Resultants:
A Tool for Chow Varieties

Dissertation

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Introduction

In [16] Lawson showed that a suitable stabilization $\mathcal{C}$ of the Chow variety of effective algebraic cycles on complex projective spaces is homotopy equivalent to the product $\prod_{i>0} K(\mathbb{Z}, 2i) =: K(\mathbb{Z}, even)$ of Eilenberg MacLane spaces $K(\mathbb{Z}, 2i)$. The proof of the equivalence is based on the complex suspension theorem, which is fundamental for Lawson homology [8]. The original proof of this theorem uses geometric measure theory. Here we provide an algebraic and, as we think, more elementary proof (Thm. 4.1) which goes back to the construction of Chow varieties via resultants [5]. Once the basic continuity statements are proved, we can stay close to the geometric intuition [16], which is behind the measure theoretic arguments, by the Chow/Van der Waerden approach to Chow varieties. The main observation is that one can compute the Chow form of a suspended cycle (Prop. 3.3, Thm. 7.1) from the original Chow form, showing that the suspension is an algebraic map.

In order to prove that the join pairing is algebraic (Thm. 3.4), hence continuous, one just has to combine the algebraicity of the complex suspension with well known results (Barlet [1]). (Independently, Barlet [2] obtained a proof of Thm. 3.4 using different methods.) As a computational complement to these arguments, Dalbec [6], describes an algorithm for computing Chow coordinates of joins.

In [4] the join pairing was used to define an infinite loop space structure on $K(\mathbb{Z}, even)$ which is compatible with the infinite loop space structure on BU induced by Whitney sum. The required continuity statements (Lemma 5.1) for the operation of the linear isometries operad [4] also follow in a natural way from the Chow/Van der Waerden picture.

The paper is divided into three parts. The first one contains three sets of results:

- The explicit description of Chow varieties over $\mathbb{C}$ and the proof for the algebraicity of the join pairing.
- Another, more algebraic, proof of the complex suspension theorem in characteristic 0 (section 4). (For a quite different algebraization see [8].)
- The continuity of the action of the linear isometries operad on the group completion of a stabilized Chow variety (section 5), which makes the total Chern class map an infinite loop map.

Throughout the first part we work in the category kTop of compactly generated, weak Hausdorff spaces [21]. The proof of the complex suspension theorem is given for algebraic cycles with support in $\mathbb{P}^n_{\mathbb{C}}$ but they directly work for cycles with support on an algebraic subset $X \subset \mathbb{P}^n_{\mathbb{C}}$ (see also [16]), except that one has to take care of π_0–phenomena (compare [8],[16, p. 265]).

In the second part we work in the general context of Chow varieties over fields of arbitrary characteristics. The problem that arises in characteristic $q \neq 0$ is the occurrence of inseparabilities which leads to the use of techniques of Galois theory. This is necessary for the definition of Chow forms and especially for their relation to resultants. The algebraicity of the complex suspension is then proved for arbitrary fields (Thm. 7.1).

The last part discusses Chow varieties of projective varieties X. Defining Chow varieties of arbitrary projective algebraic sets X now means choosing an embedding $i : X \hookrightarrow \mathbb{P}^n$ and taking the Chow variety $\mathcal{C}_{r,d}(X, i)$ of $i(X)$.

For the definition of Lawson homology one just needs to know that the homotopy type of the group completion of the Chow monoid is independent of the embedding i. However, Hoyt [14] proved the homeomorphism type of $\coprod_{d\geq 0} \mathcal{C}_{r,d}(X, i)$ to be independent of embeddings. The purpose of the third part is to clarify and straighten out Hoyt's approach for arbitrary fields. In fact by Barlet [1] one knows that $\mathcal{C}_{r,d}(X, i)$ is independent of i up to isomorphism of algebraic sets in case of $K = \mathbb{C}$. On the other hand Nagata [23] shows that the Chow variety of 0-dimensional cycles of degree 2 in $\mathbb{P}^3_{\mathbb{Z}_2}$ is not normal, but there is an embedding into some $\mathbb{P}^n_{\mathbb{Z}_2}$ such that the Chow variety of the embedded $\mathbb{P}^3_{\mathbb{Z}_2}$ becomes normal. Hence those two Chow varieties can not be isomorphic algebraic sets.

In the first section of part three (section 8) we will reduce the proof of Hoyt's theorem to the problem of determining the greatest common divisor of a set of multihomogeneous polynomials. The algebraicity of the greatest common divisor in the coefficients of the multihomogeneous polynomials is shown in section 9.

The paper constitutes a new approach to current results about Chow varieties by refocusing on the concept of resultants. Especially, the operation of the linear isometries on Chow varieties is well understood in the context of resultants as they are generalizations of determinants. In this picture the algebraicity of the complex suspension and the complex join pairing could be proved here.

1 Notations, foundations and basic definitions

There are many different ways to define the Chow form of a variety $V \subset \mathbb{P}^n$ ([5],[13],[10]). The definition of Chow and Van der Waerden is constructive and therefore very useful for proofs of algebraicity. We first introduce some notations and basic definitions:

Definition 1.1. Let $K \subset \Omega$ be a field extension.

1. A projective algebraic set $V_K \subset \mathbb{P}^n_\Omega$ *over* K is the set of common zeros of a homogeneous ideal $I \subset K[x_0, \ldots, x_n]$.

2. The *vanishing ideal* $I_K(V) \subset K[x_0, \ldots, x_n]$ of a projective algebraic set $V_K \subset \mathbb{P}^n_\Omega$ is the homogeneous ideal generated by

$$\{f \in K[x_0, \ldots, x_n] \mid f \text{ homogeneous and } f(\underline{v}) = 0 \; \forall \underline{v} \in V\} .$$

An algebraic set $V_K \subset \mathbb{P}^n_\Omega$ is called *irreducible* or *a variety* iff the homogeneous ideal $I_K(V) \subset K[x_0, \ldots, x_n]$ is prime. The dimension of a variety V_K is its dimension as a topological subspace of $\mathbb{P}^n_\Omega$ in the Zariski topology. It follows from the Noetherian Normalization Theorem [15, Chap. II, Thm. 3.1] that the dimension of $V_K \subset \mathbb{P}^n_\Omega$ is independent of the extension field Ω of K, that means

$$\dim V_K = \dim V_K \cap \mathbb{P}^n_K .$$

For a fixed ground field F from now on Ω will denote a fixed *universal field* in the sense of Van der Waerden [27], given as the algebraic closure of $F(U)$, where U is a countably infinite set of algebraically independent elements over F. K is always a subfield of Ω with finite transcendence degree over F.

Given $\underline{\xi} = (\xi_0, \ldots, \xi_n) \in \Omega^{n+1}$, one denotes $K[\underline{\xi}] := K[\xi_0, \ldots, \xi_n]$. If $\underline{\xi} \in \mathbb{P}^n_\Omega$ then $K[\underline{\xi}]$ depends on the choice of a representative of $\underline{\xi}$ in Ω^{n+1}. The definition of a special representative of points in $\mathbb{P}^n_\Omega$ makes this ring adjunction unique up to canonical isomorphisms. In the following the homogeneous coordinates of a point $\underline{\xi}$ will be denoted $[\xi_0 : \cdots : \xi_n]$.

Definition 1.2. Let $[\xi_0 : \cdots : \xi_n] = \underline{\xi} \in \mathbb{P}^n_\Omega$ and i minimal such that $\xi_i \neq 0$. Then

$$\tau\underline{\xi} := \left(\tau\frac{\xi_0}{\xi_i}, \ldots, \tau\frac{\xi_i}{\xi_i}, \ldots, \tau\frac{\xi_n}{\xi_i}\right) \in \Omega^{n+1}, \; \tau \in \Omega - \overline{K(\xi_0, \ldots, \xi_n)}.$$

For every point $\underline{\xi} \in \mathbb{P}^n_\Omega$ the ring $K[\tau\underline{\xi}]$ depends on choices of τ up to canonical isomorphisms, which extend to K–automorphisms of Ω. From now on the condition $f(\underline{\xi}) = 0$, $f \in K[x_0, \ldots, x_n]$, is understood as $f(\tau\underline{\xi}) = 0$ for $\underline{\xi} \in \mathbb{P}^n_\Omega$. Also one defines $K(\tau\underline{\xi})$ to be the quotient field of $K[\tau\underline{\xi}]$.

The advantage of viewing varieties in $\mathbb{P}^n_K$ as varieties in $\mathbb{P}^n_\Omega$ *over* K is that varieties in $\mathbb{P}^n_\Omega$ over K are completely described by single points in $\mathbb{P}^n_\Omega$.

Definition 1.3. Let $V_K \subset \mathbb{P}^n_\Omega$ be an algebraic set over K. A point $\underline{\xi}$ is a *general point* of V_K if it satisfies the following conditions:

1. $\underline{\xi} \in V_K$
2. whenever $f \in K[x_0, \ldots, x_n]$ satisfies $f(\underline{\xi}) = 0$, then it follows that $f \in I_K(V) \subset K[x_0, \ldots, x_n]$.

As with the definition of $K[\tau\underline{\xi}]$ general points of a variety V_K are only determined up to K–automorphisms of Ω, that is, if $\underline{\xi}$ is a general point of V_K and σ a K–automorphisms of Ω, then $[\sigma(\xi_0) : \cdots : \sigma(\xi_n)]$ is a general point of V_K, too. We also say the general points are conjugate to each other. It is easy to show that an algebraic set in $\mathbb{P}^n_\Omega$ has a general point iff it is irreducible. One can also prove that the projective coordinate ring of a variety V_K is naturally isomorphic to $K[\tau\underline{\xi}]$ for a general point $\underline{\xi}$ of V_K [27]. The isomorphism is given by

$$\begin{array}{rccll} E_{\tau\underline{\xi}} : & K[x_0, \ldots, x_n]/I_K(V) & \longrightarrow & K[\tau\underline{\xi}] & \\ & z & \longmapsto & z & , z \in K \\ & x_i & \longmapsto & (\tau\underline{\xi})_i & , i = 0, \ldots, n \ . \end{array}$$

Therefore a variety V has infinitely many general points iff $\dim V > 0$.

For the definition of the Chow form of a variety over K one looks at the intersections of varieties over K with K–general hyperplanes.

Definition and Notation 1.4. Let $u_i^j \in \Omega$, $i = 0, \ldots, n$, $j = 1, \ldots, r$ be algebraically independent over K. Denote

$$\mathcal{U}(n, r) := \begin{pmatrix} u_0^1 & \cdots & u_n^1 \\ \vdots & & \vdots \\ u_0^r & \cdots & u_n^r \end{pmatrix} \ .$$

The *K–general linear subspace* H_{n-r} in $\mathbb{P}^n_\Omega$ of dimension $(n-r)$ is the zero set of $\mathcal{U}(n,r)\cdot\underline{x}$ viewed as r forms in $\Omega[x_0,\ldots,x_n]$. H_{n-1} is also called a K–general hyperplane.

Observe that algebraic independence of the u_i^j implies linear independence of the $\underline{u}^j=(u_0^j,\ldots,u_n^j)$, because linear dependence can be expressed in terms of determinants of submatrices of $\mathcal{U}(n,r)$.

Lemma 1.5. *Let $V_K\subset\mathbb{P}^n_\Omega$ be a variety of dimension r and H a K–general hyperplane given by $\langle\underline{u},\underline{x}\rangle=0$. Then the intersection of V_K and H is a variety over $K(\underline{u})$ of dimension $r-1$.*

Proof. [28, p. 140]. □

Lemma 1.5 proves the following theorem:

Theorem 1.6. *Let $V_K\subset\mathbb{P}^n_\Omega$ be a variety of dimension r over K and H_{n-r} a K–general $(n-r)$–dimensional linear subspace of $\mathbb{P}^n_\Omega$. The intersection $V_K\cap H_{n-r}$ is a 0–dimensional variety over $K(\underline{u}^1,\ldots,\underline{u}^r)$.*

0–dimensional varieties over a field which is not algebraically closed, consist of finitely many points, and all of them are general points [27, §129]. This fact will be crucial in the construction of Chow varieties in section 2.

One important ingredient for the construction of Chow varieties is the notion of resultants. Resultants can be viewed as a generalization of determinants, in case of homogeneous equations of any degree, instead of fixed degree 1:

Reminder 1.7. Let S be a subring of Ω, and let $f_1,\ldots,f_k\in S[x_0,\ldots,x_n]$ be homogeneous polynomials. The *resultant of the homogeneous equations*

$$f_1=\cdots=f_k=0$$

is an ideal $J(f)\subset S$, which is generated by determinants of matrices, where one fixed row of such a matrix consists of coefficients of one f_j [27, § 130]. Therefore there is a set of elements in the polynomial ring in the coefficients of $f_1,\ldots,f_k$ over $\mathbb{Z}/(char\,\Omega)\subset S$, whose image in S generates the ideal $J(f)$. Because the ideal $J(f)$ is generated by determinants, it is homogeneous in the coefficients of each f_j, $j=1,\ldots,k$.

We also say $J(f)$ is the resultant of the homogeneous polynomials $f_1, \ldots, f_k$. The ideal $J(f)$ has the following property [27, §130]:

$$\begin{aligned} & f_1 = \cdots = f_k = 0 \text{ has a solution in } \mathbb{P}^n_\Omega \\ \iff \quad & J(f) \text{ is the zero ideal in } S \ . \end{aligned}$$

Now suppose $S = R[y_0, \ldots, y_m]$, where $y_0, \ldots, y_m \in \Omega$ are algebraically independent over R. Let $I(f) \subset R[y_0, \ldots, y_m][x_0, \ldots, x_n]$ denote the ideal which is generated by $f_1, \ldots, f_k$, where the coefficients of $f_1, \ldots, f_k$ are homogeneous polynomials in $y_0, \ldots, y_m$. The zero set of $I(f)$ in $\mathbb{P}^n_\Omega$ is empty iff $J(f) \neq 0$, but the ideal $J(f)$ depends on the choice of generators of $I(f)$. The observation is that the radical of the ideal $J(f)$ is independent of the choices of generators of $I(f)$, i.e. only depends on the radical of the ideal $I(f)$.

To see this let $g_1, \ldots, g_l \in R[y_0, \ldots, y_m][x_0, \ldots, x_n]$ be homogeneous polynomials. Let $I(g)$ be the ideal which is generated by $g_1, \ldots, g_l$. Furthermore, let $J(g) \subset R[y_0, \ldots, y_m]$ be the resultant of $g_1 = \cdots = g_l = 0$.

Suppose that $\sqrt{I(g)} = \sqrt{I(f)}$. Then $g_1 = \cdots = g_l = 0$ has a solution in $\mathbb{P}^n_\Omega$ iff $f_1 = \cdots = f_k = 0$ has a solution in $\mathbb{P}^n_\Omega$. Hence $J(g) = 0$ if and only if $J(f) = 0$.
Since $J(g)$ and $J(f)$ are ideals in the polynomial ring $R[y_0, \ldots, y_m]$ and $J(g) \neq 0 \neq J(f)$, we can determine the zero sets of $J(f)$ and $J(g)$. Suppose now $\underline{z} \in \mathbb{P}^m_\Omega$ is a common zero of the elements of $J(f)$. This zero determines a map

$$\begin{array}{rccll} \varphi: & R[y_0, \ldots, y_m][x_0, \ldots, x_n] & \longrightarrow & R[\tau\underline{z}][x_0, \ldots, x_n] & \\ & y_i & \longmapsto & (\tau\underline{z})_i & , i = 0, \ldots, m \\ & s & \longmapsto & s & , s \in R[x_0, \ldots, x_n] \end{array}$$

Now $\varphi(f_1), \ldots, \varphi(f_k)$ have a common zero in $\mathbb{P}^n_\Omega$, hence $\varphi(g_1), \ldots, \varphi(g_l)$ have a common zero in $\mathbb{P}^n_\Omega$. This again means, that $\underline{z}$ is a common zero of the elements in $J(g)$. Hence the zero sets of $J(f)$ and $J(g)$ are just the same, which in turn implies that $\sqrt{J(g)} = \sqrt{J(f)}$.

Therefore we use $\sqrt{J(f)}$ as definition for the *resultant of the ideal* $I(f)$.

Part I

Join Pairing and Complex Suspension

2 Review of the Chow – Van der Waerden approach to Chow varieties

In this section a definition for Chow varieties over $\mathbb{C}$ is given. Defining Chow varieties over arbitrary fields works analogously, but one has to be very careful in characteristic $q \neq 0$. The general case is treated in section 6.

Construction and Definition 2.1. Let $V_{\mathbb{C}} \subset \mathbb{P}^n_\Omega$ be a variety of dimension r over $\mathbb{C}$ and H_{n-r} a $\mathbb{C}$-general $(n-r)$-dimensional linear subspace of $\mathbb{P}^n_\Omega$. Further let $V_{\mathbb{C}} \cap H_{n-r} = \{\underline{q}^1, \ldots, \underline{q}^d\}$. As mentioned in section 1 the $\underline{q}^k$, $k = 1, \ldots, d$, are general points of $V_{\mathbb{C}} \cap H_{n-r}$ viewed as a variety over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$. Therefore they are conjugate to each other over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$. On the other hand it is obvious that $\underline{q}^k$, $k = 1, \ldots, d$, are general points of $V_{\mathbb{C}}$ viewed as a variety over $\mathbb{C}$.

Without loss of generality, one can fix $m \in \{0, \ldots, n\}$ such that $q^k_m \neq 0$ for all $k = 1, \ldots, d$. Now choose

$$\underline{p}^k := \left(\frac{q^k_0}{q^k_m}, \frac{q^k_1}{q^k_m}, \ldots, \frac{q^k_m}{q^k_m}, \ldots, \frac{q^k_n}{q^k_m} \right) \in \Omega^{n+1}$$

as a representation of $\underline{q}^k$, $k = 1, \ldots, d$. The p^k_i are now algebraic over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$, because $V_{\mathbb{C}} \cap H_{n-r}$ has dimension 0.

Let $u^0_0, \ldots, u^0_n$ be algebraically independent over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$ and define

$$\begin{aligned} L_k &:= \langle \underline{u}^0, \underline{p}^k \rangle && , k = 1, \ldots, d \\ G(\underline{u}^0) &:= \prod_{k=1}^{d} L_k = \prod_{k=1}^{d} \sum_{i=0}^{n} u^0_i p^k_i && \in \overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}[\underline{u}^0] \, . \end{aligned}$$

Now let Z be the splitting field of the minimal polynomials of the p^k_i, $i = 0, \ldots, n$, $k = 1, \ldots, d$, and notice that Z is a Galois extension of $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$, because the characteristic of $\mathbb{C}$ is 0. Suppose φ is an element of the Galois

group $Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))$. The general points of a variety are conjugated to each other, therefore $\varphi(\underline{p}^k)$ must be equal to $\underline{p}^l$ for some $l \in \{1, \ldots, d\}$. Hence one obtains $\varphi(G(\underline{u}^0)) = G(\underline{u}^0)$, that is, the coefficients of G remain invariant under the action of the Galois group $Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))$, and are therefore elements of $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$.

