s}{\cos \kappa_2 R} \frac{u(s)}{1 + tu^2(s)}$$

Inserting this formulas, we have

$$\begin{aligned} &\int_{-R}^{R} ds \; G(s,x) \; B(u(s)) \\ &= \; \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} ds \left[ \; \sin(\kappa_{2}x) \cos s \; - \; \cos(\kappa_{2}x) \sin s \right] \frac{u(s)}{1 + tu^{2}(s)} \\ &- \; \frac{1}{\kappa_{2}^{2}} \; \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \; \int_{0}^{\kappa_{2}R} ds \left[ \; \sin(\kappa_{2}R) \; \cos s \; - \; \cos(\kappa_{2}R) \sin s \; \right] \frac{u(s)}{1 + tu^{2}(s)} \\ &= \; \frac{1}{\kappa_{2}^{2}} \; \int_{0}^{\kappa_{2}x} ds \; \sin(\kappa_{2}x - s) \frac{u(s)}{1 + tu^{2}(s)} \\ &- \; \frac{1}{\kappa_{2}^{2}} \; \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \; \int_{0}^{\kappa_{2}R} ds \; \sin(\kappa_{2}R - s) \frac{u(s)}{1 + tu^{2}(s)} \end{aligned}$$

Taking into account the estimation

$$|B(u)| = \left|\frac{u}{1+tu^2}\right| \le \frac{1}{2\sqrt{t}} \quad \forall t,$$

we can conclude

$$\begin{aligned} \left| \int_{-R}^{R} ds \ G(s,x) \ B(u(s)) \right| \\ &\leq \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} ds \ |\sin(\kappa_{2}x-s)| \ |B(u(s))| \ + \ \frac{1}{\kappa_{2}^{2}} \ \left| \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \right| \ \int_{0}^{\kappa_{2}R} ds \ |\sin(\kappa_{2}R-s)| \ |B(u(s))| \\ &\leq \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} ds \ |\sin(\kappa_{2}x-s)| \ + \ \frac{1}{\kappa_{2}^{2}} \ \left| \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \right| \ \int_{0}^{\kappa_{2}R} ds \ |\sin(\kappa_{2}R-s)| \\ &\leq \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( \int_{0}^{\kappa_{2}x} ds \ |\sin(\kappa_{2}x-s)| \ + \ \left| \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \right| \ \int_{0}^{\kappa_{2}R} ds \ |\sin(\kappa_{2}R-s)| \right) \end{aligned}$$

The substitution  $t := \kappa_2 x - s$  yields

$$\int_0^{\kappa_2 x} ds |\sin(\kappa_2 x - s)| = \int_{\kappa_2 x}^0 (-dt) |\sin t| = \int_0^{\kappa_2 x} dt |\sin t|$$

therefore

$$\begin{aligned} \left| \int_{-R}^{R} ds \ G(s,x) \ B(u(s)) \right| &\leq \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( \int_{0}^{\kappa_{2}x} dt \left| \sin t \right| + \left| \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \right| \int_{0}^{\kappa_{2}R} dt \left| \sin t \right| \right) \\ &\leq \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( 1 + \left| \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \right| \right) \int_{0}^{\kappa_{2}R} dt \left| \sin t \right| \\ &\leq \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( 1 + \frac{1}{\left| \cos(\kappa_{2}R) \right|} \right) \int_{0}^{\kappa_{2}R} dt \left| \sin t \right| \end{aligned}$$

Let

$$\kappa_2 R = \nu \frac{\pi}{2} + \delta, \qquad \nu = 0, 2, 4, \dots, \qquad \delta < \pi/2$$
  
 $= z \cdot \pi + \delta, \qquad z = 0, 1, 2, \dots$ 

Since  $\kappa_2 R = z \cdot \pi + \delta, z = 0, 1, 2, \dots$ , we have

$$\begin{aligned} \left| \int_{-R}^{R} ds \ G(s,x) \ B(u(s)) \right| &\leq \left| \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( 1 + \frac{1}{|(-1)^{z} \cos \delta|} \right) \int_{0}^{z \cdot \pi + \delta} dt \ |\sin t| \\ &= \left| \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( 1 + \frac{1}{\cos \delta} \right) \left( 2z + \int_{0}^{\delta} dt \ \sin t \right) \\ &= \left| \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( 1 + \frac{1}{\cos \delta} \right) \left( \nu + 1 - \cos \delta \right) \\ &\leq \left| \frac{1}{2\sqrt{t}} \frac{1}{\kappa_{2}^{2}} \left( 1 + \frac{1}{\cos \delta} \right) \left( \nu + 1 \right) \end{aligned}$$