Let $c \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$ be the common denominator of the coefficients of $G(\underline{u}^0)$. Then

$$F(\underline{u}^0, \ldots, \underline{u}^r) := cG(\underline{u}^0)$$

is an element of $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$.

The form $F(\underline{u}^0, \ldots, \underline{u}^r)$ is a *Chow form* of $V_{\mathbb{C}}$. The *degree* of $V_{\mathbb{C}}$ is defined to be the degree d of F as a polynomial in $\mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r][\underline{u}^0]$. In fact F is *multihomogeneous* of *multidegree* d, which means it is homogeneous of degree d as a form in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^{j-1}, \underline{u}^{j+1}, \ldots, \underline{u}^r][\underline{u}^j]$ for all $j = 0, \ldots, r$ [5, p. 694]. It follows directly from the construction, that the degree of a variety is the number of points in which a general linear $(n-r)$–dimensional linear hyperplane meets the variety. This definition of the degree of a variety coincides with the definition of the degree by the Hilbert polynomial [12, p. 48].

The Chow form of a variety is uniquely determined up to multiplication by elements in $\mathbb{C}^*$. Viewed as an element of $\overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}[\underline{u}^0]$ the Chow form F of a variety V splits into linear factors $\langle \underline{u}^0, \underline{p}^k \rangle$, where every $\underline{p}^k$ is a general point of V. Therefore two varieties are equal iff their Chow forms coincide up to a factor in $\mathbb{C}^*$.

In the following some properties of Chow forms are described, especially their relations to resultants.

Lemma 2.2. *The Chow form $F(\underline{u}^0, \ldots, \underline{u}^r)$ of a r–dimensional variety $V_{\mathbb{C}}$ is irreducible in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$.*

Proof. [5, p. 693]. □

The Chow form of a variety $V_{\mathbb{C}}$ arises from the intersection with H_{n-r}. If one cuts this 0–dimensional set with another sufficiently general hyperplane, the intersection is empty. Therefore one gets a set of homogeneous polynomials, whose resultant is not the zero ideal. In fact the radical of this resultant is generated by the Chow form:

Theorem 2.3. *Let $V_{\mathbb{C}} \subset \mathbb{P}^n_{\Omega}$ be a r–dimensional variety, and let $f_1, \ldots, f_k$ be homogeneous generators of $I_{\mathbb{C}}(V) \subset \mathbb{C}[x_0, \ldots, x_n]$.*

The Chow form $F(\underline{u}^0, \ldots, \underline{u}^r)$ *generates the radical* $\sqrt{R} \subset \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ *of the resultant* R *of the following equations:*

$$\begin{aligned} f_1(x_0, \ldots, x_n) = \cdots = f_k(x_0, \ldots, x_n) &= 0 \\ u_0^0 x_0 + \cdots + u_n^0 x_n &= 0 \\ \mathcal{U}(n,r) \cdot \underline{x} &= 0 \end{aligned} \tag{1}$$

viewed as polynomials in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r][x_0, \ldots, x_n]$.

Proof. The first step is to show that $\sqrt{R}$ is a prime ideal in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$: The proof of this statement uses the same techniques as the proof of Lemma 1.5 in [28, p. 140]. Let

$$A := \{(\underline{p}, \underline{y}^0, \ldots, \underline{y}^r) \in \mathbb{P}_\Omega^n \times (\mathbb{P}_\Omega^n)^{r+1} \mid \underline{p} \in V_\mathbb{C}\ ,\ \langle \underline{p}, \underline{y}^0 \rangle = \cdots = \langle \underline{p}, \underline{y}^r \rangle = 0\}.$$

The vanishing ideal of A is generated by the polynomials in (1). The projection pr_1 onto the first factor is again the set $V_\mathbb{C} = pr_1(A)$ and the projection pr_2 of A onto the second factor $(\mathbb{P}_\Omega^n)^{r+1}$ gives us the zero set of R.
Now we can apply [25, I.6.3 Th. 8] to pr_1 and obtain that A is irreducible. Hence $pr_2(A)$ is irreducible, which proves that $\sqrt{R}$ is prime.

Now we can prove the theorem:
The Chow form F of $V_\mathbb{C}$ is $F(\underline{u}^0, \ldots, \underline{u}^r) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$, where $\{\underline{p}^1, \ldots, \underline{p}^d\}$ is the intersection of the zero set of $\mathcal{U}(n,r) \cdot \underline{x}$ and $V_\mathbb{C}$, $c \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$. Denote by $Z \subset \overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}$ the splitting field of the p_j^i, $i = 1, \ldots, d$, $j = 0, \ldots, n$. Now let $R \cdot Z[\underline{u}^0]$ be the resultant of the polynomials in (1), when viewed as elements in $Z[\underline{u}^0][x_0, \ldots, x_n]$. The zero set of $R \cdot Z[\underline{u}^0]$ equals the zero set of $\prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$. This means that the radical of the ideal which is generated by R in $Z[\underline{u}^0]$ is the ideal which is generated by F:

$$(F)_{Z[\underline{u}^0]} = \sqrt{R \cdot Z[\underline{u}^0]} \quad \text{in } Z[\underline{u}^0]\ .$$

Cutting down with $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ we first obtain

$$(F)_{\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]} \subset (F)_{Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]\ .$$

Now let $h \in Z[\underline{u}^0]$ such that $F \cdot h \in (F)_{Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$. And let σ be an element of the Galois group $Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))$. Then $F \cdot h = \sigma(F \cdot h) =$

$F \cdot \sigma(h)$ and this means that $h \in \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)[\underline{u}^0]$, because Z is a Galois extension. Therefore $h = \frac{h_1}{h_2}$, $h_1 \in \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r], h_2 \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$, and the common divisor of h_1 and h_2 is 1. Now h_2 must be a divisor of F, and since F is irreducible and $\underline{u}^0$ does not appear in h_2, then $h_2 \in \mathbb{C}$. Therefore

$$(F)_{Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r] = (F)_{\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]} \ .$$

Now it remains to show that $\sqrt{R}$ equals $\sqrt{R \cdot Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$. Obviously $\sqrt{R} \subset \sqrt{R \cdot Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r] = (F)_{Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r] = (F)_{\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]}$. For the inverse inclusion we have to show that $F \in \sqrt{R}$.

We have already seen that $F \in \sqrt{R \cdot Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$. This means that

$$F^k = \sum_{i=1}^{l} g_i z_i \ \ , g_i \in R \ , z_i \in Z[\underline{u}^0] \ , k \in \mathbb{N}^* \ .$$

Since Z is the splitting field of finitely many polynomials, Z is a Galois extension of $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$ and $Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))$ is finite. Furthermore, F and g_i, $i = 1, \ldots, l$ are elements in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ and therefore they stay invariant under the action of $Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))$. Now one obtains:

$$\begin{aligned} \mid Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)) \mid \cdot F^k &= \sum_{\sigma \in Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))} \sigma(\sum_{i=1}^{l} g_i z_i) \\ &= \sum_{i=1}^{l} g_i \underbrace{\sum_{\sigma \in Gal(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))} \sigma(z_i)}_{:= \tilde{z}_i} \end{aligned}$$

and the $\tilde{z}_i$ are invariant under Galois action, hence elements in the polynomial ring $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)[\underline{u}^0]$. But then there are $s_1, \ldots, s_l \in \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ and $t \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$ such that

$$t \cdot F^k = \sum_{i=1}^{l} g_i \cdot s_i \ .$$

Because g_i, $i = 1, \ldots, l$ are elements of R we have $tF^k \in R$. But that means $tF \in \sqrt{R}$. We have proved that $\sqrt{R}$ is a prime ideal, therefore $t \in \sqrt{R}$ or $F \in \sqrt{R}$.

Suppose $t \in \sqrt{R}$. We know that the polynomials (1) get common zeros $\underline{p}^1, \ldots, \underline{p}^d$ if we set $\underline{u}^0 = (0, \ldots, 0)$. That means the resultant R is mapped

onto the zero ideal if $\underline{u}^0$ is mapped to $(0,\ldots,0)$. But $t \in \mathbb{C}[\underline{u}^1,\ldots,\underline{u}^r]$ and therefore it is not mapped onto zero and hence not an element of $\sqrt{R}$. That finishes the proof of the theorem. □

If the multihomogeneous monomials in $\mathbb{C}[\underline{u}^0,\ldots,\underline{u}^r]$ are lexicographically ordered, then the coefficients of a Chow form F of degree d define a point in $\mathbb{P}^N_{\mathbb{C}}$, $N = \binom{n+d}{d}^{r+1} - 1$, because Chow forms are only determined up to a factor in $\mathbb{C}^*$. In the following the coefficients of a Chow form are also denoted *Chow coordinates*.

Until now we have thought about Chow forms of irreducible algebraic sets. Multiplication of Chow forms again yields multihomogeneous polynomials, which leads to the notion of Chow varieties:

Definition 2.4. The *Chow variety* $\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ of r-dimensional cycles of degree d in $\mathbb{P}^n_{\mathbb{C}}$ is defined as follows:

$$\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}}) := \left\{ \sum_{i=1}^{m} n_i V_i \mid \quad V_i \subset \mathbb{P}^n_{\Omega} \text{ r-dimensional varieties}, n_i \in \mathbb{N}, \right. \\ \left. i = 1,\ldots,m \text{ , and } \sum_{i=1}^{m} n_i \cdot \deg V_i = d \right\}$$

The *Chow form* of the cycle $\sum_{i=1}^{m} n_i V_i$ is defined to be

$$F(\underline{u}^0,\ldots,\underline{u}^r) = \prod_{i=1}^{m} F_i(\underline{u}^0,\ldots,\underline{u}^r)^{n_i}$$

where $F_i(\underline{u}^0,\ldots,\underline{u}^r)$ denotes the Chow form of V_i. The *degree* of $\sum_{i=1}^{m} n_i V_i$ is again the multidegree of the multihomogeneous form $F(\underline{u}^0,\ldots,\underline{u}^r)$ and hence equals $\sum_{i=1}^{m} n_i \cdot \deg V_i$.

The observation that the irreducible, multihomogeneous forms of fixed degree define points in $\mathbb{P}^N_{\mathbb{C}}$ remains correct when looking at arbitrary multihomogeneous forms of fixed degree. In fact $\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}}) \subset \mathbb{P}^N_{\mathbb{C}}$ is an algebraic set [5, Thm. 2], $N = \binom{n+d}{d}^{r+1} - 1$. Hence it is a compact topological subspace of the complex projective space with the Euclidean topology.

Example 2.5. There are three isomorphisms of algebraic sets:

1. $\mathcal{C}_{0,d}(\mathbb{P}^n_{\mathbb{C}})$ is isomorphic to the d–fold symmetric product $SP^d(\mathbb{P}^n_{\mathbb{C}})$ of $\mathbb{P}^n_{\mathbb{C}}$.

2. $\mathcal{C}_{r,1}(\mathbb{P}^n_{\mathbb{C}})$ coincides with $\mathcal{G}^{n-r}_{r+1}$ the Grassmannian of $(r+1)$–dimensional linear subspaces of $\mathbb{C}^{n+1}$. The Chow coordinates of a linear subspace coincide with its Plücker coordinates.

3. $\mathcal{C}_{n-1,d}(\mathbb{P}^n_{\mathbb{C}}) \cong \mathbb{P}^N_{\mathbb{C}}$, $N = \binom{n+d}{d} - 1$, because the Chow forms of cycles of codimension 1 and degree d coincide naturally with homogeneous forms of degree d in $\mathbb{C}[x_0, \ldots, x_n]$.

$\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ consists of algebraic cycles whose Chow forms are multihomogeneous forms of multidegree d. The following theorem of Chow and Van der Waerden demonstrates which multihomogeneous forms of multidegree d are Chow forms of algebraic cycles.

Theorem 2.6. *Let $F \in \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ be multihomogeneous of multidegree d. F is a Chow form of an algebraic cycle in $\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ iff the following conditions hold:*

There are d points $\underline{p}^i \in \overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}^{n+1} - \{0\}$ and $c \in \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$ satisfying:

1. $F(\underline{u}^0, \ldots, \underline{u}^r) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$, *in* $\overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}[\underline{u}^0]$.

2. $L_i(\underline{u}^j) := \langle \underline{p}^i, \underline{u}^j \rangle = 0$ *for* $i = 1, \ldots, d$, $j = 1, \ldots, r$.

3. *For every $i \in \{1, \ldots, d\}$ one has:*
   Let S^j be skew symmetric $(n+1) \times (n+1)$–matrices, $j = 1, \ldots, r$, whose entries $s^j_{k,l} \in \Omega$ are algebraically independent over $\mathbb{C}(\underline{p}^i)$, $k, l = 0, \ldots, n$. In this situation $L_i(\underline{u}^0)$ divides the form $F(\underline{u}^0, S^1\underline{p}^i, \ldots, S^r\underline{p}^i)$ in $\mathbb{C}(\underline{p}^i)[s^1_{0,0}, \ldots, s^r_{n,n}][\underline{u}^0]$.

Proof. [5, Thm. 1]. □

Remark 2.7. Let $\underline{p} \in \Omega^{n+1} - \{0\}$ and $L(\underline{x}) := \langle \underline{p}, \underline{x} \rangle$. Then it is easy to see that $\underline{v} \in \Omega^{n+1}$ is an element of the zero set of L iff there is a skew symmetric matrix S satisfying $\underline{v} = S\underline{p}$. Therefore the third condition can be reformulated as:
Let $v^j_k \in \Omega$, $k = 0, \ldots, n$, $j = 1, \ldots, r$ and $i \in \{1, \ldots, d\}$, such that

$$L_i(\underline{v}^j) = 0 \ \forall j = 1, \ldots, r \ .$$

Then $L_i(\underline{u}^0)$ divides $F(\underline{u}^0, \underline{v}^1, \ldots, \underline{v}^r)$ in $\mathbb{C}(\underline{p}^i)[\underline{v}^1, \ldots, \underline{v}^r][\underline{u}^0]$.

Let $V_{\mathbb{C}} = \sum_{i=1}^{m} n_i V_i \in \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$. Then the algebraic set $supp(V_{\mathbb{C}}) := \bigcup_{i=1}^{m} V_i$ is called the *support* of $V_{\mathbb{C}}$. From a given Chow form F the support of the cycle, whose Chow form is F, can be determined:

Lemma 2.8. *Let $V_{\mathbb{C}} \in \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ and $F(\underline{u}^0, \ldots, \underline{u}^r)$ be the Chow form of $V_{\mathbb{C}}$, and let S^i be skew symmetric $(n+1) \times (n+1)$–matrices whose entries $s^i_{j,k}$ are algebraically independent over $\mathbb{C}$. The coefficients of*

$$F(S^0\underline{x}, \ldots, S^r\underline{x}) \in \mathbb{C}[x_0, \ldots, x_n][s^0_{0,1}, \ldots, s^r_{n-1,n}] ,$$

are denoted $f_l(x_0, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n]$. Then the zero set of the f_l is $supp(V_{\mathbb{C}})$.

Proof. [5, Thm. 3]. □

From Theorem 2.3 one concludes directly the following corollary, which makes the connection to Šafarevič's [25, p. 65] definition of Chow varieties transparent. The equivalence of the definitions of Chow / Van der Waerden and Šafarevič is implicitly shown in the proof of 2.3.

Corollary 2.9. *Let $F(\underline{u}^0, \ldots, \underline{u}^r)$ be the Chow form of $V \in \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$, then the following statements are equivalent.*

1. *$\underline{p}$ is contained in $supp(V)$*
2. *$F(\underline{v}^0, \ldots, \underline{v}^r) = 0$ for all $\underline{v}^0, \ldots, \underline{v}^r \in \Omega^{n+1}$ such that $\langle \underline{v}^0, \underline{p} \rangle = \cdots = \langle \underline{v}^r, \underline{p} \rangle = 0$.*

3 Stabilizations and pairings of Chow varieties

In section 2 the multiplication of irreducible Chow forms was used to define Chow forms of algebraic cycles. The multiplication of arbitrary Chow forms defines a pairing on Chow varieties — the formal addition of algebraic cycles:

$$+ : \mathcal{C}_{r,d_1}(\mathbb{P}^n_{\mathbb{C}}) \times \mathcal{C}_{r,d_2}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \mathcal{C}_{r,d_1+d_2}(\mathbb{P}^n_{\mathbb{C}}) \ .$$

Looking at the Chow coordinates this pairing is obviously algebraic. It motivates the definition of the Chow monoid of effective algebraic cycles.

Definition 3.1. The *Chow monoid* $\mathcal{C}_{r,\bullet}(\mathbb{P}^n_{\mathbb{C}})$ of effective r-dimensional algebraic cycles in $\mathbb{P}^n_{\mathbb{C}}$ as a set is the free abelian semi-group generated by the r–dimensional varieties $V \subset \mathbb{P}^n_{\Omega}$ over $\mathbb{C}$.

That means $\mathcal{C}_{r,\bullet}(\mathbb{P}^n_{\mathbb{C}}) = \coprod_{d>0} \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ and therefore it is canonically topologized. The degree of algebraic cycles obviously defines an augmentation which is a proper morphism of topological monoids

$$\deg \ : \mathcal{C}_{r,\bullet}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \mathbb{Z}_+ \ .$$

There is a second pairing on Chow varieties: Let $V \subset \mathbb{P}^n_{\Omega}$ and $W \subset \mathbb{P}^m_{\Omega}$ be varieties and $I_{\mathbb{C}}(V) \subset \mathbb{C}[x_0, \ldots, x_n]$ and $I_{\mathbb{C}}(W) \subset \mathbb{C}[x_{n+1}, \ldots, x_{n+m+1}]$ be the vanishing ideals of V and W. Define $V \divideontimes W \subset \mathbb{P}^{n+m+1}_{\Omega}$ to be the zero set of the ideal $J \subset \mathbb{C}[x_0, \ldots, x_{n+m+1}]$ which is generated by the elements of $I_{\mathbb{C}}(V)$ and $I_{\mathbb{C}}(W)$ viewed as elements in $\mathbb{C}[x_0, \ldots, x_{n+m+1}]$.

Definition 3.2. The algebraic set $V \divideontimes W$ is called the *complex join* of V and W and we call the join with a point $\Sigma V := \mathbb{P}^0_{\mathbb{C}} \divideontimes V$ the *complex suspension* of V.

The join construction is an old synthetic construction in algebraic geometry, already used by the old Italian school (cp. [26, p. 8]).

There are other equivalent ways to define the join of varieties. For example let $C(V)$ and $C(W)$ be the affine cones of V and W, i.e. $C(V)$ is the zero set of $I_{\mathbb{C}}(V)$ in $\mathbb{C}^{n+1}$. Then the join of V and W is uniquely determined by

$$C(V \divideontimes W) := C(V) \times C(W) \ .$$

From the second definition we deduce that the dimension of $V \ast\!\!\ast W$ is $\dim V + \dim W + 1$, and one also obtains $V \ast\!\!\ast W = (V \ast\!\!\ast \mathbb{P}^m_{\mathbb{C}}) \cap (\mathbb{P}^n_{\mathbb{C}} \ast\!\!\ast W)$.
The join of two varieties is again a variety, and one wants to calculate the degree of the join of two varieties:
The varieties $V \ast\!\!\ast \mathbb{P}^m_{\mathbb{C}}$ and $\mathbb{P}^n_{\mathbb{C}} \ast\!\!\ast W$ intersect transversally, hence by Bezout [22, Chap. 5, Thm. 5.16]:

$$\begin{aligned} \deg V \ast\!\!\ast W &= \deg (V \ast\!\!\ast \mathbb{P}^m_{\mathbb{C}}) \cdot \deg (\mathbb{P}^n_{\mathbb{C}} \ast\!\!\ast W) \\ &= \deg V \cdot \deg W \ . \end{aligned}$$

By linear extension to algebraic cycles the join defines a pairing on Chow varieties

$$\mathcal{C}_{r_1,d_1}(\mathbb{P}^{n_1}_{\mathbb{C}}) \times \mathcal{C}_{r_2,d_2}(\mathbb{P}^{n_2}_{\mathbb{C}}) \longrightarrow \mathcal{C}_{r_1+r_2+1,d_1 d_2}(\mathbb{P}^{n_1+n_2+1}_{\mathbb{C}})$$

which is associative in the appropriate sense.

Proposition 3.3. *The complex suspension* $\Sigma\!\!\!/ : \mathcal{C}_{r-1,d}(\mathbb{P}^{n-1}_{\mathbb{C}}) \to \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ *is an algebraic map, and hence it is continuous.*

The proposition will be proved for Chow varieties in arbitrary characteristic in section 7.

As a corollary we obtain that the complex join is an algebraic map:

Theorem 3.4. *The join map*

$$\mathcal{C}_{r_1,d_1}(\mathbb{P}^{n_1}_{\mathbb{C}}) \times \mathcal{C}_{r_2,d_2}(\mathbb{P}^{n_2}_{\mathbb{C}}) \longrightarrow \mathcal{C}_{r_1+r_2+1,d_1 d_2}(\mathbb{P}^{n_1+n_2+1}_{\mathbb{C}})$$

is algebraic. (Hence it is continuous.)

Proof. The image of the algebraic map $(\ast\!\!\ast \mathbb{P}^{n_2}, \mathbb{P}^{n_1} \ast\!\!\ast)$:

$$\mathcal{C}_{r_1,d_1}(\mathbb{P}^{n_1}) \times \mathcal{C}_{r_2,d_2}(\mathbb{P}^{n_2}) \longrightarrow \mathcal{C}_{r_1+n_2+1,d_1}(\mathbb{P}^{n_1+n_2+1}) \times \mathcal{C}_{r_2+n_1+1,d_2}(\mathbb{P}^{n_1+n_2+1})$$

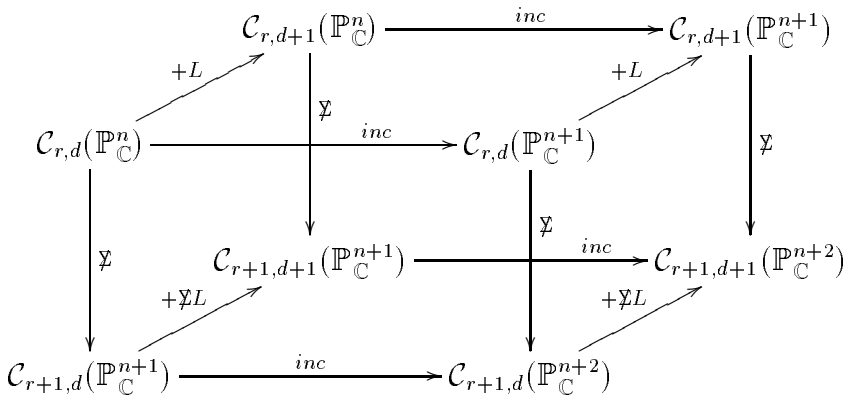
consists only on pairs of cycles whose irreducible components intersect transversally. Therefore the intersection pairing is well defined and analytic on the image of $(\ast\!\!\ast_{\mathbb{P}^{n_2},\mathbb{P}^{n_1}} \ast\!\!\ast)$ [1, Thm. VI.2.10 and p.142]. The Chow varieties are compact and hence the join is algebraic by GAGA (e. g. see [11, Chap. IV.1]). □

Remark 3.5. The continuity of the join map was known before. There are different proofs using methods from different areas of mathematics: Friedlander and Mazur [9] showed that the join is algebraic continuous (cp. Def. 8.2) and hence continuous. Paulo Lima-Filho [19] gave two different proofs, one using methods coming from geometric measure theory, and the other one using the fact, that the join can be constructed via flat pull-backs and proper push-forwards.

Now we want to define stabilizations of Chow varieties. For this purpose let $L \in \mathcal{C}_{r,1}(\mathbb{P}^n_{\mathbb{C}})$ be a fixed linear subspace. Further let inc be the map of Chow varieties, which is induced by the inclusion

$$\begin{array}{rccc} inc: & \mathbb{P}^n_\Omega & \longrightarrow & \mathbb{P}^{n+1}_\Omega \\ & (p_0,\ldots,p_n) & \longmapsto & (p_0,\ldots,p_n,0) \end{array} \quad .$$

The following diagram is commutative:



The map $+L$ is obviously algebraic. The algebraicity of the map inc follows directly from the definition of Chow forms. Hence we can define several stabilizations of Chow varieties:

Notation 3.6.

$$\mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}}) := \lim_d \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}}) \quad , \quad \mathcal{D}(d) := \lim_{r,n} \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}}) \quad ,$$
$$\mathcal{C} := \lim_{r,d,n} \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}}) \quad , \quad \mathfrak{C} := \coprod_d \mathcal{D}(d) \quad .$$

From the examples 2.5 one deduces that $\mathcal{C}_0(\mathbb{P}^n_{\mathbb{C}})$ is homeomorphic to the reduced infinite symmetric product $\widetilde{\mathrm{SP}}(\mathbb{P}^n_{\mathbb{C}})$, and that $\mathcal{D}(1)$ is homeomorphic

to BU the classifying space of U–bundles.
The spaces $\mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}})$ and $\mathcal{C}$ are commutative topological monoids by formal addition of cycles with unit L, and $\mathfrak{C}$ is a commutative topological monoid by formal addition of cycles with unit $\emptyset$ the empty cycle.
Intertwining coordinates, the join pairing extends to a multiplicative pairing on $\mathcal{C}$, $\mathfrak{C}$, and $\mathcal{D}(1)$, which on the latter space coincides with the Whitney sum on BU. In the next section the homotopy type of $\mathcal{C}$ will be determined.

4 The Complex Suspension Theorem

The following theorem is due to H. B. Lawson [16]. We are providing a more algebraic proof of the theorem, as well as giving a few more details.

The Complex Suspension Theorem 4.1. *The complex suspension*

$$\sum\!\!\!\!\!\!/ : \mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$$

is a homotopy equivalence.

As in [16] the theorem will be proved in two steps. First one can show that $\sum\!\!\!\!\!\!/\,\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ is a strong deformation retract of a space $\mathcal{T}_d \subset \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$. The second step is to show that the inclusion $\mathcal{T} \hookrightarrow \mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ induces isomorphisms on homotopy groups.

Definition 4.2. Let $\mathbb{P}^n_{\mathbb{C}} \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ denote the hyperplane given by the equation $x_0 = 0$. Then define

$$\mathcal{T}_d := \left\{ \sum_{i=1}^{m} n_i V_i \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) \mid V_i \not\subset \mathbb{P}^n_{\mathbb{C}}\,, i = 1, \ldots, m \right\} .$$

Fixing an element of $\mathcal{T}_1$, whose intersection with $\mathbb{P}^n_{\mathbb{C}}$ is the base point of $\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$, one defines $\mathcal{T} := \lim_{d\to\infty} \mathcal{T}_d$.

Let $V \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$ and F_V be the Chow form of V, and note that V is an element of $\mathcal{T}_d$ iff $F_V(\underline{u}^0, \ldots, \underline{u}^r, \underline{e}) \neq 0$, $\underline{e} = (1, 0, \ldots, 0)$. This is obviously true for varieties, since F_V generates the radical of a resultant by 2.3. Hence it is also true for cycles. This condition is polynomial in the coefficients of F_V, hence $\mathcal{T}_d$ is a Zariski open subset of $\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$.

We first construct a homotopy ϕ on Chow varieties, induced by

$$\begin{array}{ccc} \mathbb{P}^{n+1}_{\mathbb{C}} \times \mathbb{R}^*_+ & \longrightarrow & \mathbb{P}^{n+1}_{\mathbb{C}} \\ ([p_0 : \cdots : p_{n+1}], t) & \longmapsto & [tp_0 : p_1 : \cdots : p_{n+1}] \end{array} .$$

The map ϕ retracts $\mathcal{T}_d$ to $\sum\!\!\!\!\!\!/\,\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$. But first one has to check how this induced map is defined on Chow forms.

Lemma 4.3. *Let $V \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$ and $F_V(\underline{u}^0, \ldots, \underline{u}^{r+1})$ be the Chow form of the cycle V. Then the cycle $\phi(V, t)$ has the Chow form $F_V({}^t\underline{u}^0, \ldots, {}^t\underline{u}^{r+1})$, where ${}^t\underline{u}^i = (tu^i_0, u^i_1, \ldots, u^i_{n+1})$.*

Proof. Let V be a variety and $f_1, \ldots, f_k$ homogeneous generators of $I_{\mathbb{C}}(V)$. Then F_V generates the radical of the resultant of

$$f_1(x_0, \ldots, x_{n+1}) \, , \ldots , \, f_k(x_0, \ldots, x_{n+1}) \, , \, \begin{pmatrix} u_0^0 & \cdots & u_{n+1}^0 \\ \vdots & & \vdots \\ u_0^{r+1} & \cdots & u_{n+1}^{r+1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_{n+1} \end{pmatrix} \, .$$

The Chow form of $\phi(V,t)$ is the generator of the radical of the resultant of

$$f_1(\frac{1}{t}x_0, \ldots, x_{n+1}) \, , \ldots , \, f_k(\frac{1}{t}x_0, \ldots, x_{n+1}) \, , \, \begin{pmatrix} u_0^0 & \cdots & u_{n+1}^0 \\ \vdots & & \vdots \\ u_0^{r+1} & \cdots & u_{n+1}^{r+1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_{n+1} \end{pmatrix}$$

which is obviously the same as the radical of the resultant of

$$f_1(x_0, \ldots, x_{n+1}), \ldots, f_k(x_0, \ldots, x_{n+1}), \begin{pmatrix} tu_0^0 & u_1^0 & \cdots & u_{n+1}^0 \\ \vdots & \vdots & & \vdots \\ tu_0^{r+1} & u_1^{t+1} & \cdots & u_{n+1}^{r+1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_{n+1} \end{pmatrix} \, .$$

The algebraic independence of the u_i^j remains invariant under multiplication with $t \in \mathbb{C}^*$, hence the lemma is proved. □

Suppose V lies in the image of $\not\Sigma$. Then the vanishing ideal of each irreducible component is generated by polynomials in $\mathbb{C}[x_1, \ldots, x_{n+1}]$, that means x_0 does not appear in these generators. Hence one obtains directly from the proof above that

$$\phi(\quad , t) \mid_{\not\Sigma \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})} = id_{\not\Sigma \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})} \quad \forall t \in \mathbb{R}^*_+ \, .$$

On the other hand the algebraic independence of the u_i^j remains invariant under the multiplication by $t \in \mathbb{C}^*$ which implies

$$F_V(\underline{u}^0, \ldots, \underline{u}^r, \underline{e}) \neq 0 \iff F_V({}^t\underline{u}^0, \ldots, {}^t\underline{u}^r, t\underline{e}) \neq 0 \, .$$

Hence $\phi(\mathcal{T}_d, t) \subset \mathcal{T}_d$. We claim that ϕ extends continuously to $\mathcal{T}_d \times]0, \infty]$. The following lemma is very useful for this purpose:

Lemma 4.4. *Let $V \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$ be a variety and F_V be a fixed representative of the Chow form of V. Suppose*

$$F_V({}^t\underline{u}^0, \ldots, {}^t\underline{u}^{r+1}) = t^k F_V(\underline{u}^0, \ldots, \underline{u}^{r+1}) \quad \forall t \in \mathbb{C}^* ,$$

then V lies in the image of $\not\Sigma$ and $k = d$.

Proof. Let $\{\underline{p}^1, \ldots, \underline{p}^d\}$ be the intersection of V with a $\mathbb{C}$-general $(n-r)$-dimensional linear subspace H_{n-r}.
Then the Chow form F_V splits in $\overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^{r+1})}[\underline{u}^0]$:

$$\begin{aligned} F_V({}^t\underline{u}^0, \ldots, {}^t\underline{u}^{r+1}) &= t^k F_V(\underline{u}^0, \ldots, \underline{u}^{r+1}) \\ &= \frac{t^k}{t^d} \prod_{i=1}^{d} t\langle \underline{p}^i, \underline{u}^0 \rangle \\ &= \frac{t^k}{t^d} \prod_{i=1}^{d} \left(p_0^i (t u_0^0) + \sum_{j=1}^{n+1} (t p_j^i) u_j^0 \right) \\ &= \left(\tfrac{t^k}{t^d} p_0^1 (t u_0^0) + \sum_{j=1}^{n+1} (\tfrac{t^k}{t^d} t p_j^1) u_j^0 \right) \prod_{i=2}^{d} \left(p_0^i (t u_0^0) + \sum_{j=1}^{n+1} (t p_j^i) u_j^0 \right). \end{aligned}$$

This means $[p_0^i : tp_1^i : \cdots : tp_{n+1}^i]$ is a general point of V for every $t \in \mathbb{C}^*$.
Hence $[tp_0^i : p_1^i : \cdots : p_{n+1}^i]$ is a general point of V, too. One has $p_0^i \neq 0$ unless $k = 0$. Now let $f \in I_{\mathbb{C}}(V)$ and observe that tp_0^i is a zero of $f(x_0, p_1^i, \ldots, p_{n+1}^i)$ for every $t \in \mathbb{C}^*$. But this means that the polynomial has infinitely many zeros and hence there must be a set of generators of $I_{\mathbb{C}}(V)$ which are constant as polynomials in x_0. Therefore V lies in the image of $\not\nabla$. Hence $k = d$ because $(u_0^0)^d t^d$ is a monomial of the Chow form of V whose coefficient is constant in t. □

Now we want to extend ϕ: Let $V \in \mathcal{T}_d$ and F_V be a fixed representative of the Chow form of V. Let k be the degree of $F_V({}^t\underline{u}^0, \ldots, {}^t\underline{u}^{r+1})$ as a polynomial in t. For $\{t_n \in]0, \infty[\}_{n \in \mathbb{N}}$ one obtains a sequence $\{\phi(V, t_n)\}_{n \in \mathbb{N}}$ in the Chow variety. If one divides the n-th element of this sequence by $(t_n)^k$, the sequence remains invariant, because Chow forms are only determined up to a factor in $\mathbb{C}^*$. If $\lim\limits_{n \to \infty} t_n = \infty$ we obtain that the original coefficients of $(t_n)^k$ remain invariant and the original coefficients of $(t_n)^l$, $l < k$ tend to zero, showing that the sequence is convergent. Therefore $\phi(V, \infty) := \lim\limits_{t \to \infty} \phi(V, t)$ is an element of the Chow variety $\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$. Furthermore we have already seen that the Chow form $F_{\phi(V,\infty)}({}^t\underline{u}^0, \ldots, {}^t\underline{u}^{r+1})$ is homogeneous of degree k as a polynomial in t.
If one could check that this degree k in t equals d, then by 4.4 the map ϕ retracts $\mathcal{T}_d$ to $\not\nabla \mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$:
The degree k in t cannot be lower than the degree d of V, because the monomial $(u_0^0)^d t^d$ appears in the Chow form of $V \in \mathcal{T}_d$. Hence $k \geq d$.

On the other hand, one knows from Lemma 4.4 that, for each irreducible component V_i, the form $F_{\phi(V_i,\infty)}({}^t\underline{u}^0,\ldots,{}^t\underline{u}^{r+1})$ is homogeneous of degree deg V_i or of degree zero as polynomial in t. But in the case where the degree is zero for one irreducible component one obtains $k < d$. This contradicts $k \geq d$. Hence k has to be d. This proves the following lemma:

Lemma 4.5. *The map $\phi : \mathcal{T}_d \times [1,\infty] \to \mathcal{T}_d$ is a homotopy, which retracts $\mathcal{T}_d$ to $\not\exists\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$.*

The first part of the complex suspension theorem is proved. We need some preparations for the second part of the proof, where we have to check that the inclusion $\mathcal{T} \hookrightarrow \mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ (cp. Notation 3.6 and Def. 4.2) induces isomorphisms on homotopy groups:

Notation 4.6. In the following let $\underline{z}_1 := [1:1:0:\cdots:0] \in \mathbb{P}^{n+2}_{\mathbb{C}}$ and

$$\mathcal{C}'_{r+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}}) := \{V \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}}) \mid \underline{z}_1 \notin supp(V)\} .$$

Further let $pr_1 : \mathbb{P}^{n+2}_{\mathbb{C}} - \{\underline{z}_1\} \to \mathbb{P}^{n+1}_{\mathbb{C}}$ be the projection from $\underline{z}_1$ onto the hyperplane given by the equation $x_0 = 0$.