Summary:

Let 
$$\kappa_2 R = \nu \cdot \pi/2 + \delta$$
,  $\nu = 0, 2, 4, ...$  and  $|\delta| < \pi/2$ .  

$$\left| \int_{-R}^{R} ds \ G(s, x) \ B(u(s)) \right| \leq \frac{1}{\kappa_2^2} \frac{1}{2\sqrt{t}} \left( 1 + \frac{1}{\cos \delta} \right) (\nu + 1)$$
(90)

Let

$$C_{\nu} := \frac{1}{\kappa_2^2} \left( 1 + \frac{1}{\cos \delta} \right) (\nu + 1) .$$
 (127)

Then

$$\int_{-R}^{R} ds \ G(s,x) \ B(u(s)) \bigg| \leq \frac{1}{2\sqrt{t}} C_{\nu} \ .$$

### A.3.3 Operator A maps S into itself

Consider the operator

$$A: \begin{cases} S \to F(S) \\ A(u) \mapsto \alpha \int_{-R}^{R} ds \ G(s,x) \ B(u(s)) + f(x) \end{cases}$$
(128)

$$\|Au\| := \|\alpha \int_{-R}^{R} ds \, G(s, x) \, B(u(s)) + f(x)\|$$
  
$$\leq |\alpha| \left\| \int_{-R}^{R} ds \, G(s, x) \, B(u) \right\| + \|f(x)\|$$
  
$$\leq |\alpha| \frac{1}{2\sqrt{t}} C_{\nu} + \|f(x)\|$$

For even modes we have

$$\|f(x)\| = \|\frac{\cos(\kappa_2 x)}{\cos(\kappa_2 R)}\| = \|\frac{\cos(\kappa_2 x)}{(-1)^z \cos\delta}\| \le \frac{1}{\cos\delta},$$

thus

$$||Au|| \leq |\alpha| \frac{1}{2\sqrt{t}} C_{\nu} + \frac{1}{\cos \delta}$$

 $C_{\nu}$  is a constant which depends on  $\gamma$ , R and  $\nu$ .
#### A.3.4 Contractivity of operator A

Let  $u, v \in S$ .

Then

$$\begin{aligned} \|Au - Av\| &= |\alpha| \left\| \int_{-R}^{R} ds \ G(s, x) \left( B(u) - B(v) \right) \right\| \\ &= |\alpha| \left\| \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} ds \sin(\kappa_{2}x - s) \left( B(u) - B(v) \right) \right\| \\ &- \frac{1}{\kappa_{2}^{2}} \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \int_{0}^{\kappa_{2}R} ds \sin(\kappa_{2}R - s) \left( B(u) - B(v) \right) \right\| \\ &\leq |\alpha| \left\| \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} ds |\sin(\kappa_{2}x - s)| \left| B(u) - B(v) \right| \\ &+ \frac{1}{\kappa_{2}^{2}} \left| \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \right| \int_{0}^{\kappa_{2}R} ds |\sin(\kappa_{2}R - s)| \left| B(u) - B(v) \right| \right\| \\ &\leq |\alpha| \quad \|B(u) - B(v)\| \\ &\left\| \frac{1}{\kappa_{2}^{2}} \int_{0}^{\kappa_{2}x} ds |\sin(\kappa_{2}x - s)| + \frac{1}{\kappa_{2}^{2}} \left| \frac{\cos(\kappa_{2}x)}{\cos(\kappa_{2}R)} \right| \int_{0}^{\kappa_{2}R} ds |\sin(\kappa_{2}R - s)| \right\| \\ &\leq |\alpha| \quad \|B(u) - B(v)\| \\ &\leq |\alpha| \quad \|B(u) - B(v)\| \quad C_{\nu} \end{aligned}$$

where

$$C_{\nu} := \frac{1}{\kappa_2^2} \left( 1 + \frac{1}{\cos \delta} \right) (\nu + 1)$$

$$||B(u) - B(v)|| = \left\| \frac{u}{1 + tu^2} - \frac{v}{1 + tv^2} \right\|$$
  
$$\leq \left\| \frac{1 - tuv}{(1 + tu^2)(1 + tv^2)} \right\| ||u - v||$$

Since

$$\left|\frac{1-tuv}{(1+tu^2)(1+tv^2)}\right| \le 1 \qquad \forall u, v \in \mathbb{R} \quad , \tag{129}$$

we have

$$||B(u) - B(v)|| \le ||u - v||.$$
(130)