Suppose $V \in \mathcal{C}'_{r+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$, then the algebraic set $pr_1(supp(V))$ has dimension $r+1$ because the projection is an algebraic map, and the restriction $pr_1\,|_V$ has finite fibers (otherwise $\underline{z}_1 \in V$). Now we want to define a map π_1 for cycles $V \in \mathcal{C}'_{r+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$ such that $supp(\pi_1(V)) = pr_1(supp(V))$.
Let θ be the following ring homomorphism:

$$\begin{array}{rccll} \theta: & \mathbb{C}[\underline{u}^0,\ldots,\underline{u}^r] & \longrightarrow & \mathbb{C}[\underline{\tilde{u}}^0,\ldots,\underline{\tilde{u}}^r] & \\ & u_0^j & \longmapsto & -u_1^j & , j = 0,\ldots,r \\ & z & \longmapsto & z & \forall z \in \mathbb{C}[\underline{\tilde{u}}^0,\ldots,\underline{\tilde{u}}^r], \end{array}$$

where $\underline{\tilde{u}}^j := (u_1^j,\ldots,u_{n+2}^j)$ and $\underline{u}^j = (u_0^j,\ldots,u_{n+2}^j)$.

Lemma 4.7. *θ maps the Chow form of a cycle $V \in \mathcal{C}'_{r+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$ to the Chow form of a cycle, which is denoted $\pi_1(V)$, in $\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$, such that*

$$supp(\pi_1(V)) = pr_1(supp(V)) .$$

Proof. Let $V \in \mathcal{C}'_{r+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$ be a variety and F_V be the Chow form of V. Further let $f_1,\ldots,f_k \in \mathbb{C}[x_0,\ldots,x_{n+2}]$ be homogeneous generators of the

vanishing ideal of V, and let $g_1, \ldots, g_l \in \mathbb{C}[x_1, \ldots, x_{n+2}]$ be homogeneous generators of the vanishing ideal of $pr_1(V)$. F_V is the generator of the radical of the resultant of the following equations:

$$f_1(\underline{x}) = \cdots = f_k(\underline{x}) = \begin{pmatrix} u_0^0 & u_1^0 & \cdots & u_{n+2}^0 \\ \vdots & \vdots & & \vdots \\ u_0^r & u_1^r & \cdots & u_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \quad . \tag{2}$$

But the resultant of (2) is surjectively (R. 1.7) mapped under θ onto the resultant of

$$f_1(\underline{x}) = \cdots = f_k(\underline{x}) = \begin{pmatrix} -u_1^0 & u_1^0 & \cdots & u_{n+2}^0 \\ \vdots & \vdots & & \vdots \\ -u_1^r & u_1^r & \cdots & u_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \quad . \tag{3}$$

Therefore the radical of the ideal which is generated by $\theta(F_V)$in $\mathbb{C}[\underline{\tilde{u}}^0, \ldots, \underline{\tilde{u}}^r]$ is the radical of the resultant of (3). But the resultant of (3) is the same as the resultant of

$$g_1(\underline{x}) = \cdots = g_l(\underline{x}) = \begin{pmatrix} u_1^0 & \cdots & u_{n+2}^0 \\ \vdots & & \vdots \\ u_1^r & \cdots & u_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \quad . \tag{4}$$

To check this let $\underline{v}^0, \ldots, \underline{v}^r \in \Omega^{n+2}$ be a zero of the resultant of (3). That means the equations

$$f_1(\underline{x}) = \cdots = f_k(\underline{x}) = \begin{pmatrix} -v_1^0 & v_1^0 & \cdots & v_{n+2}^0 \\ \vdots & \vdots & & \vdots \\ -v_1^r & v_1^r & \cdots & v_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \quad .$$

have a common solution $\underline{q} = [q_0 : \cdots : q_{n+2}] \in V$. But $[q_1 - q_0 : q_2 : \cdots : q_{n+2}] = pr_1(\underline{q})$ is contained in $pr_1(V)$ and therefore a solution of the equations

$$g_1(\underline{x}) = \cdots = g_l(\underline{x}) = \begin{pmatrix} v_1^0 & \cdots & v_{n+2}^r \\ \vdots & & \vdots \\ v_1^r & \cdots & v_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \quad .$$

Therefore the radical of the resultant of (4) is a subideal of the radical of the resultant of (3). In the analogous way one checks the inverse inclusion.

This shows that $\theta(F_V) = F^m_{pr_1(V)}$ for some $m \in \mathbb{N} - \{0\}$. Since θ is a homomorphism of rings, we obtain for an arbitrary cycle $V = \sum_{i=1}^{s} n_i V_i$:

$$\theta(F_V) = (F^{n_1}_{V_1})^{m_1} \cdot \cdots \cdot (F^{n_s}_{V_s})^{m_s}$$

for some $m_1, \ldots, m_s \in \mathbb{N} - \{0\}$, where F_{V_i} denotes the Chow form of V_i. This proves that $\theta(F_V)$ is a Chow form of multidegree d and the support of the cycle $\pi_1(V)$, whose Chow form is $\theta(F_V)$, is $pr_1(V)$. □

Notation 4.8. The Chow variety $\mathcal{C}_{n+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$ of effective divisors of degree d can be identified with the equivalence classes of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_{n+2}]$ of degree d (Example 2.5), where $f \sim g$ iff $f = zg$ for some $z \in \mathbb{C}^*$. Let $Div'_d \subset \mathcal{C}_{n+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$ denote the set of cycles

$$\sum_{i_0+\cdots+i_{n+2}=d} a_{i_0,\ldots,i_{n+2}} x_0^{i_0} \cdot \cdots \cdot x_{n+2}^{i_{n+2}} \quad ,$$

such that $a_{d,0,\ldots,0} \neq 0$ and $\sum_{\substack{i_0+i_1=d \\ i_0<d}} a_{i_0,i_1,0,\ldots,0} = 0$.

Div'_d is a quasi projective algebraic subset of $\mathcal{C}_{n+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$ of codimension 1. Hence it has dimension $\binom{n+2+d}{d} - 2$.
Since elements in Div'_d are only determined up to a factor in $\mathbb{C}^*$, we can suppose $a_{d,0,\ldots,0} = 1$. The space Div'_d is an affine subspace of $\mathcal{C}_{n+1,d}(\mathbb{P}^{n+2}_{\mathbb{C}})$, hence it is contractible for instance by the following homotopy:
Let I be the index set $I := \{(i_0, \ldots, i_{n+2}) \in \mathbb{N}_0^{n+3} \mid i_0 + \cdots + i_{n+2} = d, i_0 \neq d\}$.

$$\begin{array}{rccc} \Phi: & Div'_d \times [0,1] & \longrightarrow & Div'_d \\ & \left(x_0^d + \sum_I a_{i_0,\ldots,i_{n+2}} x_0^{i_0} \cdots x_{n+2}^{i_{n+2}}, t\right) & \longmapsto & x_0^d + \sum_I t \cdot a_{i_0,\ldots,i_{n+2}} x_0^{i_0} \cdots x_{n+2}^{i_{n+2}}. \end{array}$$

$\Phi(Div'_d, 0)$ is the hyperplane $x_0 = 0$ with multiplicity d. Simple calculations show that $\underline{z}_0 := [1 : 0 : \cdots : 0]$ and $\underline{z}_1 := [1 : 1 : 0 : \cdots : 0]$ are not contained in the support of any cycle in Div'_d.

Now we are able to define a map which will prove the second part of the complex suspension theorem:

Definition 4.9.

$$\Psi := \pi_1 \circ \cap \circ (\not\exists, \Phi) : \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) \times Div'_k \times [0,1] \longrightarrow \mathcal{C}_{r+1,dk}(\mathbb{P}^{n+1}_{\mathbb{C}}) \quad ,$$

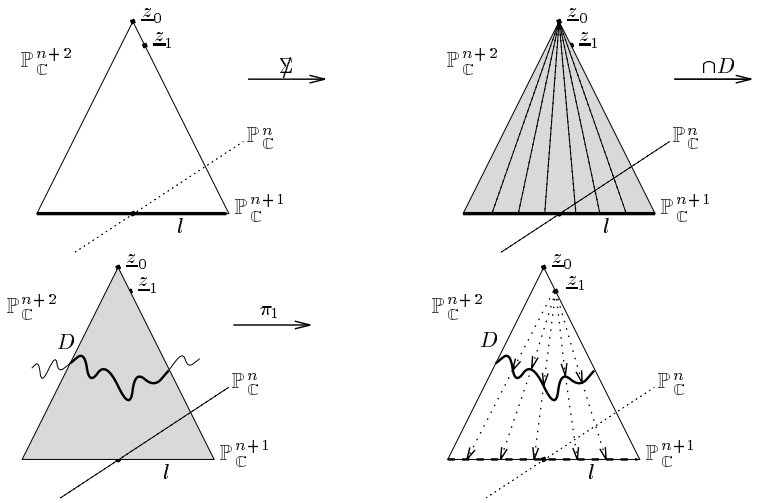
where $\cap$ denotes the intersection of cycles in the sense of Barlet [1].

We want to check that Ψ is well defined: First we have $\Phi(Div'_k, [0,1]) \subset Div'_k$.
Each irreducible component of a cycle in $\not\nabla\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$ meets the point $\underline{z}_0$, and $\underline{z}_0$ is not contained in the support of any element in Div'_k. Therefore a pair of cycles in $\not\nabla\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) \times Div'_k$ is in general position and the intersection is well defined. The support of the resulting cycle is just the geometric intersection of the supports of the original cycles. Hence the support of the resulting cycle cannot contain the point $\underline{z}_1$, because $\underline{z}_1$ is not contained in the support of any cycle in Div'_k. That means the map π_1 is well defined in this situation.

The maps $\not\nabla$, Φ, and π_1 are algebraic and $\cap$ is analytic, hence Ψ is analytic. Ψ has nice geometric properties as we will see in the following. First the map $\Psi(_, _, 0)$ is multiplication by k. On the other hand Ψ maps cycles in $\not\nabla\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ to cycles in $\not\nabla\mathcal{C}_{r,dk}(\mathbb{P}^n_{\mathbb{C}})$.

Lemma 4.10. *The map* $\Psi\,|_{\not\nabla\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}}) \times Div'_k \times [0,1]}$ *is the map which sends a cycle* $V \in \not\nabla\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ *to the cycle* $k \cdot V$.

Proof. The support of an element in $\not\nabla\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ can be viewed as a union of points lying on lines through points $[0 : p_1 : \cdots : p_{n+1}]$ and $[1 : 0 : \cdots : 0]$ in $\mathbb{P}^{n+1}_{\mathbb{C}}$. Therefore it suffices to check the case $r = 0$ and $d = 1$. Ψ first takes the complex suspension of this line l. The complex suspension is cut with a cycle $D \in Div'_k$, and the common points of D and $\not\nabla l$ are projected from $\underline{z}_1$ to the hyperplane $x_0 = 0$ in $\mathbb{P}^{n+2}_{\mathbb{C}}$ via π_1.



Since $supp(\pi_1(\not\Sigma l)) \subset l$, the support of $\Psi(l, D, t)$ is a closed subset of l which has dimension 1. But a closed 1–dimensional subset of a line is the line itself. Hence $\Psi(l, D, t) = k \cdot l$. □

For a fixed $D \in Div'_k$ the map Ψ is a homotopy. We now determine which divisors $D \in Div'_k$ move $V \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$ to an element in $\mathcal{T}_{dk}$.

Lemma 4.11. *Let π_0 denote the projection from $\underline{z}_0$ onto the hyperplane $x_0 = 0$. There is an isomorphism $F := (\pi_0 \,|_{\overline{\pi_1^{-1}(\mathbb{P}^n_{\mathbb{C}})}})^{-1} : \mathbb{P}^{n+1}_{\mathbb{C}} \to \overline{\pi_1^{-1}(\mathbb{P}^n_{\mathbb{C}})}$ such that*

$$\Psi(V, D, 1) \notin \mathcal{T}_{dk} \Longrightarrow F(supp(V)) \subset supp(D) \ .$$

A proof of this lemma is given in [16, p.286,287], where no arguments of continuity are needed, one only has to check this pointwise.

For a fixed cycle $V \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$ let $\mathcal{B}(V)$ denote the set of those cycles $D \in Div'_k$ such that $\Psi(V, D, 1) \notin \mathcal{T}_{dk}$. As one has already seen $\mathcal{B}(V) = \{D \in Div'_k \mid F(supp(V)) \subset supp(D)\}$. In general $\mathcal{B}(V)$ is a proper subset of Div'_k.

Lemma 4.12. *Given $V = \sum_{i=1}^{m} n_i V_i \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$, the complex codimension of $\mathcal{B}(V)$ in Div'_k satisfies:*

$$codim_{\mathbb{C}}\mathcal{B}(V) = \min_i \ codim_{\mathbb{C}}\mathcal{B}(V_i) \geq \binom{r+k+1}{k} - 1 \ .$$

Proof. We expand the ideas of [16, p.288] (see also [8, p. 83]). It suffices to check the case $m = 1$, i.e. to check the lemma for varieties $V \subset \mathbb{P}^{n+1}_{\mathbb{C}}$. We can view such a V — analogously to the construction of Ψ — also as a subset of $\mathbb{P}^{n+2}_{\mathbb{C}}$, where in this case the first coordinate of each point of V is equal to 0.

We look at an affine open $\mathbb{C}^{n+2} \cong \mathbb{P}^{n+2}_{\mathbb{C}} - H$, where $V \not\subset H$ and $\underline{z}_0 \in H$. The last condition implies, that H is the zero set of $\sum_{i=1}^{n+2} a_i x_i$ for some $a_1, \ldots, a_{n+2} \in \mathbb{C}$. To calculate dimensions we can think of V as an affine subvariety of $\mathbb{C}^{n+2}$.

First we want to check that if $V \not\subset H$ then $F(V)$ is not contained in H: Let $\underline{p} := [0 : p_1 : \cdots : p_{n+2}] \in V \subset \mathbb{P}^{n+2}_{\mathbb{C}}$. It follows directly from the definition of F (Lemma 4.11) that $F(\underline{p}) = [-p_1 : p_1 : \cdots : p_{n+2}]$. Hence if $\underline{p} \notin H$, which means $\sum_{i=1}^{n+2} a_i p_i \neq 0$, then $F(\underline{p}) \notin H$.

For simplicity we assume H to be the zero set of $x_{n+2} = 0$. In this situation the set Div'_k can be viewed as a subset of the polynomials of degree $\leq k$ in $x_0, \ldots, x_{n+1}$. And the set $\mathcal{B}(V) \subset Div'_k$ is the set of those polynomials, which vanish on $F(V) \cap \mathbb{C}^{n+2}$.
$F(V)$ is $r+1$–dimensional. If one projects $F(V) \cap \mathbb{C}^{n+2}$ onto a linear subspace of dimension $r+1$, then without loss of generality the image of $F(V) \cap \mathbb{C}^{n+2}$ has dimension $r+1$, too, because $F(V) \cap \mathbb{C}^{n+2}$ contains regular points. Up to linear isomorphisms one can suppose the linear subspace to be the zero set of $x_{r+1} = \cdots = x_{n+1} = 0$.
In this situation a polynomial $f \in \mathbb{C}[x_0, \ldots, x_r]$ can only vanish on the image of $F(V) \cap \mathbb{C}^{n+2}$ under the projection, if $f = 0$. This is obvious, because the zero set of f is a r–dimensional subset of the linear subspace.
Therefore the complex codimension of $\mathcal{B}(V)$ in $\mathcal{C}_{n+1,k}(\mathbb{P}^{n+1}_{\mathbb{C}})$ is greater or equal to the dimension of the space of polynomials in $\mathbb{C}[x_0, \ldots, x_r]$ of degree $\leq k$. This is $\binom{r+k+1}{k}$. Because $Div'_k \subset \mathcal{C}_{n+1,k}(\mathbb{P}^{n+2}_{\mathbb{C}})$ has codimension 1 the lemma is proved. □

Now let $\mathcal{B}$ denote the following set:

$$\begin{aligned} \mathcal{B} &:= (\Psi \mid_{\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) \times Div'_k \times \{1\}})^{-1}(\mathcal{C}_{r+1,dk}(\mathbb{P}^{n+1}_{\mathbb{C}}) - \mathcal{T}_{dk}) \\ &= \{(V, D) \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) \times Div'_k \mid \Psi(V, D, 1) \notin \mathcal{T}_{dk}\} . \end{aligned}$$

The set $\mathcal{C}_{r+1,dk}(\mathbb{P}^{n+1}_{\mathbb{C}}) - \mathcal{T}_{dk}$ is an algebraic set and Ψ is an analytic map, hence $\mathcal{B}$ is an analytic set. Let $V \in \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$ then $(V \times Div'_k) \cap \mathcal{B} \subset \mathcal{B}(V)$. Therefore the codimension of $\mathcal{B}$ is greater or equal $\binom{r+k+1}{k} - 1$. Now choose $E(r,d) \in \mathbb{N}$ such that $\binom{r+k+1}{k} - 1 > \dim \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$, $\forall k \geq E(r,d)$. Further denote by pr_2 the projection from $\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) \times Div'_k$ onto the second factor. Then one obtains

$$\begin{aligned} \dim pr_2(\mathcal{B}) &\leq \dim \mathcal{B} \\ &\leq \dim \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) + \dim Div'_k - \tbinom{r+k+1}{k} + 1 \\ &< \dim Div'_k . \end{aligned}$$

Hence the projection from $\mathcal{B}$ on Div'_k cannot be surjective $\forall k > E(r,d)$. Therefore one obtains a cycle $\mathcal{D} \in Div'_k$ for each $k \geq E(r,d)$ such that the map $\Psi : \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) \times \{\mathcal{D}\} \times [0,1] \to \mathcal{C}_{r+1,dk}(\mathbb{P}^{n+1}_{\mathbb{C}})$ has the following properties:

$$\Psi(V, \mathcal{D}, 0) = k \cdot V \quad \text{and} \quad \Psi(\mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}}), \mathcal{D}, 1) \subset \mathcal{T}_{dk} .$$

To finish the proof of the complex suspension theorem one has to check that the inclusion $i : \mathcal{T} \hookrightarrow \mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ induces isomorphisms on homotopy groups. For this purpose let $\not\exists\mathbb{P}^r_{\mathbb{C}}$ be the basepoint $*$ of $\mathcal{T}$ and $\mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$. First of all, the monoid $\mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ is connected. This is obvious from the properties of Ψ. Furthermore, the addition on homotopy groups, which is induced by the H–cogroup structure of the spheres, coincides with the addition which is induced by the topological monoid structure of $\mathcal{T}$ and $\mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$.