# **B** Cylindrical waveguide

# B.1 Cutoff limit

Let

$$a(x) := x \frac{J_0(x)}{J_1(x)},$$
  
$$b(x) := x \frac{K_0(x)}{K_1(x)}.$$

In the limit  $\gamma \to \sqrt{\varepsilon_1}$ , taking into account  $\kappa_1^2 := \gamma^2 - \varepsilon_1$ , we have

$$\delta := \frac{\kappa_1}{\kappa} \ll 1 \,.$$

Since  $\kappa_2 := \varepsilon_2 - \gamma^2$  and  $\kappa^2 := \kappa_1^2 + \kappa_2^2$ , we have

$$\frac{\kappa_2^2}{\kappa^2} = 1 - \delta^2 \,.$$

Let further

$$R := R_m(1 + \chi)$$
 where  $R_m := \frac{j_{0m}}{\kappa}$  and  $\chi \ll 1$ 

Then

$$\kappa_2 R = \kappa_2 R_m (1 + \chi)$$

$$= \frac{\kappa_2}{\kappa} \kappa R_m (1 + \chi)$$

$$= \sqrt{1 - \delta^2} j_{0m} (1 + \chi)$$

$$= j_{0m} \left( 1 - \frac{1}{2} \delta^2 + \mathcal{O}(4) \right) (1 + \chi)$$

$$= j_{0m} \left( 1 - \frac{1}{2} \delta^2 + \chi + \mathcal{O}(\chi^2) \right)$$

and

$$\kappa_1 R = \delta \kappa R_m (1+\chi) = \delta j_{01} (1+\chi).$$

We assume that

$$\mathcal{O}(\chi^2) \ < \ \mathcal{O}(\delta^2) \ < \ \mathcal{O}(\chi) \ < \ \mathcal{O}(\delta)$$

(more exactly, it is assumed that  $\mathcal{O}(\chi) = \mathcal{O}(\delta^2 \ln[\delta])).$ 

Let  $j_{0m}$  be the  $m^{th}$  zero of Bessel function  $J_0$  and  $\varepsilon \ll 1$ . Let further

$$u = j_{0m} + \varepsilon$$

Taking into account  $J_0(j_{0m}) = 0$  and  $J'_0(u) = -J_1(u)$ , we have

$$\begin{aligned} a(u) &= a(j_{0m}) + \varepsilon \left. \frac{\partial a}{\partial u} \right|_{u=j_{0m}} + \mathcal{O}(\varepsilon^2) \\ &= u \frac{J_0(u)}{J_1(u)} \right|_{u=j_{0m}} + \varepsilon \frac{u}{J_1(u)} \left. \frac{\partial J_0(u)}{\partial u} \right|_{u=j_{0m}} + \mathcal{O}(\varepsilon^2) \\ &= -\varepsilon \frac{j_{0m}}{J_1(j_{0m})} J_1(j_{0m}) + \mathcal{O}(\varepsilon^2) \\ &= -\varepsilon j_{0m} + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Combination of

$$a(j_{0m}+\varepsilon) = -\varepsilon j_{0m} + \mathcal{O}(\varepsilon^2)$$

and

$$\kappa_2 R = j_{0m} + j_{0m} (\chi - \frac{1}{2} \delta^2)$$

leads to

$$a(\kappa_2 R) = -j_{0m}^2 \left(\chi - \frac{1}{2}\delta^2\right) + \mathcal{O}(\chi^2).$$
(132a)

Taking into account the asymptotic behavior of the MacDonald functions,

$$K_0(x) \sim -\ln(\frac{\Gamma}{2}x) \qquad (x \ll 1),$$
  

$$K_1(x) \sim \frac{1}{x} \qquad (x \ll 1)$$

where  $\Gamma := e^{\gamma}$  and  $\gamma \approx 0.577$  (Euler constant), and taking into account  $\kappa_1 R = \delta \kappa R \ll 1$ , one obtains

$$b(\kappa_{1}R) = -j_{0m}^{2}\delta^{2}(1+\chi)^{2}\ln(\frac{\Gamma}{2}j_{0m}\delta(1+\chi))$$
  
$$= -j_{0m}^{2}\delta^{2}(1+\chi)^{2}\left(\ln[\frac{\Gamma}{2}j_{0m}\delta] + \ln[1+\chi]\right)$$
  
$$= -j_{0m}^{2}\delta^{2}(1+\chi)^{2}\left(\ln[\frac{\Gamma}{2}j_{0m}\delta] + \chi + \mathcal{O}(\chi^{2})\right)$$
  
$$= -j_{0m}^{2}\delta^{2}\ln[\frac{\Gamma}{2}j_{0m}\delta] + \mathcal{O}(\chi^{2})$$
(132b)

Summary:

$$\kappa_{2}R = j_{0\nu} \left( 1 - \frac{1}{2}\delta^{2} + \chi + \mathcal{O}(\chi^{2}) \right)$$
  

$$\kappa_{1}R = \delta j_{0\nu} \left( 1 + \chi + \mathcal{O}(\chi^{2}) \right)$$
  

$$a(\kappa_{2}R) = -j_{0\nu}^{2} \left( \chi - \frac{1}{2}\delta^{2} \right) + \mathcal{O}(\chi^{2})$$
(132a)  

$$b(\kappa_{1}R) = -j_{0\nu}^{2}\delta^{2} \ln[\frac{\Gamma}{2}j_{0\nu}\delta] + \mathcal{O}(\chi^{2})$$
(132b)

1

Inserting (132a) and (132b) in the linear dispersion relation a + b = 0, we have

$$-j_{0m}^{2} \left(\chi - \frac{1}{2}\delta^{2}\right) - j_{0m}^{2}\delta^{2} \ln\left[\frac{\Gamma}{2}j_{0m}\delta\right] = 0$$
$$\chi(\delta) = \left(\frac{1}{2} - \ln\left[\frac{\Gamma}{2}\delta j_{0m}\right]\right)\delta^{2}.$$

or

Therefore if  $\delta \ll 1$  (cutoff limit), the radius  $R = R(\delta)$  converges towards  $j_{0m}/\kappa$  according to

$$R(\delta) = \frac{j_{0m}}{\kappa} \left\{ 1 + \left(\frac{1}{2} - \ln[\frac{\Gamma}{2}\delta j_{0m}]\right) \delta^2 \right\} .$$
(133)

•

## **B.2** Convergence

## **B.2.1** Estimation of $\|\rho G\|$

Introducing the abbreviation

$$D(u) := J_1(u) \frac{Y_1'(\kappa_2 R)}{J_1'(\kappa_2 R)} - Y_1(u) ,$$

the Greens function G can be written

$$G(\kappa_2 \rho, \kappa_2 r) = \begin{cases} \frac{\pi}{2} J_1(\kappa_2 \rho) D(\kappa_2 r) & \rho < r \le R \\ \frac{\pi}{2} D(\kappa_2 \rho) J_1(\kappa_2 r) & r < \rho \le R \end{cases}$$

Then

$$s(\kappa_2 r) := \int_0^R d\rho |\rho G(\rho, r)| = s_1(r) + s_2(r)$$

where

$$s_1(\kappa_2 r) := \frac{\pi}{2} |D(\kappa_2 r)| \int_0^r d\rho \, \rho \, |J_1(\kappa_2 \rho)|$$

and

$$s_2(\kappa_2 r) := \frac{\pi}{2} |J_1(\kappa_2 r)| \int_r^R d\rho \ \rho |D(\kappa_2 \rho)|$$

The substitution

$$t := \kappa_2 \rho$$
$$u := \kappa_2 r$$
$$U := \kappa_2 R$$

yields

$$s(u,U) = \frac{1}{\kappa_2^2} \int_0^{\kappa_2 R} dt |t G(t,u)|$$
  
=  $\frac{\pi}{2} \frac{1}{\kappa_2^2} \left( |D(u)| \int_0^u dt t |J_1(t)| + |J_1(u)| \int_u^U dt t |D(t)| \right)$ 

For the following considerations, the condition

 $j_{01} \leq \kappa_2 R \leq j_{11}$  resp.  $j_{01} \leq U \leq j_{11}$  (134)

where  $j_{01} = 2.405...$  and  $j_{11} = 3.832...$  is assumed.