Surjectivity: Let $\alpha \in \pi_j(\mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}}))$. There is $d \in \mathbb{N}$ such that $\alpha : S^j \to \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$.
For $k \geq E(d,r)$ there exists $\alpha_k \in \pi_j(\mathcal{T}_{dk})$ such that $i_*[\alpha_k] \stackrel{\Psi}{=} [k \cdot \alpha]$. Therefore $\alpha_{k+1} - \alpha_k \in \pi_j(\mathcal{T})$ and $i_*[\alpha_{k+1} - \alpha_k] = \alpha$.

Injectivity: Let $\beta \in \pi_j(\mathcal{T})$ such that $i_*[\beta] = *$. Without loss of generality suppose $\beta : S^j \to \not\exists\mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}})$, because $\not\exists\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ is a deformation retract of $\mathcal{T}$.
Now there is a map $h : S^j \times [0,1] \to \mathcal{C}_{r+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ such that $h(\ \ ,0) = i_*(\beta)$ and $h(\ \ ,1) = *$. There exists $d \in \mathbb{N}$ such that $h(S^j \times [0,1]) \subset \mathcal{C}_{r+1,d}(\mathbb{P}^{n+1}_{\mathbb{C}})$. For each $k \geq E(d,r)$ there exists $h_k : S^j \times [0,1] \to \mathcal{T}_{dk}$ such that $i_*[h_k] \stackrel{\Psi}{=} [k \cdot h]$. Because Ψ restricted to $\not\exists\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$ is just multiplication by d one obtains $h_k(\ \ ,1) = *$ and $h_k(\ \ ,0) = k \cdot \beta$. In this situation one obtains $(h_{k+1} - h_k) : S^i \times [0,1] \to \mathcal{T}$ such that $(h_{k+1} - h_k)(\ \ ,1) = *$ and $(h_{k+1} - h_k)(\ \ ,0) = \beta$.

The complex suspension theorem enables one to determine the homotopy type of the monoids $\mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}})$.

Corollary 4.13. *The topological monoid $\mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}})$ is homotopy equivalent to a product of Eilenberg–MacLane spaces:*

$$\mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}}) \simeq \prod_{i=1}^{n-r} K(\mathbb{Z}, 2i) \ .$$

Proof. Let H be the unreduced singular homology theory. By Dold/Thom [7, Satz 7.1] we have

$$\widetilde{\mathrm{SP}}(\mathbb{P}^{n-r}_{\mathbb{C}}) \ \simeq \ \prod_{i>0} K(H_i(\mathbb{P}^{n-r}_{\mathbb{C}}, \mathbb{Z}), i)$$

Hence $\mathcal{C}_r(\mathbb{P}^n_{\mathbb{C}}) \stackrel{\not\exists}{\simeq} \ldots \stackrel{\not\exists}{\simeq} \mathcal{C}_0(\mathbb{P}^{n-r}_{\mathbb{C}}) \cong \widetilde{\mathrm{SP}}(\mathbb{P}^{n-r}_{\mathbb{C}}) \simeq \prod_{i=1}^{n-r} K(\mathbb{Z}, 2i)$. □

It follows that $\mathcal{C}$ is homotopy equivalent to the weak product $\prod_{i>0} K(\mathbb{Z}, 2i)$. Standard arguments show that the naive group completion of $\mathfrak{C} = \coprod_d \mathcal{D}(d)$ is homotopy equivalent to $\mathcal{C} \times \mathbb{Z}$ [17, Thm. 5.4] using the complex suspension theorem and Dold/Thom [7, Thm. 6.10.III].
The naive group completion of $\mathfrak{C}$ is homotopy equivalent to the homotopy theoretic group completion $\Omega B(\mathfrak{C})$ by [18, Thm. 4.4']. Hence we obtain $\Omega B(\mathfrak{C}) \simeq \prod_{i\geq 0} K(\mathbb{Z}, 2i)$.
The inclusion $BU \cong \mathcal{D}(1) \hookrightarrow \mathcal{C}$ represents the total Chern class [17, Thm. 3.6]. Boyer, Lawson, Lima–Filho, Mann and Michelsohn [4] proved that there is an $\mathcal{I}_*$–functor T [20] which induces an infinite loop space structure on $\Omega B\mathfrak{C}$ compatible with the infinite loop space structure on $\mathcal{D}(1) \cong BU$ induced by Whitney sum of vector bundles. Hence the total Chern class map is an infinite loop map.

5 The continuity of the $\mathcal{I}_*$–functor defined by Chow varieties

The authors of [4] provided a framework for producing an $\mathcal{I}_*$–functor (cp. [20, Def. I.1.8]) from Chow varieties. Here we show, how continuity assertions in [4] can be derived in the language of resultants. The continuity of the $\mathcal{I}_*$–functor T follows from the following Lemma.

Lemma 5.1. *Let $V \subset \mathbb{C}^m$ be a n–dimensional vector space and $W \subset \mathbb{C}^m$ a p–dimensional vector space. If $f : V \to W$ is a linear, isometric embedding, then the map*

$$\begin{array}{rccc} Tf: & \coprod_d \mathcal{C}_{n-1,d}(\mathbb{P}(V \oplus V)) & \longrightarrow & \coprod_d \mathcal{C}_{p-1,d}(\mathbb{P}(W \oplus W)) \\ & c & \longmapsto & \mathbb{P}(f(V)^\perp \oplus \{0_W\}) \divideontimes (f \oplus f)_*(c) \end{array}$$

is algebraic.

Proof. The complex join is algebraic. Hence it suffices to check that

$$(f \oplus f)_* : \coprod_d \mathcal{C}_{n-1,d}(\mathbb{P}(V \oplus V)) \longrightarrow \coprod_d \mathcal{C}_{n-1,d}(\mathbb{P}(W \oplus W))$$

is algebraic.
We shall calculate the Chow form $F_{(f\oplus f)_*(c)}$ of the cycle $(f \oplus f)_*(c)$.
We can extend the map $(f \oplus f)$ to an automorphism of the vector space $\mathbb{C}^{2m}$ by choice of the orthogonal complement of $V \oplus V$ in $\mathbb{C}^{2m}$. This automorphism is given by an invertible $(2m \times 2m)$–matrix A over $\mathbb{C}$.
Without loss of generality suppose c to consist only on one irreducible component. Let $g_1, \ldots, g_k \in \mathbb{C}[x_0, \ldots, x_{2m-1}]$ be homogeneous generators of the vanishing ideal of c. Then $(f \oplus f)_*(c)$ is the irreducible zero set of

$$g_1(A^{-1}\underline{x}), \ldots, g_k(A^{-1}\underline{x}) \quad .$$

The Chow form F_c of c generates the radical of the resultant of

$$g_1(\underline{x}) = \cdots = g_k(\underline{x}) = \begin{pmatrix} u_0^0 & \cdots & u_{2m-1}^0 \\ \vdots & & \vdots \\ u_0^n & \cdots & u_{2m-1}^n \end{pmatrix} \cdot \underline{x} = 0 \quad .$$

The solutions of this equations are the same as the solutions of

$$g_1(A^{-1}\underline{x}) = \cdots = g_k(A^{-1}\underline{x}) = \begin{pmatrix} u_0^0 & \cdots & u_{2m-1}^0 \\ \vdots & & \vdots \\ u_{2m-1}^n & \cdots & u_{2m-1}^n \end{pmatrix} A^{-1}\underline{x} = 0\ . \quad (5)$$

Therefore F_c generates the radical of the resultant of (5), too. The following map is a ring automorphism

$$\begin{array}{rccccl} \varphi_f : & \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^{n-1}] & \to & \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^{n-1}] & & \\ & u_j^i & \mapsto & (\underline{u}^i A)_j & , i = 0, \ldots, n-1, j = 0, \ldots, 2m-1 & \\ & z & \mapsto & z & \forall z \in \mathbb{C}\ , & \end{array}$$

because the $(\underline{u}^i A)_j$ are algebraically independent over $\mathbb{C}$, $i = 0, \ldots, n-1$, $j = 0, \ldots, 2m-1$, for the entries of A are elements of $\mathbb{C}$, and A is invertible. The equations (5) are mapped onto

$$g_1(A^{-1}\underline{x}) = \cdots = g_k(A^{-1}\underline{x}) = \begin{pmatrix} u_0^0 & \cdots & u_{2m-1}^0 \\ \vdots & & \vdots \\ u_{2m-1}^{n-1} & \cdots & u_{2m-1}^{n-1} \end{pmatrix} AA^{-1}\underline{x} = 0\ . \quad (6)$$

Hence $\varphi_f(F_c) = F_c(\underline{u}^0 A, \ldots, \underline{u}^{n-1} A)$ generates the radical of the resultant of (6). But $F_{(f\oplus f)_*(c)}$ is a generator of the radical of the resultant of (6), too. Hence $\varphi_f(F_c)$ and $F_{(f\oplus f)_*(c)}$ coincide up to a factor in $\mathbb{C}^*$. But this means that $\varphi_f(F_c)$ is the Chow form of $(f \oplus f)_*$. φ_f depends on an extension of the linear isometry $(f \oplus f)$ to a matrix A. But $\varphi_f(F_c)$ coincides with $F_{(f\oplus f)_*}$ and is therefore independent of a chosen extension.

We obtain, that $(f \oplus f)_*$ is given on Chow forms by the map φ_f, which is polynomial in the coefficients of Chow forms. Hence $(f \oplus f)_*$ and Tf are algebraic. □

Part II

Chow varieties in arbitrary characteristic

6 Construction of Chow forms

We want to generalize the definition of Chow varieties to arbitrary fields K of arbitrary characteristic q. The problem that arises in characteristic $q \neq 0$ is that the coordinates of the points in the intersection of a variety over K and a K-general hyperplane can be inseparable over K. Hence their splitting field is not a Galois extension of K.

The property of being a Galois extension is fundamental for the definition of Chow forms. So we have to pass to intermediate fields to reinvent the Galois situation.

Construction and Definition 6.1. Let $V \subset \mathbb{P}^n_\Omega$ be a variety over K and H_{n-r} a K-general $(n-r)$-dimensional linear subspace of $\mathbb{P}^n_\Omega$. Further let $V \cap H_{n-r} = \{\underline{p}^1, \ldots, \underline{p}^d\}$. As mentioned in section 1 the $\underline{p}^k$, $k = 1, \ldots, d$, are general points of $V \cap H_{n-r}$ viewed as a variety over $K(\underline{u}^1, \ldots, \underline{u}^r)$. Therefore they are conjugate to each other over $K(\underline{u}^1, \ldots, \underline{u}^r)$. On the other hand it is obvious that $\underline{p}^k$, $k = 1, \ldots, d$, are general points of V viewed as a variety over K.

Since the general points of a variety are conjugate to each other we may again assume $p^i_j = 1$ for a fixed $j \in \{0, \ldots, n\}$ and for all $i = 1, \ldots, d$. Since $V \cap H_{n-r}$ has dimension 0, the p^i_j, $j = 0, \ldots, n$, $i = 1, \ldots, d$, are now elements of $\overline{K(\underline{u}^1, \ldots, \underline{u}^r)}$.

In case of $char(K) = q \neq 0$ let $l \in \mathbb{N}$ be minimal such that $(p^i_j)^{(q^l)}$ is separable over $K(\underline{u}^1, \ldots, \underline{u}^r)$ for all $i = 1, \ldots, d$; $j = 0, \ldots, n$. Observe that l is independent of i because the $\underline{p}^i$ are conjugate to each other. Define

$$\begin{aligned} L_i(\underline{u}^0) &:= \begin{cases} \langle \underline{p}^i, \underline{u}^0 \rangle & , char(K) = 0 \\ \langle \underline{p}^i, \underline{u}^0 \rangle^{(q^l)} & , char(K) = q \neq 0 \end{cases} \\ G(\underline{u}^0) &= \prod_{i=1}^{d} L_i(\underline{u}^0) \ . \end{aligned} \tag{7}$$

Let $\widetilde{Z}$ be the splitting field of the minimal polynomials of the p_j^i, $j = 0, \ldots, n$; $i = 1, \ldots, d$, and $Z \subset \widetilde{Z}$ the separable closure of $K(\underline{u}^1, \ldots, \underline{u}^r)$ in $\widetilde{Z}$.

By definition we have $L_i(\underline{u}^0) \in Z[\underline{u}^0]$ is irreducible, for $\langle \underline{p}^i, \underline{u}^0 \rangle^{q^k} \notin Z[\underline{u}^0]$ in case $q \neq 0$, $k < l$, for all $i = 1, \ldots, d$. Z is a Galois extension of $K(\underline{u}^1, \ldots, \underline{u}^r)$, for $i = 1, \ldots, d$. Now the Galois action of this extension still permutes the $L_i(\underline{u}^0)$, $i = 1, \ldots, d$. But this means the polynomial $G(\underline{u}^0)$ remains invariant under this Galois action, hence it has coefficients in $K(\underline{u}^1, \ldots, \underline{u}^r)$. Let $c \in K[\underline{u}^1, \ldots, \underline{u}^r]$ be the common denominator of the coefficients of $G(\underline{u}^0)$. Now we define

$$F(\underline{u}^0, \ldots, \underline{u}^r) := cG(\underline{u}^0) \in K[\underline{u}^0, \ldots, \underline{u}^r] \tag{8}$$

to be the Chow form of V.

By construction we know

Lemma 6.2. *Let $V \subset \mathbb{P}^n_\Omega$ be a variety of dimension r. The Chow form $F(\underline{u}^0, \ldots, \underline{u}^r) \in K[\underline{u}^0, \ldots, \underline{u}^r]$ is uniquely determined up to a factor in K^* and irreducible.*

Since the $\underline{p}^i$ are general points of V, we also obtain different varieties having different Chow forms.

Definition 6.3. Let $V_1, \ldots, V_k \subset \mathbb{P}^n_\Omega$ be r-dimensional varieties with Chow forms $F_1, \ldots, F_k$. Then we define

$$F(\underline{u}^0, \ldots, \underline{u}^r) := F_1^{n_1} \cdot \cdots \cdot F_k^{n_k}$$

to be the Chow form of the *algebraic cycle* $V := \sum_{i=1}^{k} n_i V_i$, $n_i \in \mathbb{N}$.
The degree of $F(\underline{u}^0, \ldots, \underline{u}^r)$ as a polynomial in $K[\underline{u}^1, \ldots, \underline{u}^r][\underline{u}^0]$ is called the *degree* of V.

Analogously to section 2 we want to show the relation between Chow forms and resultants:

Theorem 6.4. *Let $V \subset \mathbb{P}^n_\Omega$ be a r-dimensional variety and $f_1, \ldots, f_k$ be homogeneous generators of the vanishing ideal of V. The Chow form F of V generates the resultant $\sqrt{R} \subset K[\underline{u}^0, \ldots, \underline{u}^r]$ of the following equations:*

$$\begin{aligned} f_1 = \cdots = f_k \\ \langle \underline{u}^0, \underline{x} \rangle = \cdots = \langle \underline{u}^r, \underline{x} \rangle \end{aligned} \tag{9}$$

Before we prove the theorem we need a technical lemma:

Lemma 6.5. *Let Z be a field of characteristic $q \neq 0$ and $u_0, \ldots, u_n \in \Omega$ algebraically independent over Z. Suppose $\widetilde{Z} \supset Z$ is a purely inseparable field extension. Let $L(\underline{u}) \in Z[\underline{u}]$ be irreducible and*

$$L(\underline{u}) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u} \rangle \in \widetilde{Z}[\underline{u}] \ , c \in \widetilde{Z}$$

Then $\underline{p}^1 = \underline{p}^i$ for all $i = 1, \ldots, d$ and $d = q^l$, where $l \in \mathbb{N}$ is minimal such that $(p_j^1)^{q^l} \in Z$ for all $j = 0, \ldots, n$.

Proof. The first step is to prove that if we have $p_j^i = 0$ for a fixed $j \in \{0, \ldots, n\}$ and for one $i \in \{1, \ldots, d\}$, then $p_j^i = 0$ for all $i = 1, \ldots, d$. Suppose $j = 0$ and $p_0^1 = \cdots = p_0^k = 0$, $p_0^{k+1} = \cdots = p_0^d = 1$. Now determine the coefficient $C \in Z$ of the monomial $(u_0)^{d-k} u_{i_1} \cdots u_{i_k}$ of $L(\underline{u})$ for $i_1, \ldots, i_k \in \{1, \ldots, n\}$:

$$C = c \sum_{J(d)} p_{i_1}^{j_1} p_{i_2}^{j_2} \cdots p_{i_k}^{j_k} p_0^1 \cdots \widehat{p_0^{j_1}} \cdots \widehat{p_0^{j_k}} \cdots p_0^d$$

$J(d) = \{(j_1, \ldots, j_k) \in \{1, \ldots, d\}^k \mid j_s \neq j_t \text{ for } s \neq t\}$, and $\widehat{p} := 1$. Since $p_0^1 = \cdots = p_0^k = 0$ we obtain

$$C = c \sum_{J(k)} p_{i_1}^{j_1} p_{i_2}^{j_2} \cdots p_{i_k}^{j_k} \underbrace{p_0^{k+1} \cdots p_0^d}_{=1}$$

But this implies

$$c \prod_{i=1}^{k} \langle \underline{p}^i, \underline{u} \rangle$$

is a polynomial in $Z[\underline{u}]$, which implies $k = d$ because of the irreducibility of $L(\underline{u})$.

Now we can assume without loss of generality $p_0^i = 1$ for $i = 1, \ldots, d$ and therefore $c \in Z$. As a primitive polynomial $L(\underline{u})$ is irreducible in $Z[\underline{u}] = Z[u_1, \ldots, u_n][u_0]$ if and only if it is irreducible in $Z(u_1, \ldots, u_n)[u_0]$.
The d zeros of $L(\underline{u})$ in $Z(u_1, \ldots, u_n)$ are given by $z(i) = -p_1^i u_1 - \cdots - p_n^i u_n \in \widetilde{Z}(u_1, \ldots, u_n)$ for $i = 1, \ldots, d$. But since $\widetilde{Z} \supset Z$ is purely inseparable, we know $\widetilde{Z}(u_1, \ldots, u_n) \supset Z(u_1, \ldots, u_n)$ is purely inseparable. That means $z(i)$ are purely inseparable and zeros of an irreducible polynomial, therefore $z(1) = z(i)$ for all $i = 1, \ldots, d$ and $d = q^l$ for some $l \in \mathbb{N}$. □

Proof of Theorem 6.4. The first step is to show $\sqrt{R}$ is a prime ideal in $K[\underline{u}^0, \ldots, \underline{u}^r]$: Let

$$A := \{(\underline{p}, \underline{y}^0, \ldots, \underline{y}^r) \in \mathbb{P}^n_\Omega \times (\mathbb{P}^n_\Omega)^{r+1} \mid \underline{p} \in V \ , \ \langle \underline{p}, \underline{y}^0 \rangle = \cdots = \langle \underline{p}, \underline{y}^r \rangle = 0\}.$$

The vanishing ideal of A in $K[\underline{u}^0, \ldots, \underline{u}^r, \underline{x}]$ is generated by the polynomials in (9). The projection pr_1 onto the first factor is V and the projection pr_2 of A onto the second factor $(\mathbb{P}^n_\Omega)^{r+1}$ gives us the zero set of $R \subset K[\underline{u}^0, \ldots, \underline{u}^r]$. Now we can apply [25, I.6.3 Th. 8] to pr_1 and obtain A being irreducible. Hence $pr_2(A)$ is irreducible, which proves $\sqrt{R}$ being prime.

Now we can prove the theorem:

The Chow form F of V is $F(\underline{u}^0, \ldots, \underline{u}^r) = \prod_{i=1}^{d} L_i(\underline{u}^0)$ as defined in (8) and (7). We can assume the $\underline{p}^1, \ldots, \underline{p}^d$ to be normed such that the factor c in (8) equals 1. Denote by $\widetilde{Z} \subset \overline{K(\underline{u}^1, \ldots, \underline{u}^r)}$ the splitting field of the minimal polynomials of p^i_j, $i = 1, \ldots, d$, $j = 0, \ldots, n$. Further let Z be the separable closure of $K(\underline{u}^1, \ldots, \underline{u}^r)$ in $\widetilde{Z}$.

Now view the polynomials in (9) as polynomials in $Z[\underline{u}^0][x_0, \ldots, x_n]$ and determine the resultant $\sqrt{R \cdot Z[\underline{u}^0]}$. The zero set of $R \cdot Z[\underline{u}^0]$ in $\mathbb{P}^n_\Omega$ equals the zero set of $\prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$. We know by Lemma 6.5 that if $B \in Z[\underline{u}^0]$ is divisible by $\prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$, then B is divisible by F. This means that the radical of the ideal which is generated by R in $Z[\underline{u}^0]$ is the ideal which is generated by F, i.e.

$$(F)_{Z[\underline{u}^0]} = \sqrt{R \cdot Z[\underline{u}^0]} \quad \text{in } Z[\underline{u}^0] \ .$$

Cutting down with $K[\underline{u}^0, \ldots, \underline{u}^r]$ we first obtain

$$(F)_{K[\underline{u}^0, \ldots, \underline{u}^r]} \subset (F)_{Z[\underline{u}^0]} \cap K[\underline{u}^0, \ldots, \underline{u}^r] \ .$$

Now let $h \in Z[\underline{u}^0]$ such that $F \cdot h \in (F)_{Z[\underline{u}^0]} \cap K[\underline{u}^0, \ldots, \underline{u}^r]$. And let σ be an element of the Galois group of Z over $K(\underline{u}^1, \ldots, \underline{u}^r)$. Then $F \cdot h = \sigma(F \cdot h) = F \cdot \sigma(h)$, i.e. $h \in K(\underline{u}^1, \ldots, \underline{u}^r)[\underline{u}^0]$, for Z is a Galois extension. Therefore $h = \frac{h_1}{h_2}$, $h_1 \in K[\underline{u}^0, \ldots, \underline{u}^r], h_2 \in K[\underline{u}^1, \ldots, \underline{u}^r]$, and the greatest common divisor of h_1 and h_2 is 1. Now h_2 must be a divisor of F, and since F is irreducible in $K[\underline{u}^0, \ldots, \underline{u}^r]$ and $\underline{u}^0$ does not appear in h_2, then $h_2 \in K^*$. Therefore

$$(F)_{Z[\underline{u}^0]} \cap K[\underline{u}^0, \ldots, \underline{u}^r] = (F)_{K[\underline{u}^0, \ldots, \underline{u}^r]} \ .$$

Now it remains to show $\sqrt{R}$ equals $\sqrt{R \cdot Z[\underline{u}^0]} \cap K[\underline{u}^0, \ldots, \underline{u}^r]$. Obviously $\sqrt{R} \subset \sqrt{R \cdot Z[\underline{u}^0]} \cap K[\underline{u}^0, \ldots, \underline{u}^r] = (F)_{Z[\underline{u}^0]} \cap K[\underline{u}^0, \ldots, \underline{u}^r] = (F)_{K[\underline{u}^0, \ldots, \underline{u}^r]}$. For the inverse inclusion we have to show $F \in \sqrt{R}$.

Suppose $g_1, \ldots, g_s \in K[\underline{u}^0, \ldots, \underline{u}^r]$ generate the ideal $\sqrt{R}$. Since $\sqrt{R} \subset (F)_{K[\underline{u}^0, \ldots, \underline{u}^r]}$ we know

$$g_i = F^{n_i} \cdot G_i \ , \ F \not| \, G_i \text{ in } K[\underline{u}^0, \ldots, \underline{u}^r]$$

for some $n_i > 0$, $G_i \in K[\underline{u}^0, \ldots, \underline{u}^r]$, $i = 1, \ldots, s$. For the zero sets in $(\mathbb{P}^n_\Omega)^{r+1}$ we obtain

$$\begin{aligned} Z((F)) \subset Z(\sqrt{R}) &= Z((F^{n_i} \cdot G_i \mid i = 1, \ldots, s)) \\ &= Z((F)) \cup Z((G_i \mid i = 1, \ldots, s)) \ . \end{aligned}$$

But $\sqrt{R}$ is a prime ideal, which means $Z(\sqrt{R}) = Z((F))$ or $Z(\sqrt{R}) = Z((G_i \mid i = 1, \ldots, s))$. The second case would imply $Z((F)) \subset Z((G_i \mid i = 1, \ldots, s))$, hence the G_i would be divisible by F, which contradicts the properties of the G_i. That finishes the proof of the theorem. □

Lemma 6.5 now permits us to transfer the theorem of Chow and Van der Waerden [5, Satz 1], which characterizes those multihomogeneous form that actually are Chow forms, in a formal way to the case of arbitrary characteristic:

Theorem 6.6. *Let $F \in K[\underline{u}^0, \ldots, \underline{u}^r]$ be multihomogeneous of multidegree d. F is a Chow form of an algebraic cycle in $\mathcal{C}_{r,d}(\mathbb{P}^n_K)$ iff the following conditions hold:*

There are d points $\underline{p}^i \in \overline{K(\underline{u}^1, \ldots, \underline{u}^r)}^{n+1} - \{0\}$ not necessarily different and $c \in K(\underline{u}^1, \ldots, \underline{u}^r)$ satisfying:

1. $F(\underline{u}^0, \ldots, \underline{u}^r) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$, *in* $\overline{K(\underline{u}^1, \ldots, \underline{u}^r)}[\underline{u}^0]$.

2. $L_i(\underline{u}^j) := \langle \underline{p}^i, \underline{u}^j \rangle = 0$ *for* $i = 1, \ldots, d$, $j = 1, \ldots, r$.

3. *For every $i \in \{1, \ldots, d\}$ one has:*
   Let S^j be skew symmetric $(n+1) \times (n+1)$–matrices, $j = 1, \ldots, r$, whose entries $s^j_{k,l} \in \Omega$ are algebraically independent over $K(\underline{p}^i)$, $k, l = 0, \ldots, n$. In this situation $L_i(\underline{u}^0)$ divides the form $F(\underline{u}^0, S^1\underline{p}^i, \ldots, S^r\underline{p}^i)$ in $K(\underline{p}^i)[s^1_{0,0}, \ldots, s^r_{n,n}][\underline{u}^0]$.

The three conditions of the theorem can be reformulated in homogeneous equations [5]. Hence we obtain for an algebraic set $X \subset \mathbb{P}^n_\Omega$ over K the Chow variety $\mathcal{C}_{r,d}(X) \subset \mathbb{P}^N_\Omega$ being an algebraic set over K.

7 Algebraicity of complex suspension

The aim of this section is to prove that the complex suspension is an algebraic map.

Recall that for a variety $V \subset \mathbb{P}^{n-1}_{\Omega}$, whose vanishing ideal $I_K(V)$ is generated by $f_0, \ldots, f_k \in K[x_1, \ldots, x_n]$, the vanishing ideal $I_K(\not\exists V)$ is generated by the same polynomials $f_0, \ldots, f_k$ but viewed as polynomials in $K[x_0, \ldots, x_n]$, i.e. in one more variable.

Theorem 7.1. *The complex suspension $\not\exists : \mathcal{C}_{r-1,d}(\mathbb{P}^{n-1}_K) \to \mathcal{C}_{r,d}(\mathbb{P}^n_K)$ is algebraic.*

Proof. Let $F(\underline{u}^1, \ldots, \underline{u}^r)$ be the Chow form of $V = \sum_{m=1}^{k} n_i V_i \in \mathcal{C}_{r-1,d}(\mathbb{P}^{n-1}_K)$. Let $\underline{u}^i = (u^i_1, \ldots, u^i_n)$, $i = 1, \ldots, r$ and let $v^j_k \in \Omega$ be algebraically independent over $K(\underline{u}^1, \ldots, \underline{u}^r)$, $k = 0, \ldots, n$, $j = 0, \ldots, r$.
For suitable $\underline{p}^l \in \left(\overline{K(\underline{u}^2, \ldots, \underline{u}^r)}\right)^n$, $l = 1, \ldots, d$, the Chow form of V splits:

$$\begin{aligned} F_V(\underline{u}^1, \ldots, \underline{u}^r) &= \prod_{l=1}^{d} \sum_{k=1}^{n} p^l_k u^1_k \qquad\qquad \in \overline{K(\underline{u}^2, \ldots, \underline{u}^r)}[\underline{u}^1] \\ &= \sum_{1 \le i_1 \le \cdots \le i_d \le n} u^1_{i_1} \cdot \ldots \cdot u^1_{i_d} \underbrace{\sum_{\sigma \in \Sigma(i_1, \ldots, i_d)} p^1_{\sigma(i_1)} \cdot \ldots \cdot p^d_{\sigma(i_d)}}_{=: G_{i_1, \ldots, i_d}(\underline{u}^2, \ldots, \underline{u}^r)} , \end{aligned}$$

such that $G_{i_1,\ldots,i_d} \in K[\underline{u}^2, \ldots, \underline{u}^r]$. The group $\Sigma(i_1, \ldots, i_d)$ is that subgroup of Σ_n which only permutes the numbers $i_1, \ldots, i_d \in \{1, \ldots, n\}$. Each $\underline{p}^l$ is contained in one of the V_j and is a solution of the following equations:

$$\langle \underline{u}^i, \underline{x} \rangle = 0 \ , i = 2, \ldots, r \ .$$

On the other hand the Chow form of $\not\exists V$ is obtained from points $\underline{q}^l$ of one of the $\not\exists V_j$ which are solutions of the equation (cp. Construction 6.1):

$$\begin{pmatrix} v^1_0 & v^1_1 & \ldots & v^1_n \\ \vdots & \vdots & & \vdots \\ v^r_0 & v^r_1 & \ldots & v^r_n \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad .$$

This equation is equivalent to

$$\underbrace{\begin{pmatrix} v_0^1 & v_1^1 & \dots & v_n^1 \\ 0 & v_1^2v_0^1 - v_1^1v_0^2 & \dots & v_n^2v_0^1 - v_n^1v_0^2 \\ \vdots & \vdots & & \vdots \\ 0 & v_1^rv_0^1 - v_1^1v_0^r & \dots & v_n^rv_0^1 - v_n^1v_0^r \end{pmatrix}}_{=: A} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad ,$$

and the entries of the matrix A are algebraically independent over the field $K(\underline{u}^1, \dots, \underline{u}^r)$.

Step 1: Observe the following injective ring homomorphism:

$$\begin{array}{rccll} \tilde{\varphi}: & K[\underline{u}^2, \dots, \underline{u}^r, \underline{v}^0, \underline{v}^1] & \longrightarrow & K[\underline{v}^0, \dots, \underline{v}^r] & \\ & u_k^i & \longmapsto & v_k^iv_0^1 - v_k^1v_0^i & , i = 2, \dots, r \ , k = 1, \dots, n \\ & c & \longmapsto & c & \forall c \in K[\underline{v}^0, \underline{v}^1] \ . \end{array}$$

It is extendable to an automorphism φ of Ω and we obtain:

$$\varphi^{-1}(A) = \begin{pmatrix} v_0^1 & v_1^1 & \dots & v_n^1 \\ 0 & u_1^2 & \dots & u_n^2 \\ \vdots & \vdots & & \vdots \\ 0 & u_1^r & \dots & u_n^r \end{pmatrix} \ .$$

Now one can calculate the points $\underline{q}^l$ from which the Chow form of $\nexists V$ arises:

$$\begin{array}{l} \underline{q}^i = (t_i, \varphi(p_1^i), \dots, \varphi(p_n^i)) \qquad\qquad , i = 1, \dots, d \\ t_i = -\dfrac{v_1^1\varphi(p_1^i) + \dots + v_n^1\varphi(p_n^i)}{v_0^1} \ . \end{array}$$

This procedure determines the Chow form, according to Construction 6.1. We start with the form G using the notations of 2.1 multiplied by $(v_0^1)^d$. Let I_k be the index set

$I_k := \{(i_1, \dots, i_d) \in \mathbb{N}^d \mid 1 \le i_1 \le \dots \le i_k \le n, 1 \le i_{k+1} \le \dots \le i_d \le n\}$ for

$0 \leq k \leq d$.

$$G(\underline{v}^0,\ldots,\underline{v}^r) := \quad (v_0^1)^d \prod_{i=1}^{d}\sum_{j=0}^{n} q_j^i v_j^0 \quad = \quad \prod_{i=1}^{d}\left(v_0^1 t_i v_0^0 + \sum_{j=1}^{n} v_0^1\varphi(p_j^i)v_j^0\right) \quad =$$

$$\prod_{i=1}^{d}\left(-(v_1^1\varphi(p_1^i)+\cdots+v_n^1\varphi(p_n^i))v_0^0 + \sum_{j=1}^{n} v_0^1\varphi(p_j^i)v_j^0\right) \quad =$$

$$\prod_{i=1}^{d}\left(-v_0^0\sum_{j=1}^{n} v_j^1\varphi(p_j^i) + v_0^1\sum_{j=1}^{n}\varphi(p_j^i)v_j^0\right) \quad =$$

$$\sum_{k=0}^{d}\sum_{I_k}(-v_0^0)^k v_{i_1}^1\cdots v_{i_k}^1 (v_0^1)^{d-k} v_{i_{k+1}}^0\cdots v_{i_d}^0 \sum_{\sigma\in\Sigma(i_1,\ldots,i_d)} \varphi(p^1_{\sigma(i_1)})\cdots\varphi(p^d_{\sigma(i_d)}) \quad =$$

$$\sum_{k=0}^{d}\sum_{I_k}(-v_0^0)^k v_{i_1}^1\cdots v_{i_k}^1 (v_0^1)^{d-k} v_{i_{k+1}}^0\cdots v_{i_d}^0 \cdot \varphi\left(\sum_{\sigma\in\Sigma(i_1,\ldots,i_d)} p^1_{\sigma(i_1)}\cdots p^d_{\sigma(i_d)}\right) =$$

$$\sum_{k=0}^{d}\sum_{I_k}(-v_0^0)^k v_{i_1}^1\cdot\cdots\cdot v_{i_k}^1 (v_0^1)^{d-k} v_{i_{k+1}}^0\cdot\cdots\cdot v_{i_d}^0 \cdot \varphi(G_{i_1,\ldots,i_d}(\underline{u}^2,\ldots,\underline{u}^r)) \quad .$$

This calculation shows that the coefficients of the form G are polynomials in the coefficients of F_V. These polynomials depend only on the dimension, the codimension and the degree of V.

On the other hand this calculation proves, that the degree of an algebraic cycle is stable under complex suspension, so the complex suspension is a well defined map.

Step 2: The form $G \in K[\underline{v}^0,\ldots,\underline{v}^r]$ is homogeneous of degree rd in $\underline{v}^1$ and homogeneous of degree d in $\underline{v}^0,\underline{v}^2,\ldots,\underline{v}^r$. Therefore G is not the Chow form of $\not\exists V$, but as in Construction 2.1 there is a $T \in K(\underline{v}^1,\ldots,\underline{v}^r)$ such that

$$G(\underline{v}^0,\ldots,\underline{v}^r) = F_{\not\exists V}(\underline{v}^0,\ldots,\underline{v}^r)T \ ,$$

where $F_{\not\exists V}$ denotes the Chow form of $\not\exists V$. Let $T_1 \in K[\underline{v}^1,\ldots,\underline{v}^r]$ be the common denominator of T. Then T_1 has to be a divisor of $F_{\not\exists V}$ in $K[\underline{v}^0,\ldots,\underline{v}^r]$. But this can only happen if $T_1 \in K^*$, because the irreducible factors of $F_{\not\exists V}$ are polynomials of positive degree in $\underline{v}^0$. This proves that $T \in K[\underline{v}^1]$ and T is homogeneous of degree $(r-1)d$.