 $\frac{\text{Properties of function } s = s(u, U)}{\text{Let}}$ 

$$D(u,U) := J_1(u) \frac{Y_1'(U)}{J_1'(U)} - Y_1(u)$$

Then

$$D'(u,U) = J'_1(u) \frac{Y'_1(U)}{J'_1(U)} - Y'_1(u)$$

and therefore

$$D'(U,U) = 0. (135)$$

By means of a 2D-plot, it is easy to show that if  $u \in [0, U]$  and  $U \in [j_{01}, j_{11}]$ , then there exists exactly one zero  $d_0(U)$  of D(u, U) and D(u, U) < 0 if  $u > d_0(U)$ .

Let  $u > d_0(U)$ . Since  $J_1(u) \ge 0$  if  $0 \le u \le j_{11}$ , it follows

$$s(u,U) = -\frac{\pi}{2} \frac{1}{\kappa_2^2} \left( D(u) \int_0^u dt \, t \, J_1(t) + J_1(u) \int_u^U dt \, t \, D(t) \right)$$

The derivative is given by

$$\begin{aligned} -\frac{2}{\pi}\kappa_2^2 \,s'(u,U) &= D'(u) \,\int_0^u dt \,t \,J_1(t) &+ D(u) \cdot u J_1(u) \\ &+ J_1'(u) \,\int_u^U dt \,t \,D(t) &- J_1(u) \cdot u D(u) \\ &= D'(u) \,\int_0^u dt \,t \,J_1(t) &+ J_1'(u) \,\int_u^U dt \,t \,D(t) \end{aligned}$$

With D'(U, U) = 0 it follows immediately that

$$s'(U,U) = 0 (136)$$

The following 3D-Plot of s=s(u,U) (Abb. 26) clearly shows that if  $j_{01} \leq U \leq j_{11}$  then

$$s(u, U) \leq s(j_{01}, j_{01})$$



Figure 25:  $s = s(u, U); \quad 0 \le u \le U; \quad j_{01} \le U \le j_{11}$ 

Abb. 26 confirms in accordance with (136) that the maximum s = s(u, U) is build up if u = U and furthermore if  $U = j_{01}$ .



Figure 26:  $s = s(u, j_{01}); \quad 0 \le u \le U; \quad U = j_{01}$ 

Then the following estimation holds:

$$\begin{aligned} \|\rho \ G(\rho, r)\| &:= \max_{r \in [0, R]} s(\kappa_2 r, \kappa_2 R) \\ &= \frac{1}{\kappa_2^2} \max_{u \in [0, U]} s(u, U) \quad \text{where } j_{01} \le U \le j_{11} \\ &\le \frac{1}{\kappa_2^2} s(j_{01}, j_{01}) \\ &= \frac{1}{\kappa_2^2} s_1(j_{01}, j_{01}) \quad \left( s_2(j_{01}, j_{01}) = 0 \text{ due to } u = U \right) \\ &= \frac{1}{\kappa_2^2} \frac{\pi}{2} |D(j_{01}, j_{01})| \int_0^{j_{01}} dt \ t \ J_1(t) \end{aligned}$$

Taking into account

$$D(U,U) = \left(J_1(U) \frac{Y_1'(U)}{J_1'(U)} - Y_1(U)\right)$$
  
=  $\frac{1}{J_1'(U)} \left(J_1(U) Y_1'(U) - Y_1(U)J_1'(U)\right)$   
=  $\frac{1}{J_1'(U)} \frac{2}{\pi} \frac{1}{U}$  (Snyder/Love, (37-76a), S. 714) (137)

and  $U = j_{01}$ , it follows

where  

$$\begin{aligned} \|\rho \ G(\rho, r)\| &\leq \frac{c_1}{\kappa_2^2} \end{aligned} (138) \\
c_1 &:= \frac{1}{|J_1'(j_{01})|} \frac{1}{j_{01}} \int_0^{j_{01}} dt \ t \ J_1(t) \approx 2.83214 \,. \end{aligned}$$

#### B.2.2 Contraction

The contractivity can be shown with the help of

$$\left|\frac{1 - tu_1 u_2}{(1 + tu_1^2)(1 + tu_2^2)}\right| \le 1 \qquad \forall u_1, u_2, \quad t > 0.$$
(139)