For the case $r = 1$ the theorem is proved. For the case $r > 1$ one has to show that $T(\underline{v}^1) = (v_0^1)^{(r-1)d}$. Without loss of generality assume that V is a variety:

Step 3: Determine the zero set of $T(\underline{v}^1)$ as a polynomial in $K[\underline{v}^1]$:
Let $\underline{w} = (0, v_1^1, \ldots, v_n^1)$ and $\underline{\tilde{w}} = (v_1^1, \ldots, v_n^1)$. Then one obtains:

$$
\begin{aligned}
&G(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) = \\
&\sum_{k=0}^{d} \sum_{\substack{1 \le i_1 \le \cdots \le i_k \le n \\ 1 \le i_{k+1} \le \cdots \le i_d \le n}} (-v_0^0)^k v_{i_1}^1 \cdots v_{i_k}^1 0^{d-k} v_{i_{k+1}}^0 \cdots v_{i_d}^0 \cdot \varphi(G_{i_1,\ldots,i_d}(\underline{u}^2, \ldots, \underline{u}^r)) = \\
&\sum_{1 \le i_1 \le \cdots \le i_d \le n} (-v_0^0)^d v_{i_1}^1 \cdot \cdots \cdot v_{i_d}^1 \cdot G_{i_1,\ldots,i_d}(v_0^2 \cdot \underline{\tilde{w}}, v_0^3 \cdot \underline{\tilde{w}}, \ldots, v_0^r \cdot \underline{\tilde{w}}) = \\
&(-v_0^0)^d \sum_{1 \le i_1 \le \cdots \le i_d \le n} v_{i_1}^1 \cdot \cdots \cdot v_{i_d}^1 \cdot (v_0^2)^d \cdot \cdots \cdot (v_0^r)^d \cdot G_{i_1,\ldots,i_d}(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}}) = \\
&(-v_0^0)^d \cdot (v_0^2)^d \cdot \cdots \cdot (v_0^r)^d \sum_{1 \le i_1 \le \cdots \le i_d \le n} v_{i_1}^1 \cdot \cdots \cdot v_{i_d}^1 \cdot G_{i_1,\ldots,i_d}(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}}) = \\
&(-v_0^0)^d \cdot (v_0^2)^d \cdot \cdots \cdot (v_0^r)^d \cdot F_V(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}}) \ .
\end{aligned}
$$

But $F_V(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}})$ must be zero, because the dimension of V is greater than 0, and $F_V(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}})$ is a divisor of a resultant which has to be 0. It is easy to show that the Chow form $F_{\not\exists V}(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) \neq 0$ using the property that $F_{\not\exists V}$ generates the radical of a resultant. Therefore $T(0, v_1^1, \ldots, v_n^1) = 0$ for all $v_1^1, \ldots, v_n^1 \in \Omega$ and hence v_0^1 divides T.
Now we claim that v_0^1 is the only irreducible factor of T: Suppose $T(\underline{w}) = 0$ such that $w_0 \neq 0$, $\underline{w} \in \mathbb{P}_\Omega^n$. We may assume $w_0 = 1$. For the Chow form of $\not\exists V$ one has

$$F_{\not\exists V}(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) = 0 \text{ iff } \langle \underline{p}, \underline{w} \rangle = 0 \text{ for all } \underline{p} \in \not\exists V \cap H_{n-(r-1)} \ .$$

But this would mean $\langle \underline{x}, \underline{w} \rangle \in \Omega[\underline{x}]$ is an element of $\sqrt{I_K(\not\exists V) \cdot \Omega[\underline{x}]}$, since $\not\exists V \cap H_{n-(r-1)}$ contains general points of $\not\exists V$. This implies $w_0 = 0$, because $\langle \underline{x}, \underline{w} \rangle$ is irreducible and $I_K(\not\exists V)$ is generated by elements in $K[x_1, \ldots, x_n]$. This is a contradiction to the assumption $w_0 = 1$.

On the other hand we have $G(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) = 0$, because $T(\underline{w}) = 0$. That means there is a point $\underline{p} \in \not\exists V$ satisfying the following equation:

$$
\begin{pmatrix}
v_0^0 & v_1^0 & \ldots & v_n^0 \\
1 & w_1^1 & \ldots & w_n^1 \\
0 & v_1^2 - w_1^1 v_0^2 & \ldots & v_n^2 - w_n^1 v_0^2 \\
\vdots & \vdots & & \vdots \\
0 & v_1^r - w_1^1 v_0^r & \ldots & v_n^r - w_n^1 v_0^r
\end{pmatrix}
\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \ ,
$$

which is equivalent to

$$\begin{pmatrix} v_0^0 & v_1^0 & \dots & v_n^0 \\ 1 & w_1^1 & \dots & w_n^1 \\ v_0^2 & v_1^2 & \dots & v_n^2 \\ \vdots & \vdots & & \vdots \\ v_0^r & v_1^r & \dots & v_n^r \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad ,$$

because in the case where V is irreducible, the form G equals $\varphi(F_{\not\Sigma V})$ up to a factor in $K[v_0^1]$. But this means that $F_{\not\Sigma V}(\underline{v}^0, \underline{w}, \underline{v}^2, \dots, \underline{v}^r) = 0$, which leads to a contradiction as above. □

Part III
Topological Invariance of Chow varieties

8 Independence of embeddings

We want to generalize the notion of the Chow variety of cycles in $\mathbb{P}^n_\Omega$ to cycles on an arbitrary projective variety. This is done via chosen embeddings $X \hookrightarrow \mathbb{P}^n_\Omega$.

Nagata [23] showed that the Chow variety of 0-dimensional cycles of degree 2 in $\mathbb{P}^3$ over $\mathbb{Z}_2$ is not normal, but that there is an embedding into some $\mathbb{P}^n$ such that the Chow variety of the embedded $\mathbb{P}^3$ becomes normal. Hence those two Chow varieties can not be isomorphic algebraic sets.

The previous discussion shows that (in general) building Chow varieties in the CATEGORY of algebraic sets and algebraic maps is not possible. Therefore our definition of Chow varieties of cycles in X will depend on a chosen embedding.

Hoyt [14] proved the homeomorphism type of $\coprod_{d\geq 0} \mathcal{C}_{r,d}(X,i)$ to be independent of embeddings. The purpose of this part is to clarify and straighten out Hoyt's approach for arbitrary fields.

Definition 8.1. Let X be a projective variety over K with a K-embedding $j : X \hookrightarrow \mathbb{P}^n_\Omega$. Denote by $\mathcal{C}_{r,d}(X,j)$ the Chow variety of cycles V, such that $\deg V = d$ and $supp(V) \subset j(X)$. Further let $\mathcal{C}_r(X,j) := \coprod_{d\geq 0} \mathcal{C}_{r,d}(X,j)$.

As already mentioned the Chow varieties of projective varieties depend on embeddings. But in the case of ground field $\mathbb{C}$ the independence of the chosen embedding arises from analytic arguments as we will see below. In the case of characteristic $q \neq 0$ the algebraic sets can be topologized using the methods of Friedlander [8]. Then algebraic maps become continuous. This leads to the following definition.

Definition 8.2. A *continuous algebraic map* is a map $\varphi : V \to W$ — in the category of sets — between projective algebraic sets, which is continuous in

the Zariski topology, and whose graph is an algebraic subset of the product $V \times W$.

In fact every algebraic map of projective subsets is continuous algebraic, but a continuous algebraic map need not be algebraic. We want to prove the following

Theorem 8.3. *Let X be a projective variety over an arbitrary field. The definition of $\mathcal{C}_r(X) := \mathcal{C}_r(X, j)$ is independent of a chosen embedding $j : X \hookrightarrow \mathbb{P}^n_\Omega$ up to algebraic continuous homeomorphism.*

By Barlet [1, Thm. 6,p. 109] we know the space of analytic cycles to be independent of embeddings up to analytic isomorphism. Since the space of analytic cycles and the space of algebraic cycles are analytically isomorphic in case $K = \mathbb{C}$ ([1, Thm. 7]), we know by GAGA that $\mathcal{C}_r(X)$ is independent of embeddings up to isomorphism in case $K = \mathbb{C}$. This already proves Theorem 8.3 in the case of complex numbers.

In the following we want to prove Theorem 8.3 for K an arbitrary field.

Definition 8.4. Let X be a projective variety together with two embeddings $i : X \hookrightarrow \mathbb{P}^n_\Omega$ and $j : X \hookrightarrow \mathbb{P}^m_\Omega$. Define the embedding

$$i * j : X \xrightarrow{\Delta} X \times X \xrightarrow{i \times j} \mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega \xrightarrow{S} \mathbb{P}^{nm+n+m}_\Omega$$

where S denotes the Segre embedding.

To prove theorem 8.3 it suffices to check that the projection

$$S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega) \cong \mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega \xrightarrow{pr_1} \mathbb{P}^n_\Omega$$

induces an algebraic map

$$\mathcal{C}_r(X, i * j) \xrightarrow{(pr_1)_*} \mathcal{C}_r(X, i) \ .$$

Since the map is bijective, onto and proper [25, I.5.2 Th. 2], it is a homeomorphism. The graph of the inverse map is just the transposed graph of the map itself, hence the inverse map is algebraic continuous, too. In this situation the two algebraic continuous homeomorphism

$$\mathcal{C}_r(X, i) \xleftarrow{(pr_1)_*} \mathcal{C}_r(X, i * j) \xrightarrow{(pr_2)_*} \mathcal{C}_r(X, j)$$

give the algebraic continuous homeomorphism $\mathcal{C}_r(X,i) \cong \mathcal{C}_r(X,j)$.

The strategy of the proof of theorem 8.3 is the following:

Strategy of proving theorem 8.3. At first we will evaluate the Chow form of $i * j(V)$ on special vectors, where the evaluated form contains $F_{i(V)}$ as a factor (Prop. 8.9). To obtain this result we need some reformulations on the properties of $i * j(V)$ and its Chow form (Rem. 8.6, Rem. 8.7, Cor. 8.12).

Then we will have to check that the chosen evaluation does not produce unwanted multiplicities, which means we never leave general position during the evaluation (Lem. 8.10, Thm. 8.11). That reduces the proof of theorem 8.3 to the determination of the greatest common divisor of polynomials (Cor. 8.12). The algebraicity of the determination of greatest common divisors is at last proved in the following section.

As a first part of the proof of theorem 8.3 we want to determine the map $(pr_1)_*$ for a fixed cycle V. We need some notations:

Notation 8.5. For $\underline{a} \in \mathbb{P}^n_\Omega$ and $\underline{b} \in \mathbb{P}^m_\Omega$ let

- $H(\underline{a})$ denote the hyperplane in $\mathbb{P}^n_\Omega$ given by $a_0x_0 + \cdots + a_nx_n = 0$ and
- $H(S(\underline{a},\underline{b}))$ the hyperplane in $\mathbb{P}^{nm+n+m}_\Omega$ given by $a_0b_0z_{0,0} + a_0b_1z_{0,1} + \cdots + a_nb_mz_{n,m} = 0$.

Remark 8.6. Obviously

$$S(\underline{p},\underline{q}) \in H(S(\underline{a},\underline{b})) \Longleftrightarrow \underline{p} \in H(\underline{a}) \text{ or } \underline{q} \in H(\underline{b})$$

as we have

$$\begin{aligned} & a_0b_0 \cdot p_0q_0 + a_0b_1 \cdot p_0q_1 + \cdots + a_nb_m \cdot p_nq_m = 0 \\ \Longleftrightarrow \quad & (a_0p_0 + \cdots + a_np_n)(b_0q_0 + \cdots + b_mq_m) = 0 \end{aligned}$$

To determine $(pr_1)_*$ for a fixed cycle we will make some set theoretic rewritings using several times the following fundamental fact which arises from the property of Chow forms being the generator of resultants (Thm. 6.4):

Remark 8.7. For a variety $W \subset \mathbb{P}^n_\Omega$ with Chow form F_W and $\underline{a}^0, \ldots, \underline{a}^r \in \mathbb{P}^n_\Omega$ one has:

$$F_W(\underline{a}^0, \ldots, \underline{a}^r) = 0 \Longleftrightarrow W \cap H(\underline{a}^0) \cap \cdots \cap H(\underline{a}^r) \neq \emptyset\,.$$

Step 1. Without loss of generality let V be a r-dimensional subvariety of X. Further let $F_{i(V)}$ be the Chow form of $i(V)$ and $F_{i*j(V)}$ the Chow form of $i*j(V)$.

For $i*j(V)$ we now obtain by 8.7 for fixed $\underline{a}^0,\ldots,\underline{a}^r \in \mathbb{P}^n_\Omega$:

$$\begin{aligned} & F_{i*j(V)}(S(\underline{a}^0,\underline{b}^0),S(\underline{a}^1,\underline{b}^1),\ldots,S(\underline{a}^r,\underline{b}^r)) = 0 \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \\ \iff & \bigcap_{k=0}^{r} H(S(\underline{a}^k,\underline{b}^k)) \cap (i*j(V)) \neq \emptyset \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \end{aligned}$$

Suppose $\underline{P} \in i*j(V)$, that means $\underline{P} = S(\underline{p},\underline{q})$ such that $j(i^{-1}(\underline{p})) = \underline{q}$. Using 8.6 we obtain

$$\begin{aligned} & \bigcap_{k=0}^{r} H(S(\underline{a}^k,\underline{b}^k)) \cap (i*j(V)) \neq \emptyset \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \\ \iff & \bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) \neq \emptyset \ \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega. \end{aligned}$$

Lemma 8.8.

$$\begin{aligned} & \bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) \neq \emptyset \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \\ \iff & \bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V) \neq \emptyset \ . \end{aligned}$$

Proof. For $\underline{p} \in \bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V)$ we know that $(\underline{p}, j(i^{-1}(\underline{p})))$ is an element of $\bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V))$ for any $\underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega$.

Hence it remains to show, if $\bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V) = \emptyset$ there exist $\underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega$ such that $\bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) = \emptyset$.

Now $\bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V) = \emptyset$ means

$$\dim \left(H(\underline{a}^k) \cap i(V) \cap \bigcap_{l \in L_k} H(\underline{a}^l) \right) \neq \dim \left(i(V) \cap \bigcap_{l \in L_k} H(\underline{a}^l) \right)$$

for each $L_k \subset \{0,\ldots,r\} - \{k\}$. Now choose b^k_l , $l = 0,\ldots,m$, $k = 0,\ldots,r$ algebraically independent over $K(\underline{a}^0,\ldots,\underline{a}^r)$. Then $\mathbb{P}^m_\Omega \times H(\underline{b}^k)$ meets

$$S^{-1}(i*j(V)) \cap \left(\bigcap_{l \in L_k} H(\underline{a}^l) \times \mathbb{P}^m_\Omega \right) \cap \left(\bigcap_{l \in \overline{L}_k} \mathbb{P}^n_\Omega \times H(\underline{b}^l) \right)$$

transversally for each $L_k \subset \{0,\dots,r\}-\{k\}$ and $\overline{L}_k = \{0,\dots,r\}-(\{k\}\cup L_k)$. But this means $\bigcap_{k=0}^{r} \left(H(\underline{a}^k)\times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) = \emptyset$ for the chosen $\underline{b}^0,\dots,\underline{b}^r$. □

Using again 8.7 we have already proved

$$\begin{aligned} & F_{i*j(V)}(S(\underline{a}^0,\underline{b}^0),S(\underline{a}^1,\underline{b}^1),\dots,S(\underline{a}^r,\underline{b}^r)) = 0 \quad \forall \underline{b}^0,\dots,\underline{b}^r \in \mathbb{P}^m_\Omega \\ \iff\ & F_{i(V)}(\underline{a}^0,\dots,\underline{a}^r) = 0 \end{aligned}$$

Summarizing the first step we have proved the following proposition:

Proposition 8.9. *Let $u^s_k, v^s_l \in \Omega$ be algebraically independent over K, $s = 0,\dots,r$, $k = 0,\dots,n$, $l = 0,\dots,m$. For a r-dimensional cycle V we have*

$$F_{i*j(V)}(S(\underline{u}^0,\underline{v}^0),\dots,S(\underline{u}^r,\underline{v}^r)) = F_{i(V)}(\underline{u}^0,\dots,\underline{u}^r)^t G(\underline{u}^0,\dots,\underline{u}^r,\underline{v}^0,\dots,\underline{v}^r)$$

for some $t \in \mathbb{N}^*$, and $G \in K[\underline{u}^0,\dots,\underline{u}^r,\underline{v}^0,\dots,\underline{v}^r]$ contains no factor in $K[\underline{u}^0,\dots,\underline{u}^r]$.

If we could check $t = 1$, the map $(pr_1)_*$ would be given pointwise by determining a unique factor of the evaluated Chow form. To prove $t = 1$ it suffices to show that $H(S(\underline{u},\underline{v}))$ and $i*j(V)$ meet transversally.

First we will check that $H(S(\underline{u},\underline{v}))$ meets $S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ transversally in all "sufficiently general" points of $H(S(\underline{u},\underline{v})) \cap S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$. If $S(\underline{p},\underline{q})$ is an element of the intersection, then we know by 8.6 that $\underline{p} \in H(\underline{u})$ or $\underline{q} \in H(\underline{v})$. By a sufficiently general point we think of a point $S(\underline{p},\underline{q})$ of the intersection satisfying the following condition: In case $\underline{p} \in H(\underline{u})$ the elements $v_0,\dots,v_m$ are still algebraically independent over $K(\underline{p})$. Or symmetrically in case $\underline{q} \in H(\underline{v})$ the $u_0,\dots,u_n$ are algebraically independent over $K(\underline{q})$.

Lemma 8.10. *Let $u_0,\dots,u_n,v_0,\dots,v_m \in \Omega$ be algebraically independent over K. Suppose $S(\underline{p},\underline{q}) \in H(S(\underline{u},\underline{v})) \cap S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ such that $v_0,\dots,v_m$ are algebraically independent over $\overline{K(\underline{q})}$. Then $H(S(\underline{u},\underline{v}))$ meets $S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ transversally in $S(\underline{p},\underline{q})$.*

Proof. Without loss of generality let $p_0 q_0 \neq 0$.
Let $f_{i,j} := z_{0,0}z_{i,j} - z_{i,0}z_{0,j}$, $i = 1,\dots,n$, $j = 1,\dots,m$.

Assertion: The zero set of $A \cdot \underline{z}$, with

$$A := \left(\frac{\delta f_{i,j}}{\delta z_{k,l}} \left(S(\underline{p}, \underline{q}) \right) \right)_{(i,j),(k,l)} = \begin{pmatrix} \frac{\delta f_{1,1}}{\delta z_{0,0}}(S(\underline{p}, \underline{q})) & \cdots & \frac{\delta f_{1,1}}{\delta z_{n,m}}(S(\underline{p}, \underline{q})) \\ \vdots & & \vdots \\ \frac{\delta f_{n,m}}{\delta z_{0,0}}(S(\underline{p}, \underline{q})) & \cdots & \frac{\delta f_{n,m}}{\delta z_{n,m}}(S(\underline{p}, \underline{q})) \end{pmatrix}$$

is the tangent space of $S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ in $S(\underline{p}, \underline{q})$.

To check this assertion we need A to have rank nm. We look at the nm columns (k,l), where $k = 1, \ldots, n$, $l = 1, \ldots, m$. Then we obtain the following submatrix of A:

$$\begin{pmatrix} p_0 q_0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_0 q_0 \end{pmatrix}$$

which has rank nm.

For the transversality of the intersection it now remains to prove that adding the row $(u_0 v_0, \ldots, u_n v_m)$ to A enlarges the rank to $nm + 1$. We look at the columns (k,l) where $k = 1, \ldots, n$, $l = 1, \ldots, m$ and $(n, 0)$:

$$B := \begin{pmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_{n-1} v_m & u_n v_1 & \cdots & \cdots & u_n v_m & u_n v_0 \\ p_0 q_0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & & & \vdots \\ \vdots & & \ddots & p_0 q_0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & 0 & p_0 q_0 & 0 & \cdots & 0 & -p_0 q_1 \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \vdots & -p_0 q_2 \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & p_0 q_0 & -p_0 q_m \end{pmatrix}$$

Now it suffices to show $\det B \neq 0$:
Solving this determinant along the rows $2, \ldots, m(n-1)+1$ leads us to

$$\det B = -(p_0q_0)^{m(n-1)} \cdot \det \begin{pmatrix} u_nv_1 & \cdots & \cdots & u_nv_m & u_nv_0 \\ p_0q_0 & 0 & \cdots & 0 & -p_0q_1 \\ 0 & \ddots & \ddots & \vdots & -p_0q_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & p_0q_0 & -p_0q_m \end{pmatrix}$$

$$= -(p_0q_0)^{m(n-1)}u_np_0^m \cdot \det \begin{pmatrix} v_1 & \cdots & \cdots & v_m & v_0 \\ q_0 & 0 & \cdots & 0 & -q_1 \\ 0 & \ddots & \ddots & \vdots & -q_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & q_0 & -q_m \end{pmatrix}$$

$$\neq 0$$

because $v_0, \ldots v_m$ are algebraically independent over $K(\underline{q})$. □

Now we can prove the multiplicity t in 8.9 to be actually one:

Theorem 8.11. *Let X be a projective variety together with two embeddings $i : X \hookrightarrow \mathbb{P}^n_\Omega$ and $j : X \hookrightarrow \mathbb{P}^m_\Omega$ and V a r-dimensional cycle on X. Further let $u^s_k, v^s_l \in \Omega$ be algebraically independent over K, $s = 0, \ldots, r$, $k = 0, \ldots, n$, $l = 0, \ldots, m$. Then we have*

$$F_{i*j(V)}(S(\underline{u}^0, \underline{v}^0), S(\underline{u}^1, \underline{v}^1), \ldots, S(\underline{u}^r, \underline{v}^r)) = F_{i(V)}(\underline{u}^0, \ldots, \underline{u}^r) \cdot G(\underline{u}^0, \ldots, \underline{u}^r, \underline{v}^0, \ldots, \underline{v}^r)$$

and G contains no factor in $K[\underline{u}^0, \ldots, \underline{u}^r]$.

Proof. $i * j(V) \cap H(S(\underline{u}^1, \underline{v}^1)) \cap \cdots \cap H(S(\underline{u}^r, \underline{v}^r))$ consists of finitely many points. It suffices to check transversality of the intersection in one of these points which we denote $S(\underline{p}, \underline{q})$. During the proof we have to use transcendence degrees. To keep the argument transparent we work in fixed affine subsets of $i(X), j(X)$. Therefore let $\underline{p} \in \Omega^{n+1}$ and $\underline{q} \in \Omega^{m+1}$ such that $p_i = 1$ and $q_j = 1$ for one $i = 0, \ldots, n$ and $j = 0, \ldots, m$.
Assertion: Let $\{l_1, \ldots, l_r\}$ be an order of $\{1, \ldots, r\}$ such that

$$\underline{p} \in i(V) \cap \bigcap_{t=1}^{k} H(\underline{u}^{l_t}) \quad \text{and} \quad \underline{q} \in j(V) \cap \bigcap_{t=k+1}^{r} H(\underline{v}^{l_t})$$

then the coefficients of $\underline{u}^{k+1}, \ldots, \underline{u}^r, \underline{v}^1, \ldots, \underline{v}^k$ are algebraically independent over $K(\underline{u}^1, \ldots, \underline{u}^k, \underline{v}^{k+1}, \ldots, \underline{v}^r, \underline{p}, \underline{q})$.

To prove this assertion we use again 8.6. It says

$$\underline{p} \in i(V) \cap \bigcap_{t=1}^{k} H(\underline{u}^{l_t}) \text{ and } \underline{q} \in j(V) \cap \bigcap_{t=k+1}^{r} H(\underline{v}^{l_t})$$

for $S(\underline{p}, \underline{q}) \in i * j(V) \cap H(S(\underline{u}^t, \underline{v}^t))$ for all $t \in \{1, \ldots, r\}$.
Since $H(\underline{u}^{l_1}), \ldots, H(\underline{u}^{l_k})$ meet $i(V)$ transversally we know

$$\text{transdeg } \left[K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})(\underline{p}) : K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})\right] \leq r - k \, .$$

Analogously transdeg $\left[K(\underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})(\underline{q}) : K(\underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})\right] \leq k$.
The K-morphism $i \circ j^{-1}$ maps $\underline{q}$ to $\underline{p}$ for $S(\underline{p}, \underline{q}) \in i * j(V)$. That means $q_0, \ldots, q_m$ are expressible as fractions of homogeneous polynomials in $p_0, \ldots, p_n$ with coefficients in K and vice versa. Therefore we obtain also for $\underline{q}$

$$\text{transdeg } \left[K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})(\underline{q}) : K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})\right] \leq r - k \, .$$

This implies $\underline{q}$ is a general point of a subvariety of $j(V)$ of dimension at most $r - k$ over the ground field $K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})$. Since the coefficients of $\underline{v}^{k+1}, \ldots, \underline{v}^r$ are algebraically independent over $K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})$ the intersection of $\bigcap_{t=k+1}^{r} H(\underline{v}^t)$ and this subvariety has dimension at most 0. Hence

$$\text{transdeg } \left[K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k}, \underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})(\underline{q}) : K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k}, \underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})\right] = 0.$$

But now the coordinates of $\underline{u}^{k+1}, \ldots, \underline{u}^r, \underline{v}^1, \ldots, \underline{v}^k$ must be algebraically independent over $K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k}, \underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})(\underline{p}, \underline{q})$. This proves the assertion.

For the proof of transversality of $i * j(V)$ and $\bigcap_{t=1}^{r} H(S(\underline{u}^t, \underline{v}^t))$ in $S(\underline{p}, \underline{q})$ it now suffices to check the transversality in case $r = 1$ because $i * j(V) \cap H(S(\underline{u}^1, \underline{v}^1))$ is again a variety but over a larger field $K(\underline{u}^1, \underline{v}^1)$, which allows us to repeat the argument for the intersection with $H(S(\underline{u}^r, \underline{v}^r))$ for $r > 1$.

For simplicity we may now assume $\underline{u} = \underline{u}^1$, $\underline{v} = \underline{v}^1$ and $\underline{p} \in H(\underline{u})$ which implies $v_0, \ldots, v_m$ to be algebraically independent over $K(\underline{q})$ as seen before.

We want to determine a line in the tangent space of $i * j(V)$ which meets $H(S(\underline{u}, \underline{v}))$ transversally.

Since $H(\underline{u})$ meets $i(V)$ transversally there is a line $g \subset T_{i(V),\underline{p}}$ in the tangent space of $i(V)$ in $\underline{p}$ which meets the tangent space $T_{H(\underline{u}),\underline{p}} = H(\underline{u})$ transversally. But this implies

$$T_{S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} = T_{S(g\times\{\underline{q}\}),S(\underline{p},\underline{q})} \oplus T_{S(H(\underline{u})\times\mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} \ .$$

By 8.6 we know $S(H(\underline{u}) \times \mathbb{P}^m_\Omega) \subset H(S(\underline{u},\underline{v}))$. On tangent space level this says

$$T_{S(H(\underline{u})\times\mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} \subset T_{H(S(\underline{u},\underline{v})),S(\underline{p},\underline{q})} = H(S(\underline{u},\underline{v})) \ .$$

On the other hand we can apply Lemma 8.10 which says

$$T_{S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} \cap H(S(\underline{u},\underline{v})) \neq T_{S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} \ .$$

So we conclude $T_{S(g\times\{\underline{q}\}),S(\underline{p},\underline{q})}$ and $H(S(\underline{u},\underline{v}))$ intersect transversally. Since $T_{S(g\times\{\underline{q}\}),S(\underline{p},\underline{q})} \subset T_{i*j(V),S(\underline{p},\underline{q})}$ the transversality of $i*j(V)$ and $H(S(\underline{u},\underline{v}))$ is shown and the theorem proved. □

Reformulating the theorem we obtain

Corollary 8.12. *Let V be a r–dimensional cycle in X and $F_{i*j(V)}$ the Chow form of $i*j(V)$ of multidegree d.*

$$F_{i*j(V)}(S(\underline{u}^1,\underline{v}^1),\ldots,S(\underline{u}^r,\underline{v}^r)) = \sum_{k=0}^{M} A_k(\underline{u}^0,\ldots,\underline{u}^r)M_k(v) \ ,$$

*where $M = \binom{m+d}{d}^{r+1} - 1$ and $M_k(v)$ are the multihomogeneous monomials in $\underline{v}^0,\ldots,\underline{v}^r$ of multidegree d. In this situation $F_{i(V)}$ is the **g**reates **c**ommon **d**ivisor of $A_0,\ldots,A_M$.*

The coefficients of $(A_0,\ldots,A_M)$ are exactly the coefficients of the form $F_{i*j(V)}(S(\underline{u}^1,\underline{v}^1),\ldots,S(\underline{u}^r,\underline{v}^r)) = \sum_{k=0}^{M} A_k(\underline{u}^0,\ldots,\underline{u}^r)M_k(v)$, which coincide by definition with the Chow coordinates of $i*j(V)$ respectively with the coefficients of $F_{i*j(V)}$. Hence by Theorem 9.4, which says determination of gcd is an algebraic map in this special case, we get a morphism

$$\begin{array}{rccc} (pr_1)_* : & \mathcal{C}_r(X, i*j) & \longrightarrow & \mathcal{C}_r(X,i) \\ & \sum_k n_k(i*j)(V_k) & \longmapsto & \sum_k n_k i(V_k) \end{array}$$

of algebraic sets, which leads us (cp. p. 46) to an algebraic continuous homeomorphism

$$\mathcal{C}_r(X, i) \cong \mathcal{C}_r(X, j)$$

and Theorem 8.3 is proved.

In the next section the algebraicity of forming the greatest common divisor in the case at hand is shown.

9 Determination of GCDs

In the previous section we embedded a cycle diagonally into the Segre embedding to prove topological invariance of Chow varieties. We reformulated the problem of determining the original Chow form from the Chow form of the embedded cycle to the determination of greatest common divisors in a special situation.

In this section we want to prove that in this specific situation the determination of greatest common divisors is an algebraic map.

Notation 9.1. Let $\underline{u}^j = (u_0^j, \ldots, u_n^j)$ for $j = 0, \ldots, r$. Further let $A_k \in K[\underline{u}^0, \ldots, \underline{u}^r]$ be a multihomogeneous polynomial of multidegree d in $\underline{u}^0, \ldots, \underline{u}^r$ or $A_k = 0$; $k = 0, \ldots, M$. At least one of $A_0, \ldots, A_M$ should not be equal to 0 (Compare Cor. 8.12).

Notation 9.2. Let $N = \binom{n+d}{d}^{r+1} - 1$ and $A_k = \sum_{l=0}^{N} a_{k,l} M_l(u)$, where $M_l(u)$ are homogeneous monomials of multidegree d in $\underline{u}^0, \ldots, \underline{u}^r$, $a_{k,l} \in K$; $k = 0, \ldots, M$; $l = 0, \ldots, N$.

For the algebraicity of the determination of *gcd*s we have to interpret the different sets of polynomials as points in some projective space:

Notation 9.3. The coefficients $(a_{k,l})$ define a point in $\mathbb{P}_\Omega^a := \mathbb{P}_\Omega^{NM+N+M}$. In the situation of corollary 8.12 they are just the Chow coordinates of the diagonally embedded cycle $i * j(V)$.

Let $U \subset \mathbb{P}_\Omega^a$ denote the set of those points $(A_0, \ldots, A_M)$ where the greatest common divisor $gcd(A_0, \ldots, A_M)$ has degree $\widetilde{d}$. Further the multihomogeneous polynomials in $K[\underline{u}^0, \ldots, \underline{u}^r]$ of multidegree $\widetilde{d}$ correspond to points in $\mathbb{P}_\Omega^{\widetilde{N}}$, $\widetilde{N} = \binom{n+\widetilde{d}}{\widetilde{d}}^{r+1} - 1$. In the analogous way a set of $M+1$ multihomogeneous polynomials in $K[\underline{u}^0, \ldots, \underline{u}^r]$ of multidegree $d - \widetilde{d}$ corresponds to a point in $\mathbb{P}_\Omega^b$, where $b = (M+1)\binom{n+d-\widetilde{d}}{d-\widetilde{d}}^{r+1} - 1$.

To summarize the situation in a sloppy way:

$U \subset \mathbb{P}_\Omega^a$	the space of $A_0, \ldots, A_M$
$\mathbb{P}_\Omega^{\widetilde{N}}$	the space of the *gcd*s D
$\mathbb{P}_\Omega^b$	the space of the remaining factors $\frac{A_k}{D}$

Theorem 9.4. *The map* $gcd : U \to \mathbb{P}_\Omega^{\widetilde{N}}$ *which maps* $A_0, \ldots, A_M$ *to their greatest common divisor is algebraic.*

For the proof of this theorem we need some technical results. First observe that a locally closed subset of a (quasi) projective variety is the intersection of an open and a closed subset hence a quasi projective variety.

Lemma 9.5. *U is an irreducible, locally closed subset of $\mathbb{P}_\Omega^a$.*

Proof. The greatest common divisor of $A_0, \ldots, A_M$ has multidegree $\widetilde{d}$ if

1. There exist polynomials B_k such that $A_k B_l = A_l B_k$ for all $k, l = 0, \ldots, M$ and B_k has multidegree $d - \widetilde{d}$ if $A_k \neq 0$; $k = 0, \ldots, M$.

2. There exist no polynomials C_k such that $A_k C_l = A_l C_k$ for all $k, l = 0, \ldots, M$ and C_k has multidegree lower than $d - \widetilde{d}$ if $A_k \neq 0$; $k = 0, \ldots, M$.

In this situation the B_k are unique up to a common factor $c_k \in K$ and $A_k = c_k B_k D$, where $D = gcd(A_0, \ldots, A_M)$.

Let

$$\begin{array}{rccc} \varphi : & \mathbb{P}_\Omega^{\widetilde{N}} \times \mathbb{P}_\Omega^b & \longrightarrow & \mathbb{P}_\Omega^a \\ & (F, (G_0, \ldots, G_M)) & \longmapsto & (FG_0, \ldots, FG_M) \end{array}$$

where we think of $F, G_0, \ldots, G_M$ as multihomogeneous polynomials as in Notation 9.3.

Obviously φ is a regular map. Since $\mathbb{P}_\Omega^{\widetilde{N}} \times \mathbb{P}_\Omega^b$ is a projective variety, $Im(\varphi)$ is an algebraically closed [25, Thm 2, p. 57], irreducible set. Now the set U is the intersection of $Im(\varphi)$ and the open set defined by condition 2, which is the complement of a finite union of algebraically closed sets. Hence U is locally closed and irreducible. □

Condition 1 of the proof of Lemma 9.5 defines an algebraic subset of $U \times \mathbb{P}_\Omega^b$. We know for $(A_0, \ldots, A_M) \in U$ the polynomials $(B_0, \ldots, B_M) \in \mathbb{P}_\Omega^b$ are uniquely determined, i.e. we obtain a map $\psi : U \to U \times \mathbb{P}_\Omega^b$.

Lemma 9.6.

$$\begin{array}{rccc} \psi : & U & \longrightarrow & U \times \mathbb{P}_\Omega^b \\ & (A_0, \ldots, A_M) & \longmapsto & ((A_0, \ldots, A_M), (B_0, \ldots, B_M)) \end{array} \quad \textit{is algebraic.}$$

Proof. The B_k are uniquely determined by $B_kA_l = B_lA_k$ for all $k, l = 0, \ldots, M$. The polynomial B_kA_l is multihomogeneous of degree $2d - \widetilde{d}$ or equal to 0. If it is multihomogeneous we can check the condition $B_kA_l = B_lA_k$ in the coefficient of each monomial. Hence we get a finite set of polynomials $f_0, \ldots, f_Q$ (over $\mathbb{Z}/(char\ \Omega)$) which are homogeneous in the coordinates of the point $(A_0, \ldots, A_M) \in U$ of degree 1 and homogeneous in the coordinates of the point $(B_0, \ldots, B_M)$ of degree 1. If we now fix the point $(A_0, \ldots, A_M) \in U$ we obtain a system of linear equations in the coordinates of $(B_0, \ldots, B_M)$. All coefficients of this linear equations are 0 or coordinates of $(A_0, \ldots, A_M)$. Since $(B_0, \ldots, B_M)$ are uniquely determined we can apply Cramer's rule and obtain that each coordinate of $(B_0, \ldots, B_M)$ can locally be written as a fraction of the determinant of matrices in the coefficients of $(A_0, \ldots, A_M)$. □

Analogously to the proof of Lemma 9.6 it is seen that

$$\begin{array}{rccc} \Psi: & \psi(U) & \longrightarrow & \psi(U) \times \mathbb{P}_{\Omega}^{\widetilde{N}} \\ & ((A_0, \ldots, A_M), (B_0, \ldots, B_M)) & \longmapsto & ((A_0, \ldots, A_M), (B_0, \ldots, B_M), D) \end{array}$$

such that $A_k = cB_kD$ for fixed $c \in K$ and $k = 0, \ldots, M$ is algebraic. Now projection onto the second factor is again an algebraic map. Hence we obtain the morphism

$$gcd : U \xrightarrow{\psi} \psi(U) \xrightarrow{\Psi} \psi(U) \times \mathbb{P}_{\Omega}^{\widetilde{N}} \xrightarrow{pr_2} \mathbb{P}_{\Omega}^{\widetilde{N}}$$

such that $(A_0, \ldots, A_M)$ is mapped to $gcd(A_0, \ldots, A_M)$ and Theorem 9.4 is proved.

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Versicherung nach §3 Absatz 2a) der Promotionsordnung

Ich, Judith Plümer, versichere an Eides statt, daß ich diese Abhandlung selbständig und ohne unerlaubte Hilfe verfaßt und die benutzten Hilfsmittel vollständig angegeben habe.

Osnabrück, 11. Mai 1999